Bitopological Spaces and Quasiuniformities

By

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It seems that every property of a uniform space has its analog for quasiuniform spaces. That quasiuniform spaces have a certain kind of symmetry can be inferred from the symmetry exhibited by their bitopological spaces; the problem appears to be the recognition and utilization of this symmetry.

That a quasiuniformity is generated by the family of all uniformly continuous quasimetrics is one of the results proved in this paper.

Nachbin (4) and Tamari (7) have observed that a quasiuniform space gives rise naturally to a bitopological space. Kelly (1), Lane (3), and Stoltenberg (6) have proved several results on quasiuniform and bitopological spaces. Pervin (5) has proved a quasiuniformization theorem for a topological space. Thampuran (8) has proved that a bitopological space is completely regular iff it is homeomorphic to a subspace of a product of quasimetric spaces. But there does not seem to have been a systematic account of the interrelationships between quasiuniform spaces and their bitopological spaces; such an account is attempted here. In the terminology of this paper the analogous nature of the results for uniform and quasiuniform spaces can be clearly exhibited.

Unless otherwise specified the terms used in this paper have the same meaning as in Kelley (2).

Let M be a set and L the cartesian product of M with itself. If $U \subset L$ then $(x,y) \in U$ or xUy will be used to denote the same fact. For $U,V \subset L$ the composition UV will stand for the set of all pairs (x,z) such that xVy and yUz for some y. For $A \subset M$ we will write $AU = \{ y : xUy$ for some x in $A\}$ and $UA = \{ y : yUx$ for some x in $A\}$; if A contains only one point x then we will write xU and Ux for AU and UA. The diagonal which is the set of all pairs (x,x) for x in M will be denoted by Δ. For a subset A of M denote by cA the complement of A.

Definition 1. A nonempty family $\mathcal{U}$ of subsets of L is said to be a quasiuniformity for M iff

(i) each member of $\mathcal{U}$ contains the diagonal

(ii) U in $\mathcal{U}$ implies there is V in $\mathcal{U}$ such that $VV \subset U$

(iii) $U,V \in \mathcal{U}$ imply $U \cap V \in \mathcal{U}$

(iv) $U \subset V \subset L$ and $U \in \mathcal{U}$ imply $V \in \mathcal{U}$.

We will call the ordered pair $(M, \mathcal{U})$ a quasiuniform space.

Definition 2. A subfamily $\mathcal{B}$ of a quasiuniformity $\mathcal{U}$ is called a base for $\mathcal{U}$ iff each member of $\mathcal{U}$ contains a member of $\mathcal{B}$. A subfamily $\mathcal{S}$ of $\mathcal{U}$ is said to be a subbase for $\mathcal{U}$ iff finite intersections of members of $\mathcal{S}$ is a base for $\mathcal{U}$.

Theorem 1. A family $\mathcal{B}$ of subsets of L is a base for a quasiuniformity for M iff

(i) each member of $\mathcal{B}$ contains the diagonal

(ii) $U \in \mathcal{B}$ implies there is V in $\mathcal{B}$ such that $VV \subset U$

(iii) the intersection of two members of $\mathcal{B}$ contains a member.

Theorem 2. A family $\mathcal{S}$ of subsets of L is a subbase for a quasiuniformity for M iff

(i) each member of $\mathcal{S}$ contains the diagonal

(ii) U in $\mathcal{S}$ implies there is V in $\mathcal{S}$ such that $VV \subset U$

Definition 3. Let $\mathcal{T}$, $\mathcal{T}'$ be two topologies for M. Then the ordered triple $(M, \mathcal{T}, \mathcal{T}')$ is said to be a bitopological space. We will call $\mathcal{T}$ the left and $\mathcal{T}'$ the right topology of the bitopological space $(M, \mathcal{T}, \mathcal{T}')$.

When there can be no ambiguity we will denote this bitopological space by M.

Let $\mathcal{U}$ be a quasiuniformity for M. Denote by $\mathcal{T}$ the family of all subsets T of M such that x in T implies $Ux \subset T$ for some U in $\mathcal{U}$; it is clear $\mathcal{T}$ is a topology for M. Let $\mathcal{T}'$ be the family of all subsets T of M such that x in T implies $xU \subset T$ for some U in $\mathcal{U}$. Then $\mathcal{T}'$ is also a topology for M.

Definition 4. We will call $\mathcal{T}$ the left topology of $\mathcal{U}$, $\mathcal{T}'$ the right topology of $\mathcal{U}$, $(M, \mathcal{T})$ the left and $(M, \mathcal{T}')$ the right topological space of $\mathcal{U}$ and $(M, \mathcal{U}, \mathcal{T}, \mathcal{T}')$

the bitopological space of $\mathcal{U}$.

When there is no risk of confusion we will denote both the quasiuniform space $(M, \mathcal{U})$ and its bitopological space by M or $(M, \mathcal{U})$.

Theorem 3. Let $(M, \mathcal{U}, \mathcal{T}, \mathcal{T}')$ be a bitopological space and let A be a subset of M. Then the $\mathcal{T}$-interior of A is $\{x : Ux \subset A \text{ for some } U \text{ in } \mathcal{U}\}$ and the $\mathcal{T}'$-interior of A is $\{x : xU \subset A \text{ for some } U \text{ in } \mathcal{U}\}$.

Proof. Let B be the set of all points x such that $Ux \subset A$ for some U in $\mathcal{U}$. Then the $\mathcal{T}$-interior of A is a subset of B and B is a subset of A. Hence the interior of A is equal to B if B is open. Let $x \in B$. There is then a U in $\mathcal{U}$ such that $Ux \subset A$. Now there is V in $\mathcal{U}$ such that $VV \subset U$. If $y \in Vx$ then $Vy \subset VVx \subset Ux \subset A$. Hence $y \in B$ and so $Vx \subset B$ which implies B is open.

The other part of the theorem can be proved in the same way.

Corollary. Relative to $\mathcal{T}$ the set Ux is a neighborhood of x and the family of all Ux for U in $\mathcal{U}$ is a base for the neighborhood system of x. Also, relative to $\mathcal{T}'$ the set xU is a neighborhood of x and the family of all xU for $U \in \mathcal{U}$ is a base for the neighborhood system of x.

Corollary. If $\mathcal{B}$ is a base (or subbase) for the quasiuniformity $\mathcal{U}$ then for each x, relative to $\mathcal{T}$ and

$\mathcal{T}'$, the families of sets Ux and xU for U in $\mathcal{B}$ are respectively bases (or subbases) for the neighborhood system of x.

Let $\mathcal{U}$ be a quasiuniformity for M and $\mathcal{U}'$ the family of all inverses of members of $\mathcal{U}$. Then the left and right topologies of $\mathcal{U}'$ are respectively the right and left topologies of $\mathcal{U}$. Hence we get no new topologies from $\mathcal{U}'$.

Theorem 4. Let $(M, \mathcal{U})$ be a quasiuniform space and A a subset of M. Then the $\mathcal{T}$-closure of A is $\bigcap \{AU : U \in \mathcal{U}\}$ and the $\mathcal{T}'$-closure of A is $\bigcap \{UA : U \in \mathcal{U}\}$.

Proof. Relative to $\mathcal{T}$, a point x is in the closure of A iff Ux intersects A for each $U \in \mathcal{U}$ and this happens iff $x \in AU$ for each $U \in \mathcal{U}$. The proof for the second part is similar.

Lemma 1. Let V,Y be subsets of L. Then

$$VYV = \bigcup \{Vx \times yV : xYy\} \text{ and}$$

$$V^{-1}YV^{-1} = \bigcup \{xV \times Vy : xYy\}.$$

Let $(M, \mathcal{U}, \mathcal{T}, \mathcal{T}')$ be a bitopological space, $\mathcal{L}$ the product of the topologies $\mathcal{T}, \mathcal{T}'$ in this order and $\mathcal{L}'$ the product of the topologies $\mathcal{T}', \mathcal{T}$ in this order. Then $(L, \mathcal{L}, \mathcal{L}')$ is a bitopological space.

Theorem 5. Let Y be a subset of L. Then the $\mathcal{L}'$-closure of Y is $\bigcap \{VYV : V \in \mathcal{U}\}$ and the $\mathcal{L}$-closure of Y is $\bigcap \{V^{-1}YV^{-1} : V \in \mathcal{U}\}$.

Proof. Let Z be the $\mathcal{L}'$-closure of Y, V a member of $\mathcal{U}$ and xZy. Now $xV \times Vy$ intersects Y iff there is (u,v) in Y such that $(x,y) \in Vu \times vV$, that is, iff $(x,y) \in \bigcup \{ Vu \times vV : uYv \} = VYV$. Hence the result follows.

The other part can be proved in the same way.

Theorem 6. Let $\mathcal{U}$ be a quasiuniformity for M. Then the family of all $\mathcal{L}'$-closed members of $\mathcal{U}$ is a base for $\mathcal{U}$ and the family of all $\mathcal{L}$-closed members of $\mathcal{U}'$ is a base for $\mathcal{U}'$.

Proof. Let U be a member of $\mathcal{U}$. There is then V in $\mathcal{U}$ such that $VVV \subset U$. But VVV contains the $\mathcal{L}'$-closure of V and the first part of the theorem follows. The proof of the second part is similar.

Theorem 7. The $\mathcal{L}$-interior of a member of a quasiuniformity $\mathcal{U}$ is a member of $\mathcal{U}$ and the $\mathcal{L}'$-interior of a member of $\mathcal{U}'$ is a member of $\mathcal{U}'$. Hence the family of all $\mathcal{L}$-open members of $\mathcal{U}$ is a base for $\mathcal{U}$ and the family of all $\mathcal{L}'$-open members of $\mathcal{U}'$ is a base for $\mathcal{U}'$.

Proof. Let U be a member of $\mathcal{U}$. Then the $\mathcal{L}$-interior of U is the set of all (x,y) such that $Wx \times yW \subset U$ for some W in $\mathcal{U}$. Now there is $V \in \mathcal{U}$ such that $VVV \subset U$. It then follows from Lemma 1 that V is a subset of the $\mathcal{L}$-interior of U.

Corollary. Every member of a quasiuniformity $\mathcal{U}$ is a $\mathcal{L}$-neighborhood of the diagonal and every member of $\mathcal{U}'$ is a $\mathcal{L}'$-neighborhood of the diagonal.

If a member U of $\mathcal{U}$ is $\mathcal{L}$-open then $xU \in \mathcal{T}'$ and $Ux \in \mathcal{T}$ for each x. There is a similar result for closed members of $\mathcal{U}$. Hence we have the following relation. If A is a $\mathcal{T}$-neighborhood of x then there is a $\mathcal{T}$-neighborhood B of x such that $B \subset A$ and B is $\mathcal{T}'$-closed; there is a similar result for $\mathcal{T}'$-neighborhoods of x.

Definition 5. A bitopological space $(M, \mathcal{T}, \mathcal{T}')$ is said to be regular iff

(i) A is $\mathcal{T}$-closed and $x \in cA$ imply there are disjoint sets X, X' such that $X \in \mathcal{T}$, $X' \in \mathcal{T}'$, $A \subset X'$ and $x \in X$ and

(ii) B is $\mathcal{T}'$-closed and $y \in cB$ imply there are disjoint sets Y, Y' such that $Y \in \mathcal{T}$, $Y' \in \mathcal{T}'$, $y \in Y'$ and $B \subset Y$.

It is obvious a quasiuniform space is regular.

Definition 6. Let $(M, \mathcal{T}, \mathcal{T}')$ be a bitopological space. It is said to be T_1 iff each one-point set is both $\mathcal{T}$-closed and $\mathcal{T}'$-closed. It is said to be Hausdorff iff x,y are two distinct points imply

(i) there is a $\mathcal{T}$-neighborhood of x and a $\mathcal{T}'$-neighborhood of y which are disjoint and

(ii) there is a $\mathcal{T}'$-neighborhood of x and a $\mathcal{T}$-neighborhood of y which are disjoint.

A quasiuniform space is said to be T_1 or Hausdorff iff its bitopological space has the corresponding property.

It is easily seen that a Hausdorff space is T_1 and that a regular T_1 space is Hausdorff. A quasiuniform space is Hausdorff iff it is T_1. Also a quasiuniform space is Hausdorff iff the intersection of all the members of the quasiuniformity is the diagonal.

Let $(M, \mathcal{T}, \mathcal{T}')$ be regular. If x,y are distinct and the $\mathcal{T}$-closure of x contains y then the $\mathcal{T}'$-closure of y contains x; also the $\mathcal{T}'$-closure of x contains y implies the $\mathcal{T}$-closure of y contains x. If the $\mathcal{T}$-closure of x intersects a $\mathcal{T}'$-closed set B then $x \in B$ and if the $\mathcal{T}'$-closure of x intersects a $\mathcal{T}$-closed set A then $x \in A$.

Uniform Continuity

Definition 7. Let $(M, \mathcal{U})$, $(N, \mathcal{V})$ be quasiuniform spaces and f a function from M to N. Then f is said to be uniformly continuous relative to $\mathcal{U}$ an $\mathcal{V}$ iff for every V in $\mathcal{V}$ the set $\{(x,y) : (f(x),f(y)) \in V\}$ is a member of $\mathcal{U}$. We will say f is a uniform isomorphism iff it is one to one and both f and its inverse are uniformly continuous.

Given a function f from M to N define the function f' by $f'(x,y) = (f(x),f(y))$ for all x,y in M. Then f is uniformly continuous iff for each V in $\mathcal{V}$ there is U in $\mathcal{U}$ such that $f'U \subset V$. If $\mathcal{S}$ is a subbase for $\mathcal{V}$ then f is uniformly continuous iff the inverse under f' of each member of $\mathcal{S}$ is a member of $\mathcal{U}$. It is clear that the composition of two uniformly continuous functions is also uniformly continuous. Let $\mathcal{U}', \mathcal{V}'$ be the families of inverses of members of $\mathcal{U}$ and $\mathcal{V}$; then f is uniformly continuous relative to $\mathcal{U}$ and $\mathcal{V}$ implies f is also uniformly continuous relative to $\mathcal{U}'$ and $\mathcal{V}'$.

Definition 8. Let $(M, \mathcal{T}, \mathcal{T}')$, $(N, \mathcal{N}, \mathcal{N}')$ be two bitopological spaces and f a function from M to N. We will say f is continuous iff f is both $\mathcal{T}$-$\mathcal{N}$ and $\mathcal{T}'$-$\mathcal{N}'$ continuous and f is said to be a homeomorphism iff f is one to one, is continuous and its inverse is continuous. Two bitopological spaces are said to be homeomorphic iff there is a homeomorphism from one to the other.

Theorem 8. A uniformly continuous function is continuous relative to the bitopological spaces of the quasiuniformities and a uniform isomorphism is a homeomorphism.

Definition 9. Let $\mathcal{U}$ and $\mathcal{V}$ be two quasiuniformities for a set. We will say $\mathcal{U}$ is finer than $\mathcal{V}$ or $\mathcal{V}$ is coarser than $\mathcal{U}$ iff $\mathcal{V}$ is a subfamily of $\mathcal{U}$.

Let f be a function from a set M to a quasiuniform space $(N, \mathcal{V})$. Then the family of all inverses under f' of members of $\mathcal{V}$ is a base for a quasiuniformity for M and this is the coarsest quasiuniformity for which f is uniformly continuous.

Let $(M, \mathcal{U})$ be a quasiuniform space and N a subset of M. There is then a coarsest uniformity $\mathcal{V}$ for N such that the identity map of N into M is uniformly continuous. It is obvious $\mathcal{V}$ is the family of all intersections of members of

$\mathcal{U}$ with $N \times N$. Then the topologies of $\mathcal{V}$ are the relativizations of the topologies of $\mathcal{U}$.

Definition 10. Let $(M, \mathcal{U})$ be a quasiuniform space and N a subset of M. Then the $\mathcal{V}$ of the preceding paragraph is called the relativization of $\mathcal{U}$ to N or the relative quasiuniformity for N and $(N, \mathcal{V})$ is called a quasiuniform subspace of $(M, \mathcal{U})$.

Definition 11. Let $(M, \mathcal{T}, \mathcal{T}')$ be a bitopological space and N a subset of M. Let $\mathcal{N}, \mathcal{N}'$ be the relativizations of $\mathcal{T}, \mathcal{T}'$ to N. We will call $(N, \mathcal{N}, \mathcal{N}')$ a subspace of $(M, \mathcal{T}, \mathcal{T}')$.

Definition 12. Let J be an index set and $(M_j, \mathcal{T}_j, \mathcal{T}_j')$, $j \in J$ a family of bitopological spaces. Denote by M the product of the sets M_j, by $\mathcal{T}$ the product of the topologies $\mathcal{T}_j$ and by $\mathcal{T}'$ the product of the topologies $\mathcal{T}_j'$. Then $(M, \mathcal{T}, \mathcal{T}')$ is said to be the product of the spaces $(M, \mathcal{T}_j, \mathcal{T}_j')$, $j \in J$.

Definition 13. Let J be an index set, $(M_j, \mathcal{U}_j)$, $j \in J$ a family of quasiuniform spaces and M the cartesian product of the sets M_j. Then the coarsest quasi-uniformity $\mathcal{U}$ for M such that projection into each coordinate space is uniformly continuous is said to be the product quasiuniformity and we will say $(M, \mathcal{U})$ is the product quasiuniform space.

That the product quasiuniformity exists is obvious. Let f denote projection into the j-th coordinate space and

U a member of $\mathcal{U}_j$. Take V to be the set of all (x,y) such that (f(x),f(y)) ε U. Let $\mathcal{V}_j$ be the family of all sets of the form V as U varies over $\mathcal{U}_j$ and let $\mathcal{V}$ be the union of the families $\mathcal{V}_j$. Then the product quasiuniformity $\mathcal{U}$ has $\mathcal{V}$ as a subbase. Also the bitopological space of $\mathcal{U}$ is the product of the bitopological spaces of $\mathcal{U}_j$.

Theorem 9. A function f from a quasiuniform space to a product of quasiuniform spaces is uniformly continuous iff the composition of f with every projection into a coordinate space is uniformly continuous.

Quasimetrics

Definition 14. A function d from L to the nonnegative reals is said to be a quasimetric for M iff for all x,y,z in M

(i) $d(x,y) = 0$ if $x = y$ and

(ii) $d(x,z) \leq d(x,y) + d(y,z)$.

(M,d) is called a quasimetric space. The quasiuniformity $\mathcal{U}$ having as base the family of all sets of the form $\{(x,y) : d(x,y) < r\}$, $r > 0$ is said to be the quasiuniformity of d. The topologies, etc. of $\mathcal{U}$ are called the topologies, etc., of d. The product of quasimetric spaces is defined to be the product of their quasiuniform spaces.

Let R be the reals. Define a quasimetric m for R as follows:

$$m(x,y) = \begin{cases} y - x & x \leq y \\ 0 & x > y \end{cases}$$

Definition 15. We will call m the usual quasimetric for the reals, the quasiuniformity, etc., of m the usual quasiuniformity, etc., for the reals.

Theorem 10. Let $(M, \mathcal{U})$ be a quasiuniform space, $\mathcal{U}'$ the family of all inverses of members of $\mathcal{U}$ and $\mathcal{P}$, $\mathcal{P}'$ the products respectively of $\mathcal{U}$, $\mathcal{U}'$ (in this order) and $\mathcal{U}'$, $\mathcal{U}$ (in this order). Let d be a quasimetric for M and $L = M \times M$. For $r > 0$ let $V(r) = \{(x,y) : d(x,y) < r\}$. Then

(i) d, from $(L, \mathcal{P}')$ to (R,m), is uniformly continuous iff $V(r)$ is a member of $\mathcal{U}$ for each $r > 0$ and

(ii) d, from $(L, \mathcal{P})$ to (R,m), is uniformly continuous iff $V(r)$ is a member of $\mathcal{U}'$ for each $r > 0$.

Proof.

(i) The family of all sets of the form $\{((x,y),(u,v)) : (u,x),(y,v) \in U\}$, $U \in \mathcal{U}$ is a base for $\mathcal{P}'$. If d, from $(L, \mathcal{P}')$ to (R,m), is uniformly continuous then there is $U \in \mathcal{U}$ such that (u,x), (y,v) in U imply $d(u,v) - d(x,y) < r$. Hence $(u,v) \in U$ implies $d(u,v) - d(v,v) < r$ and so $U \subset V(r)$. This proves necessity. Next let $V(r)$ be a member of $\mathcal{U}$ for each $r > 0$. If (u,x), (y,v) are in $V(r)$ then $d(u,v) \leq d(u,x) + d(x,y) + d(y,v)$ and so $d(u,v) - d(x,y) < 2r$ from which the uniform continuity of d follows.

(ii) Proof is similar to that of (i).

Hereafter, uniform continuity of a quasimetric is used in the sense of part (i) of Theorem 10.

Let Q be a family of quasimetrics for M. Then the family of all sets of the form $V(q,r) = \{(x,y) : q(x,y) < r\}$, for q in Q and $r > 0$ is a subbase for a quasiuniformity $\mathcal{U}$ for M.

Definition 16. The quasiuniformity $\mathcal{U}$, having as subbase the family of all sets of the form $V(q,r)$, for q in Q and $r > 0$, is said to be the quasiuniformity generated by Q.

It follows from Theorem 10 that the quasiuniformity generated by Q is the coarsest one such that each member of Q is uniformly continuous. For a q in Q the family of all sets of the form $V(q,r)$ for $r > 0$ is a base for the quasiuniformity of q. Hence $\mathcal{U}$ is the coarsest uniformity such that the identity map of M into (M,q) is uniformly continuous for each q in Q. We also have the following result. Let P be the product of M with itself as many times as there are members of Q. Assign to the q-th coordinate space the quasiuniformity of q and to P the product quasiuniformity. Define the function f from M to P by $f(x)_q = x$ for each x in M and each q in Q. Then the projection of P into the q-th coordinate space is the identity map of M into (M,q). By virtue of Theorem 9, the uniformity generated by Q is the coarsest which makes f uniformly continuous. Now f is one to one and so it is a uniform isomorphism of M onto a subspace of a product of quasimetric spaces.

Lemma 2. Let $\{V_n : n = 0,1,2,\ldots\}$ be a sequence of subsets of $L = M \times M$ such that $V_0 = L$, every V_n contains Δ and $V_{n+1} V_{n+1} V_{n+1} \subset V_n$ for every n. There is then a quasimetric d for M such that $V_n \subset \{ (x,y) : d(x,y) < 2^{-n} \} \subset V_{n-1}$ for each positive integer n.

A proof of this may be found in Kelley (2).

Theorem 11. Every quasiuniformity for M is generated by the family of all quasimetrics, for M, which are uniformly continuous.

Proof. Let $\mathcal{U}$ be a quasiuniformity for M and let Q be the family of all quasimetrics, for M, which are uniformly continuous. It follows from Theorem 10 that the quasiuniformity generated by Q is coarser than $\mathcal{U}$. But we get from Lemma 2 that if $U \varepsilon \mathcal{U}$ then there is q in Q such that $V(q,\frac{1}{4}) \subset U$. Hence the quasiuniformity generated by Q is finer than $\mathcal{U}$.

Theorem 12. Let M be a quasiuniform space and Q the family of all quasimetrics uniformly continuous on M. Then the quasiuniform space M is uniformly isomorphic to a subspace of the product of the quasimetric spaces (M,q) for q in Q.

Definition 17. A bitopological space is said to be quasi-Hausdorff iff it satisfies at least one of the two conditions, for a Hausdorff space, specified in Definition 6. A quasiuniform space is said to be quasi-Hausdorff iff its bitopological space is quasi-Hausdorff.

Theorem 13. Let (M,d) be a quasimetric space. For x in M denote by x' the intersection of the closures of x in the left and right topologies of d. Let N be the set of all x' for x in M and for x', y' in N let $e(x',y') = d(x,y)$. Then (N,e) is a quasi-Hausdorff quasimetric space and the mapping f from M to N defined by $f(x) = x'$ is an isometry.

Proof. If $y \varepsilon x'$ then $y' \subset x'$. Also $y \varepsilon x'$ implies $x \varepsilon y'$ and so $x' \subset y'$. Hence $y \varepsilon x'$ implies $x' = y'$. Hence $x' \neq y'$ implies x' and y' are disjoint and so $x \not\varepsilon y'$ from which it follows that $d(x,y)$ or $d(y,x) > 0$.

Let $u \varepsilon x'$, $v \varepsilon y'$. Then $d(u,x) = 0 = d(x,u)$ and $d(v,y) = 0 = d(y,v)$. Hence $d(u,v) = d(x,y)$. Therefore $e(x',y')$ is identical with $d(u,v)$ for every u in x' and every v in y'. It follows that (N,e) is a quasi-Hausdorff quasimetric space and that the mapping f is an isometry.

Theorem 14. A quasi-Hausdorff quasiuniform space $(M, \mathcal{U})$ is uniformly isomorphic to a subspace of a product of quasi-Hausdorff quasimetric spaces.

Proof. Let Q be the family of all uniformly continuous quasimetrics on M. Then $\mathcal{U}$ is generated by Q and it is the coarsest quasiuniformity which makes the identity map of M into the quasimetric space (M,q) uniformly continuous for each q in Q. Now there is an isometry f_q from (M,q) to a quasi-Hausdorff quasimetric space (M_q, q^*) and so $\mathcal{U}$ is the coarsest quasiuniformity making each of the maps f_q uniformly continuous. Let P be the product of the quasiuniform space M with itself as many times as there are q in Q and let f be the map of M into P defined by $f(x)_q = f_q(x)$. Then $\mathcal{U}$ is the coarsest quasiuniformity such that f is uniformly continuous. Now f is one to one, since M is quasi-Hausdorff, and in this case f is a uniform isomorphism.

Let I denote the closed unit interval $[0,1]$; we will also denote by I the usual bitopological space for this interval.

Definition 18. A bitopological space $(M, \mathcal{T}, \mathcal{T}')$ is said to be completely regular iff

(i) A is $\mathcal{T}$-closed and x is in cA imply there is a continuous function f from M to I such that $fA = 0$ and $f(x) = 1$ and

(ii) B is $\mathcal{T}'$-closed and y is in cB imply there is a continuous function g from M to I such that $g(y) = 0$ and $gB = 1$.

It is obvious that complete regularity implies regularity.

Definition 19. A bitopological space $(M, \mathcal{T}, \mathcal{T}')$ is said to be quasiuniformizable iff there is a quasiuniformity $\mathcal{U}$ for M such that $\mathcal{T}$, $\mathcal{T}'$ are respectively the left and right topologies of $\mathcal{U}$.

Thampuran (8) has proved that a bitopological space is completely regular iff it is homeomorphic to a subspace of a product of quasimetric spaces. Hence we have the result:

Theorem 15. A bitopological space is quasiuniformizable iff it is completely regular.

Theorem 16. Let $(M, \mathcal{T})$ be a topological space.

There is then a quasiuniformity $\mathcal{U}$ for M such that $\mathcal{T}$ is the left topology of $\mathcal{U}$ and the right topology $\mathcal{T}'$ of $\mathcal{U}$ has the family of all $\mathcal{T}$-closed sets as base.

Proof. If $\mathcal{T}$ is indiscrete the result is obvious; so consider the case where it is not indiscrete. Let A be a nonempty open proper subset of M. Define a quasimetric d for M by:

$$d(x,y) = \begin{cases} 0 & \text{if } x \in M,\ y \in cA \\ 0 & \text{if } x, y \in A \\ 1 & \text{if } x \in cA,\ y \in A \end{cases}$$

Let Q be the family of all such quasimetrics for M, a member of Q being obtained in this manner from each nonempty open proper subset of M. Let $\mathcal{U}$ be the quasiuniformity generated by Q.

The bitopological space of Theorem 13 has another property too. If A, B are disjoint $\mathcal{T}$-closed and $\mathcal{T}'$-closed subsets of M then there are disjoint $\mathcal{T}$-open and $\mathcal{T}'$-open sets X, X′ such that $A \subset X'$ and $B \subset X$. Such a space may be called normal.

Definition 20. A quasiuniform space $(M, \mathcal{U})$ is said to be quasimetrizable iff there is a quasimetric d for M such that $\mathcal{U}$ is the quasiuniformity of d.

It then follows from Theorem 11 that a quasiuniform space is quasimetrizable iff its quasiuniformity has a countable base. If we call a quasimetric d with the property that $d(x,y) = 0$ iff $x = y$ a Hausdorff quasimetric then we have the result: a quasiuniform space is Hausdorff quasi-metrizable iff it is Hausdorff and has a countable base.

Definition 21. A family G of quasimetrics for a set M is said to be a gage iff there is a quasiuniformity $\mathcal{U}$ for M such that G is the family of all quasimetrics which are uniformly continuous. We will also say G is the gage of $\mathcal{U}$ and $\mathcal{U}$ is the quasiuniformity of G. A family Q of quasimetrics for a set M generates a quasiuniformity and Q is also said to generate the gage of this quasiuniformity.

Let Q be a family of quasimetrics for M and let G be the gage generated by Q. The family of all sets of the form V(q,r) for q in Q and r positive is a subbase for the quasiuniformity of G and so a quasimetric g is in G or is uniformly continuous iff the set V(g,s), for each positive s, contains some finite intersection of sets V(q,r) for q in Q.

Every property of a quasiuniformity can be expressed in terms of its gage. We will write

$g\text{-dist}\,(x,A) = \inf\{g(x,y) : y \in A\}$ and

$g\text{-dist}\,(A,x) = \inf\{g(y,x) : y \in A\}$.

Theorem 17. Let $(M, \mathcal{U}, \mathcal{T}, \mathcal{T}')$ be a quasiuniform space and G the gage of $\mathcal{U}$. Then

(i) the family of all sets V(g,r) for g in G and r positive is a base for $\mathcal{U}$

(ii) the $\mathcal{T}$-closure of a subset A of M is the set of all x such that g-dist (A,x) = 0 for each g in G;

the $\mathcal{T}'$-closure of A is the set of all points x such that g-dist (x,A) = 0 for each g in G

(iii) the $\mathcal{T}$-interior of A is the set of all points x such that $V(g,r)x \subset A$ for some g in G and some positive r; the $\mathcal{T}'$-interior of A is the set of all x such that $xV(g,r) \subset A$ for some g in G and some positive r

(iv) a function f from M to a quasiuniform space (N, $\mathcal{V}$) is uniformly continuous iff for each member q of the gage of $\mathcal{V}$ and each positive s there is g in G and r positive such that $g(x,y) < r$ implies $q(f(x), f(y)) < s$

(v) if $(M_j, \mathcal{U}_j)$ is a quasiuniform space and G_j is the gage of $\mathcal{U}_j$ for each j in an index set J then the gage of the product quasiuniformity is generated by all quasimetrics of the form $q(x,y) = g_j(x_j, y_j)$ for j in J and g_j in G_j.

It is shown in the preceding pages that many of the properties of uniform spaces have their analogs for quasi-uniform spaces. For instance the result, that a quasi-Hausdorff quasiuniform space is uniformly isomorphic to a subspace of a product of quasi-Hausdorff quasimetric spaces, takes the place of the similar theorem for Hausdorff uniform spaces.

Other analogous properties of quasiuniform spaces such as completeness, compactness, characterization by coverings, nets, etc., are dealt with in separate papers.

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