# Bitopological Spaces and Complete Regularity

By

D. V. Thampuran

The main object of this paper is to prove that a bitopological space is completely regular iff it is homeomorphic to a subspace of a product of quasimetric spaces. While proving this main result it has also been proved that a quasimetric space is completely regular and the product of completely regular bitopological spaces is completely regular.

It is not difficult to deduce from the main result that a quasiuniform space is completely regular.

Kelly (1) has proved several results on bitopological spaces. But our terminology is different and in our terminology, the results for bitopological spaces can be expressed in the same way as for topological spaces.

We have also proved that a subspace of a regular bitopological space is regular and that the product of regular bitopological spaces is regular.

For topological spaces we follow the terminology of Kelley (2).

Definition 1. Let M be a set and $\mathfrak{J}, \mathfrak{J}'$ two topologies for M. The ordered triple $(M,\mathfrak{J},\mathfrak{J}')$ is said to be a bitopological space. We will say $\mathfrak{J}$ is the left topology, $(M,\mathfrak{J})$ is the left topological space, $\mathfrak{J}'$ is the right topology and $(M,\mathfrak{J}')$ is the right topological space of the bitopological space $(M,\mathfrak{J},\mathfrak{J}')$.

When there is no risk of confusion we will denote this bitopological space by M. Take $cA = M - A$ for $A \subset M$.

Definition 2. A function d on the cartesian product of M with itself to the nonnegative reals is said to be a quasimetric for the set M iff d is such that for all points x, y, z of M

(i) $d(x,y) = 0$ if $x = y$ and

(ii) $d(x,y) \leq d(x,z) + d(z,y)$.

We will say (M,d) is a quasimetric space.

Let $\mathcal{T}$ be the family of all subsets T of M such that $x \in T$ implies $\{y: d(y,x) < r\} \subset T$ for some $r > 0$. Then $\mathcal{T}$ is a topology for M.

Denote by $\mathcal{T}'$ the family of all subsets T of M such that $x \in T$ implies $\{y: d(x,y) < r\} \subset T$ for some $r > 0$. Then $\mathcal{T}'$ is also a topology for M.

Definition 3. We will call $\mathcal{T}$ the left topology, $\mathcal{T}'$ the right topology and $(M, d, \mathcal{T}, \mathcal{T}')$ the bitopological space of the quasimetric d.

We will usually denote the bitopological space $(M, d, \mathcal{T}, \mathcal{T}')$ by (M,d) and also call it a quasimetric space.

For the reals, define a quasimetric m as follows: $m(x,y) = y - x$ for all real x,y.

Definition 4. We will call m the usual quasimetric for the reals and the bitopological space of m the usual bitopological space for the reals.

Definition 5. Let $(M, \mathfrak{T}, \mathfrak{T}')$, $(N, \mathfrak{N}, \mathfrak{N}')$ be two bitopological spaces and f a function from M to N. We will say f is continuous iff it is $\mathfrak{T}$-$\mathfrak{N}$ continuous and $\mathfrak{T}'$-$\mathfrak{N}'$ continuous and f is a homeomorphism iff it is one to one, it is continuous and its inverse is continuous. The two spaces are said to be homeomorphic iff there exists a homeomorphism from one space to the other.

Definition 6. Let J be an index set and let $(M_j, \mathfrak{T}_j, \mathfrak{T}_j')$, $j \varepsilon J$ be a family of bitopological spaces. Denote by $M$, $\mathfrak{T}$, $\mathfrak{T}'$ the product respectively of the sets $M_j$, the topologies $\mathfrak{T}_j$ and the topologies $\mathfrak{T}_j'$. Then the bitopological space $(M, \mathfrak{T}, \mathfrak{T}')$ is said to be the product of the bitopological spaces $(M_j, \mathfrak{T}_j, \mathfrak{T}_j')$, $j \varepsilon J$.

Definition 7. Let $(M, \mathfrak{T}, \mathfrak{T}')$ be a bitopological space and let N be a subset of M. Let $\mathfrak{N}$, $\mathfrak{N}'$ be the relativizations respectively of $\mathfrak{T}$, $\mathfrak{T}'$ to N. Then $(N, \mathfrak{N}, \mathfrak{N}')$ is said to be a subspace of $(M, \mathfrak{T}, \mathfrak{T}')$.

Definition 8. Let $(M, \mathfrak{T}, \mathfrak{T}')$, $(N, \mathfrak{N}, \mathfrak{N}')$ be two bitopological spaces and f a function from M to N. We will say f is an open map iff

(i) $B \varepsilon \mathfrak{T}$ implies the image fB, of B under f, is in $\mathfrak{N}$ and
(ii) $B' \varepsilon \mathfrak{T}'$ implies $fB' \varepsilon \mathfrak{N}'$

Let $(M, \mathfrak{T}, \mathfrak{T}')$ be a bitopological space and F a family of functions f such that f maps M into a bitopological space $N_f$. Denote by $(N, \mathfrak{N}, \mathfrak{N}')$ the product of the bitopological spaces $N_f$, $f \in F$. Define the function e from M to N by $e(x)_f = f(x)$, i.e. the f-th coordinate of e(x) is f(x).

Definition 9. The function e defined above is called the evaluation map of M into N. The family F of functions is said to distinguish points iff for each pair of distinct points x, y of M there is f in F such that $f(x) \neq f(y)$. The family F is said to distinguish points and closed sets iff

(i) a subset B of M is closed in $\mathcal{J}$ and $x \in cB$ imply there is f in F such that f(x) is not in the closure of fB in the left topology of $N_f$ and

(ii) a subset $B'$ of M is closed in $\mathcal{J}'$ and $y \in cB'$ imply there is g in F such that g(y) is not in the closure of $gB'$ in the right topology of $N_g$.

Embedding Lemma. Let F be a family of continuous functions f such that f maps the bitopological space M into the bitopological space $N_f$. Denote by N the product of the bitopological spaces $N_f$, $f \in F$. Then

(i) the evaluation map e is a continuous function from M to N

(ii) e is an open map of M onto eM if F distinguishes points and closed sets

(iii) e is one to one iff F distinguishes points

Proof. The last part is obvious. The first part follows since e followed by projection $P_f$ into the f-th coordinate space is continuous since $P_f e(x) = f(x)$. The second part can be proved a follows. Let B be a set open in the left topology of M and let x be a point of B. There is then a member f of F such that $f(x)$ is not in the closure $C$ of $fcB$ in the left topology of $N_f$. Now the set of all $y \in N$ such that $y_f \notin C$ is open in the left topology of $N$ and its intersection with $eM$ is a subset of $eB$. Hence $eB$ is a neighborhood of each of its points and so is open in the left topology of $N$. Similarly $e$ maps sets open in the right topology of $M$ into sets open in the right topology of $N$. Therefore $e$ is an open map if $F$ distinguishes points and closed sets.

Let $I$ denote the closed unit interval $[0, 1]$ of the reals. Then the usual quasimetric $m$ for the reals is also a quasimetric for $I$ and so $(I,m)$ is a quasimetric space.

Definition 10. A bitopological space $(M, \mathcal{T}, \mathcal{T}')$ is said to be completely regular iff

(i) $B \in \mathcal{T}$ and $x \in B$ imply there is a continuous function $f$ from $M$ to $I$ such that $fcB = 0$, $f(x) = 1$ and

(ii) $B' \in \mathcal{T}'$, $y \in B'$ imply there is a continuous function $f$ from $M$ to $I$ such that $f(y) = 0$ and $fcB' = 1$

Theorem 1. Let $(M, \mathcal{T}, \mathcal{T}')$ be a completely regular space. Then $M$ is homeomorphic to a subspace of a product of quasimetric spaces.

Proof. Let $B \in \mathcal{T}$ and $x \in B$. There is then a continuous function $f_{B,x}$ from $M$ to $I$ mapping $cB$ into $0$ and $x$ into $1$. And $B' \in \mathcal{T}'$, $y \in B'$ imply there is a continuous function $f_{y,B'}$ from $M$ to $I$ mapping $y$ into $0$ and $cB'$ into $1$. Finally let $(M,d)$ be the quasimetric space where $d(x,y) = 0$ for all $x,y$ in $M$ and let f be the identity mapping from $(M, \mathcal{T}, \mathcal{T}')$ to $(M,d)$ defined by $f(x) = x$. Let F be the family of functions consisting of f, all functions of the form $f_{B,x}$ and all functions of the form $f_{y,B'}$ . Then F distinguishes points and points and closed sets. Hence $(M, \mathcal{T}, \mathcal{T}')$ is homeomorphic to a subspace of a product of quasimetric spaces and this completes the proof.

Let x be a point and A a subset of M. We will write $D(x,A) = \inf\{d(x,y) : y \in A\}$ and $D(A,x) = \inf\{d(y,x) : y \in A\}$.

Theorem 2 Let $(M,d, \mathcal{T}, \mathcal{T}')$ be a quasimetric space and A a subset of M. Then $\{x : D(A,x) = 0\}$ is the closure of A in $(M, \mathcal{T})$ and $\{x : D(x,A) = 0\}$ is the closure of A in $(M, \mathcal{T}')$.

Proof. Let B be the closure of A in $(M, \mathcal{T})$ and let $C = \{x : D(A,x) = 0\}$. Then $x \in C$ and $r > 0$ imply there is $y \in A$ such that $d(y,x) < r$ and so every neighborhood of x in $(M, \mathcal{T})$ intersects A. Hence $x \in B$ which implies $C \subset B$. Next $x \in B$ implies every neighborhood of x in $(M, \mathcal{T})$ intersects A and so for each $r > 0$ there is $y \in A$ such that $d(y,x) < r$. Therefore $D(A,x) = 0$ or $x \in C$. Hence $B \subset C$. Then $B = C$.

The second part of the theorem can be proved in the same way.

Lemma. Let $(M, \mathcal{T})$ be a topological space, $(R, \mathcal{R}, \mathcal{R}')$ the usual bitopological space for the reals and f a function from M to R. Then f is $\mathcal{T}$-$\mathcal{R}$ continuous (lower semi-continuous) iff $x \in M$ and $r > 0$ imply there is a neighborhood A of x such that $f(x) - f(y) < r$ for all $y \in A$. Also f is $\mathcal{T}$-$\mathcal{R}'$ continuous (upper semi-continuous) iff $x \in M$, $r > 0$ imply there is a neighborhood B of x such that $f(y) - f(x) < r$ for all $y \in B$.

Theorem 3. Let $(M, d, \mathcal{T}, \mathcal{T}')$ be a quasimetric space and $(R, \mathcal{R}, \mathcal{R}')$ the usual bitopological space for the reals. Let $f(x) = D(A,x)$ and $g(x) = D(x,A)$ where $x \in M$ and $A \subset M$. Then f and $-g$ are continuous functions from M to R.

Proof. Let x, y, z be points of M. We know $d(z,x) \leq d(z,y) + d(y,x)$. Taking infima for z in A we get $D(A,x) \leq D(A,y) + d(y,x)$. If y is in the neighborhood $\{y: d(y,x) < r\}$ for $r > 0$ of x then $D(A,x) - D(A,y) < r$ and so f is $\mathcal{T}$-$\mathcal{R}$ continuous. In the same way we can prove f is also $\mathcal{T}'$-$\mathcal{R}'$ continuous.

Proceeding similarly we can prove g is $\mathcal{T}$-$\mathcal{R}'$ and $\mathcal{T}'$-$\mathcal{R}$ continuous. Hence $-g$ is $\mathcal{T}$-$\mathcal{R}$ and $\mathcal{T}'$-$\mathcal{R}'$ continuous.

Lemma. Let f, g be continuous functions from a bitopological space M to R and $k > 0$. Then $f + g$ and $kf$ are also continuous.

Theorem 4. A quasimetric space is completely regular.

Proof. Let $(M, d, \mathcal{T}, \mathcal{T}')$ be a quasimetric space, A a closed subset of $(M, \mathcal{T})$ and x a point of cA. Let $D(A,x) = k$. Then k is positive. Define e by $e(u,v) = \min\{k, d(u,v)\}$

for all u,v in M. Then e is a quasimetric for M whose left and right topologies are the same as those of d. Take $E(A,u) = \inf \{ e(v,u) ; v \in A \}$. Define $f(u) = E(A,u)/k$. Then f is a continuous function from M to I such that $fA = 0$ and $f(x) = 1$.

Next let B be a closed subset of $(M, \mathcal{T}')$ and let y be in cB. Let $n = D(y,B)$. Then n is positive. Let $p(u,v) = \min \{ n, d(u,v) \}$. This will make p a quasimetric for M whose left and right topologies coincide with those of d. Let $P(u,B) = \inf \{ p(u,v) : v \in B \}$. Take $g(u) = 1 - P(u,B)/n$. Then g is a continuous function from M to I such that $g(y) = 0$ and $g(B) = 1$.

Theorem 5. The product of completely regular bitopological spaces is completely regular.

Proof. Let J be an index set and $(M_j, \mathcal{T}_j, \mathcal{T}_j')$ a family of completely regular spaces. Denote the product space by $(M, \mathcal{T}, \mathcal{T}')$. Let A be a closed subset of $(M, \mathcal{T})$ and x a point of cA. Then cA is a neighborhood of x in $(M, \mathcal{T})$ and so there is a finite number of open sets $S_i \in \mathcal{T}_i, \ldots, S_k \in \mathcal{T}_k$ such that $x \in P_i^{-1} S_i \cap \ldots \cap P_k^{-1} S_k \subset cA$ where $P_j$ denotes projection into the j - th coordinate space. Denote by $x_j$ the projection of x into the j - th coordinate space. There are then functions $f_i, \ldots, f_k$ such that $f_i$ is continuous from $M_i$ to I, $f_i(M_i - S_i) = 0$, $f_i(x_i) = 1$, etc. Define the function $g_i$ from M to I by $g_i = f_i P_i$, etc. Take

$g(x) = \min \{ g_1(x), \ldots, g_k(x) \}$ . Then $g$ is a continuous function from $M$ to $I$ such that $gA = 0$ and $g(x) = 1$.

It can similarly be proved that $B$ is closed in $(M, \mathcal{T}')$, $y \in cB$ imply there is a continuous function on $M$ mapping $y$ into $0$ and $B$ into $1$.

Lemma. A subspace of a completely regular bitopological space is completely regular.

Hence we get the result: if a bitopological space $M$ is homeomorphic to a subspace of a product of quasimetric spaces then $M$ is completely regular. Combining this with Theorem 1 we thus have

Theorem 6. A bitopological space is completely regular iff it is homeomorphic to a subspace of a product of quasimetric spaces.

Definition 11. Let $(M, \mathcal{T}, \mathcal{T}')$ be a bitopological space. It is said to be regular iff

(i) $A$ is a closed subset of $(M, \mathcal{T})$, $x \in cA$ imply there are disjoint open sets $X, X'$ such that $X \in \mathcal{T}$, $X' \in \mathcal{T}'$, $A \subset X'$, $x \in X$ and

(ii) $B$ is a closed subset of $(M, \mathcal{T}')$, $y \in cB$ imply there are disjoint open sets $Y, Y'$ such that $Y \in \mathcal{T}$, $Y' \in \mathcal{T}'$, $y \in Y'$, $B \subset Y$

It is obvious from the definition that a completely regular bitopological space is regular. But the converse is not necessarily true.

Theorem 7. The product of regular bitopological spaces is regular.

Proof. Let $J$ be an index set. Let $(M_j, \mathcal{T}_j, \mathcal{T}_j')$, $j \in J$ be a family of regular bitopological spaces and $(M, \mathcal{T}, \mathcal{T}')$ their product. Suppose $A$ is a closed subset of $(M, \mathcal{T})$ and $x \in cA$. Then $cA$ is a neighborhood of $x$ in $(M, \mathcal{T})$ and so there are open sets $S_i \in \mathcal{T}_i, \ldots, S_k \in \mathcal{T}_k$ such that $x \in P_i^{-1} S_i \cap \ldots \cap P_k^{-1} S_k \subset cA$ where $P_j$ is the projection from $M$ to $M_j$. Denote by $x_j$ the projection of $x$ into the $j$-th coordinate space. There are then disjoint open sets

$T_i \in \mathcal{T}_i$, $T_i' \in \mathcal{T}_i'$ such that $M_i - S_i \subset T_i'$, $x_i \in T_i$, etc. Let $T$ be the intersection of the inverse projections of $T_i, \ldots, T_k$ and $T'$ the union of the inverse projections of $T_i', \ldots, T_k'$. Then $T$, $T'$ are disjoint, $T \in \mathcal{T}$, $T' \in \mathcal{T}'$, $A \subset T'$ and $x \in T$.

We can prove similarly that $B$ is a closed subset of $(M, \mathcal{T}')$, $y \in cB$ imply there are disjoint sets $Y$, $Y'$ such that $Y \in \mathcal{T}$, $Y' \in \mathcal{T}'$, $x \in Y'$ and $B \subset Y$. This completes the proof.

It is easy to prove that a subspace of a regular bitopological space is also regular.

Definition 12. A bitopological space $(M, \mathcal{T}, \mathcal{T}')$ is said to be normal iff $A$, $B$ are disjoint sets closed respectively in $(M, \mathcal{T})$ and $(M, \mathcal{T}')$ imply there are disjoint sets $C \in \mathcal{T}$, $D \in \mathcal{T}'$ such that $A \subset D$ and $B \subset C$.

Definition 13. A bitopological space $(M, \mathcal{T}, \mathcal{T}')$ is said to be completely normal iff A, B are disjoint sets closed respectively in $(M, \mathcal{T})$ and $(M, \mathcal{T}')$ imply there is a continuous function f from M to I such that $fA = 0$ and $fB = 1$.

It is obvious that complete normality implies normality.

The following results were proved by Kelly (1) but using a different terminology: A normal bitopological space is completely normal and a quasimetric space is normal. But these results are not difficult to prove using the view-point of this paper.

It is clear that normality is not hereditary and that the product of normal bitopological spaces need not be normal.

A topological space $(M, \mathcal{T})$ can be considered to be the bitopological space $(M, \mathcal{T}, \mathcal{T}')$ and then the preceding definitions and results apply to regular, normal, completely regular and completely normal topological spaces. The results of this paper are thus generalizations of the corresponding results for topological spaces.

If $(M, \mathcal{T})$ is a topological space then there is a topology $\mathcal{T}'$ such that $(M, \mathcal{T}, \mathcal{T}')$ is normal and completely regular; it is only necessary to take $\mathcal{T}'$ as the topology having as base the family of all closed subsets of (M, $\mathcal{J}$ ).

## References

1. Kelly, J. C., Bitopological Spaces, Proc. London Math. Soc. (3) 13 (1963) 71 - 89.

2. Kelley, J. L., General Topology, Princeton (1968).