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THE GALAXIES OF NONSTANDARD ENLARGEMENTS OF TRANSFINITE GRAPHS OF HIGHER RANKS: II

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Abstract — This report is an improvement of a prior report [4, Report 814]. It sharpens the principal theorem (Theorem 5.1 of Report 814) and also simplifies its proof. There are also several minor changes involving clarifications and corrections of misprints. The prior abstract remains the same as follows:

In a previous work, the galaxies of the nonstandard enlargements of connected, conventionally infinite graphs and also of connected transfinite graphs of the first rank of transfiniteness were defined, examined, and illustrated by some examples. In this work it is shown how the results of the prior work extend to transfinite graphs of higher ranks. Among those results are following: Any such enlargement either has exactly one galaxy, its principal one, or it has infinitely many such galaxies. In the latter case, the galaxies are partially ordered by there "closeness" to the principal galaxy. Also, certain sequences of galaxies whose members are totally ordered by that "closeness" criterion are identified.

Key Words: Nonstandard graphs, enlargements of graphs, transfinite graphs, galaxies in nonstandard graphs, graphical galaxies.

1 Introduction

In some previous works, the ideas of "nonstandard graphs" [2, Chapter 8] and the "galaxies of nonstandard enlargements of graphs" [3], [5] were defined and examined. However, all this was done only for conventionally infinite graphs and transfinite graphs of the first rank of transfiniteness. The purpose of this work is to define and examine nonstandard transfinite graphs of higher ranks of transfiniteness.

This paper is written as a sequel to [5] and uses a symbolism and terminology consistent with that prior work. We also use a variety of results concerning transfinite graphs, and these may all be found in [2]. We refer the reader to those sources for such information. For instance, the "hyperordinals" are constructed in much the same way as are the hypernaturals, and their definitions are given in [5, Section 5]. Furthermore, we use herein the idea of "wgraphs," which are transfinite graphs based upon walks rather than paths. This avoids some of the difficulties associated with path-based transfinite graphs and is in fact both simpler and more general than the path-based theory of transfinite graphs. Wgraphs and their transfinite extremities are defined and examined in [2, Sections 5.1 to 5.6]. Lengths of walks on wgraphs and "wdistances" based on such lengths are discussed in [2, Sections 5.7 and 5.8]. All this is assumed herein as being known. It is also assumed that all the wgraphs considered herein are wconnected.

Our arguments will be based on ultrapower constructions, and, to this end, we assume throughout that a free ultrafilter $\mathcal{F}$ has been chosen and fixed. Finally, when adding ordinals, we always take it that ordinals are in normal form and that the natural summation of ordinals is being used [1, pages 354-355].

It is a fact about a transfinite wgraph G^{ν} of rank ν that it contains wsubgraphs of all ranks ρ with $0 \leq \rho \leq \nu$, called ρ-wsections, that at each rank ρ the ρ-wsections are ρ-wgraphs by themselves and induce a partitioning of the branch set of G^{ν}, and that the one and only ν-wsection is G^{ν} itself. We define the "enlargements" ${}^{*}G^{\nu}$ of a transfinite wgraph G^{ν} and of its ρ-wsections in the next section. The galaxies of all ranks in ${}^{*}G^{\nu}$ are defined in Section 3. A galaxy of rank ρ $(0 \leq \rho \leq \nu)$ is called a "ρ-galaxy."

Within the enlargement ${}^{*}S^{\rho}$ of an ρ-wsection S^{ρ} of G^{ν}, there is either exactly one ρ-galaxy, the "principal ρ-galaxy," or infinitely many ρ-galaxies in addition to the principal ρ-galaxy. The latter case arises when G^{ν} is locally finite in a certain way (Section 4), but it may arise in other ways as well. Moreover, the enlargements of all the ρ-wsections within a $(\rho+1)$-wsection lie within the principal $(\rho+1)$-galaxy of the enlargement of that $(\rho+1)$-wsection, and so on through the wsections of higher ranks. In that latter case still, there will be a two-way infinite sequence of ρ-galaxies that are totally ordered according to their "closeness to the principal ρ-galaxy," and there may be many such totally ordered sequences of ρ-galaxies. When there are many ρ-galaxies in ${}^*S^\rho$, they are partially ordered, again according to their closeness to the principal ρ-galaxy (Section 5).

2 Enlargements of ν-Graphs and Hyperdistances in the Enlargements

First of all, the enlargement ${}^*G^0 = \{{}^*X^0, {}^*B\}$ of a conventionally infinite 0-graph and the enlargement ${}^*G^1 = \{{}^*X^0, {}^*B, {}^*X^1\}$ of a transfinite 1-graph are discussed in [3, Sections 2 and 8]. These prior constructs will be encompassed by the more general development we now undertake. We shall assume that the rank ν is no larger than ω. The extensions to higher ranks of transfiniteness proceeds in much the same way.

Consider a wconnected transfinite wgraph of rank ν $(0 \leq \nu \leq \omega)$:

$$G^\nu \;=\; \{X^0, B, X^1, \ldots, X^\nu\}$$

where X^0 is a set of 0-nodes, B is a set of branches (i.e., two-element sets of 0-nodes), and X^ρ $(\rho = 1, \ldots, \nu)$ is a set of ρ-wnodes. It is assumed that each X^ρ $(\rho = 0, \ldots, \nu)$ is nonempty except possibly for $\rho = \vec{\omega}$. In general, $X^{\vec{\omega}}$ may be empty.

The "enlargement" ${}^*G^\nu$ of G^ν is defined as follows: Two sequences $\langle x_n^\rho \rangle$ and $\langle y_n^\rho \rangle$ of ρ-wnodes in G^ν (i.e., $x_n^\rho, y_n^\rho \in X^\rho$) are taken to be equivalent if $\{n : x_n^\rho = y_n^\rho\} \in \mathcal{F}$. This is truly an equivalence relation on the set of sequences of ρ-wnodes, as is easily shown. Each equivalence class $\mathbf{x}^\rho$ will be called a *ρ-hypernode* and will be represented by $\mathbf{x}^\rho = [x_n^\rho]$ where the x_n^ρ are elements of any one of the sequences in that equivalence class. We let ${}^*X^\rho$ denote the set of all such equivalence classes (i.e., the set of all ρ-hypernodes). Then, the *enlargement* ${}^*G^\nu$ of G^ν is the set

$${}^*G^\nu \;=\; \{{}^*X^0, {}^*B, {}^*X^1 \ldots, {}^*X^\nu\}.$$

The elements of ${}^*B$ are called *hyperbranches* and have been defined in [2, Section 8.1]. Here, too, ${}^*X^\rho$ is nonempty if $\rho \neq \vec{\omega}$; ${}^*X^{\vec{\omega}}$ may be empty.

Next, we wish to define the "hyperdistances" between the hypernodes of ${}^*G^\nu$. The "length" $|W_{xy}|$ of any two-ended walk W_{xy} terminating at two wnodes x and y of any ranks in G^ν is defined in [2, Section 5.7]. Also, the wdistance $d(x,y)$ between those two wnodes is

$$d(x,y) \;=\; \min\{|W_{x,y}|\}$$

where the minimum is taken over the lengths $|W_{xy}|$ of all the walks W_{xy} in G^ν that terminate at x and y. The minimum exists because those lengths comprise a well-ordered set of ordinals. Furthermore, a wnode is said to be *maximal* if it is not embraced by a wnode of higher rank. In these definitions, x and y may be either maximal or nonmaximal wnodes. Note that the wdistance measured from any nonmaximal wnode z is the same as that measured from the maximal wnode x that embraces z. We also set $d(x,x) = 0$. Thus, d is an ordinal-valued metric defined on the maximal wnodes in $\cup_{\rho=0}^{\nu} X^\rho$. (The axioms of a metric are readily verified.)

Given two hypernodes $\mathbf{x} = [x_n]$ and $\mathbf{y} = [y_n]$ of any ranks in ${}^*G^\nu$, we defined the *hyperdistance* $\mathbf{d}$ between them as the internal function

$$\mathbf{d}(\mathbf{x},\mathbf{y}) \;=\; [d(x_n,y_n)].$$

We say that a hypernode $\mathbf{x}^\rho = [x_n^\rho]$ is *maximal* if it is not embraced by a hypernode $\mathbf{y}^\gamma = [y_n^\gamma]$ of higher rank ($\gamma > \rho$) (i.e., x_n^ρ is not embraced by y_n^γ for almost all n). Upon restricting $\mathbf{d}$ to the maximal hypernodes in ${}^*G^\nu$, we have that this restricted $\mathbf{d}$ satisfies the metric axioms except that it is hyperordinal-valued. In particular, we have by the transfer principle that the triangle inequality holds for any three maximal hypernodes $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{z}$, namely,

$$\mathbf{d}(\mathbf{x},\mathbf{z}) \;\leq\; \mathbf{d}(\mathbf{x},\mathbf{y}) \,+\, \mathbf{d}(\mathbf{y},\mathbf{z}). \tag{1}$$

3 The Galaxies of ${}^*G^\nu$

We continue to assume that the rank ν is no larger than ω. Also, we assume at first that the rank ρ is a natural number no larger than ν. Consider the ν-wgraph G^ν and its enlargement ${}^*G^\nu$. Two hypernodes $\mathbf{x} = [x_n]$ and $\mathbf{y} = [y_n]$ of any ranks in ${}^*G^\nu$ will be said to be in the same *nodal ρ-galaxy* $\dot{\Gamma}^\rho$ if there exists a natural number μ_{xy} depending on $\mathbf{x}$ and $\mathbf{y}$ such that $\{n\colon d(x_n,y_n) \leq \omega^\rho \cdot \mu_{xy}\} \in \mathcal{F}$. In this case, we say that $\mathbf{x}$ and $\mathbf{y}$ are *ρ-limitedly distant.* This defines an equivalence relation on the set $\cup_{\gamma=0}^{\nu} {}^*X^\gamma$ of all the hypernodes in ${}^*G^\nu$, and thus $\cup_{\gamma=0}^{\nu} {}^*X^\gamma$ is partitioned into nodal ρ-galaxies. The proof of this is the same as that given in [3, Section 9] except that the rank 1 therein is now replaced by ρ. By the same arguments, we have that, for $\alpha \leq \rho$, the nodal α-galaxies provide a finer partitioning of $\cup_{\gamma=0}^{\nu} {}^*X^\gamma$ than do the nodal γ-galaxies. Moreover, any nodal ρ-galaxy is partitioned by the nodal α-galaxies ($\alpha < \rho$) in that nodal ρ-galaxy.

Corresponding to each nodal ρ-galaxy $\dot{\Gamma}^\rho$, we define a *ρ-galaxy* Γ^ρ of ${}^*G^\nu$ as the nonstandard wsubgraph of ${}^*G^\nu$ induced by all the ρ-hypernodes in $\dot{\Gamma}^\rho$; that is, along with the hypernodes of $\dot{\Gamma}^\rho$, we have hyperbranches whose incident 0-hypernodes are in $\dot{\Gamma}^\rho$. Note that every hyperbranch must lie in a single ρ-galaxy for every ρ because their incident 0-hypernodes are at a hyperdistance of 1.

Let us now turn to the case where $\rho = \vec{\omega}$. Now, ν is either $\vec{\omega}$ or ω. The definition of the $\vec{\omega}$-galaxies is rather different. Two hypernodes $\mathbf{x} = [x_n]$ and $\mathbf{y} = [y_n]$ in ${}^*G^\nu$ will be said to be in the same *$\vec{\omega}$-galaxy* $\dot{\Gamma}^{\vec{\omega}}$ if there exists a natural number μ_{xy} depending on $\mathbf{x}$ and $\mathbf{y}$ such that $\{n : d(x_n, y_n) \leq \omega^{\mu_{xy}}\} \in \mathcal{F}$. Thus, $\mathbf{x}$ and $\mathbf{y}$ are in $\dot{\Gamma}^{\vec{\omega}}$ if and only if $\{n : d(x_n, y_n) < \omega^\omega\} \in \mathcal{F}$. In this case, we say that $\mathbf{x}$ and $\mathbf{y}$ are *$\vec{\omega}$-limitedly distant*. Here, too, the property of being $\vec{\omega}$-limitedly distant defines an equivalence relation on the set of all hypernodes in ${}^*G^\nu$. So, the nodal $\vec{\omega}$-galaxies partition the set of all hypernodes. Then, the $\vec{\omega}$-galaxy $\Gamma^{\vec{\omega}}$ corresponding to any nodal $\vec{\omega}$-galaxy $\dot{\Gamma}^{\vec{\omega}}$ consists of the hypernodes in $\dot{\Gamma}^{\vec{\omega}}$ along with the hyperbranches whose 0-hypernodes are in $\dot{\Gamma}^{\vec{\omega}}$.

Finally, the ω-galaxies of a nonstandard ω-wgraph ${}^*G^\omega$ are defined just as are the ρ-galaxies of natural-number ranks. We now require that $\{n : d(x_n, y_n \leq \omega^\omega \cdot \mu_{xy}\} \in \mathcal{F}$ for some natural number μ_{xy} depending upon $\mathbf{x} = [x_n]$ and $\mathbf{y} = [y_n]$ in order for $\mathbf{x}$ and $\mathbf{y}$ to be in the same nodal ω-galaxy. When this is so, we again say that $\mathbf{x}$ and $\mathbf{y}$ are *ω-limitedly distant.* The same partitioning properties hold.

In general now, let G^ν be a ν-wgraph where $0 \leq \nu \leq \omega$, possibly $\nu = \vec{\omega}$. The *principal ν-galaxy* Γ_0^ν of ${}^*G^\nu$ is that ν-galaxy whose hypernodes are ν-limitedly distant from a standard hypernode of ${}^*G^\nu$. We shall show later on that ${}^*G^\nu$ either has exactly one ν-galaxy, its principal one, or has infinitely many of them.

Let us now recall another definition concerning standard transfinite wgraphs. A ρ-

not pass through any wnode of rank greater than ρ.) Consequently, $\mathbf{x}$ and $\mathbf{y}$ are ρ-limitedly distant. Also, for every hypernode $\mathbf{z} = [z_n]$ in ${}^*S^\alpha$, $\mathbf{y}$ and $\mathbf{z}$ are α-limitedly distant, which implies that they are ρ-limitedly distant. So, by the triangle inequality (1), $\mathbf{x}$ and $\mathbf{z}$ are ρ-limitedly distant. Thus, $\mathbf{z}$ is in $\Gamma_0^\rho(S^\rho)$. □

Note that ${}^*S^\alpha$ may (but need not) have other α-galaxies besides its principal one $\Gamma_0^\alpha(S^\alpha)$, and ${}^*S^\rho$ may have still other α-galaxies not in $\Gamma_0^\rho(S^\rho)$. Also, there may be a ρ-galaxy (possibly many of them) consisting of a single λ-hypernode when $\lambda > \rho$.

Furthermore, it is possible of ${}^*G^\nu$ to have exactly one ρ-galaxy. This occurs, for instance, when there is a single node in G^ν to which all the other nodes of G^ν are connected through two-ended ρ-paths, with each such path being connected to the rest of G^μ only at its terminal nodes. When ${}^*G^\nu$ has exactly one ρ-galaxy, then ${}^*G^\nu$ has exactly one σ-galaxy for every σ such that $\rho < \sigma \leq \nu$ because, if two hypernodes are ρ-limitedly distant, then they are also σ-limitedly distant.

In view of all this, we again observe that the galaxies of ${}^*G^\nu$ can have rather complicated structures and dissimilarities.

4 Locally Finite Sections and a Property of Their Enlargements

In this and the next section, the rank ρ is not allowed to be $\vec{\omega}$. We now establish a sufficient (but not necessary) condition under which the enlargement ${}^*S^\rho$ has at least one ρ-galaxy different from its principal galaxy $\Gamma_0^\rho(S^\rho)$. Let us first recall some definitions for standard wgraphs.

Assume initially that ρ is a natural number. Two ρ-wnodes of S^ρ will be called *ρ-wadjacent* if they are incident to the same $(\rho - 1)$-wsection. A ρ-wnode will be called a *boundary ρ-wnode* if it is incident to two or more $(\rho - 1)$-wsections. A ρ-wsection S^ρ will be called *locally ρ-finite* if each of its $(\rho - 1)$-wsections has only finitely many incident boundary ρ-wnodes. These same definitions hold when $\rho = \omega$ except that $\rho - 1$ is understood to be $\vec{\omega}$. The case where $\rho = \vec{\omega}$ is prohibited in the statements of this section.

In the following, we let ρ be a natural number or $\rho = \omega$. When $\rho = \omega$, $\rho - 1$ denotes $\vec{\omega}$.

Lemma 4.1. *Let x^ρ be a boundary ρ-wnode. Then, any ρ-walk that passes through x^ρ from one $(\rho-1)$-wsection $S_1^{\rho-1}$ incident to x^ρ to another $(\rho-1)$-wsection $S_2^{\rho-1}$ incident to x^ρ must have a length no less that ω^ρ (thus, when $\rho=\omega$, a length no less than ω^ω).*

Proof. The only way such a walk can have a length less than ω^ρ is if it avoids traversing a $(\rho-1)$-wtip in x^ρ. But, this means that it passes through two wtips embraced by x^ρ of ranks less than ρ. But, that in turn means that $S_1^{\rho-1}$ and $S_2^{\rho-1}$ cannot be different $(\rho-1)$-wsections. □

An immediate consequence of Lemma 4.1 is

Lemma 4.2. *Any two ρ-wnodes x^ρ and y^ρ that are ρ-wconnected but not ρ-wadjacent must satisfy $d(x^\rho, y^\rho) \geq \omega^\rho$.*

Theorem 4.3. *Let the ρ-wsection S^ρ of G^ν be locally ρ-finite and have infinitely many boundary ρ-wnodes. Then, given any ρ-wnode x_0^ρ in S^ρ, there is a one-ended ρ-walk W^ρ starting at x_0^ρ:*

$$W^\rho \;=\; \langle x_0^\rho, W_0^{\alpha_0}, x_1^\rho, W_1^{\alpha_1}, \ldots, x_m^\rho, W_m^{\alpha_m}, \ldots \rangle$$

such that there is a subsequence of ρ-wnodes $x_{m_k}^\rho$, $k=1,2,3,\ldots$, satisfying $d(x_0^\rho, x_{m_k}^\rho) \geq \omega^\rho \cdot k$.

The proof of this theorem is just like that of Theorem 10.3 in [3] except that the rank 1 therein is replaced by the rank ρ herein. In the same way, Corollary 10.4 of [3] generalizes into the following assertion.

Corollary 4.4 *Under the hypothesis of Theorem 4.3, the enlargement ${}^*S^\rho$ of S^ρ has at least one ρ-hypernode not in its principal galaxy $\Gamma_0^\rho(S^\rho)$ and thus has at least one ρ-galaxy Γ^ρ different from its principal ρ-galaxy $\Gamma_0^\rho(S^\rho)$.*

5 When the Enlargement ${}^*S^\rho$ of a ρ-Wsection Has a ρ-Hypernode Not in the Principal ρ-Galaxy of ${}^*S^\rho$

As always, we take G^ν to be a ν-wgaph with $2 \leq \nu \leq \vec{\omega}$, possibly $\nu = \vec{\omega}$. We continue to asssume that the rank ρ of a ρ-wsection S^ρ of G^ν is either a natural number or ω, but not $\vec{\omega}$. Let Γ_a^ρ and Γ_b^ρ be two ρ-galaxies in the enlargement ${}^*S^\rho$ of a ρ-wsection S^ρ that are different from the principal ρ-galaxy Γ_0^ρ of ${}^*S^\rho$. We shall say that Γ_a^ρ *is closer to* Γ_0^ρ *than is*

Γ_b^ρ and that Γ_b^ρ *is further away from* Γ_0^ρ *than is* Γ_a^ρ if there are a hypernode $\mathbf{y} = [y_n]$ in Γ_a^ρ and a hypernode $\mathbf{z} = [z_n]$ in Γ_b^ρ such that, for some $\mathbf{x} = [x_n]$ in Γ_0^ρ and for every $m \in \boldsymbol{N}$, we have

$$N_0(m_0) \;=\; \{n\colon d(z_n, x_n) - d(y_n, x_n) \,\geq\, \omega^\rho \cdot m\} \;\in\; \mathcal{F}.$$

(The ranks of $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{z}$ may have any values no larger than ν other than $\vec{\omega}$.) Any set of ρ-galaxies for which every two of them, say, Γ_a^ρ and Γ_b^ρ satisfy this condition will be said to be *totally ordered according to their closeness to* Γ_0^ρ. That the conditions for a total ordering (reflexivity, antisymmetry, transitivity, and connectedness) are fulfilled are readily shown. For instance, the proof of Theorem 5.2 below establishes transitivity.

These definitions are independent of the representative sequences $\langle x_n\rangle$, $\langle y_n\rangle$, and $\langle z_n\rangle$ chosen for $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{z}$; the proof of this is exactly the same as the proof of Lemma 4.1 of [3].

We will say that a set A is a *totally ordered, two-way infinite sequence* if there is a bijection from the set $\boldsymbol{Z}$ of integers to the set A that preserves the total ordering of $\boldsymbol{Z}$.

Theorem 5.1. *If the enlargement* $^*S^\rho$ *of a* ρ*-wsection* S^ρ *of* G^ν *has a hypernode* $\mathbf{v} = [v_n]$ *(of any rank* α *with* $0 \leq \alpha \leq \rho$*) that is not in the principal* ρ*-galaxy* Γ_0^ρ *of* $^*S^\rho$*, then there exists a two-way infinite sequence of* ρ*-galaxies totally ordered according to their closeness to* Γ_0^ρ *and with* $\mathbf{v}$ *being in one of those galaxies.*

Note. Here, too, the proof of this is much like that of Theorem 4.2 of [3], but, since this is the main result of this work, let us present a detailed argument. As always, we choose and fix upon a free ultrafilter $\mathcal{F}$. (Let us also mention that a somewhat different version of this theorem with a rather longer proof can be found in the archival website, www.arxiv.gov, under Mathematics, Zemanian.)

Proof. In this proof, we use the fact that between any two nodes in a ν-graph there exists a geodesic walk terminating at those nodes; that is, the length of the walk is equal to the wdistance between those nodes. This is a consequence of the facts that the walks terminating at those nodes have ordinal lengths and that any set of ordinals is well-ordered and thus has a least ordinal. That least ordinal must be the length of at least one walk terminating at those nodes, for otherwise the minimum of the walk-lengths would be larger.

As before, let $\mathbf{x} = [\langle x, x, x, \ldots \rangle]$ be a standard hypernode in Γ_0^ρ. ($\mathbf{x}$ can be of any rank from 0 to ρ.) Since $\mathbf{v}$ is not in Γ_0^ρ, we have that for each $m \in \boldsymbol{N}$

$$\{n\colon d(x, v_n) \,>\, \omega^\rho \cdot m\} \,\in\, \mathcal{F} \tag{2}$$

For each $n \in \boldsymbol{N}$, if $d(x, v_n) < \omega^\rho \cdot 6$, set $u_n = x$, but, if $d(x, v_n) \geq \omega^\rho \cdot 6$, choose u_n such that

$$d(x, v_n) \,\leq\, d(x, u_n) \cdot 3 \,\leq\, d(x, v_n) \cdot 2 \tag{3}$$

That the latter can be done can be seen as follows.

Choose a geodesic ρ-walk W^ρ terminating at x and v_n. Remember that W^ρ is incident to each of its nonterminal ρ-nodes through at least one $\omega^{\rho-1}$-wtip. (See Sections 5.3 and 5.5 in [2] in this regard.) Moreover, the transition through each $\omega^{\rho-1}$-wtip contributes ω^ρ to the length of W^ρ. Upon tracing W^ρ from x toward v_n, we must encounter at least two ρ-nodes, both of which are neither closer to x by one-third of the number of $\omega^{\rho-1}$-wtips traversed by W^ρ nor further away from x by two-thirds of the number of $\omega^{\rho-1}$-wtips traversed by W^ρ. A node on W^ρ between those two ρ-nodes can be chosen as u_n.

Suppose there is a $k \in \boldsymbol{N}$ such that $\{n : d(x, u_n) \leq \omega^\rho \cdot k\} \in \mathcal{F}$. By the left-hand inequality of (3),

$$\{n\colon d(x, v_n) \,\leq\, (\omega^\rho \cdot k) \cdot 3\} \,\supseteq\, \{n\colon d(x, u_n) \,\leq\, \omega^\rho \cdot k\} \,\in\, \mathcal{F}.$$

Hence, the left-hand set is a member of $\mathcal{F}$, in contradiction to (2). (These sets cannot both be in the ultrafilter $\mathcal{F}$.) Therefore, $\mathbf{u} = [u_n]$ satisfies (2) for every $m \in \boldsymbol{N}$ when v_n is replaced by u_n; that is, $\mathbf{u}$ is in a galaxy different from the principal ρ-galaxy Γ_0^ρ.

Furthermore, by the right-hand inequality of (3),

$$d(x, u_n) \,\leq\, (d(x, v_n) \,-\, d(x, u_n)) \cdot 2.$$

Suppose there exists a $j \in \boldsymbol{N}$ such that

$$\{n\colon d(x, v_n) \,-\, d(x, u_n) \,\leq\, \omega^\rho \cdot j\} \,\in\, \mathcal{F}.$$

Then,

$$\{n\colon d(x, u_n) \,\leq\, (\omega^\rho \cdot j) \cdot 2\} \,\supseteq\, \{n\colon d(x, v_n) \,-\, d(x, u_n) \,\leq\, \omega^\rho \cdot j\} \,\in\, \mathcal{F}$$

So, the left-hand set is in $\mathcal{F}$, in contradiction to our previous conclusion that $\mathbf{u}$ satisfies (2) with v_n replaced by u_n. We can conclude that $\mathbf{u}$ and $\mathbf{v}$ are in different ρ-galaxies Γ_a^ρ and Γ_b^ρ respectively, with Γ_a^ρ closer to Γ_0^ρ than is Γ_b^ρ.

We can now repeat this argument with Γ_b^ρ replaced by Γ_a^ρ to find still another ρ-galaxy $\tilde{\Gamma}_a^\rho$ of ${}^*S^\rho$ different from Γ_0^ρ and closer to Γ_0^ρ than is Γ_a^ρ. Continual repetitions yield an infinite sequence of ρ-galaxies indexed by, say, the negative integers and totally ordered by their closeness to Γ_0^ρ.

The conclusion that there is an infinite sequence of ρ-galaxies progressively further away from Γ_0^ρ than is Γ_b^ρ is easier to prove. With $\mathbf{v} \in \Gamma_b^\rho$ as before, we have (2) again for every $m \in \mathbb{N}$. Therefore, for each $n \in \mathbb{N}$, we can choose w_n as an element of $\langle v_n \rangle$ such that

$$d(x, w_n) \geq d(x, v_n) + \omega^\rho \cdot n \tag{4}$$

Hence, for each $m \in \mathbb{N}$,

$$\{n \colon d(x, w_n) > \omega^\rho \cdot m\} \supseteq \{n \colon d(x, v_n) > \omega^\rho \cdot m\}.$$

Since the right-hand side is a member of $\mathcal{F}$, so too is the left-hand side. Thus, $\mathbf{w} = [w_n]$ is in a galaxy different from Γ_0^ρ. Moreover, from (4) we have that, for each $m \in \mathbb{N}$, $\{n \colon d(x, w_n) - d(x, v_n) > \omega^\rho \cdot m\}$ is a cofinite set and therefore is a member of $\mathcal{F}$. Consequently, $\mathbf{w}$ is in a galaxy Γ_c^ρ that is further away from Γ_0^ρ than is Γ_b^ρ.

We can repeat the argument of the last paragraph with Γ_c^ρ in place of Γ_b^ρ to find still another ρ-galaxy $\tilde{\Gamma}_c^\rho$ further away from Γ_0^ρ than is Γ_c^ρ. Repetitions of this argument show that there is an infinite sequence of ρ-galaxies indexed by, say, the positive integers and totally ordered by their closeness to Γ_0^ρ. The conjunction of the two infinite sequences along with Γ_b^ρ yields the conclusion of the theorem. □

By virtue of Corollary 4.4, the conclusion of Theorem 5.1 holds whenever G is locally finite.

In general, the hypothesis of Theorem 5.1 may or may not hold. Thus, ${}^*S^\rho$ either has exactly one ρ-galaxy, its principal one Γ_0^ρ, or has infinitely many ρ-galaxies.

Instead of the idea of "totally ordered according to closeness to Γ_0^ρ," we can define the idea of "partially ordered according to closeness to Γ_0^ρ" in much the same way. Just drop the connectedness axiom for a total ordering.

Theorem 5.2. *Under the hypothesis of Theorem 5.1, the set of ρ-galaxies of $^*S^\rho$ is partially ordered according to the closeness of the ρ-galaxies to the principal ρ-galaxy Γ_0^ρ.*

Proof. Reflexivity and antisymmetry are obvious. Consider transitivity: Let Γ_a^ρ, Γ_b^ρ, and Γ_c^ρ be ρ-galaxies different from Γ_0^ρ. (The case where $\Gamma_a^\rho = \Gamma_0^\rho$ can be argued similarly.) Assume that Γ_a^ρ is closer to Γ_0^ρ than is Γ_b^ρ and that Γ_b^ρ is closer to Γ_0^ρ than is Γ_c^ρ. Thus, for any $\mathbf{x}$ in Γ_0^ρ, $\mathbf{u}$ in Γ_a^ρ, $\mathbf{v}$ in Γ_b^ρ, and $\mathbf{w}$ in Γ_c^ρ and for each $m \in \mathbf{N}$, we have

$$N_{uv} \;=\; \{n \colon d(v_n, x_n) - d(u_n, x_n) \geq \omega^\rho \cdot m\} \;\in\; \mathcal{F}$$

and

$$N_{vw} \;=\; \{n \colon d(w_n, x_n) - d(v_n, x_n) \geq \omega^\rho \cdot m\} \;\in\; \mathcal{F}.$$

We also have

$$d(w_n, x_n) - d(u_n, x_n) \;=\; d(w_n, x_n) - d(v_n, x_n) + d(v_n, x_n) - d(u_n, x_n).$$

So,

$$N_{uw} \;=\; \{n \colon d(w_n, x_n) - d(u_n, x_n) \geq \omega^\rho \cdot 2m\} \;\supseteq\; N_{uv} \cap N_{vw} \;\in\; \mathcal{F}.$$

Thus, $N_{uw} \in \mathcal{F}$. Since m can be chosen arbitrarily, we can conclude that Γ_a^ρ is closer to Γ_0^ρ than is Γ_c^ρ. $\Box$

References

[1] A. Abian, *The Theory of Sets and Transfinite Arithmetic*, W.B. Saunders Company, Philadelphia, Pennsylvania, 1965.

[2] A.H. Zemanian, *Graphs and Networks: Transfinite and Nonstandard*, Birkhauser-Boston, Cambridge, Massachsetts, 2004.

[3] A.H. Zemanian, *The Galaxies of Nonstandard Enlargements of Infinite and Transfinite Graphs*, CEAS Technical Report 813, University at Stony Brook, September 1, 2004.

[4] A.H. Zemanian, *The Galaxies of Nonstandard Enlargements of Transfinite Graphs of Higher Ranks*, CEAS Technical Report 814, University at Stony Brook, September 15, 2004.

[5] A.H. Zemanian, *The Galaxies of Nonstandard Enlargements of Infinite and Transfinite Graphs: II*, CEAS Technical Report 819, University at Stony Brook, April 6, 2005.