Quasiproximity Spaces

By

D. V. Thampuran

The study of proximity spaces was pioneered by Efremovich (2,3) and Smirnov (4). A quasiproximity does not have the full symmetry of a proximity but has a certain kind of symmetry as can be deduced from its definition and from the fact that the bitopological space of a quasiproximity is completely regular. Quasiproximities are equivalent to the topogenous structures of Császár (1).

Quasiproximity spaces have properties analogous to those of proximity spaces and quasiuniform spaces. For instance, two sets distant from one another, in a quasiproximity space can be functionally separated by a δ-function and a quasiproximity is generated by the family of all the quasiproximities, of quasimetrics, which are finer.

Let M be a set. For a subset A of M we will write cA for the complement of A. If $A = \{x\}$, $B = \{y\}$, C are subsets of M denote cA by cx and the pairs (A,B), (A,C), (C,B) by (x,y), (x,C), (C,y) respectively. If b is a binary relation on the power set of M and A,B are subsets of M let $(A,B) \in cb$ and $(A,B) \notin b$ denote the same fact.

Let q be a binary relation on the power set of M such that for all subsets A, B, C, D of M

Q(1). $(M, \emptyset), (\emptyset, M) \in cq$

Q(2). if A, B intersect then $(A,B) \in q$

Q(3). $A \subset C$, $B \subset D$, $(A,B) \in q$ imply $(C,D) \in q$

Q(4). $(A \cup B, C) \in q$ implies $(A,C) \in q$ or $(B,C) \in q$ and
$(A, B \cup C) \in q$ implies $(A,B) \in q$ or $(A,C) \in q$

Q(5). $(A,B) \in cq$ implies there is a subset X of M such that
$(A,X) \in cq$ and $(cX,B) \in cq$.

Definition 1. We will call q a quasiproximity for M. The ordered pair (M, q) is said to be a quasiproximity space.

A symmetric q is a proximity. If we define $<$ by $A < B$ iff $(A, cB) \in cq$ then $<$ is obviously a topogenous structure as defined by Császár (1).

Definition 2. Let $\mathcal{T}, \mathcal{T}'$ be two topologies for M. Then the ordered triple $(M, \mathcal{T}, \mathcal{T}')$ is said to be a bitopological space. $\mathcal{T}$ and $\mathcal{T}'$ are said to be the left and right topologies of this bitopological space.

When there is no ambiguity we will also denote this bitopological space by M.

Let $\mathcal{T}$ be the family of all subsets T of M such that x in T implies $(cT, x) \in cq$; it is obvious $\mathcal{T}$ is a topology for M. Let $\mathcal{T}'$ be the family of all subsets T of M such that x in T implies $(x, cT) \in cq$; then $\mathcal{T}'$ is also a topology for M.

Definition 3. We will say $\mathcal{T}$ is the left and $\mathcal{T}'$ the right topology and $(M, q, \mathcal{T}, \mathcal{T}')$ the bitopological space of q.

When (M,q) is considered as a bitopological space we will denote it by $(M, q, \mathcal{T}, \mathcal{T}')$.

Let k, k' be the Kuratowski closure functions of $\mathcal{T}, \mathcal{T}'$. Take $i = ckc$, $i' = ck'c$. Then i, i' are the interior functions of k, k'. We will denote the bitopological space $(M, \mathcal{T}, \mathcal{T}')$ also by (M, k, k').

Theorem 1. Let A be a subset of M. Then $iA = \{x : (cA, x) \in cq\}$ and $i'A = \{x : (x, cA) \in cq\}$.

Proof. Let $B = \{x : (cA, x) \in cq\}$. Then $iA \subset B \subset A$. Hence if $iB = B$ then $iA = B$. Let $x \in B$. Then $(cA, x) \in cq$. Hence there is a subset C of M such that $(cA, C) \in cq$ and $(cC, x) \in cq$. Then $y \in C$ implies $(cA, y) \in cq$ and so $y \in B$. This means $C \subset B$ and therefore $cB \subset cC$ which implies $(cB, x) \in cq$. Hence $iB = B$.

The second part can be proved similarly.

Corollary. $kA = \{ x : (A,x) \in q \}$ and $k'A = \{ x : (x,A) \in q \}$.

Theorem 2. $(A,B) \in q$ iff $(kA, k'B) \in q$.

Proof. Let $(A,B) \in cq$. Then $x \in A$ implies $(x,B) \in cq$ and so $x \in i'cB$. Hence $A \subset i'cB = ck'B$. Similarly $B \subset ckA$ and so $kA \subset cB$. Now $(A,B) \in cq$ implies (A,C), $(cC,B) \in cq$ for some subset C of M. Then $kA \subset cC$ and so $(kA,B) \in cq$. Also $cC \subset ck'B$ or $k'B \subset C$ and so $(A,k'B) \in cq$. It now easily follows $(kA,k'B) \in cq$.

The converse is obvious. This completes the proof.

Given a quasiproximity q, define q' by $(A,B) \in q'$ iff $(B,A) \in q$. It is clear q' is also a quasiproximity. The left and right topologies of q' are respectively the right and left topologies of q. Hence we do not get any new topologies from q'.

Definition 4. A bitopological space (M,k,k') is said to be regular iff

(i) $A = kA$, $y \in cA$ imply there are disjoint sets $X = iX$, $X' = i'X'$ such that $A \subset X'$, $y \in X$ and

(ii) $B = k'B$, $x \in cB$ imply there are disjoint sets $Y = iY$, $Y' = i'Y$ such that $x \in Y'$, $B \subset Y$.

A quasiproximity space is obviously regular.

Theorem 3. Let (M,k,k') be a regular bitopological space. Then

(i) a k-closed set A is the intersection of all the k'-open sets containing A

(ii) a k'-closed set B is the intersection of all the k-open sets containing B.

Proof.

(i) Let C be the intersection of all the k'-open sets containing A. If there is a point x in $C - A$ then there is a k'-open set D containing A such that x is in cD but this is a contradiction.

(ii) Proof is similar.

Corollary. A k-open set A is the union of all the k'-closed sets contained in A and a k'-open set B is the union of all the k-closed sets contained in B.

Let (M,k,k') be a regular space. Then $x \in ky$ iff $y \in k'x$. Also $x \notin k'B$ implies $kx \cap k'B = \emptyset$ and $x \in kB$ implies $k'x \cap kB = \emptyset$.

Theorem 4. Let (M,q,k,k') be a bitopological space. Then $kA = \bigcap \{ cB : (A,B) \in cq \}$ and $k'A = \bigcap \{ cB : (B,A) \in cq \}$.

Proof. Let $C = \bigcap \{ cB : (A,B) \in cq \}$. Then $x \in C - kA$ implies $(A,x) \in cq$ and so $C \subset cx$ which is a contradiction. The second part can be proved similarly.

Corollary. $iA = \bigcup \{ B : (cA,B) \in cq \}$ and $i'A = \bigcup \{ B : (B,cA) \in cq \}$.

Definition 5. A bitopological space (M,k,k') is said to be T_1 iff each one-point set is both k-closed and k'-closed.

It is easily seen that a quasiproximity space (M,q) is T_1 iff $(x,y) \in q$ implies $x = y$.

Definition 6. A bitopological space (M,k,k') is said to be Hausdorff iff x,y are distinct points imply

(i) x and y have disjoint k and k'-neighborhoods respectively and

(ii) x and y have disjoint k' and k-neighborhoods respectively.

It is obvious that a quasiproximity space is Hausdorff iff it is T_1.

Let $\mathcal{U}$ be a quasiuniformity for M. For U in $\mathcal{U}$ let $xU = \{ y : xUy \}$ and $Ux = \{ y : yUx \}$. Let $\mathcal{T}$ be the family of all subsets T of M such that x in T implies $Ux \subset T$ for some U in $\mathcal{U}$; then $\mathcal{T}$ is a topology for M. Denote by $\mathcal{T}'$ the family of all subsets T of M such that x in T implies $xU \subset T$ for some U in $\mathcal{U}$; then $\mathcal{T}'$ is also a topology for M. For subsets A,B of M write $(A,B) \in q$

iff each U in $\mathcal{U}$ intersects $A \times B$; then q is a quasiproximity for M.

Definition 7. $\mathcal{T}$ is said to be the left topology, $\mathcal{T}'$ the right topology and $(M, \mathcal{U}, \mathcal{T}, \mathcal{T}')$ the bitopological space of $\mathcal{U}$. We will call q the quasiproximity of $\mathcal{U}$.

It is obvious a quasiuniformity and its quasiproximity have the same bitopological space. In this sense we can say a quasiuniform space is a quasiproximity space.

Definition 8. A function d from the Cartesian product of M with itself to the nonnegative reals is said to be a quasimetric for M iff for all x,y,z in M

(i) $d(x,y) = 0$ if $x = y$ and

(ii) $d(x,y) \leq d(x,z) + d(z,y)$.

The ordered pair (M,d) is said to be a quasimetric space.

Let (M,d) be a quasimetric space. Let $\mathcal{U}$ be the family of all subsets U of $M \times M$ such that $\{(x,y) : d(x,y) < r\} \subset U$ for some $r > 0$. Then $\mathcal{U}$ is a quasiuniformity for M.

Definition 9. $\mathcal{U}$ is said to be the quasiuniformity of d. The quasiproximity, left topology, etc., of $\mathcal{U}$ are also said to be the quasiproximity, left topology, etc., of d.

Let (M,d) be a quasimetric space. For subsets A,B of M if $D(A,B) = \inf\{d(x,y) : x \in A, y \in B\}$, then the quasiproximity q of d is given by $(A,B) \in q$ iff $D(A,B) = 0$.

Definition 10. A subset B of a topological space is said to be compact iff every open cover of B has a finite subcover.

Theorem 5. Let (M,q,k,k') be a quasiproximity space such that

(i) each k-closed set is k'-compact or

(ii) each k'-closed set is k-compact.

Then $(A,B) \in q$ iff $kA \cap k'B \neq \phi$. Hence q is unique.

Proof.

(i) Let $kA \cap k'B = \phi$. Then $x \in kA$ implies $(x,B) \in cq$. Hence there is a C such that $(x,C) \in cq$ and $(cC,B) \in cq$. Therefore $x \in ck'C$ and the family of all such sets $ck'C$, for x in kA, is a cover of kA by k'-open sets and so has a finite subcover $D_1, \ldots, D_n$, say. Let D be the union of $D_1, \ldots, D_n$. Now $(D_j, B) \in cq$ for each $j = 1, \ldots, n$ and so $(D,B) \in cq$. Hence $(A,B) \in cq$. The converse is obvious.

(ii) Proof is similar to that of (i).

The q of Theorem 5 is unique in the sense that if p is a quasiproximity for M such that p has the same bitopological space as that of q then $p = q$.

Define a quasimetric m' for the reals as follows: for all real x,y

$$m'(x,y) = \begin{cases} y - x, & x \leq y \\ 0, & x > y \end{cases}$$

Definition 11. We will call m' the usual quasimetric for the reals and the quasiproximity of m' the usual quasiproximity for the reals. The left topology, etc., of m' are called the usual left topology, etc., for the reals.

Let I be the closed unit interval $[0,1]$ of the reals. The usual quasimetric m' restricted to I is a quasimetric for I. We will also denote by I the bitopological space of m' for I.

Definition 12. Let (M,k,k'), (N,n,n') be two bitopological spaces and f a function from M to N. We will say f is continuous iff f is both k-n continuous and k'-n' continuous.

Definition 13. A bitopological space (M,k,k') is said to be completely regular iff

(i) $A = kA$ and $y \in cA$ imply there is a continuous function f from M to I such that $fA = 0$ and $f(y) = 1$ and

(ii) $B = k'B$ and $x \in cB$ imply there is a continuous function g from M to I such that $g(x) = 0$ and

$gB = 1$.

A completely regular space is obviously regular. Thampuran (5) has proved that the product of completely regular bitopological spaces is completely regular.

Thampuran (6,7) has proved that a bitopological space is completely regular iff it is quasiuniformizable or topogenizable. Hence we have the following result.

Theorem 6. A bitopological space is completely regular iff it is the bitopological space of a quasiproximity.

Let (M,k,k') be a Hausdorff space such that M is both k-compact and k'-compact. Thampuran (8) has proved that then $k = k'$. Hence the bitopological space reduces to a compact Hausdorff topological space and so this is the topological space of a unique proximity.

Let (M,q,k,k'), (N,p,n,n') be two quasiproximity spaces and f a function from M to N. Then f is continuous iff $(A,x) \in q$ implies $(fA,f(x)) \in p$ and $(y,B) \in q$ implies $(f(y),fB) \in p$.

Definition 14. Let (M,q), (N,p) be two quasiproximity spaces and f a function from M to N. Then f is said to be a δ-function (relative to q,p) iff $(A,B) \in q$ implies $(fA,fB) \in p$.

A δ-function is obviously continuous. Also f is a δ-function iff $(X,Y) \varepsilon$ cp implies $(f^{-1}X, f^{-1}Y) \varepsilon$ cq. Let q' and p' be the inverses of q and p; then f is a δ-function relative to q,p iff f is a δ-function relative to q', p'.

Let $(M, \mathcal{U})$, $(N, \mathcal{V})$ be quasiuniform spaces, q,p the quasiproximities of $\mathcal{U}$ and $\mathcal{V}$ and f a function from M to N. Then f is uniformly continuous implies f is a δ-function.

Theorem 7. Let (M,q,k,k') be a quasiproximity space such that

(i) each k-closed set is k'-compact or

(ii) each k'-closed set is k-compact.

Let f be a continuous function from (M,q,k,k') to (N,p,n,n'). Then f is a δ-function.

Proof. Let $(A,B) \varepsilon$ q. Then there is a point x in $kA \cap kB$. Since f is continuous this means f(x) is in both nfA and $n'fB$ and so $(fA,fB) \varepsilon$ p.

Lemma 1. Let (M,q) be a quasiproximity space. For each t in a dense subset D of the positive reals let S(t) be a subset of M such that

(i) $(S(t), cS(u)) \varepsilon$ cq if $t < u$ and

(ii) $\bigcup \{ S(t) : t \varepsilon D \} = M$.

For x in M take $f(x) = \inf \{ t : x \varepsilon S(t) \}$. Then

$\{x : f(x) < u\} \subset S(u)$ and $\{x : f(x) > u\} \subset cS(u)$ for every real u.

Proof. Let x be such that $f(x) < u$. Then $x \in S(t)$ for some $t < u$ and so $x \in S(u)$. This proves the first relation. Next, let y be such that $f(y) > u$. If $y \in S(u)$ then $f(y) \leq u$ which is a contradiction and the second relation follows.

Lemma 2. Let (M,q) be a quasiproximity space. For each t in a dense subset D of the positive reals let S(t) be a subset of M such that

(i) $(S(t), cS(u)) \in cq$ if $t < u$ and

(ii) $\bigcup \{ S(t) : t \in D\} = M$.

Then the function f from M to the reals R defined by $f(x) = \inf \{t : x \in S(t)\}$ is a δ-function relative to q, m′.

Proof. Let A,B be subsets of the nonnegative reals such that $(A,B) \in cm'$. Then there are u,v in D such that s in A and t in B imply $s < u < v < t$. Then $f^{-1}A \subset S(u)$ and $f^{-1}B \subset cS(v)$.

Theorem 8. Let (M,q) be a quasiproximity space, (I,m′) the usual quasiproximity space for the closed unit interval [0,1] and A,B subsets of M such that $(A,B) \in cq$. There is then a δ-function f from M to I such that $fA = 0$ and $fB = 1$.

Proof. Let D be the set of all numbers of the form $a2^{-b}$ where a and b are positive integers. Take $S(t) = M$ for t in D and $t > 1$, take $S(1) = cB$ and take S(0) such that $(A, cS(0)) \varepsilon$ cq and $(S(0), B) \varepsilon$ cq. For t in D and $0 < t < 1$ take t in the form $t = (2m + 1)2^{-n}$ and choose, inductively on n, S(t) to be a set such that $(S(2m2^{-n}), cS(t)) \varepsilon$ cq and $(S(t), cS((2m + 2)2^{-n})) \varepsilon$ cq. Such choice is possible since q is a quasiproximity. Take $f(x) = \inf \{ t : x \varepsilon S(t) \}$.

Definition 15. Let p, q be two binary relations on the power set of M. Then p is said to be finer than q or q is said to be coarser than p iff $q \subset p$.

Let s be the intersection of a family of quasiproximities for M. Then s satisfies all the conditions of a quasiproximity except Q(4). Let u be the union of all the quasiproximities coarser than s. Then u satisfies all the conditions of a quasiproximity except Q(5).

Definition 16. Let A be a subset of M. Then a finite family $A_1, \ldots, A_n$ of nonempty subsets, of A, whose union is A is said to be a partition of A.

The next Theorem is easy to prove.

Theorem 9. Let s be the intersection of a family F of quasiproximities for M and u the union of all the quasiproximities coarser than s. Define a binary relation $q \subset 2^M \times 2^M$ as follows: $(A,B) \in q$ iff $A_1, \ldots, A_a$ and $B_1, \ldots, B_b$ are partitions of A and B imply $(A_j, B_r) \in s$ for some j and some r where $1 \leqslant j \leqslant a$ and $1 \leqslant r \leqslant b$. Then q is a quasiproximity coarser than s and $q = u$; hence q is the finest quasiproximity coarser than each member of F.

Definition 17. The quasiproximity q of Theorem 9 is said to be generated by F.

Let M be a set and (R, m') the usual quasimetric space for the reals. Denote by u the quasiproximity of m'. Let f be a function from M to R and for x,y in M write $d(x,y) = m'(f(x), f(y))$. Then d is a quasimetric for M and if p is the quasiproximity of d then $(A,B) \in p$ iff $(fA, fB) \in u$.

Theorem 10. Let (M,q) be a quasiproximity space. There is then a family of quasimetrics, for M, whose quasiproximities generate q.

Proof. Let $(A,B) \in cq$. By Theorem 8 there is a δ-function f from M to R such that $(fA, fB) \in cu$. Define the quasimetric d for M by $d(x,y) = m'(f(x), f(y))$ and let p be the quasiproximity of d. Then $(A,B) \in cp$. Also p is finer than q since $(X,Y) \in q$ implies $(fX, fY) \in u$ and so $(X,Y) \in p$.

For each pair A,B of subsets of M such that $(A,B) \in cq$ there is a quasiproximity p for M with the properties $(A,B) \in cp$ and $q \subset p$. Let P be the family of all such p for M and g the quasiproximity generated by P. Let $(X,Y) \in q$; then $X_1, \ldots, X_a$ and $Y_1, \ldots, Y_b$ are partitions of X and Y imply there are j,r, $1 \leq j \leq a$, $1 \leq r \leq b$ such that $(X_j, Y_r) \in q \subset p$ for every p in P and so $(X,Y) \in g$. Hence g is finer than q. Also $(A,B) \in cq$ implies $(A,B) \in cp$ for some p in P and so $(A,B) \in cg$; therefore q is finer than g and the proof is complete.

Let (M,q) be a quasiproximity space. Let G be the family of all the quasiproximities of quasimetrics for M such that each member of G is finer than q. It is obvious that q is generated by G. Thampuran (9) has proved that if a quasimetric d is a δ-function then the quasiproximity p of d is finer than q; hence G will contain all such quasiproximities. G is somewhat similar to the gage of a quasiuniformity.

It has thus been shown that quasiproximities have properties similar to those of proximities and quasiuniformities.

References

1. Á. Császár, Foundations of general topology, New York, (1963).

2. V. A. Efremovich, Infinitesimal Spaces, Dokl. Akad. Nauk SSSR 76(1951) 341 - 343 (Russian).

3. V. A. Efremovich, Geometry of proximity, I, Mat. Sb. 31 (73) (1952) 189 - 200 (Russian)

4. Yu. M. Smirnov, On proximity spaces, Mat. Sb. 31 (73) (1952) 543 - 574 (Russian).

5. D. V. Thampuran, Bitopological spaces and complete regularity (to appear)

6. ------ , Bitopological spaces and quasiuniformities (to appear).

7. ------ , Syntopogenous structures and complete regularity (to appear).

8. ------ , Bitopological spaces and compactness (to appear).

9. ------ , Products of quasiproximities (to appear).