Neighborhood Spaces and Convergence

By

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A few of the properties of neighborhood spaces have been given by Mamuzić [2] . The main object of this paper is to use convergence to characterize neighborhood spaces as well as a few others more general than topological spaces.

Kelley [1] has characterized topological spaces by convergence. He makes use of the iterated theorem of convergence, in topological spaces, which is related to the fact that the Kuratowski closure function is idempotent. Hence this method does not seem to be applicable to the case of spaces defined by functions which need not be idempotent. But for topological spaces there is a convergence property which depends on the fact that the Kuratowski closure function is isotone. This property can be used instead of iterated convergence to characterize topological spaces and moreover this makes the proof simpler. Also this property can be used in the characterization of more general spaces by convergence.

Neighborhood Spaces.

Let M be a set with null subset N. Denote by g a set-valued set-function mapping the power set, of M, into itself such that

1. $gN = N$
2. $A \subset gA$ for $A \subset M$
3. $gA \subset gB$ when $A \subset B \subset M$

Definition 1. The ordered pair (M,g) is said to be a neighborhood space.

Definition 2. A nonempty family λ of nonempty subsets of M is said to be a generalized filterbase. We will say λ is in a subset A, of M, iff every member of λ is a subset of A. Define a generalized filterbase λ to be finer than a generalized filterbase μ iff every member of μ contains a member of λ.

For $A \subset M$ denote by cA the complement of A relative to M. Composition of functions will be denoted by juxtaposition; thus cgA will denote c(gA) for $A \subset M$. Take r = cgc.

Definition 3. A subset A of M is said to be a neighborhood of a point x of M iff $x \in rA$. The generalized filterbase λ is said to converge to the point x iff for each neighborhood A of x there is a member B of λ such that $B \subset A$.

It is obvious that if a generalized filterbase λ converges to a point x then every generalized filterbase μ finer than λ also converges to x.

Lemma. $x \in gA$ iff every neighborhood of x intersects A.

Proof. Let $x \in gA$, $x \in rB$. Then B intersects A for if not $B \subset cA$ and so $x \in rB \subset rcA = cgA$ which contradicts $x \in gA$.

Next if every neighborhood of x intersects A then x is in gA. For if not $x \in cgA = rcA$ and so cA is a neighborhood of x which leads to a contradiction.

Theorem 1. $x \in gA$ iff there is a generalized filterbase, in A, which converges to x.

Proof. If $x \in gA$ and B is a neighborhood of x then A and B intersect. The family λ of all intersections of neighborhoods of x with A is a generalized filterbase in A and λ evidently converges to x.

To prove the converse let λ be a generalized filterbase, in A, which converges to x. Then every neighborhood of x contains a member of λ and therefore intersects A. This proves the theorem.

Theorem 2. Let λ be a generalized filterbase which does not converge to x. There is then a subset X of M such that

(a) there is in X a generalized filterbase μ finer than λ

(b) no generalized filterbase in X converges to x.

Proof. Since λ does not converge to x there is a neighborhood A of x such that cA intersects every member of λ. Take $X = cA$. Then the family of all intersections of members of λ with cA is a generalized filterbase, in X, which is finer than λ. Moreover it is obvious that no filterbase in X converges to x.

Definition 4. We will say R is a convergence relation, for a neighborhood space, iff it satisfies the following conditions:

(a) If a generalized filterbase λ consists of the single set $\{x\}$ then $(\lambda, x) \in R$

(b) μ is finer than λ and $(\lambda, x) \in R$ imply $(\mu, x) \in R$

(c) If $(\lambda, x) \notin R$ for a generalized filterbase λ then there is a subset X of M such that there is a generalized filterbase, in X, finer than λ and that $(\mu, x) \notin R$ for all generalized filterbases μ in X

It has already been shown that convergence in (M, g) satisfies these conditions.

Theorem 3. Let R be a convergence relation for a neighborhood space. Define the set-valued set-function g as follows: if $A \subset M$ then gA is the set of all points x such that $(\lambda, x) \in R$ for some generalized filterbase λ in A. Then (M, g) is a neighborhood space and $(\lambda, x) \in R$ iff λ converges to x in the neighborhood space (M,g).

Proof. Evidently $gM = M$. It is also obvious that $A \subset gA$ for all $A \subset M$ and that $gA \subset gB$ if $A \subset B \subset M$. Hence (M,g) is a neighborhood space.

Next let a generalized filterbase λ converge to x but $(\lambda, x) \notin R$. There is then a set X such that there is in X a generalized filterbase μ finer than λ and that τ in X implies $(\tau, x) \notin R$. Hence μ converges to x and so $x \in gX$. But $x \notin gX$ since $(\tau, x) \notin R$ for all generalized filters τ in X.

Finally let $(\lambda, x) \in R$ and let λ not converge to x. It then follows from Theorem 2 that there is a subset X of M such that there is in X a generalized filterbase μ finer than λ and that no generalized filterbase in X converges to x. By this last condition $x \notin gX$. But $(\mu, x) \in R$ since μ is finer than λ and hence $x \in gX$ by definition.

Topological Spaces

By proceeding along the same lines topological spaces can be characterized by convergence without making use of iterated convergence and it makes the proof simpler. Now the Kuratowski closure function is additive and so we can use filterbases instead of generalized filterbases. And since the Kuratowski closure function is idempotent, convergence here has an additional property.

As usual, a nonempty family λ of nonempty subsets of M is said to be a filterbase iff A, B ε λ imply there is C ε λ such that $C \subset A \cap B$.

Let h be a Kuratowski closure function for M; then (M,h) is a topological space. Since a filterbase is a generalized filterbase and a topological space is a neighborhood space it follows that definitions and results (except Theorem 2) on being finer, convergence etc., given before hold for filterbases and topological spaces.

Theorem 4. If no filterbase in A converges to x then there exist sets X,Y such that (a) $X \cup Y = M$

(b) no filterbase in X converges to x

(c) no filterbase in A converges to a point of Y.

Proof. Obviously x ε chA. Take X = hA and Y = chA.

Theorem 2'. If a filterbase λ does not converge to x then there is a subset X of M such that there is in X a filterbase finer than λ and that no filterbase in X converges to x.

This is the analog, of Theorem 2, for topological spaces and can easily be proved along the same lines.

Definition 5. We will say R is a convergence relation, for a topological space, iff it satisfies the following conditions:

(a) If a filterbase λ consists of the single set $\{x\}$ then (λ, x) ε R

(b) If a filterbase μ is finer than a filterbase λ and (λ, x) ε R then (μ, x) ε R

(c) If $(\lambda, x) \notin$ R for a filterbase λ then there is a subset X of M such that there is in X a filterbase finer than λ and that

$(\mu, x) \notin R$ for all filterbases μ in X

(d) If λ is a filterbase in a subset A, of M, implies $(\lambda, x) \notin R$ then there exist two subsets X,Y of M such that $X \cup Y = M$, $(\mu, x) \notin R$ for all filterbases μ in X and $(\tau, y) \notin R$ for all filterbases τ in A and for all $y \in Y$.

Theorem 5. Let R be a convergence relation for a topological space. Define the set-valued set-function h as follows: if $A \subset M$ then hA is the set of all points x such that $(\lambda, x) \in R$ for some filterbase λ in A. Then (M,h) is a topological space and $(\lambda, x) \in R$ iff λ converges to x in the topological space (M,h).

Proof. It is only necessary to prove that h is additive and idempotent; the rest of the proof is exactly the same as in Theorem 3.

Since h is isotone, to prove $h(A \cup B) = hA \cup hB$ it is only necessary to show $h(A \cup B) \subset hA \cup hB$. If $x \in h(A \cup B)$ then there is a filterbase λ in $A \cup B$ such that $(\lambda, x) \in R$; there is then a filterbase μ finer than λ such that μ is either in A or in B. Hence $(\mu, x) \in R$ and so $x \in hA \cup hB$.

Finally we want to prove $h^2 = h$; this will follow if we can show $h^2 \subset h$. If $x \in chA$ then λ is a filterbase in A implies $(\lambda, x) \notin R$. There are then sets X,Y such that $X \cup Y = M$, μ is a filterbase in X implies $(\mu, x) \notin R$ and $y \in Y$, τ is a filterbase in A imply $(\tau, y) \notin R$; from this last condition we get $Y \subset chA$, from the one before we get $x \in chX$ and from the first we get $cX \subset Y$. Hence $cX \subset chA$ or $hA \subset X$ and so $h^2A \subset hX$. Therefore $chX \subset ch^2A$ whence $x \in ch^2A$. Thus $x \in chA$ implies $x \in ch^2A$ and so $chA \subset ch^2A$ or $h^2A \subset hA$.

More General Spaces

Let us consider a few of the more general kinds of spaces possible.

Case 1. Let (M,g) be a neighborhood space such that $g(A \cup B) = gA \cup gB$ for all subsets A,B of M. This case can be treated in the same way as that of neighborhood spaces but use filterbases instead of generalized filterbases.

Case 2. Let the neighborhood space (M,g) have the property that g is idempotent. The analog of Theorem 4 for generalized filterbases will hold in this case. So proceed as for neighborhood spaces but in the definition of convergence relation impose as additional condition the analog, of (d) of Definition 5, for generalized filterbases.

Case 3. Consider a neighborhood space (M,g). If we remove the condition that $gN = N$ then (M,g) will be a space more general than a neighborhood space. From the foregoing it is obvious how to characterize such a space by convergence. Here we will define a convergence relation R exactly as in Definition 4 but the definition of g in Theorem 3 will be changed as follows:

Take a convenient subset S of M and take $gN = S$ and for $A \subset M$, define $x \in gA$ iff $x \in S$ or $(\lambda, x) \in R$ for some generalized filterbase λ in A.

Case 4. Let g satisfy the conditions $gN = N$ and $gA \subset gB$ for $A \subset B$. Here we will define a convergence relation as in Definition 4 but will remove condition (a) and g will be defined in the same way as in Theorem 3.

Case 5. Suppose the only condition imposed on g is that it be isotone, i.e., $gA \subset gB$ for $A \subset B$. It is now obvious how to treat this case.

It is clear how to deal with other possible generalizations of topological spaces. It is also clear how to use convergence of filters, nets or generalization of nets to characterize these spaces.

In the foregoing pages it has been shown that there is a general technique available which can be used to characterize topological and more general spaces by convergence.

References

1. Kelley, J. L., Convergence in Topology, Duke Math. J. 17 (1950) 277 - 283.
2. Mamuzić, Z. P., Introduction to General Topology, Groningen (1963).