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The theory of deep holograms

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A simple theory of the first-order properties of holograms in thick photographic emulsions is presented. Following a combination by Denisyuk of the basic method of wavefront reconstruction with Lippmann's method of colour photography, such holograms are produced by the interference of an object wave with a coherent reference wave, falling on the emulsion from opposite sides. It is shown, by using the first Born approximation, that such deep holograms have three properties which distinguish them from two-dimensional or 'plane' holograms: (1) They have directional selectivity, that is to say the image will appear only if the hologram is illuminated in the reconstruction within a certain angular zone. (2) They have colour selectivity, that is to say they will reflect only within a certain narrow waveband close to the original wavelength. (3) The second wave, which is a disturbance in two-dimensional holograms is as good as completely suppressed. All three are of great practical value.

It is shown that holograms which are produced by strongly diffused, wide-angle illumination, and which have a random, noise-like appearance, contain the information in the form of the auto-correlation function of densities or scattering powers between different space-elements in the emulsion.

Introduction

Photography by wavefront reconstruction now called 'holography' was initiated by one of us (Gabor 1948, 1949, 1951) starting from the realization that a light wave can be 'frozen' into a photographic plate simply by the superposition of a 'background' or 'reference' wave, coherent with the first. When the processed plate is illuminated by the reference wave alone, the frozen wave will be revived; it will reappear as a component of the light wave transmitted and diffracted by the photograph, called a 'hologram'.

Let A_0, A_1 be the complex amplitudes of the reference wave and of the object wave in the plane of the emulsion. (We can drop the factor $e^{-i\omega t}$ in all equations.) The resulting amplitude is A_0+A_1, but the plate records only the intensity

$$I = (A_0+A_1)(A_0^*+A_1^*).$$

For simplicity of explanation, assume that the plate is processed with a γ of -2, so that its intensity transmission is proportional to I^2 and its transmission amplitude to I. If we now illuminate the plate with A_0 alone, the transmitted amplitude in the plane of the emulsion will be proportional to

$$A_0 I = A_0(A_0A_0^*+A_1A_1^*)+A_0A_0^*A_1+A_0^2A_1^*. \tag{1}$$

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If the reference wave A_0 is plane or spherical, $A_0A_0^*$ will be exactly or approximately constant over the plate. This will be a good approximation also for the bracket expression in the first term, in all but rather exceptional cases (Stroke, Restrick, Funkhouser & Brumm 1965*a*, *b*). The first term is therefore essentially the transmitted reference wave. The second term, on the other hand, is the original object-wave amplitude in the plane of the hologram, therefore, by Huygens's principle it appears to issue from the original object, in its original spatial position. The last term is the 'twin wave'. It can be interpreted (Gabor 1952) as issuing from the mirror image of the original object, with respect to the illuminating wavefront considered as a mirror, at the plate. It can be also shown (Gabor 1949, 1951) that there is no need to illuminate with the original reference wave, or even with the original wavelength. So long as the illuminating wave is spherical, the displacement of its origin produces only certain optical transformations of the object and its twin.

Wavefront reconstruction had a spectacular revival in 1963 when the laser was first applied to it (Leith & Upatnieks 1962, 1963, 1964; Stroke 1964, 1966). It now became possible to separate the three terms in equation (1) into three waves going in different directions, by using reference waves at a substantial angle to the plate normal. This was not possible with the light of a high-pressure mercury lamp, as used in the original experiments, which has a coherence length of only about 0·1 mm. Moreover, it is no longer necessary to observe the rule '$\gamma = -2$', because the higher-order terms which then appear in the transmission, additionally to those in equation (1), produce diffracted beams of higher order which are angularly separated from the others. This had the great advantage that one could use for the reconstruction the original, negative hologram, which is markedly superior to prints, for reasons which will become evident later on. Another great progress was achieved by the previously mentioned authors when they introduced diffuse illumination of the object. While in the case of a plane or spherical illuminating wave the information is recorded only on an area of the plate corresponding to its diffraction pattern, illumination coming from all sides makes it possible to spread the information on any part of the object over the whole plate. This made it possible to view three-dimensional objects with two eyes, instead of with short-focus optical viewing aids.

A further important idea was added to holography by Denisyuk (1962) even before the laser became available. This was the combination of the basic idea of wavefront reconstruction with Lippmann's (1894) method of colour photography. In both methods an interference pattern is recorded, that is to say standing waves. In transmission holograms the nodal surfaces of the standing waves are nearly at right angles to the surface of the emulsion, therefore one can consider these as approximately plane patterns. In Lippmann's colour photography the object wave was brought to interference with the wave reflected by a plane mirror backing the emulsion, hence the nodal surfaces were planes, parallel to the surface of the plate. Denisyuk combined these two methods, by letting the reference wave fall on the emulsion from the opposite side as the object wave, thus creating, in a fine-grain emulsion, scattering strata spaced by approximately half a wavelength at variable acute angles to the surface. The result was a reflecting or 'deep' hologram (sometimes referred to as 'volume' hologram). As Denisyuk had no laser available, he could give experimentally only an 'existence proof'. The first successful reflecting holograms in 'natural' colours were produced by Stroke & Labeyrie (1966), following the work by Denisyuk (1962, 1963) and interesting ancillary work on transmission colour holograms by Pennington & Lin (1965). Stroke & Labeyrie (1966), moreover, initiated white-light reconstruction of holographic images by showing that the reflecting volume holograms could be made to reconstruct multi-colour images upon illumination with ordinary white light.

Though several further papers have appeared on the subject of deep holograms (for instance Denisyuk 1962, 1963; Friesem 1965; Leith, Kozma, Upatnieks, Marks & Massey 1966; Upatnieks, Marks & Fedorowicz 1966; Kogelnik 1967; Lin, Pennington, Stroke & Labeyrie 1966; Stroke & Zech 1966; Stroke 1967; and many others), a comprehensive theory is still outstanding. We propose to give one below, by a unified method, which reveals the essential features, in particular the absence of the twin object, the colour selectivity and the directional selectivity of deep holograms. These are of particular interest for an important potential application of deep holograms; as projection screens for three-dimensional pictures. For completeness we will discuss also a question which has not yet been satisfactorily elucidated. 'In what form is the information stored in the apparently random pattern of diffuse holograms?'

General formulas for recording and reconstructing deep holograms

We assume an object consisting of a discrete set of numbered points O_n, illuminated in coherent (e.g. laser) light. The points O_n are assumed sufficiently spaced to be resolvable according to diffraction theory, and their spatial phases, relative to a point on the hologram, may be independent. Operating with discrete points greatly simplifies the calculations and it does not restrict the discussion. With sufficiently large holograms and diffusely scattering objects ('diffuse illumination') the diffraction-limited resolution is in most cases so fine that it cannot be quite realized owing to technical reasons, such as photographic grain-noise and laser 'speckle'-noise, which we do not propose to discuss here.

Deep, reflecting holograms can be made and used in several ways, two of which are illustrated in figure 1. In the first method (so far exclusively used), the object is at one side of the emulsion, and the point source from which the reference wave is issuing is at the other side. In the reconstruction the hologram is illuminated with a point source in the original position, and a *virtual* image of the object appears on the other side, and can be viewed through the hologram. In the second method a lens system or the like is used for producing a point focus, that is to say a *virtual* source at the same side of the object. In the reconstruction a real source is placed into this point, and one obtains a *real* image of the object, in the original position but with all rays reversed. In figure 2, which explains the notation, we have assumed the second method, so as to have all coordinates at the same side, but the equations apply equally to both cases.

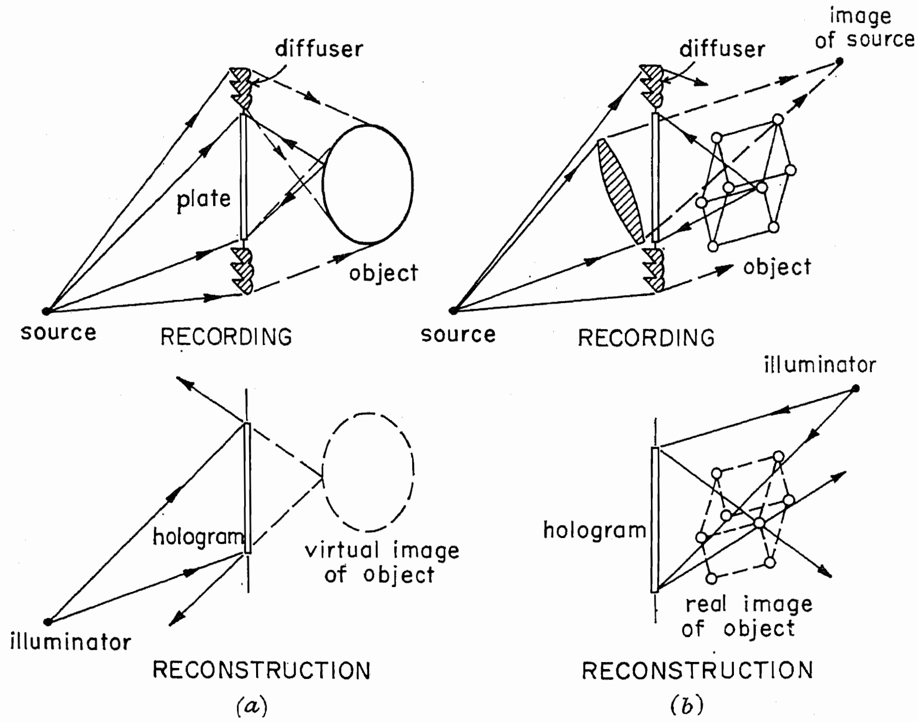

FIGURE 1. Two methods for recording and reconstructing deep holograms.

The *notation*, as illustrated in figure 2, is as follows:

$P(X, Y, Z)$ is the convergence point of the reference wave in the taking.

$P'(X', Y', Z')$ is the illuminating point in the reconstruction.

$O_n(x_n, y_n, z_n)$ is the nth object point in the recording.

$O'_n(x'_n, y'_n, z'_n)$ is the image of the nth object point in the reconstruction when P is shifted to P'.

$Q(\xi, \eta, \zeta)$ is an arbitrary point in the reconstruction space.

$S(x, y, z)$ is a scattering point in the emulsion.

When the points P, P', O_n, O'_n are at such large distances that their waves can be considered as plane in the areas of the emulsion which we take into consideration, we describe their direction by the wave vectors $\mathbf{K}$, $\mathbf{K}'$, $\mathbf{k}_n$ and $\mathbf{k}'_n$ whose components are the direction cosines, multiplied by $k = 2\pi/\lambda$. The components in the x, y, z directions will be numbered 1, 2, 3. The time factor will be assumed as usual to be $e^{(-i\omega t)}$, so that $e^{(ikz)}$ is a forward wave, e^{-ikz} a backward wave. The distances R, R' R_n, R'_n and the direction cosines are always positive.

Photography

The amplitude of the reference wave passing through S to the convergence point P is proportional to $(1/R)\, e^{ikR}$.

From here on we neglect the variations of R inside the small regions of x, y, z, and write for the amplitudes:

$$\textit{reference wave } A\, e^{ikR}, \textit{ wave from nth object point } a_n\, e^{-ikR_n}.$$

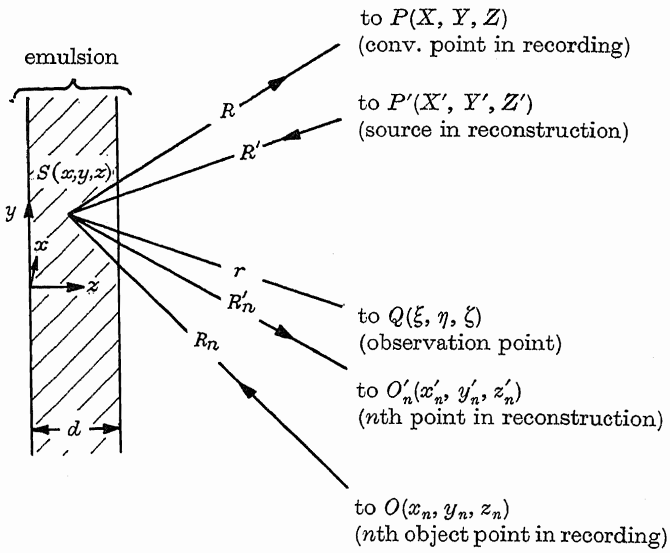

FIGURE 2. Explanation of notation.

The intensity I at the point x, y, z of the emulsion is then

$$I = (AA^* + \Sigma a_n a_n^*) + [A\Sigma a_n^* \, e^{ik(R+R_n)} + A^*\Sigma a_n \, e^{-ik(R+R_n)}]. \tag{2}$$

The effect of this intensity is to produce, in a fine-grain emulsion an absorption which is a function of the sum of the two terms, and scattering proportional to the last term only. If the emulsion is bleached, the bracketed first term, s_0, causes only an almost uniform change of the refractive index (in practice with some haze) and only the square-bracketed second term s_1 will cause scattering. This can be written in the real form

$$s_1 = \Sigma(Aa_n^* + A^*a_n) \cos k(R+R_n). \tag{3}$$

Taking this as a measure of the scattering power, that is to say neglecting the non-linearity of the emulsion is a good approximation in the photographic materials at present available, in which the change in the refractive index is at most a few parts per cent of the mean. A somewhat larger error has been made by assuming the waves to penetrate the emulsion without a loss. In fact the amplitude absorption of red light in Eastman Kodak 649 F emulsions of 20 μm thickness is of the order of 15 %, much less for green and blue. We can neglect this for the present, and also the dimensional change which emulsions suffer during the processing, which can be kept within tolerable limits.

Reconstruction

We illuminate with $A' e^{-ik'R'}$, a wave coming from P'. In the first Born approximation this produces a secondary scattered spherical wavelet

$$(s_1 A'/R'r) \, e^{-ik'(R'+r)}.$$

Again neglecting the small variation of $R'r$, the scattering centre at x, y, z will produce at the space point $Q(\xi, \eta, \zeta)$ an amplitude proportional to

$$A'A\Sigma a^* \, e^{i[k(R+R_n)-k'(R'+r)]} + A'A\Sigma \, e^{-i[k(R+R_n)+k'(R'+r)]}. \tag{4}$$

If we make $k = k'$, $R' = R$, we see that it is the first term which gives a wave converging upon $r = R_n$. Comparing this with equation (1) we see that in the method shown in figure 1*b* it is the 'twin' wave, with complex amplitude a_n^* which is reconstructed. This method gives a real image. If we had followed the method in figure 1(*a*), which gives a virtual image, we would have reconstructed the complex amplitude a_n. This makes no difference, as either of these methods gives a luminous point in the correct position, one real, the other virtual, with an intensity $a \,.\, a^*$.

So far the formulae were the same as in holography with vanishingly thin emulsions, and they give the obvious result that it does not matter where we put the scattering points, so long as we consider them singly. Important differences between 'plane' and 'deep' holography arise only when we sum the wavelets, that is to say integrate over the emulsion volume. We then find three important features of deep holograms which distinguish them from plane ones:

(1) Directional selectivity, that is to say the illuminating point can be displaced only in a certain limited region without the intensity of the reconstruction dropping to a small value or zero.

(2) Colour selectivity, that is to say the intensity of the reconstruction drops off with the departure of the illuminating wavelength from the original.

(3) The second, unwanted wave, which in plane holography always contains the same energy as the wave which reconstructs the original, is almost completely suppressed.

All three are very valuable properties. The third is a practically ideal method for getting rid of the unwanted second image. The second gives holographic images in natural colours. The first has been proposed by van Heerden (1963) for multiple data storage and has been used by several authors (for example, Leith *et al.* 1966), for multiple images in the same emulsion. It may become even more important for three-dimensional projection by means of holographically produced screens.

Directional selectivity

In order to separate directional and colour selectivity, we make $k' = k$, that is to say we reconstruct with the original wavelength, assuming of course that the dimensions of the emulsion have not changed in the processing. We use again the first Born approximation, that is to say we neglect the loss of the illuminating beam as it penetrates into the processed emulsion by absorption and by scattering, and we neglect multiple scattering Both are admissible at the present stage of Lippmann emulsions, which so far have given maximum (intensity) reflectances of only 5 to 6 %, but more accurate calculations would be necessary if further development leads to reflectances of 20 % or more.

We must integrate the first term in equation (4) which gives rise to the reconstruction over the volume of the emulsion. We use the approximation

$$kR = k[(X-x)^2+(Y-y)^2+(Z-z)^2]^{\frac{1}{2}}$$

$$\approx k\left[R_0 - \frac{xX+yY+zZ}{R_0} + \frac{1}{2}\,\frac{(xX+yY+zZ)^2-(x^2+y^2+z^2)R_0}{R_0}\right]$$

where $R_0^2 = X^2 + Y^2 + Z^2$. The last term becomes important only at small distances, and it makes the difference between 'Fresnel' and 'Fourier' holograms (Stroke & Falconer 1965; Stroke 1965). We neglect it here, because we can learn all the essentials from Fourier holograms. We put therefore, introducing the wave-vector components K_1, K_2, K_3

$$kR = kR_0 - \frac{xX + yY + zZ}{R_0} = kR_0 - (K_1 x + K_2 y + K_3 z), \tag{5}$$

and similarly for R', R_n and r. The wave-vector components of r will be called k_1, k_2, k_3. Dropping a phase factor independent of the position x, y, z of the emulsion point, the wavelet amplitude becomes

$$A'A\Sigma a_n^* \exp i[(K_1' - K_1 + k_1 - k_{n1})x + (K_2' - K_2 + k_2 - k_{n2})y + (K_3' - K_3 + k_3 - k_{n3})z]. \tag{6}$$

This has to be integrated over x, y, z to give the reconstructed wave amplitude as a function of k_1, k_2, k_3, that is to say of the direction.

The integration interval in x, y is practically unlimited, hence after integration the factors in x, y become delta functions. This means that light corresponding to the nth object point will be emitted only in the directions in which

$$\left.\begin{aligned} k_1 - k_{n1} &= -(K_1' - K_1), \\ k_2 - k_{n2} &= -(K_2' - K_2). \end{aligned}\right\} \tag{7}$$

This means that the bisector of $\mathbf{K}'$ and $\mathbf{k}_n'$ in the reconstruction is the same as the bisector of $\mathbf{K}$ and $\mathbf{k}_n$ in the recording, that is to say it is as if they were reflected at the same mirror. But if k_1 and k_2 are given by equation (7), k_3 is also determined by the equation

$$k_1^2 + k_2^2 + k_3^2 = k^2 = (2\pi/\lambda)^2; \tag{8}$$

hence the third factor in equation (6) is also determined. After integration through the emulsion thickness d this gives the *directional selection factor*

$$S_d = \frac{\sin \frac{1}{2}(K_3' - K_3 + k_3 - k_{n3})\, d}{\frac{1}{2}(K_3' - K_3 + k_3 - k_{n3})d}. \tag{9}$$

The selection factor is therefore a function of the argument

$$s = (K_3 + k_{n3}) - (K_3' + k_3), \tag{10}$$

and it has the form of a 'slit' or 'sinc' function

$$S_d = \sin(\tfrac{1}{2}sd)/\tfrac{1}{2}sd.$$

The first term in equation (10) relates to data in the recording, the second in the reproduction. Note that all the $\mathbf{K}-s$ and $\mathbf{k}-s$ are vectors of the same length; one can visualize them as unit vectors. By equations (7) the two interfering rays in the recording and the incident and the reflected ray in the reconstruction have the same bisector. In equation (10) the first term is the sum of the projections of the two vectors used in the taking on the plate normal, and this is equal to twice their projection on the bisector B, projected on the plate normal N. The second term is

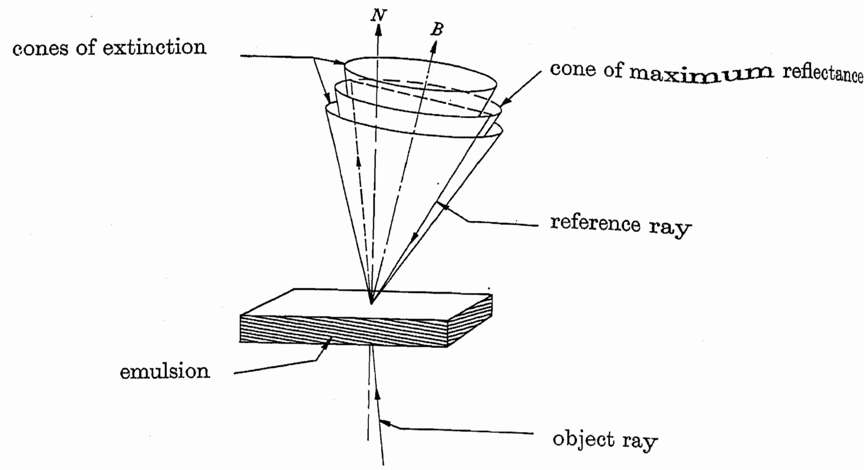

FIGURE 3. Illustrating the directional selectivity of the reflectance of deep holograms.

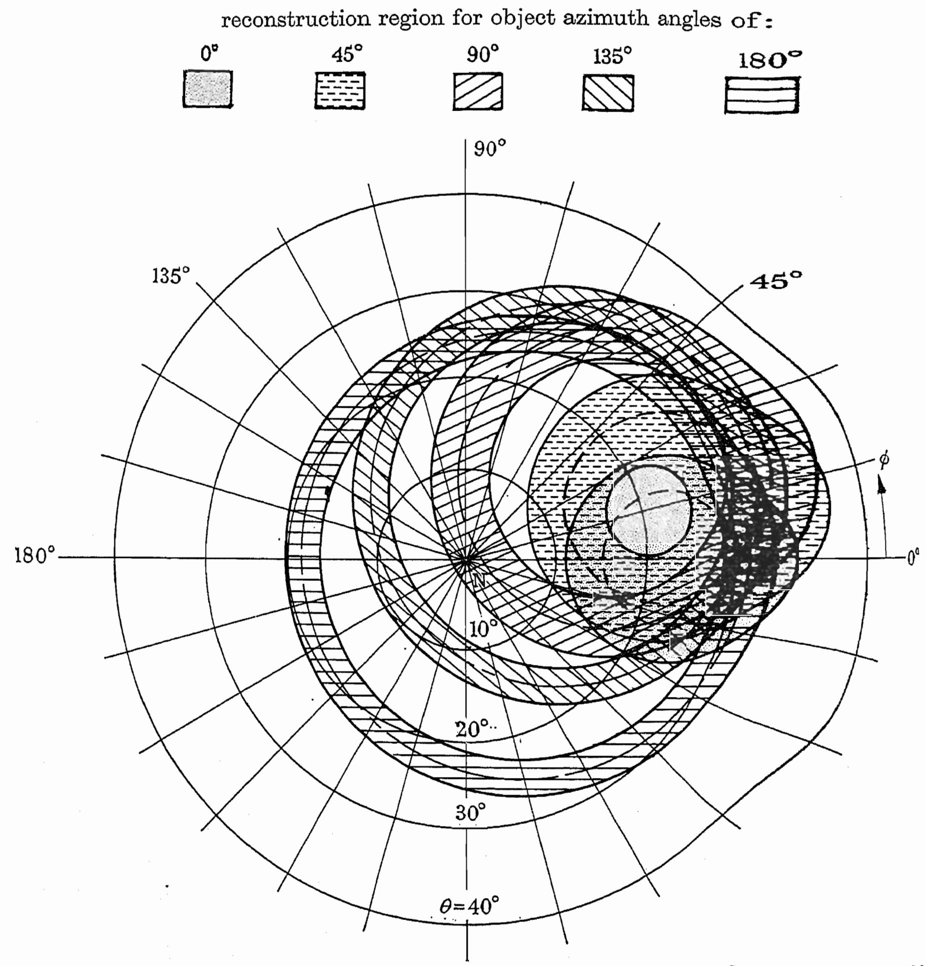

FIGURE 4. Extinction curves for the case of an object ray at 15° and reference ray at 30 plate normal (or vice versa) at various azimuths. Stereographic projection.

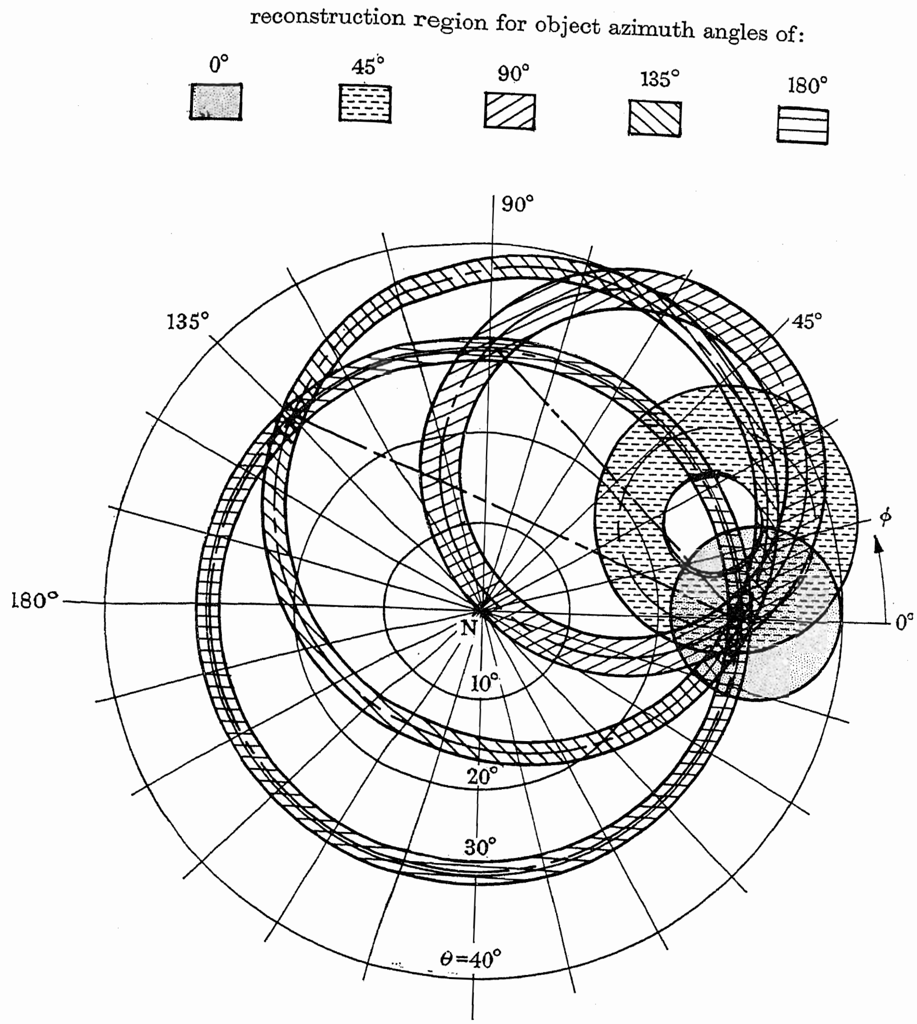

FIGURE 5. Extinction curves for the case of object ray and reference ray both at 30° to plate normal at various azimuths between them. Stereographic projection.

the same, in the reconstruction. We have therefore the simple geometrical interpretation of the argument

$$\tfrac{1}{2}sd = kd[\cos(R, B) - \cos(r, B)]\cos(B, N). \qquad (11)$$

The *loci* of constant selection factor are therefore *cones*, coaxial with the bisector B; $\cos(r, B) = \text{const}$. The geometrical conditions are illustrated in figure 3.

Of chief interest in the practical applications are the *extinction contours*; the *loci* on which $\tfrac{1}{2}sd = \pm\pi$. These are two cones, coaxial with, and at the two sides of the cone $s = 0$, which is the cone of maximum reflectance. Under certain conditions, when the cone of maximum reflectance is narrow, only one of these two extinction cones will be real. In order to obtain a representation useful for practical applications, we have shown in figures 4 and 5, the intersections of these cones with the unit sphere, in stereographic projection, so that the cones are represented by circles. W have assumed here, as a practical example $kd = 2\pi/\lambda = 100$, that is to say an emul sion thickness of 15·9 wavelengths. (For instance, red light $\lambda = 0{\cdot}63$ μm, refractiv index of emulsion 1·5, $d = 15{\cdot}9 \times 0{\cdot}42 = 6{\cdot}7$ μm.) This is a very moderate thicknes for deep emulsions, but even so it is seen that when the angles between the referenc ray and the object ray are not too small, the zone of visibility will extend over con angles of a few degrees only. For larger d/λ ratios the width of these zones is propor tionally reduced. Note that all polar angles θ in figures 4 and 5 are *in* the medium o the emulsion, hence these will appear magnified in the object space.

Our thanks are due to Mr Carl Leonard for the extensive computations which le to these graphs.

Colour selectivity

In equation (4) we now put $\mathbf{K}' = \mathbf{K}$, i.e. we illuminate in the reconstruction fro the original position, but we make $k' \neq k$, that is to say we illuminate with a differe wavelength. Integration through the emulsion thickness d now leads to a colou selection factor

$$S_c = \frac{\sin \frac{1}{2}(k'-k)(\cos\theta + \cos\theta_n)\, d}{\frac{1}{2}(k'-k)(\cos\theta + \cos\theta_n)\, d}, \quad (1$$

where θ and θ_n are the polar angles (angles with the plate normal) of the reference ra and of the object ray, both in the medium of the emulsion. This formula is we known. For small incidence angles the first zeros of the reflectance are at

$$\left.\begin{aligned} \frac{1}{\lambda'} - \frac{1}{\lambda} &= \pm\frac{1}{2d}, \\ \text{or} \quad \frac{\lambda'-\lambda}{\lambda'} &= \mp\frac{\lambda}{2d}. \end{aligned}\right\} \quad (1$$

For practical applications it is of interest to note that a high degree of direction selectivity can be achieved at the same time as a reasonably high reflectance f white light. For instance in the previous example $kd = 100$, $d/\lambda = 15{\cdot}9$, the zeros S_c are at $\pm 0{\cdot}03$ of the original wavelength, which corresponds approximately to 'window' of 3 %, or about 180 Å for the middle of the visible spectrum. Hence wi 'white laser light' with about 16 wavenumbers evenly spaced, we could achie uniform reflectance for white light, and yet sufficient colour purity to satisfy the ey This can be approached by combining neon, argon-ion and krypton-ion lasers the recording. Of course, with the photographic media at present easily availab the total reflectance for white light will still be of the order of 3–6 % only, but the is no fundamental reason why these media should not be further improved.

The second wave

By equation (4) the second wave has a phase factor

$$\exp\left[-\mathrm{i}k(R + R_n + R' + r)\right].$$

Integration over x, y gives for the direction of emission, instead of the equations (

$$\left.\begin{aligned} k_1 + k_{n1} &= -(K_1' + K_1), \\ k_2 + k_{n2} &= -(K_2' + K_2). \end{aligned}\right\} \quad ($$

We have now to distinguish two cases. Equations (14) give the projection of the (k_1, k_2, k_3) vector (the reflected ray) on the x, y plane. With $\mathbf{K}' = \mathbf{K}$, i.e. with the illuminator in the original position, this projection is

$$(k_1^2 + k_2^2)^{\frac{1}{2}} = [(2K_1 + k_{n1})^2 + (2K_2 + k_{n2})^2]^{\frac{1}{2}}.$$

So long as this is smaller than $k = 2\pi/\lambda$, the z component k_3 will be real, otherwise it will be imaginary. A simplified discussion will be sufficient to show up the essentials.

First case, k_3 real. For simplicity let the bisector of $\mathbf{K}, \mathbf{k}_n$ be in the z direction, so that

$$K_1 + k_{n1} = 0, \quad K_2 + k_{n2} = 0.$$

In this case

$$k_3 = k \cos\theta$$

and integrating over the thickness d of the emulsion we obtain a selection factor

$$S_2 = \frac{\sin(2k\cos\theta d)}{2k\cos\theta d} \approx \frac{\sin 2kd}{2kd}. \tag{15}$$

The maximum amplitude reflectance for the first wave must be multiplied by this factor to give the reflectance for the second wave. As kd is of the order of 100 or more, we obtain intensity reflectances of the order of 10^{-5} or less, that is to say practically complete extinction of the second wave. Moreover, this weak wave moves into the emulsion, away from the observer.

Second case, k_3 imaginary. We now obtain an imaginary k_3, that is to say a complex argument in the z factor of the amplitude

$$\exp[-\mathrm{i}(K_3 + K_3' + k_{n3} + k_3)z]$$

that is to say exponential damping in the z direction. This means an *evanescent* second wave, and complete extinction.

In what form is the information contained in a random hologram?

Holograms recorded with sufficiently diffused illumination have, to the eye, the appearance of completely random 'noise'. It is implicit in our previous formulae that they contain the full information on the object, because they have only to be illuminated with the original wavelength, or in the case of deep holograms, with white light, in order to make an image of the original object reappear. There must be therefore some order in the apparently random, noise-like pattern.† We propose to show that the information is contained in the autocorrelation function of the density pattern in the x, y planes, while the colour information is, as may be expected, contained in the third dimension, in the thickness.

The signal energy

The light intensity at any point of the hologram is by equations (2) and (3)

$$I = (AA^* + \Sigma a_n a_n^*) + \Sigma(A a_n^* + A^* a_n)\cos k(R + R_n). \tag{16}$$

The first term, which may be called I_0, is very nearly uniform over the hologram area, in the case of sufficiently wide-angle diffuse illumination, and nearly uniform also

† The grating-like nature of holograms has been previously discussed by one of us (Stroke 1964, 1966); see also Stroke (1967).

over its depth, if one neglects the absorption and haze of the unexposed emulsio The second term, i, is the signal intensity. Its mean value is zero, but we can co veniently define its 'energy' as its mean square fluctuation around this value. Th is perfectly in keeping with communication theory, where the signal energy power is defined as the sum of the powers of the independent, orthogonal sinusoid Fourier components of which it is composed. We wish to calculate therefore

$$\overline{\delta i^2} = \overline{i^2} - (\overline{i})^2.$$

This we do in two stages, first summing over the object, then averaging over t plane. Assuming random distribution of the phases between the object points, th is, as $\overline{i} = 0$

$$\overline{\delta i^2} = \Sigma(Aa_n^* + A^*a_n)^2 \, \overline{\cos^2 k(R+R_n)}. \quad (1$$

We now average this over the hologram. As a first step we can replace the cosi square factor by its mean value $\frac{1}{2}$, because the phase varies much more rapidly th the amplitude. Hence

$$\overline{\delta i^2} = \tfrac{1}{2}\Sigma(Aa_n^* + A^*a_n)^2 = \tfrac{1}{2}[A^2\Sigma\overline{a_n^2} + 2AA^*\Sigma\overline{a_n a_n^*} + A^{*2}\Sigma\overline{a_n^{*2}}]$$
$$= AA^*\Sigma\overline{a_n a_n^*}. \quad (1$$

In the second step we have taken into consideration that the first and last terms a as often positive as negative. We obtain therefore the simple and general result th the signal energy is the product of the reference intensity AA^* and the mean obj intensity $\Sigma\overline{a_n a_n^*}$. At a given total intensity (reference plus object intensities) this a maximum when the two are equal, and then it is just one quarter of the square the total intensity. This is the optimum adjustment, which has to be rather carefu maintained in deep holography, though it is not at all critical in thin hologran where the reference beam can be 3 to 5 times stronger than the object beam witho appreciable loss in diffraction efficiency (Stroke 1964, 1966; Friesem, Kozma Adams 1967).

The autocorrelogram of a random hologram

In a sufficient approximation we can consider the density and/or scattering pov $d(x,y,z)$ in a hologram proportional to the signal intensity at this point. We n wish to calculate the autocorrelation function

$$\overline{d(x,y,z)\,d(x+\Delta x, y+\Delta y, z+\Delta z)} \equiv \Phi(\Delta x, \Delta y, \Delta z),$$

where the averaging is over the volume in the correlogram, or a plane in it. We ha apart from an unimportant proportionality constant

$$d(x,y,z) = \tfrac{1}{2}\Sigma(Aa_n^* + A^*a_n)\{\exp[ik(R+R_p)] + \text{conj.}\}, \quad ($$

with

$$kR = kR_0 - (K_1x + K_2y + K_3z), \quad kR_n = kR_{n0} - (k_{n1}x + k_{n2}y + k_{n3}z)$$

and, in sufficient approximation,

$$K_3 = [k^2 - (K_1^2 + K_2^2)]^{\frac{1}{2}} \approx k - (K_1^2 + K_2^2)/2k, \quad ($$

and similarly for k_{n3}.

Substitution into equation (20) gives

$$\Phi = \frac{1}{4}\Big\langle\Big\{\Sigma(Aa_n^* + A^*a_n)\exp[ik(R_0 + R_{n0} - 2z)]\exp - i\Big[(K_1 + k_{n1})x + (K_2 + k_{n2})y - \frac{K_1^2 + K_2^2 + k_{n_1}^2 + k_{n_2}^2}{2k}z\Big] + \text{conj.}\Big\}$$

$$\times\Big\{\Sigma(Aa_n^* + A^*a_n)\exp[ik\{R_0 + R_{n0} - 2(z + \Delta z)\}]\exp - \Big[(K_1 + k_{n1})(x + \Delta x) + (K_2 + k_{n2})(y + \Delta y) - \frac{K_1^2 + K_2^2 + k_{n_1}^2 + k_{n_2}^2}{2k}(z + \Delta z)\Big] + \text{conj.}\Big\}\Big\rangle_{\text{av.}}. \tag{21}$$

After averaging over x, y (not over z), this expression simplifies greatly. Only conjugate terms in the two factors give non-zero averages, and only those for identical object points. The autocorrelation function is therefore

$$\Phi(\Delta x, \Delta y, \Delta z) = \overline{d(x, y, z)\,d(x + \Delta x, y + \Delta y, z + \Delta z)}$$

$$= AA^*\Sigma a_n a_n^* \cos\Big\{(K_1 + k_{n1})\Delta x + (K_2 + k_{n2})\Delta y + 2k\Big[1 - \frac{K_1^2 + K_2^2 + k_{n_1}^2 + k_{n_2}^2}{4k^2}\Big]\Delta z\Big\}. \tag{22}$$

This result will be best understood if we consider it separately in the x, y plane and in the z direction, and if we now go over to a continuous object. We then obtain in any one plane $z = \text{const.}$

$$\overline{d(x, y, z)\,d(x + \Delta x, y + \Delta y, z)}$$

$$= AA^*\iint a(k_1, k_2)\,a^*(k_1, k_2)\cos[(K_1 + k_1)\Delta x + (K_2 + k_2)\Delta y]\,dk_1\,dk_2. \tag{23}$$

Here $a(k_1, k_2)\,a^*(k_1, k_2)$ is the intensity emitted by the object in the direction k_1, k_2. In this sense, *the autocorrelation function of density in a plane hologram is the cosine Fourier transform of the intensity distribution of the object*, but with the zero shifted to $k_1 = -K_1$, $k_2 = -K_2$. The bisector of these rays with the reference ray is in the z direction, normal to the plate. The information in an apparently random hologram, obtained with diffuse illumination is thus contained in its autocorrelation function. Any area of such a 'random' hologram large enough to contain the autocorrelation function contains the full information on the object, except for diffraction limitations and of course with a lower signal-to-noise ratio than the whole hologram. The autocorrelogram extends over the area which the Fourier hologram would occupy, if the object were transparent (diffracting, not diffusing), and if it were illuminated with parallel light. Diffuse illumination distributes this interference pattern in a random way over the whole plate area. The random-looking hologram does not therefore consist of random points, but of randomly distributed patterns, each of which contains the full information.

Consider now the correlation in the z direction separately, again with a continuous object. This is

$$d(x,y,z)\,d(x,y,z+\Delta z)$$

$$= AA^* \iint a(k_1,k_2)\,a^*(k_1,k_2)\cos\left\{2k\left[1-\frac{K_1^2+K_2^2+k_{n1}^2+k_{n2}^2}{4k^2}\right]\Delta z\right\}\mathrm{d}k_1\,\mathrm{d}k_2. \quad (24)$$

This is not a Fourier transform, but a type of 'Fresnel transform'. It differs from the 'shadow transforms' (Gabor 1949) now usually called 'Fresnel transforms' only in the factor $\exp(2ik\Delta z)$, [$\exp(-2ik\Delta z)$ in the conjugate]. One could call this the 'Lippmann factor'; a modulation in depth with one-half of the period of wavelength. With all rays nearly at right angles to the plate this factor is dominant, and the depth then contains essentially the colour information. With complicated and directionally extended objects it contains also spatial information, in a complicated form. As in the x, y planes, the regularity is revealed only by the correlation function.

For simplicity we have carried out these calculations only for distant objects, for Fourier holograms; but these general conclusions apply equally well to near objects and Fresnel holograms.

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