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Iterative Maximum Likelihood Locating Method Based on RSS Measurement

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Abstract

We present a localization technique based on RSS measurement. We apply iterative maximum (ML) likelihood method to the problem where ML estimator can not be computed directly. The location of a object can be estimated by "least squares" method in determining the node locations in ad-hoc sensor networks or a cellular location system where received signal strength (RSS) measurement model is employed. Furthermore We can use that initialized estimate as the "initial guess" in the iterative ML method. We show the iterative ML method outperforms least squares, and compare the performance to Cramer-Rao bound.

I. Introduction

In certain specific problems regarding locating system, network-based localization solution using received signal strength (RSS) is quite pertinent. As the prevalent examples, we can meet the problems as in determining the node locations in ad-hoc sensor networks [1], cellular locating system by networked base stations [2], and so forth. In the problem of ad-hoc wireless sensor networks, locating sensor node is an actively studied issue, and locating cellular phone using networked base stations is also actively studied in various aspects of methods. In the U.S., the Federal Communications Commission (FCC) mandated wireless service providers to locate mobile phones of emergency 911 dialers (the enhanced 911, E-911) in 1996. Several methods for locating object have been proposed and discussed recent years such as time of arrival (TOA) [3], time difference of arrival (TDOA) [4], angle of arrival (AOA) [5], and so on. All methods estimate the location of object (node or cellular handset) relating the "object" and "networked sensors or base stations". Least squares method is always a good choice of the solution for these methods because it does not require the knowledge of noise distribution and relatively simple and easy to apply, and also it works well unless the problem is non-linear structured problem. Therefore, linear least squares method is generally a good solution. Recently, other than location estimation methods mentioned above, received signal strength (RSS) measurement-based locating method attracts attention of researchers. This article also focuses on locating system based on RSS measurement model. RSS measurement already has been used in mobile assisted handoff procedure (IS-136, IS-54B). In this article, we provide an accurate solution for locating method based on RSS measurement. Other than RSS based locating method, every method requires more complicated process to estimate parameters such as the time delay or angle at the step before estimating distance or location of source target (sensor node or mobile station). RSS model is relatively quite simple to estimate the distance of source target compared to the other methods before we take the LS method step. We assume the one to one mapping between target source and sensor node or base station in order to apply RSS model which is achievable by a grid spacing of the field [6]. Once we are provided with LS method solution, we apply numerical ML method (Newton-Raphson method) to achieve a finer accuracy of estimate.

ML estimator is always popular practical estimator especially when we are not sure if minimum variance unbiased (MVU) estimator exist or not, or when it is impossible to find it even if it exists. However, sometimes it is not easy or possible to apply ML method because maximum likelihood function is difficult to compute or by some other reason. Least squares method is a very good choice for TOA, TDOA, AOA, and RSS method. However, ML method is not pertinent to all but RSS because the location of source target is not directly related with the measurement we receive in all methods except for RSS method. After estimating pre-parameter estimation of delay, direction, or distance, we can easily apply least squares method to all, but we can not apply ML method to all except for RSS method. RSS measurement model has direct relation between "location of source target" and "measurement" as it can be seen from (1) in Section II. Therefore, we take advantage of RSS measurement model to apply the most popular estimator. However, sometimes we need to use numerical ML method because it is not possible to compute maximum likelihood function directly as in the case of (1). Another problem of iterative ML method is finding the initial guess of the estimate, and that is the most important factor in iterative ML method. We can acquire a good "initial guess" by LS method (see Section III) to apply numerical ML method (see Section IV) so that combining LS and iterative ML results in very accurate locating estimate of the source target. We show the accuracy of the result of combining those two popular methods comparing with Cramer-Rao bound (see Section V) in this article.

II. RSS Measurement Model

Consider the situations either a mobile station observes N control channels transmitted from N base stations or distributed nodes with some "anchor" nodes of which positions with respect to a certain global coordinate system are known. Our approach of locating system can be applicable to these kind of problems. Anchor sensor nodes or Mobile station receive RSS measurement from the target sensor node or base stations. From here on, we consider and focus only one situation of locating mobile station, but the solution can be applicable to locating sensor nodes too in wireless sensor networks. We assume that one to one mapping between measurement and the base station is performed without error to apply our solution. Received signal strength is described in non-linear model as follows according to [7]:

$$y_n = 10\log_{10}\left(\frac{\Psi d_0^\alpha}{|\boldsymbol{r}_n - \boldsymbol{l}|^\alpha}\right) + v_n, \quad n = 1, 2, \ldots, N \tag{1}$$

where **l** is the location of a source target (here, mobile station), n is the base station index of which location is known, Ψ is the received power between the source and mobile station at the reference distance d_0, $\boldsymbol{r}$ is the location of base station, α is the attenuation factor ($\alpha \geq 1$), v is background zero-mean Gaussian noise, and N is the total number of base stations surrounding the mobile station. Therefore, the received signal strength depends on the distance between the mobile station (MS) and the base station (BS). We may solve this equation by maximum likelihood (ML) method, but it can not be solved directly. Even if we try to apply least squares method, the equation has to be modified to apply linear LS method (see Section III). We can still have option to apply ML method using iterative method such as “Newton-Raphson” method. In applying Newton-Raphson method, it can never be over-emphasized, the importance of the selection of the initial guess. Therefore, we use the solution of least squares (LS) method as the initial guess of iterative ML method for further improvement of locating system in this article.

From the received measurement, we can estimate or compute the distance (required for LS method) between MS and BS as follows when we assume d_0 is 1 m and α is 2:

$$|\boldsymbol{r}_n - \boldsymbol{l}|^2 = \Psi 10^{-(\frac{y_n}{10})} \tag{2}$$

The strength of RSS measurement drops very quickly as the distance increases according to (1). Because of that, we do not use information from the BSs that are located relatively far from the MS and does not transmit very good information because it takes big perturbation even by the small noise. We use 3 best measurement, which means 3 strongest received power when we apply LS method. Note that if (1) is linear and background noise is Gaussian, then LS method is the same as ML method [8].

III. Least Squares Method

We adopt *least squares* method to obtain initial guess of the ML iterative (Newton-Raphson) method for locating system. Least Squares method is widely used and produce a very good estimate close to optimum in many respects [9]. Least squares method is frequently applied in the literature regarding locating system for a long time [1], [10], [11]. We adopt it and apply it for initial guess of the ML iterative method with improved performance. We present details of LS method in locating mobile station using RSS measurement in this section.

A. Least Squares

From (2), theoretically, we need exactly 3 distance information between the MS and the BSs, and 3 circles that are found from the 3 distances are supposed to cross at one point each other without considering noise. However, we receive measurement with noise, and we apply least squares (LS) method to estimate the true location of MS and LS can be applied regardless of knowing noise information. We estimate 3 distances first from the noisy measurement according to (2), and they are denoted by r_A, r_B, and r_C respectively. If we denote the known locations of 3 base stations by $A(a_1, a_2)$, $B(b_1, b_2)$, and $C(c_1, c_2)$ respectively, least squares method is performed as follows.

In X and Y, cartesian coordinate, three circles are expressed as,

$$\begin{aligned}
(x-a_1)^2+(y-a_2)^2-{r_A}^2 &= x^2+y^2+2a_1x+2a_2y+{a_1}^2+{a_2}^2-{r_A}^2=0 \\
(x-b_1)^2+(y-b_2)^2-{r_B}^2 &= x^2+y^2+2b_1x+2a_2y+{b_1}^2+{b_2}^2-{r_B}^2=0 \\
(x-c_1)^2+(y-c_2)^2-{r_C}^2 &= x^2+y^2+2c_1x+2c_2y+{c_1}^2+{c_2}^2-{r_C}^2=0
\end{aligned}$$

After manipulating three equations, we have two linear equations as,

$$\begin{aligned}
{a_1}^2-{c_1}^2-2(a_1-c_1)x+{a_2}^2-{c_2}^2-2(a_2-c_2)y &= {r_A}^2-{r_c}^2 \\
{b_1}^2-{c_1}^2-2(b_1-c_1)x+{b_2}^2-{c_2}^2-2(b_2-c_2)y &= {r_B}^2-{r_c}^2
\end{aligned}$$

Linear least squares [9] solves these linear equations as follows:

$$H\mathbf{x}=d, \text{ then } \quad \hat{\mathbf{x}}=(H^\top H)^{-1}H^\top d. \tag{3}$$

where

$$H=\begin{bmatrix} 2(a_1-c_1) & 2(a_2-c_2) \\ 2(b_1-c_1) & 2(b_2-c_2) \end{bmatrix}, \; d=\begin{bmatrix} {a_1}^2-{c_1}^2+{a_2}^2-{c_2}^2+{r_C}^2-{r_A}^2 \\ {b_1}^2-{c_1}^2+{b_2}^2-{c_2}^2+{r_C}^2-{r_B}^2 \end{bmatrix}, \text{ and } \mathbf{x}=[x \;\; y]^\top.$$

When the distances are estimated from the measurement data, least squares find the point which gives the least sum of differences between the function of data and function of estimated point. When H is a singular matrix, there is not a solution or the solution will be imaginary. We may use more data measurement to solve more dimensional linear equations. However, in our RSS measurement model, the received power at the MS that are very far from the BS is not that good quality of measurement. Therefore we use only 3 best measurement, which means we use 3 strongest measurement received for this initial guess of the iterative ML method.

IV. Maximum Likelihood Approach

Maximum likelihood estimator (MLE) is the most popular practical estimator as a alternative to the minimum variance unbiased (MVU) estimator regardless of its existence or not for a given problem. However, sometimes it is very difficult to find the MLE from the maximum likelihood (ML) function, or sometimes it is even more difficult to find the ML function itself depending on the complexity of the problems. In those situations, we can approach in the way of numerical determination of MLE, e.g., Newton-Raphson or scoring method when we can find at least ML function, and expectation-maximization (EM) algorithm is appropriate when it is very difficult even to find ML function itself. From (1), we find that we can not find the solution which maximizes likelihood function directly. Therefore, we have to use numerical approach to perform the ML method. We derive Newton-Raphson [12] method to apply ML estimator following.

A. Newton-Raphson Method Initialized by Least Squares Method

From (1), the received measurement from the BS n is

$$y_n = 10\log_{10}\left(\frac{\Psi d_0^{\alpha}}{|\boldsymbol{r}_n - \boldsymbol{l}|^{\alpha}}\right) + v_n \tag{4}$$

To reduce the complexity of derivation, we set $\Psi d_0^{\alpha} = C$, $\alpha = 2$, and $|\boldsymbol{r}_n - \boldsymbol{l}|^{\alpha} = \boldsymbol{D}_n$. Then, the log likelihood function will be

$$\ln p_y(\boldsymbol{y};\boldsymbol{l}) = \ln\left\{\frac{1}{(2\pi\sigma^2)^{N/2}}\exp\left\{-\frac{1}{2\sigma^2}\sum\left[y_n - 10\log 10\left(\frac{C}{\boldsymbol{D}_n}\right)\right]^2\right\}\right\} \tag{5}$$

where $\boldsymbol{y} = \{y_1, y_2, y_3\}$ because we use only 3 measurement, and it can be rewritten as

$$K - \frac{1}{2\sigma^2}\sum\left[y_n - 10\log_{10}\left(\frac{C}{\boldsymbol{D}_n}\right)\right]^2 \tag{6}$$

where K is a constant which does not depend on the parameter we want to estimate, and we can rewrite it as

$$g(\boldsymbol{y}) + \frac{1}{\sigma^2}\sum\left\{y_n 10\log\left(\frac{C}{\boldsymbol{D}_n} - \frac{1}{2}\left[10\log_{10}\left(\frac{C}{\boldsymbol{D}_n}\right)\right]\right)\right\} \tag{7}$$

where $g(\boldsymbol{y})$ is a function of y, and parameter does not depend on it. Therefore,

$$\ln p_y(\boldsymbol{y};\boldsymbol{l}) = g(\boldsymbol{y}) + \frac{1}{\sigma^2}\sum(A) \tag{8}$$

where A is defined as

$$\sum(A) \triangleq \boldsymbol{d}^{\top}\boldsymbol{y} - \frac{1}{2}\boldsymbol{d}^{\top}\boldsymbol{d} \tag{9}$$

, and

$$\boldsymbol{d} = \left[10\log_{10}\left(\frac{C}{\boldsymbol{D}_1}\right)\;\; 10\log_{10}\left(\frac{C}{\boldsymbol{D}_2}\right)\;\; 10\log_{10}\left(\frac{C}{\boldsymbol{D}_3}\right)\right]^{\top} \tag{10}$$

Therefore,

$$\ln p_y(\boldsymbol{y};\boldsymbol{l}) = g(\boldsymbol{y}) + \frac{1}{\sigma^2}\sum\left(\boldsymbol{d}^{\top}\boldsymbol{y} - \frac{1}{2}\boldsymbol{d}^{\top}\boldsymbol{d}\right) \tag{11}$$

and maximizing likelihood function is the same as maximizing $\boldsymbol{d}^{\top}\boldsymbol{y} - \frac{1}{2}\boldsymbol{d}^{\top}\boldsymbol{d}$. In order to solve

$$\frac{\partial \ln p(\boldsymbol{y};\boldsymbol{l})}{\partial \boldsymbol{l}} = 0, \;\; \Rightarrow \frac{\partial}{\partial \boldsymbol{l}}\left(\boldsymbol{d}^{\top}\boldsymbol{y} - \frac{1}{2}\boldsymbol{d}^{\top}\boldsymbol{d}\right) = 0, \tag{12}$$

We need to apply iterative method to find $\boldsymbol{l}$ that satisfies (12) as follows,

$$\boldsymbol{l}_{k+1} = \boldsymbol{l}_k - \left[\frac{\partial^2}{\partial \boldsymbol{l}^2}\left(\boldsymbol{d}^{\top}\boldsymbol{y} - \frac{1}{2}\boldsymbol{d}^{\top}\boldsymbol{d}\right)\right]^{-1}\frac{\partial}{\partial \boldsymbol{l}}\left(\boldsymbol{d}^{\top}\boldsymbol{y} - \frac{1}{2}\boldsymbol{d}^{\top}\boldsymbol{d}\right)\Bigg|_{\boldsymbol{l}=\boldsymbol{l}_k}. \tag{13}$$

Before we proceed, let us make some abbreviations as follows,

$$\frac{1}{|\boldsymbol{r}_n - \boldsymbol{l}|^2} = \frac{1}{(r_{nx}-l_x)^2 + (r_{ny}-l_y)^2} = \frac{1}{\boldsymbol{D}_n} = R_n \tag{14}$$

$$X_n = r_{nx} - l_x, \;\; Y_n = r_{ny} - l_y \tag{15}$$

$$\boldsymbol{Q} \triangleq \boldsymbol{d}^\top \boldsymbol{y} - \frac{1}{2}\boldsymbol{d}^\top \boldsymbol{d} \tag{16}$$

Then, $\boldsymbol{d}^\top \boldsymbol{y} - \frac{1}{2}\boldsymbol{d}^\top \boldsymbol{d}$ can be modified as follows,

$$\boldsymbol{Q} = \sum y_n 10\log_{10}\left(\frac{C}{\boldsymbol{D}_n}\right) - \frac{1}{2}\sum\left[10\log_{10}\left(\frac{C}{\boldsymbol{D}_n}\right)\right]^2 \tag{17}$$

$$= \sum y_n 10\log_{10}(R_n C) - \frac{1}{2}\sum\left[10\log_{10}(R_n C)\right]^2 \tag{18}$$

Then, the first and second derivatives of $\boldsymbol{Q}$ can be computed as follows,

$$\frac{\partial \boldsymbol{Q}}{\partial l_x} = \sum\left(y_n \frac{\partial R_n}{\partial l_x}\frac{1}{R_n}\right) - \sum\left[10\log_{10}(R_n C)\frac{\partial R_n}{\partial l_x}\frac{1}{R_n}\right] \tag{19}$$

$$= 2\sum(y_n X_n R_n) - 20\sum\left[\log_{10}(R_n C)\cdot X_n R_n\right]. \tag{20}$$

Similarly, we can compute first derivative with respect to y coordinate,

$$\frac{\partial \boldsymbol{Q}}{\partial l_y} = 2\sum(y_n Y_n R_n) - 20\sum\left[\log_{10}(R_n C)\cdot Y_n R_n\right]. \tag{21}$$

Next, the second derivative can be computed as follows,

$$\frac{\partial \boldsymbol{Q}}{\partial {l_x}^2} = \frac{\partial}{\partial l_x}\left\{2\sum(X_n R_n y_n) - 20\sum\left[X_n R_n \cdot \log_{10}(R_n C)\right]\right\} \tag{22}$$

$$= 2\sum\left[\frac{\partial}{\partial l_x}(X_n R_n y_n)\right] - 20\sum\left\{\frac{\partial}{\partial l_x}\left[X_n R_n \cdot \log_{10}(R_n C)\right]\right\} \tag{23}$$

The first and second term can be computed as follows respectively,

$$\frac{\partial}{\partial l_x}(X_n R_n y_n) = \frac{\partial X_n}{\partial l_x}(R_n y_n) + X_n\left(y_n \frac{\partial R_n}{\partial l_x}\right) \tag{24}$$

$$= -R_n y_n + X_n \cdot 2X_n {R_n}^2 y_n = R_n y_n\left(2{X_n}^2 R_n - 1\right) \tag{25}$$

and the second term can be computed as,

$$\frac{\partial}{\partial l_x}\left[X_n R_n \cdot \log_{10}(R_n C)\right] = \frac{\partial}{\partial l_x}(X_n R_n)\cdot\left[\log_{10}(R_n C)\right] + (X_n R_n)\frac{\partial R_n}{\partial l_x}\frac{1}{R_n}$$

$$= R_n\left(2{X_n}^2 R_n - 1\right)\cdot\left[\log_{10}(R_n C)\right] + 2{X_n}^2 {R_n}^2.$$

Therefore,

$$\frac{\partial \boldsymbol{Q}}{\partial {l_x}^2} = 2\sum\left[R_n y_n\left(2{X_n}^2 R_n - 1\right)\right] - 20\sum\left\{R_n\left(2{X_n}^2 R_n - 1\right)\cdot\left[\log_{10}(R_n C)\right] + 2{X_n}^2 {R_n}^2\right\}. \tag{26}$$

Similarly,

$$\frac{\partial \boldsymbol{Q}}{\partial {l_y}^2} = 2\sum \left[R_n y_n \left(2{Y_n}^2 R_n - 1\right)\right] - 20\sum \left\{R_n \left(2{Y_n}^2 R_n - 1\right) \cdot \left[\log_{10}\left(R_n C\right)\right] + 2{Y_n}^2 {R_n}^2\right\} \tag{27}$$

where we used

$$\frac{\partial R_n}{\partial l_x} = \frac{2(r_{nx} - l_x)}{\left[(r_{nx} - l_x)^2 + (r_{ny} - l_y)^2\right]^2} = 2X_n {R_n}^2, \tag{28}$$

$$\frac{\partial R_n}{\partial l_y} = \frac{2(r_{ny} - l_y)}{\left[(r_{nx} - l_x)^2 + (r_{ny} - l_y)^2\right]^2} = 2Y_n {R_n}^2. \tag{29}$$

We use the the initial guess of $\boldsymbol{l}_1$ from the solution of "least squares method" using 3 measurement (see Section III).

V. Simulations

In this section, we show a simple simulation where we apply LS method at the first step, and apply ML method using the solution of LS method. The Cramer-Rao bound (CRB) is derived in the appendix and we compare the bound with the two methods. When we apply iterative method, we have to be very careful because it can diverge sometimes and it can be fatal error especially when the second derivative of the log-liklihood function is small [13]. Therefore, in the simulation, we set the threshold for the second derivative so that when inverse of derivative is larger than the threshold, it stops the iteration and plug in the estimate the same as the LS estimate. We chose 10 as the iteration number. The 2-dimension locating system field is depicted in Fig. 1. The reference power is selected as $10,000$ [J/s], the source target (MS or sensor node) is located at (20,

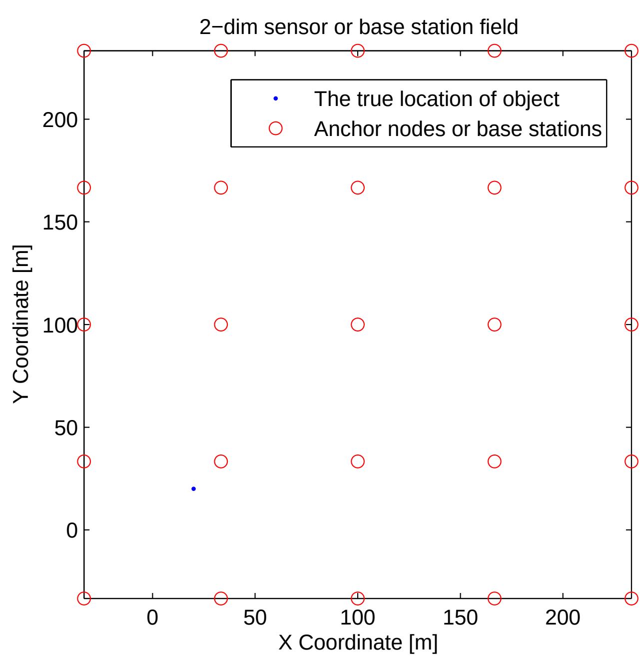
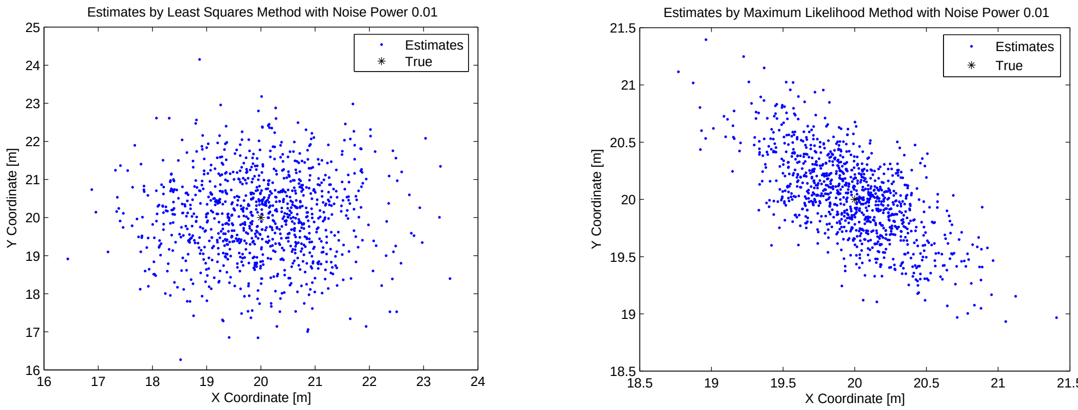


Fig. 1. 2-dimensional field of locating system.

20), the number of networked sensor nodes or BSs is 5×5, the number of simulation runs is 1000, and background noises are chosen as 3 different values (0.01, 0.1, and 0.2). If we compare each pair of the result in Fig. 2, 3, and 4, ML method

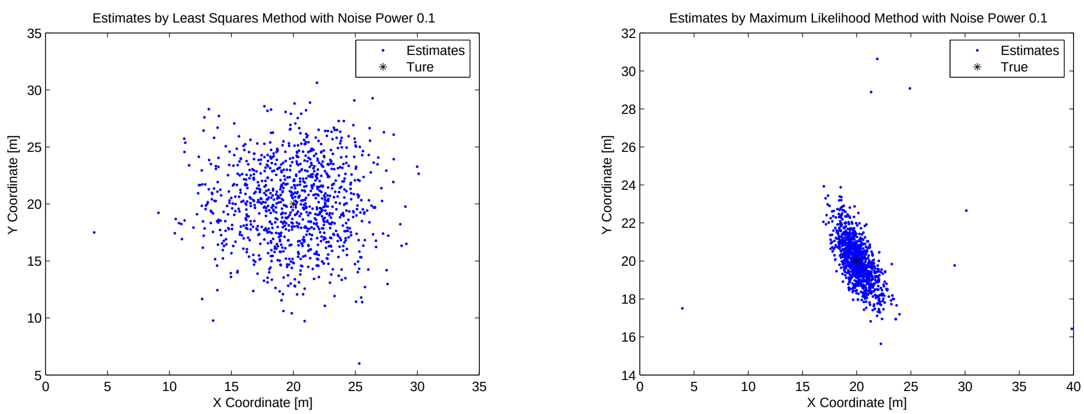


(a) By LS method

(b) By iterative ML method

Fig. 2. Estimated positions of the target when noise power is 0.01 dB

compresses the distribution of LS estimates. Note some ML iterative estimates that have relatively larger errors. Because the ML iterative estimates diverge sometimes, we stop the iteration and give the LS estimate in that case. As the noise power

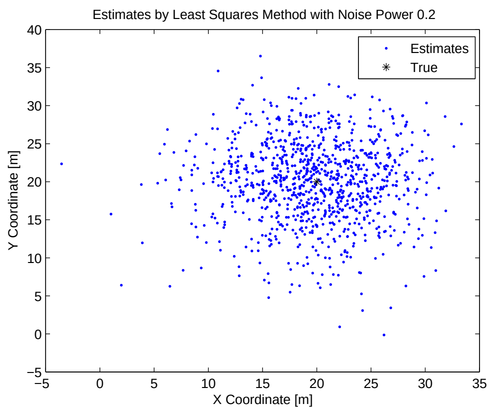


(a) By LS method

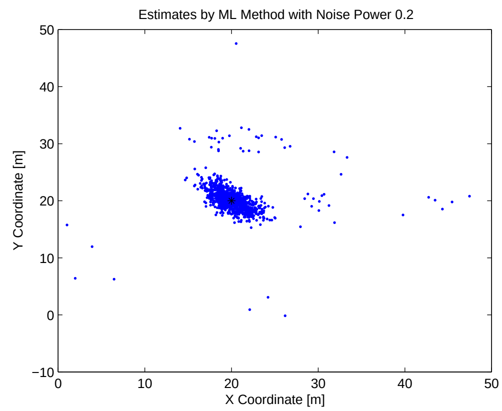
(b) By iterative ML method

Fig. 3. Estimated positions of the target when noise power is 0.1 dB

increases, the results show that, even though ML iterative method provides better performance clearly, the worst estimates in both methods do not seem to be very different, so the overall sizes of the both field occupied by estimates are almost the same. Fig. 5 shows the mean of distance error by two methods. As it shows, ML iterative methods well outperforms LS method. In Fig. 6 and 7, the performances of two methods are compared to Cramer-Rao bound. When the noise power is low (0.01 dB), ML iterative method performs almost the same as CRB. The exact value can be seen in Table I where ML iterative outperforms the lower bound of unbiased estimator. Nevertheless, as the noise power increases, the performances of two methods are by more falling down compared to the CRB.



(a) By LS method



(b) By iterative ML method

Fig. 4. Estimated positions of the target when noise power is 0.2 dB

Noise Power [dB]	MED-LS [m]	MED-ML [m]	CRB-X [m^2]	CRB-Y [m^2]	Var-LS-X [m^2]	Var-LS-Y [m^2]	Var-ML-X [m^2]	Var-ML-Y [m^2]	SNR [dB]
0.01	1.38	0.44	0.16	0.16	1.19	1.25	0.13	0.14	38.38
0.1	4.52	1.45	1.58	1.58	12.79	12.88	2.12	1.66	28.38
0.2	14.19	10.57	16.77	16.76	138.21	131.39	111.14	99.70	18.29

TABLE I

THE RESULT OF SIMULATION, MED: MEAN ERROR DISTANCE, CRB: CRAMER-RAO BOUND, X: X COORDINATE, Y: Y COORDINATE, LS: LEAST SQUARES, ML: MAXIMUM LIKELIHOOD.

VI. CONCLUSIONS

In this article, we showed the combined technique of least squares method and iterative maximum likelihood method for locating any source target in RSS measurement based system. The technique can be applied to the prevalent examples of applications such as “locating mobile station in wireless cellular system” or “locating sensor node in ad-hoc wireless sensor networks”. RSS measurement model has an advantage for us to apply maximum likelihood method because its measurement model has direct relationship between location and measurement contrary to other methods, e.g., TOA, TDOA, and AOA. The practical RSS measurement models usually does not allow for us to apply direct maximum likelihood method because it is impossible to compute the maximum likelihood function from the model. That is why we combine two popular statistical estimators so that least squares method works for the initial guess of the Newton-Raphson method in order to apply numerical ML method. Contrary to other methods, RSS measurement model requires simple and quick step for the process to acquire the estimated parameter before we apply least squares method. Therefore, adding up additional process of numerical ML method

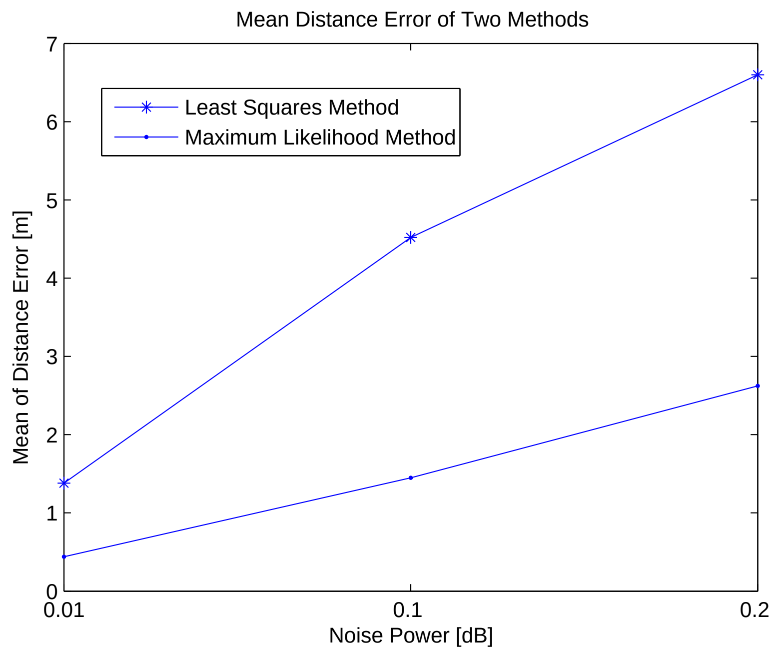


Fig. 5. Mean of distance error by two methods.

does not degrade the overall expense of the process in using RSS measurement system for localizing a source target. We compared the the result of "LS method-only" to "iterative ML method beyond LS method" along with Cramer-Rao bound. When background noise power is very small, ML method outperforms even CRB shown in the simulation result. This article presented only general idea and in rather theoretical respect of locating technique. In the future, the method presented in this paper can be implemented into direct practical stage.

In multi-target tracking system, initializing the location of target is a crucial issue. The locating technique presented in this article can be applied to the initialization of the target. However, iterative ML method beyond LS method may not be effective if particle filtering is employed in the tracking system [14].

APPENDIX

We derive the CRB of the estimator of the parameter, the location of a "source target" when we use only 3 measurement that forms right triangle in this section. The parameter is denoted by $\boldsymbol{\theta} = \mathbf{l} = [x\ y]^\top$. From (1), the likelihood function is

$$p(\mathbf{y};\boldsymbol{\theta}) = \left(\frac{1}{\sqrt{2\pi\sigma^2}}\right)^3 \exp\left\{-\frac{1}{2\sigma^2}\sum_{n=1}^{3}\left[y_n - f_n(\boldsymbol{\theta})\right]\right\}$$

where

$$f_n(\boldsymbol{\theta}) = 10\log_{10}\left[\frac{\Psi}{g_n(\boldsymbol{\theta})}\right] \tag{30}$$

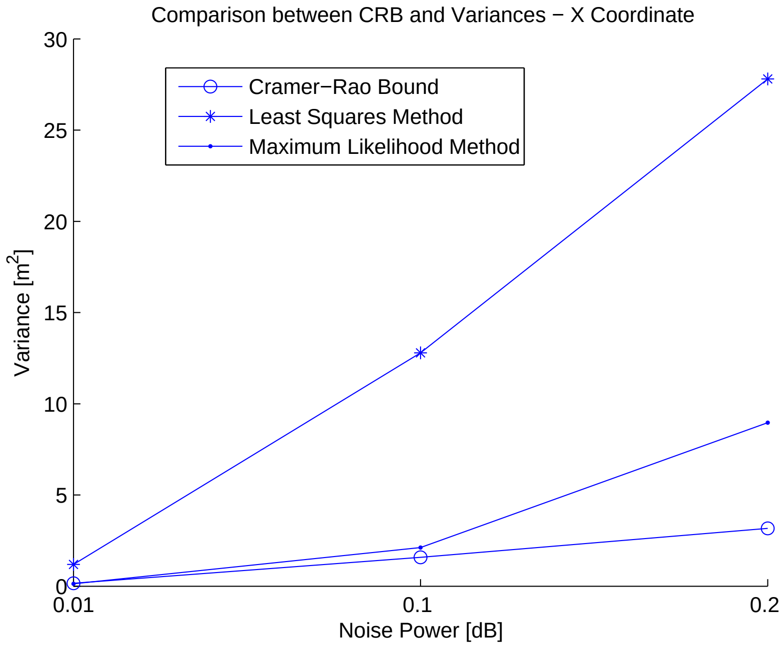


Fig. 6. Comparison between CRB and variances by two methods - X coordinate.

and

$$g_n(\boldsymbol{\theta}) = g_n(x,y) = |\mathbf{s}_n - \mathbf{l}|^{\alpha} = (s_{n_x} - x)^2 + (s_{n_y} - y)^2. \tag{31}$$

The log-likelihood function is,

$$\ln p(\mathbf{y};\boldsymbol{\theta}) = \ln\left(\frac{1}{\sqrt{2\pi\sigma^2}}\right)^3 + \left[-\frac{1}{2\sigma^2}\sum_{n=1}^{3}(y_n - f_n)\right] \tag{32}$$

from which the derivative of x coordinate follows as

$$\frac{\partial \ln p}{\partial x} = \frac{\partial}{\partial x}\left[-\frac{1}{2\sigma^2}\sum_{n=1}^{3}(y_n - f_n)^2\right] = -\frac{1}{2\sigma^2}\sum_{n=1}^{3}\underbrace{\left\{\frac{\partial}{\partial x}\left[(y_n - f_n)^2\right]\right\}}_{\mathcal{A}}. \tag{33}$$

From $\mathcal{A}$,

$$\mathcal{A} = \frac{\partial}{\partial x}\left[(y_n - f_n)^2\right] = 2\left[y_n - f_n(\boldsymbol{\theta})\right]\underbrace{\left[-\frac{\partial f_n(x,y)}{\partial x}\right]}_{\mathcal{B}}. \tag{34}$$

From $\mathcal{B}$,

$$\begin{aligned}
\mathcal{B} &= \frac{\partial f_n(x,y)}{\partial x} = \frac{\partial}{\partial x}\left\{10\log_{10}\left[\frac{\Psi}{g_n(x,y)}\right]\right\} && (35)\\
&= \frac{\partial}{\partial x}\left[10\log_{10}\Psi - 10\log_{10} g_n(x,y)\right] = \frac{\partial}{\partial x}\left[10\log_{10} g_n(x,y)\right] && (36)\\
&= -10\frac{\partial}{\partial x}\left[\log_{10} g_n(x,y)\right] = -\frac{10}{\ln 10}\frac{\left[\partial g_n(x,y)/\partial x\right]}{g_n(x,y)} && (37)\\
&= \frac{20}{\ln 10}\frac{(s_{n_x} - x)}{g_n}\quad\left(\because \frac{\partial g_n}{\partial x} = -2(s_{n_x} - x)\right). && (38)
\end{aligned}$$

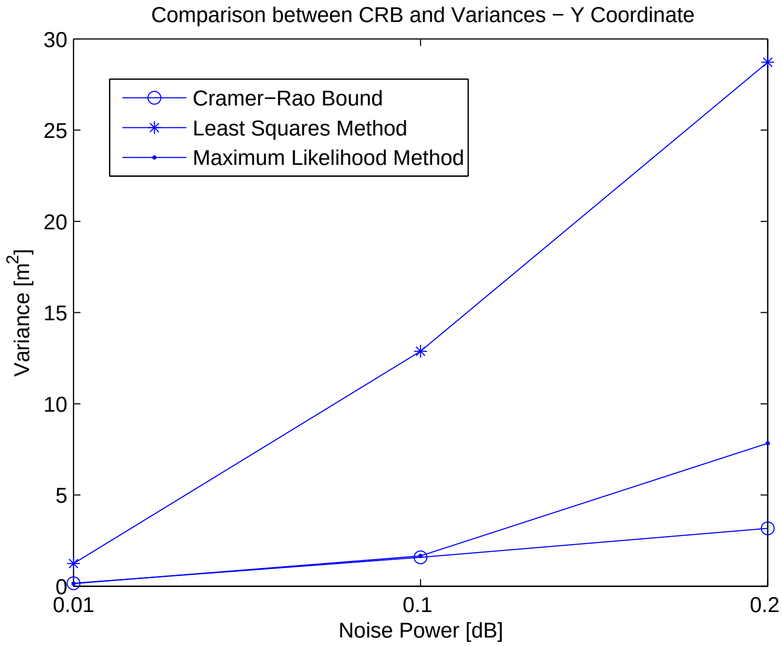


Fig. 7. Comparison between CRB and variances by two methods - Y coordinate.

If we plug $\mathcal{B}$ into $\mathcal{A}$,

$$\mathcal{A} = 2[y_n - f_n(\boldsymbol{\theta})]\left[-\frac{20}{\ln 10}\frac{(s_{n_x} - x)}{g_n}\right]. \tag{39}$$

Plugging $\mathcal{A}$ into (33),

$$\begin{aligned}
\frac{\partial \ln p}{\partial x} &= -\frac{1}{2\sigma^2}\sum_{n=1}^{3}\left\{2[y_n - f_n(\boldsymbol{\theta})]\left[-\frac{20}{\ln 10}\frac{(s_{n_x} - x)}{g_n(x,y)}\right]\right\} && (40)\\
&= \frac{20}{\sigma^2 \ln 10}\sum\left\{[y_n - f_n(\boldsymbol{\theta})]\left[\frac{(s_{n_x} - x)}{g_n(x,y)}\right]\right\} && (41)\\
&= \frac{20}{\sigma^2 \ln 10}\sum\left\{\frac{[y_n - f_n(\boldsymbol{\theta})](s_{n_x} - x)}{g_n(x,y)}\right\}. && (42)
\end{aligned}$$

Similarly, we can derive derivative of y coordinate as

$$\frac{\partial \ln p}{\partial y} = \frac{20}{\sigma^2 \ln 10}\sum\left\{\frac{[y_n - f_n(\boldsymbol{\theta})](s_{n_y} - y)}{g_n(x,y)}\right\}. \tag{43}$$

The second derivative of x coordinate follows as

$$\begin{aligned}
\frac{\partial^2 \ln p}{\partial x^2} &= \frac{\partial}{\partial x}\left\{\frac{20}{\sigma^2 \ln 10}\sum\left[\frac{(y_n - f_n)(s_{n_x} - x)}{g_n(x,y)}\right]\right\} && (44)\\
&= \frac{20}{\sigma^2 \ln 10}\sum\frac{\partial}{\partial x}\left[\frac{(y_n - f_n)(s_{n_x} - x)}{g_n(x,y)}\right]. && (45)
\end{aligned}$$

If we define

$$P_x(x,y) \triangleq (s_{n_x} - x)(y_n - f_n) \tag{46}$$

we have

$$\frac{\partial^2 \ln p}{\partial x^2} = \frac{20}{\sigma^2 \ln 10} \sum \underbrace{\left[\frac{\partial}{\partial x} \left(\frac{P_x}{g_n} \right) \right]}_{\mathcal{G}}. \tag{47}$$

From $\mathcal{G}$,

$$\mathcal{G} = \frac{\partial}{\partial x} \left(\frac{P_x}{g_n} \right) = \frac{{P_x}' g_n - P_x {g_n}'}{{g_n}^2} \tag{48}$$

where

$$\begin{aligned} {P_x}' &= -(y_n - f_n) - (s_{n_x} - x){f_n}' && (49) \\ {f_n}' &= \frac{20}{\ln 10} \frac{(s_{n_x} - x)}{g_n} \quad \text{from } \mathcal{B}, && (50) \\ {g_n}' &= -2(s_{n_x} - x) && (51) \end{aligned}$$

then

$$\begin{aligned} {P_x}' &= -(y_n - f_n) - (s_{n_x} - x) \left(\frac{20}{\ln 10} \cdot \frac{(s_{n_x} - x)}{g_n} \right) && (52) \\ &= -(y_n - f_n) - \frac{20}{\ln 10} \cdot \frac{(s_{n_x} - x)^2}{g_n}. && (53) \end{aligned}$$

Plugging ${P_x}'$ into $\mathcal{G}$,

$$\begin{aligned} \mathcal{G} &= \frac{-(y_n - f_n) \cdot g_n - \frac{20}{\ln 10} \cdot (s_{n_x} - x)^2 + 2(s_n - x)(y_n - f_n)(s_{n_x} - x)}{{g_n}^2} && (54) \\ &= \frac{2(s_n - x)^2 (y_n - f_n) - (y_n - f_n) \cdot g_n - \frac{20}{\ln 10} \cdot (s_{n_x} - x)^2}{{g_n}^2}. && (55) \end{aligned}$$

Plugging $\mathcal{G}$ into (47),

$$\frac{\partial^2 \ln p}{\partial x^2} = \frac{20}{\sigma^2 \ln 10} \sum \left[\frac{2(s_n - x)^2 (y_n - f_n) - (y_n - f_n) \cdot g_n - \frac{20}{\ln 10} \cdot (s_{n_x} - x)^2}{{g_n}^2} \right]. \tag{56}$$

Similarly, we can drive

$$\frac{\partial^2 \ln p}{\partial y^2} = \frac{20}{\sigma^2 \ln 10} \sum \left[\frac{2(s_n - y)^2 (y_n - f_n) - (y_n - f_n) \cdot g_n - \frac{20}{\ln 10} \cdot (s_{n_y} - y)^2}{{g_n}^2} \right]. \tag{57}$$

To completely find the elements of the Fisher information matrix, we have to find

$$\begin{aligned} \frac{\partial^2 \ln p}{\partial y \partial x} &= \frac{\partial}{\partial y} \left\{ \frac{20}{\sigma^2 \ln 10} \sum \left[\frac{(y_n - f_n)(s_{n_x} - x)}{g_n} \right] \right\} && (58) \\ &= \frac{20}{\sigma^2 \ln 10} \sum \left\{ \frac{\partial}{\partial y} \left[\frac{(y_n - f_n)(s_{n_x} - x)}{g_n} \right] \right\} && (59) \\ &= \frac{20}{\sigma^2 \ln 10} \sum \underbrace{\left[\frac{\partial}{\partial y} \left(\frac{P_x}{g_n} \right) \right]}_{Q_n}. && (60) \end{aligned}$$

$$\frac{\partial Q_n}{\partial y} = \frac{{P_x}' g_n - P_x {g_n}'}{{g_n}^2} = \frac{2(s_{n_x}-x)(s_{n_y}-y)(y_n-f_n)}{{g_n}^2} - \frac{\frac{20}{\ln 10}(s_{n_x}-x)(s_{n_y}-y)}{{g_n}^2} \tag{61}$$

where $\frac{\partial g_n}{\partial y} = -2(s_{n_y}-y)$, $\frac{\partial f_n}{\partial y} = \frac{20}{\ln 10}\frac{(s_{n_x}-x)(s_{n_y}-y)}{{g_n}^2}$. Therefore

$$\frac{\partial^2 \ln p}{\partial y \partial x} = \frac{20}{\sigma^2 \ln 10}\sum\left[\frac{2(s_{n_x}-x)(s_{n_y}-y)(y_n-f_n) - \frac{20}{\ln 10}(s_{n_x}-x)(s_{ny}-y)}{{g_n}^2}\right]. \tag{62}$$

Similarly,

$$\frac{\partial^2 \ln p}{\partial x \partial y} = \frac{20}{\sigma^2 \ln 10}\sum\left[\frac{2(s_{n_x}-x)(s_{n_y}-y)(y_n-f_n) - \frac{20}{\ln 10}(s_{n_x}-x)(s_{ny}-y)}{{g_n}^2}\right]. \tag{63}$$

To find the Fisher information matrix,

$$\mathbf{I}(\boldsymbol{\theta}) = \begin{bmatrix} -E\left(\frac{\partial^2 \ln p}{\partial x^2}\right) & -E\left(\frac{\partial^2 \ln p}{\partial x \partial y}\right) \\ -E\left(\frac{\partial^2 \ln p}{\partial y \partial x}\right) & -E\left(\frac{\partial^2 \ln p}{\partial y^2}\right) \end{bmatrix} \tag{64}$$

note [expectation of $f_n] = y_n$, and from (56) and (62), we can compute

$$\begin{aligned} E\left(\frac{\partial^2 \ln p}{\partial x \partial y}\right) &= \frac{20}{\sigma^2 \ln 10}\sum\left[\frac{-\frac{20}{\ln 10}\cdot(s_{n_x}-x)(s_{n_y}-y)}{{g_n}^2}\right] \\ &= -\left(\frac{20}{\sigma \ln 10}\right)^2 \sum\left[\frac{(s_{n_x}-x)(s_{n_y}-y)}{{g_n}^2}\right]. \end{aligned} \tag{65}$$

Similarly,

$$E\left(\frac{\partial^2 \ln p}{\partial y \partial x}\right) = -\left(\frac{20}{\sigma \ln 10}\right)^2 \sum\left[\frac{(s_{n_y}-y)(s_{n_x}-x)}{{g_n}^2}\right]. \tag{66}$$

$$E\left(\frac{\partial^2 \ln p}{\partial x^2}\right) = \frac{20}{\sigma^2 \ln 10}\sum\left[\frac{-\frac{20}{\ln 10}\cdot(s_{n_x}-x)^2}{{g_n}^2}\right] = -\left(\frac{20}{\sigma \ln 10}\right)^2 \sum\left[\frac{(s_{n_x}-x)^2}{{g_n}^2}\right]. \tag{67}$$

Similarly,

$$E\left(\frac{\partial^2 \ln p}{\partial y^2}\right) = \frac{20}{\sigma^2 \ln 10}\sum\left[\frac{-\frac{20}{\ln 10}\cdot(s_{n_y}-y)^2}{{g_n}^2}\right] = -\left(\frac{20}{\sigma \ln 10}\right)^2 \sum\left[\frac{(s_{n_y}-y)^2}{{g_n}^2}\right]. \tag{68}$$

Therefore,

$$\mathbf{I}(\boldsymbol{\theta}) = \left(\frac{20}{\sigma \ln 10}\right)^2 \cdot \begin{bmatrix} \sum\left[\frac{(s_{n_x}-x)^2}{{g_n}^2}\right] & \sum\left[\frac{(s_{n_x}-x)(s_{n_y}-y)}{{g_n}^2}\right] \\ \sum\left[\frac{(s_{n_x}-x)(s_{n_y}-y)}{{g_n}^2}\right] & \sum\left[\frac{(s_{n_y}-y)^2}{{g_n}^2}\right] \end{bmatrix} \triangleq \left(\frac{20}{\sigma \ln 10}\right)^2 \begin{bmatrix} a & b \\ c & d \end{bmatrix} \tag{69}$$

then

$$\mathbf{I}^{-1}(\boldsymbol{\theta}) = \left(\frac{20}{\sigma \ln 10}\right)^2 \times \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}. \tag{70}$$

Since

$$var\left(\hat{\theta}_i\right) \geq \left[\mathbf{I}^{-1}\left(\boldsymbol{\theta}\right)\right]_{ii} \tag{71}$$

$$var(\hat{x}) \geq \frac{\left(\frac{\sigma \ln 10}{20}\right)^2 \cdot \sum\left[\frac{(s_{n_y}-y)^2}{{g_n}^2}\right]}{\sum\left[\frac{(s_{n_x}-x)^2}{{g_n}^2}\right]\sum\left[\frac{(s_{n_y}-y)^2}{{g_n}^2}\right] - \left\{\sum\left[\frac{(s_{n_x}-x)(s_{n_y}-y)}{{g_n}^2}\right]\right\}^2} \tag{72}$$

$$var(\hat{y}) \geq \frac{\left(\frac{\sigma \ln 10}{20}\right)^2 \cdot \sum\left[\frac{(s_{n_x}-x)^2}{{g_n}^2}\right]}{\sum\left[\frac{(s_{n_x}-x)^2}{{g_n}^2}\right]\sum\left[\frac{(s_{n_y}-y)^2}{{g_n}^2}\right] - \left\{\sum\left[\frac{(s_{n_x}-x)(s_{n_y}-y)}{{g_n}^2}\right]\right\}^2} \qquad \square \tag{73}$$

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