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STATE UNIVERSITY OF NEW YORK AT STONY BROOK

COLLEGE OF ENGINEERING

Report No. 19

THE TIME DOMAIN SYNTHESIS OF DISTRIBUTIONS

by

A. H. ZEMANIAN

Third Scientific Report

Contract No. AF 19(628)-2981

Project No. 5628
Task No. 562801

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1. A. H. Zemanian, "A Time-Domain Characterization of Rational Positive-Real Matrices," First Scientific Report, AFCRL-63-390, College of Engineering Tech. Rep., State University of New York at Stony Brook; August 5, 1963.

2. A. H. Zemanian, "A Time-Domain Characterization of Positive-Real Matrices," Second Scientific Report, AFCRL-63-391, College of Engineering Tech. Rep., State University of New York at Stony Brook; August 15, 1963.

Papers

1. A. H. Zemanian, "A Characterization of the Inverse Laplace Transforms of Positive-Real Matrices," SIAM Journal (submitted).

2. A. H. Zemanian, "An Elementary Proof That Locally Every Distribution Is a Finite-Order Derivative of a "Continuous Function," SIAM Review (submitted).

3. A. H. Zemanian, "The Time-Domain Synthesis of Distributions," Proceedings of the First Allerton Conference on Circuit Theory, University of Illinois; 1963.

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ABSTRACT

A new dimension to the time-domain synthesis problem is proposed. In particular, the objective is to find a realizable signal that approximates a given distribution (i.e., generalized function). A solution is then presented, several convergence criteria are discussed, and some examples are given. A feature of the present technique is that, if the Laplace transform of the given distribution is known, the realizable approximating signal can be written down without any computation. It only requires values of the Laplace transform at various points in its region of convergence. The method also yields a convenient technique for "visualizing" those distributions that cannot be plotted. Since ordinary locally integrable functions are special cases of distributions, the technique is significant for the customary time-domain synthesis problem as well.

INTRODUCTION

An outstanding question in electrical network theory has been the time-domain synthesis problem. Given a prescribed function $f(t)$ of time t, the objective is to develop an approximation technique whereby another function $f_a(t)$, whose Laplace transform $F_a(s)$ is a real rational function with poles in the left-half s-plane and with more poles than zeros, approximates $f(t)$ in some sense. Simple imaginary poles for $F_a(s)$ are allowed. A great deal of attention has been paid to this problem and a number of solutions have been expounded. References [1] to [53] comprise a certainly incomplete list of pertinent papers. The purpose of this paper is to suggest a new dimension to this problem and to present a solution to it. An early and abbreviated version of this work was presented at the First Allerton Conference on Circuit Theory.

Schwartz's distributions [54,55] and equivalent types of generalized functions [56-61] are now becoming used in various branches of network theory [62-66]. In view of their effect in other physical and mathematical sciences, it seems safe to predict that the use of distribution theory will lead to simplifications and additional results in network theory. For example, in this paper we pose and solve the problem of synthesizing a lumped passive electrical network whose response to a delta functional input approximates a prescribed disbribution. In view of certain known network synthesis techniques, the issue degenerates into the following approximation problem. "Given a distribution f(t), approximate it by a real finite linear combination of terms of the form,

$$1_+(t)\ t^n\ e^{-at}\ \cos(bt+c),$$

where $1_+(t)$ is the Heaviside unit step function, n is a non-negative integer, and a, b, and c are real constants with $a \geq 0$. In any such term, if $a = 0$, then $n = 0$. The Laplace transform of the resulting approximation will be a rational function with real coefficients whose poles outnumber its zeros and have nonpositive real parts; furthermore, the imaginary poles are simple. Such a function is realizable as the transfer function of a lumped passive electrical network.

The method that is proposed here is quite simple and yields a convergent approximation to any distribution f(t) whose support is bounded and contained in the open semi-infinite interval, $0 < t < \infty$. Under certain circumstances the technique can be extended to distributions having supports that are unbounded on the right and are contained in the semi-closed interval, $0 \leq t < \infty$. Of course, ordinary locally integrable functions can be approximated in the very same way.

An important facet of this method is that no computation is needed in order to write down the realizable approximating signal so long as the Laplace transform of the given distribution or function is known. All that is required is the values of this Laplace transform at various points in its region of convergence and this can be ascertained in many cases by referring to a table of Laplace transforms. (For tables of distributional Laplace transforms, refer to [55] and [67].)

A byproduct of this technique is that it yields a convenient means of "visualizing" distributions. For example, one crudely thinks of the delta functional as a pulse of unit area having an extremely small base and an extremely large height. Similarly, one such positive pulse followed immediately by such a negative pulse is one of the possible approximations for the first derivative of the delta functional multiplied by some constant. In general, however, it has not been clear how to visualize arbitrary distributions. The approximations generated by the method presented here can be used as picturizations of various types of distributions. Several examples of this are given below.

A DELTA FORMING SEQUENCE

In this discussion we shall employ various concepts and results developed in [54,55]. A brief sketch of the most important ideas is given in the appendix of the technical report 400-55 of [64]. Readers not familiar with distribution theory can follow the discussion in a formal way by skipping over all proofs and by interpreting the symbol $\langle f, \phi \rangle$ as the integral,

$$\int_{-\infty}^{\infty} f(t)\,\phi(t)\,dt.$$

The approximation procedure, which we shall discuss subsequently, is based upon a sequence $\{g_\nu\}_{\nu=1}^{\infty}$ of functions that converges to the Dirac delta functional $\delta(t)$ in the topology of the space $\mathcal{D}'$ of Schwartz's distributions. (We shall refer to such convergence as <u>convergence in</u> $\mathcal{D}'$.) Each term g_ν of this sequence is given by

$$g_\nu(t) = \frac{1}{\nu}\sum_{k=-\nu^p}^{\nu^p} \exp \frac{i2\pi kt}{\nu} \qquad (1)$$

$$(p>1;\ \nu = 1, 2, 3, \ldots).$$

(When p is not an integer, it shall be understood that ν^p is replaced by the largest integer that is no larger than the principal-branch value of ν^p. In our examples we shall always set p = 2.)

That (1) truly converges in $\mathcal{D}'$ to $\delta(t)$ can be shown as follows. A well-known Fourier series that converges in $\mathcal{D}'$ to a row of delta functionals is

$$\sum_{k=-\infty}^{\infty} \delta(t-k\nu) = \frac{1}{\nu} \sum_{k=-\infty}^{\infty} \exp \frac{i2\pi kt}{\nu} \tag{2}$$

(See example 2-4-4 of [55].) Now, let ϕ be an arbitrary function in the space $\mathcal{D}$ of infinitely differentiable functions of bounded support. Applying both sides of (2) to ϕ and letting ν be so large that the interval, $-\nu < t < \nu$, contains the support of $\phi(t)$, we obtain the following form of Poisson's summation formula.

$$\phi(0) = \frac{1}{\nu} \sum_{k=-\nu^p}^{\nu^p} \tilde{\phi}\left(\frac{2\pi k}{\nu}\right) + \frac{1}{\nu} \sum_{|k| > \nu^p} \tilde{\phi}\left(\frac{2\pi k}{\nu}\right) \tag{3}$$

Here, $\tilde{\phi}$ denotes the Fourier transform of ϕ and is, therefore, of rapid descent (i.e., $\tilde{\phi}(\omega) = o(1/|\omega|^n)$ for every integer n . (See Sec. 7-3 of [55].) Since $p > 1$, we may write for a suitably large constant K

$$\left| \frac{1}{\nu} \sum_{|k| > \nu^p} \tilde{\phi}\left(\frac{2\pi k}{\nu}\right) \right| \leq \frac{1}{\nu} \sum_{|k| > \nu^p} \frac{K}{\left(\frac{k}{\nu}\right)^2} \leq \int_{|x| > \nu^{p-1}} \frac{K}{x^2}\, dx$$

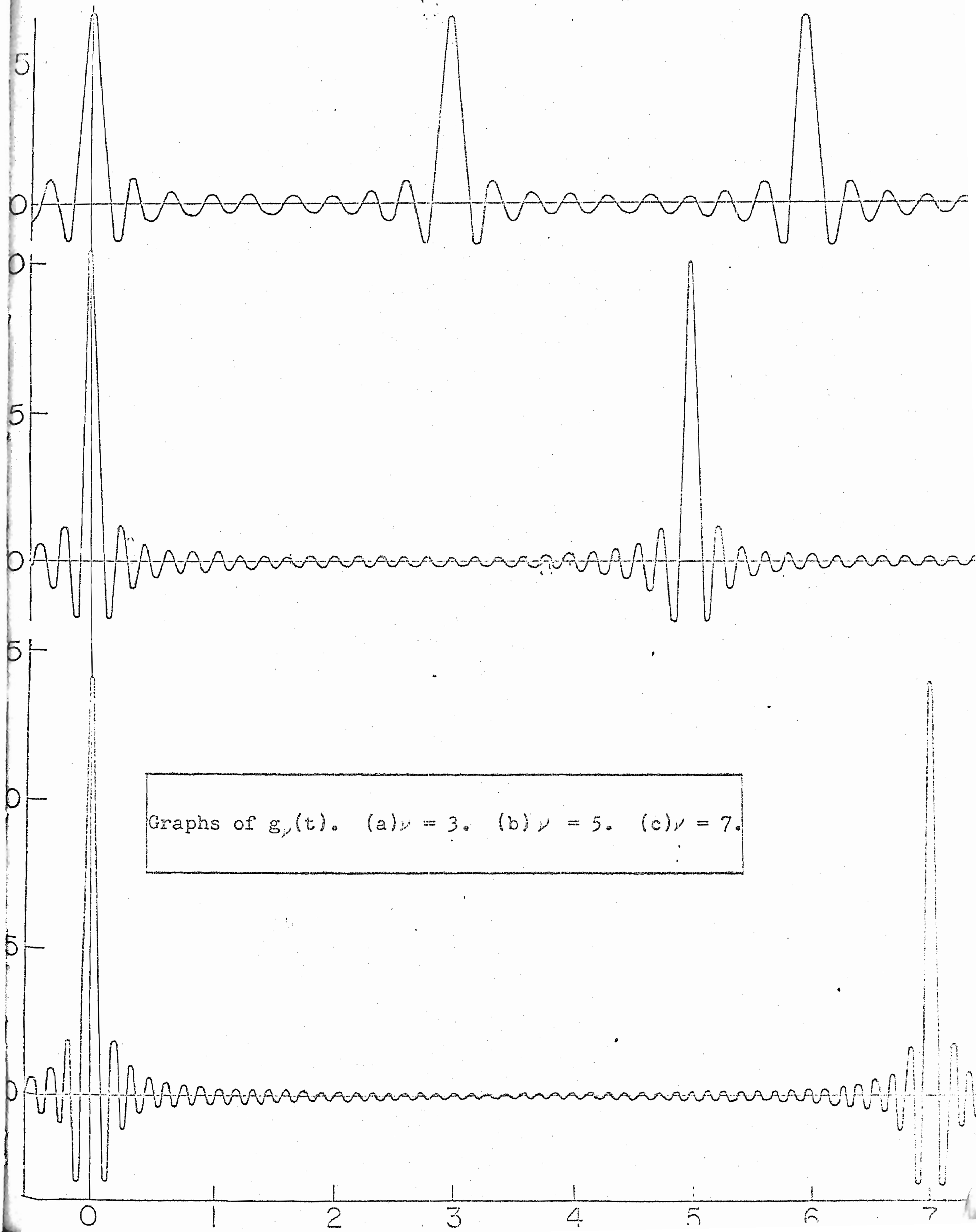
The right-hand side converges to zero as $\nu \to \infty$. This proves our assertion.

Graphs of $g_\nu(t)$ for several choices of ν are shown in Fig. 1. It can be seen that each g_ν consists of a row of pulses. As $\nu \to \infty$, each pulse approaches a delta functional; moreover, the pulse at the origin remains there whereas all the others move steadily out toward $t = +\infty$ or $t = -\infty$.

Let $g_{\nu\rho}(t) = e^{-\rho t} g_\nu(t)$, where ρ is a constant. (Hence, $g_{\nu 0} = g_\nu$.) Since $\{g_\nu\}_{\nu=1}^{\infty}$ converges in $\mathcal{D}'$ to $\delta(t)$, $\{g_{\nu\rho}\}_{\nu=1}^{\infty}$ also converges in $\mathcal{D}'$ to $\delta(t)$. For, if θ is in $\mathcal{D}$, then

$$\langle g_{\nu\rho}, \theta \rangle = \langle g_\nu, e^{-\rho t}\theta(t) \rangle \to \theta(0).$$

We shall also make use of the sequence $\{g_{\nu\rho}\}_{\nu=1}^{\infty}$. The graphs for $g_{\nu\rho}$ appear the same as those for g_ν except that all magnitudes are modulated by $e^{-\rho t}$.



Graphs of $g_\nu(t)$. (a) $\nu = 3$. (b) $\nu = 5$. (c) $\nu = 7$.

THE APPROXIMATION METHOD

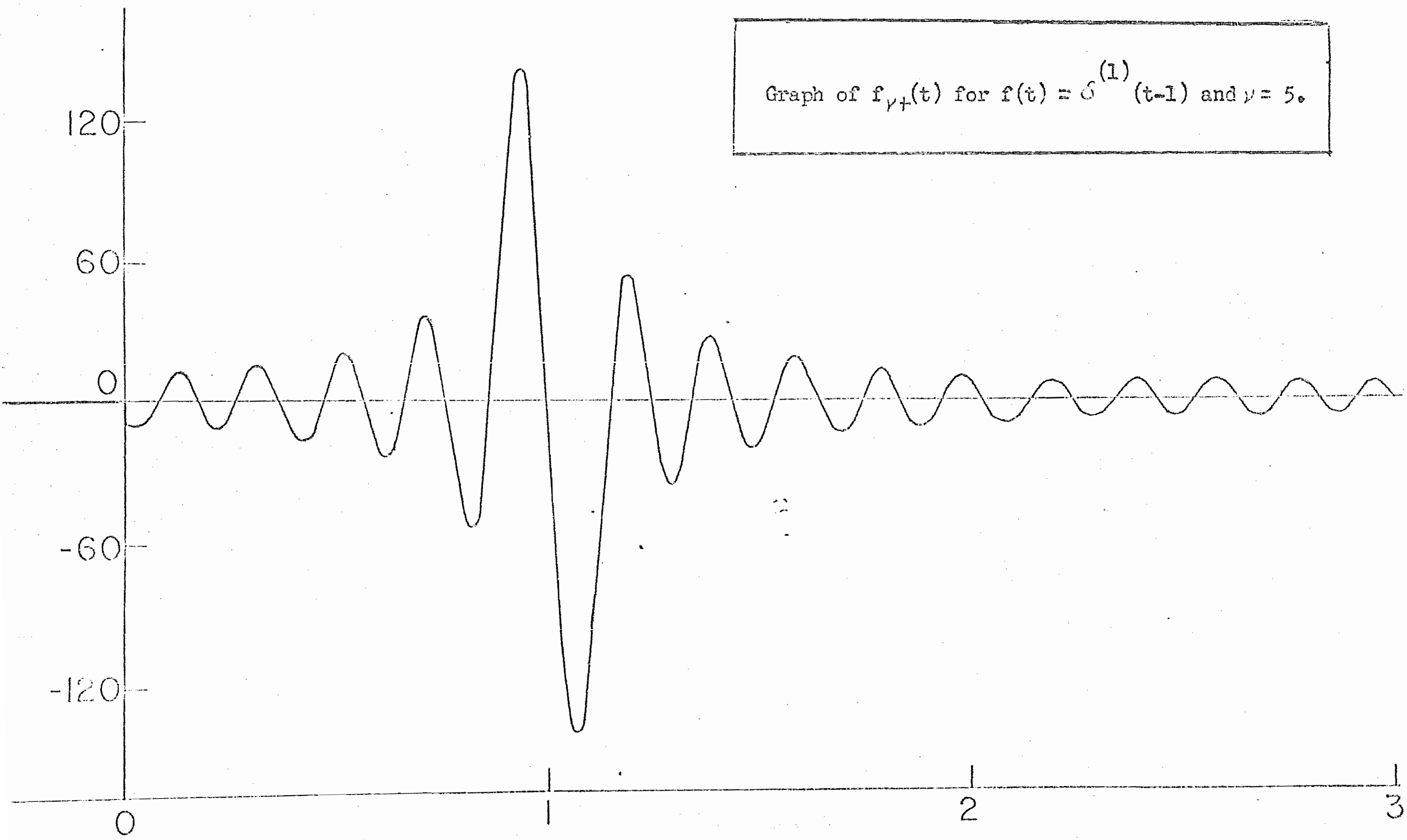
Let us assume for the moment that the distribution f(t), which we shall approximate by a realizable signal, has a bounded support contained in the interval, $\varepsilon \leq t < \infty$, where $\varepsilon > 0$. An approximation f_ν to f can be obtained by convolving f with some g_ν in the distributional sense. This convolution will certainly exist because f is of bounded support. (See theorem 5-4-1 of [55].)

$$
\begin{aligned}
f_\nu &= f * g_\nu \\
&= \frac{1}{\nu} \sum_{k=-\nu p}^{\nu p} \left\langle f(\tau) \, , \exp \frac{i 2\pi k (t-\tau)}{\nu} \right\rangle \\
&= \frac{1}{\nu} \sum_{k=-\nu p}^{\nu p} F\left(\frac{i 2\pi k}{\nu}\right) \exp \frac{i 2\pi k t}{\nu}
\end{aligned}
\qquad (4)
$$

Here, F denotes the distributional Laplace transform of f.

$$F(s) = \left\langle f(\tau) \, , e^{-s\tau} \right\rangle$$

Because of the continuity of the convolution process (See theorem 5-6-1 of [55].) $\{f_\nu\}_{\nu=1}^{\infty}$ converges in $\mathcal{D}'$ to f. Moreover, since f(t) equals the zero distribution over $-\infty < t < \varepsilon > 0$, we can truncate the approximation (4) at t=0 without upsetting the convergence. In short, the sequence $\{f_{\nu+}\}_{\nu=1}^{\infty}$, whose terms are given by $f_{\nu+} = f_\nu(t)1_+(t)$, converges in $\mathcal{D}'$ to f. The approximation $f_{5+}(t)$ for $f(t) = \delta^{(1)}(t-1)$, where p = 2, is



Graph of $f_{\nu t}(t)$ for $f(t) = \delta^{(1)}(t-1)$ and $\nu = 5$.

FIG. 2

shown in Fig. 2. (Henceforth, $h^{(n)}$ shall denote the nth distributional derivative of the distribution h.)

IF THE $F(\frac{i2\pi k}{\nu})$ ARE ALL POSITIVE our approximating signal $f_{\nu+}$ ~~is~~ WILL BE realizable by a passive lossless network since the poles of its Laplace transform are all simple and occur on the imaginary axis. However, (4) consists of a one-sided periodic wave which approximates not f(t) but rather

$$\sum_{n=0}^{\infty} f(t-n\nu).$$

This objectionable feature can be eliminated if we convolve f by some $g_{\nu\rho}$, where $\rho > 0$. Moreover, the resulting approximation will be realizable by a passive dissipative network.

In particular, we start with the approximation,

$$\begin{aligned} f_{\nu\rho}(t) &= f * g_{\nu\rho} \\ &= \frac{1}{\nu} \sum_{k=-\nu\rho}^{\nu\rho} \left\langle f(\tau), \exp\left[\left(-\rho + \frac{i2\pi k}{\nu}\right)(t-\tau)\right] \right\rangle \\ &= \frac{1}{\nu} \sum_{k=-\nu\rho}^{\nu\rho} F\left(-\rho + \frac{i2\pi k}{\nu}\right) \exp\left(-\rho t + \frac{i2\pi k t}{\nu}\right). \end{aligned} \quad (5)$$

Once again, a realizable approximation can be generated by truncating (5) at t = 0. That is, set

$$f_{\nu\rho+}(t) = f_{\nu\rho}(t) 1_+(t) \quad (6)$$

The corresponding Laplace transform is

$$F_{\nu\rho+}(s) = \frac{1}{\nu} \sum_{k=-\nu P}^{\nu P} \frac{F(-\rho + \frac{i2\pi k}{\nu})}{s + \rho - \frac{i2\pi k}{\nu}} \quad (\mathrm{Re}\ s > -\rho) \qquad (7)$$

Referring to (5) and (6), we see that $f_{\nu\rho+}(t)$ is exponentially damped so that the succeeding repetitions of the desired approximation can be made as small as one wishes merely by choosing ρ and/or ν large enough. This behavior is shown in Figs. 5 and 6, where a Cesaro mean (which we shall discuss later) has also been used.

From standard synthesis techniques, we know that there exists some 2-port network composed of only a finite number of resistors, inductors, and capacitors, whose response to a delta functional input is (6). Again by the continuity of convolution and by the fact that f has a bounded support in $0 < \varepsilon \leq t < \infty$ it follows that $f_{\nu\rho+}$ converges in $\mathcal{D}'$ to f.

Equations (4) and (5) contain the essence of our approximation procedure. Note that the only quantities that need be computed are the coefficients, $F(i2\pi k/\nu)$ or $F(-\rho + i2\pi k/\nu)$. Even this calculation may be eliminated if F(s) can be found in a table of Laplace transforms. Furthermore, since f has a bounded support, F(s) is analytic over the entire s-plane. (See theorem 8-3-2 of [55].) Thus, these coefficients can always be evaluated once F(s) is known.

Let us now remove some of the restrictions on the support of f. First of all, assume that f(t) still has a bounded support, which extends up to and includes the origin, t = 0. If f is an integrable function in a neighborhood of the origin, then the above procedure will still yield a realizable approximating sequence that converges in $\mathcal{D}'$ to f. However, if f is a singular distribution in every neighborhood of the origin (this will be the case, for example, when f has a delta functional or a singular point of a pseudofunction at the origin), then an error may be generated when the approximation $f_{\nu\rho}$ is truncated by multiplying it by $1_+(t)$. For instance, if $f(t) = \delta(t)$, then the approximation $f_{\nu+}$ equals $g_\nu(t)1_+(t)$ and as $\nu \to \infty$ this converges in $\mathcal{D}'$ to $\delta(t)/2$ rather than to $\delta(t)$.

We can overcome this difficulty, if we first translate f(t) to the right somewhat before applying our approximation technique. More specifically, instead of approximating f(t) we approximate f(t-x) where the positive number x can be chosen as small as we wish. Since f(t-x) converges in $\mathcal{D}'$ to f(t) as $x \to \infty$, we can approximate f by choosing ν sufficiently large and x sufficiently small.

We can relax our restriction on the support of f in still another way. Assume that the support of f(t) is bounded on the left at $t = \varepsilon > 0$ but unbounded on the right (i.e., it extends infinitely toward $t = \infty$). Then, $f_{\nu\rho} = f * g_{\nu\rho}$ will not in general exist since the support of $g_{\nu\rho}$ is unbounded on both sides. (See Sec. 5-4 of [55].) Nevertheless, let us formally compute $f_{\nu\rho}$. We get

$$f_{\nu\rho}(t) = \frac{1}{\nu} \sum_{k=-\nu\rho}^{\nu\rho} \left\langle f(\tau) , \exp(\rho\tau - \frac{i2\pi k\tau}{\nu}) \right\rangle \exp(-\rho t + \frac{i2\pi kt}{\nu}).$$

If f is such that its distributional Laplace transform F(s) has a region of convergence, Re $s > \sigma_1$, where $\sigma_1 < 0$, then for $0 < \rho < -\sigma_1$ the applications of the distribution f to the exponential functions in the right-hand side of (8) will all have sense. Indeed, the approximation (8) is the same as (5) except that now ρ cannot be chosen larger than $-\sigma_1$. The truncated signal $f_{\nu\rho}(t)1_+(t)$ is once again the realizable approximation to f.

We can verify that $f_{\nu\rho}(t)$ converges in $\mathcal{D}'$ to f(t) in the case where f is Laplace-transformable and has a support unbounded to the right, as follows. Let ϕ be an arbitrary function in $\mathcal{D}$ and let $\sigma_1 < -\rho$. As $\nu \to \infty$

$$\langle f_{\nu\rho}, \phi \rangle = \frac{1}{\nu} \sum_{k=-\nu\rho}^{\nu\rho} \Phi(\rho - \frac{i2\pi k}{\nu}) F(-\rho + \frac{i2\pi k}{\nu})$$

$$\to \int_{-\infty}^{\infty} \Phi(\rho - i2\pi x) F(-\rho + i2\pi x)\, dx.$$

Here, Φ denotes the Laplace transform of ϕ. By definition of the distributional Fourier transform, the last expression equals

$$\langle f(t)e^{\rho t} , \phi(t)e^{-\rho t} \rangle = \langle f, \phi \rangle.$$

This establishes our assertion.

THE USE OF CESARO MEANS

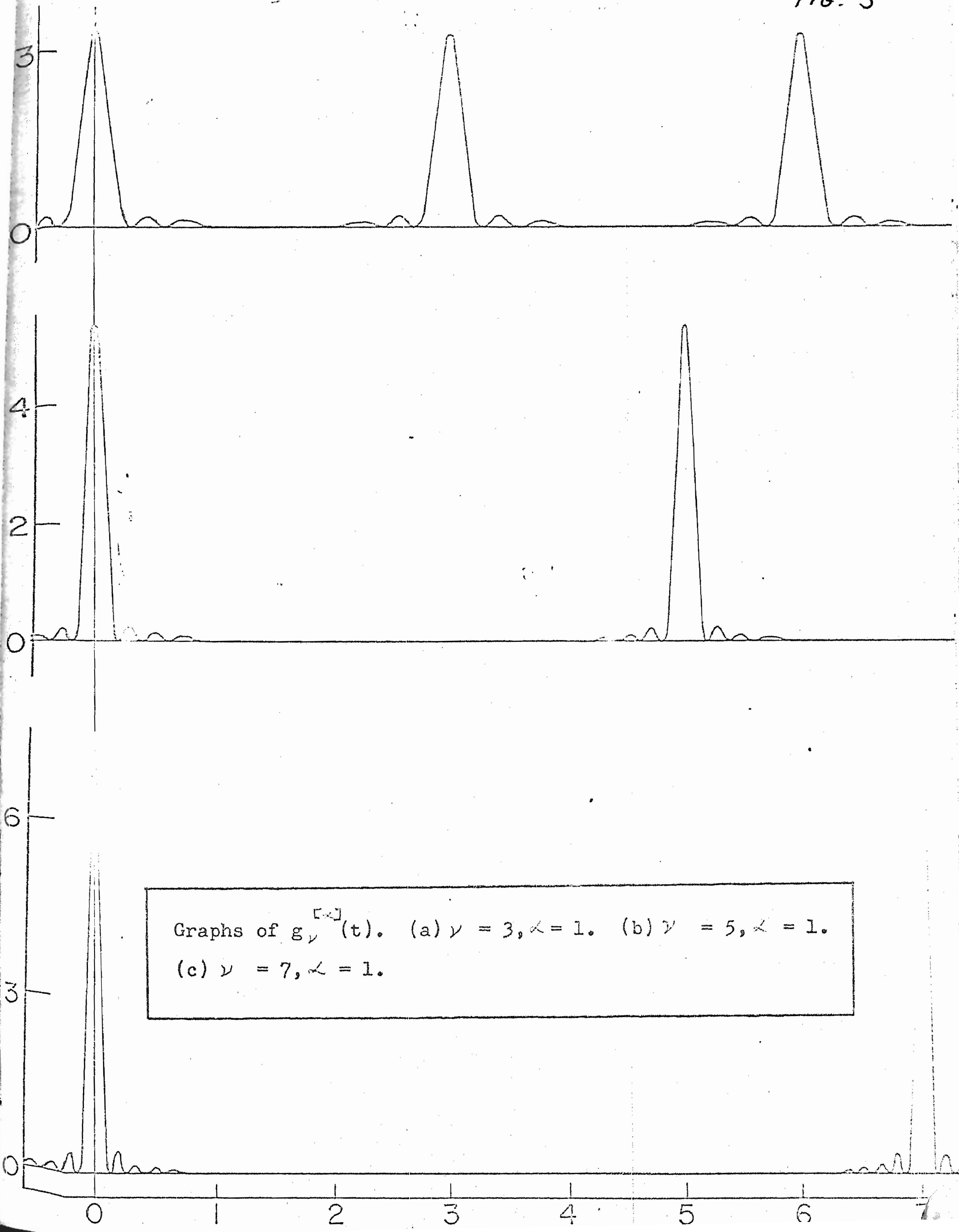
As is indicated in Figs. 1 and 2, the approximating functions are oscillatory. In fact, it turns out that over intervals, where the distribution f is a continuous function, the approximations $f_{\nu\rho}$ may not even converge uniformly. This is a generalization of the Gibb's phenomena, which is the inevitable oscillatory behavior of a truncated Fourier series in the vicinity of an ordinary discontinuity of the limit function. Just as the Gibb's phenomena can be eliminated by employing a first-order Cesaro mean, our present oscillatory behavior can also be removed in the same way. But now, the singularities in f are in general considerably more severe than an ordinary discontinuity and we must resort to the higher-order Cesaro means in order to eliminate the unnecessary oscillations. (For a discussion of these Cesaro means, see pages 76-77 of Vol. I and p. 60 of Vol. II of [68].) Unfortunately, it also turns out that the rate of convergence slows down as we go to the higher-order Cesaro means.

The α-order Cesaro mean ($\alpha = 1, 2, 3, \ldots$) of the finite sume g_ν shall be denoted by $g_\nu^{[\alpha]}$ and is given by

$$g_\nu^{[\alpha]}(t) = \frac{2}{\nu}\left\{\frac{1}{2} + \sum_{k=1}^{n} \frac{(n+1-k)(n+2-k)\ldots(n+\alpha-k)}{(n+1)(n+2)\ldots(n+\alpha)} \cos\frac{2\pi k t}{\nu}\right\} \quad (4)$$

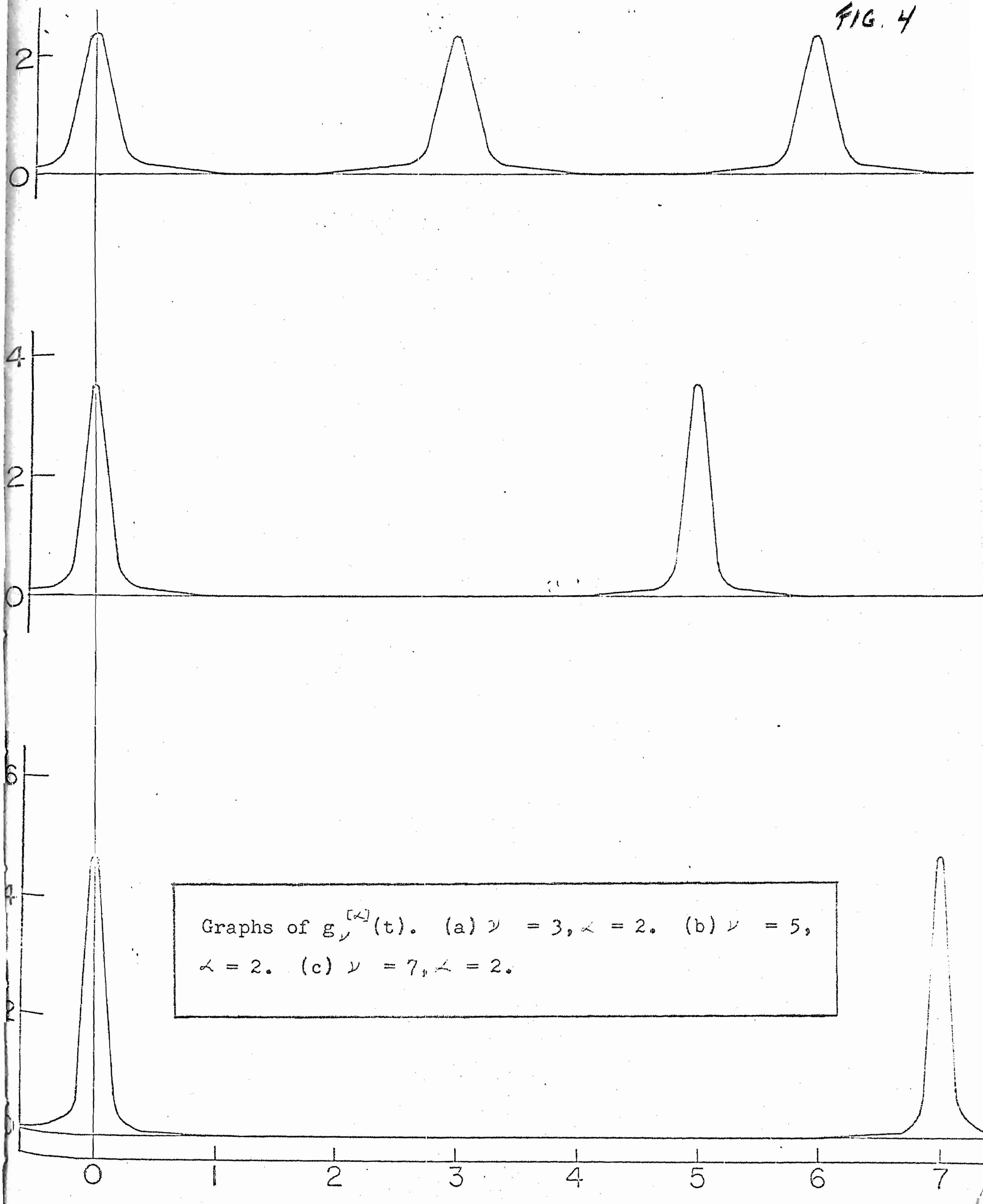
where $n = \nu^p$. These functions are plotted in Figs. 3 and 4 for $\alpha = 1, 2$ and for $\nu = 3, 5, 7$. As before, p = 2. Note that

FIG. 3



Graphs of $g_{\nu}^{[\alpha]}(t)$. (a) $\nu = 3, \alpha = 1$. (b) $\nu = 5, \alpha = 1$. (c) $\nu = 7, \alpha = 1$.

FIG. 4



Graphs of $g_{\nu}^{[\alpha]}(t)$. (a) $\nu = 3, \alpha = 2$. (b) $\nu = 5, \alpha = 2$. (c) $\nu = 7, \alpha = 2$.

the approximating functions become smoother while the convergence becomes slower as we proceed to the higher-order Cesaro means.

We shall now prove that, as $\nu \to \infty$, $g_\nu^{[\alpha]}(t)$ converges in $\mathcal{D}'$ to $\delta(t)$. For every ϕ in $\mathcal{D}$,

$$\langle g_\nu^{[\alpha]}, \phi \rangle = \frac{1}{\nu} \sum_{k=-n}^{n} \frac{(n+1-|k|)\cdots(n+\alpha-|k|)}{(n+1)\cdots(n+\alpha)} \tilde{\phi}\left(\frac{2\pi k}{\nu}\right). \quad (10)$$

Restricting ν such that the support of $\phi(t)$ is contained in $-\nu < t < \nu$ and then invoking (3), we may write

$$\langle \delta - g_\nu^{[\alpha]}, \phi \rangle = \frac{1}{\nu} \sum_{k=-n}^{n} \left[1 - \frac{(n+1-|k|)\cdots(n+\alpha-|k|)}{(n+1)\cdots(n+\alpha)}\right] \tilde{\phi}\left(\frac{2\pi k}{\nu}\right) + \frac{1}{\nu} \sum_{|k|>n} \tilde{\phi}\left(\frac{2\pi k}{\nu}\right) \quad (11)$$

We have already seen that the second sum on the right-hand side of (11) converges to zero as $\nu \to \infty$. Furthermore, for $|k| \le n$, we have the inequality,

$$1 - \frac{(n+1-|k|)\cdots(n+\alpha-|k|)}{(n+1)\cdots(n+\alpha)} \le 1 - \left(\frac{n+1-|k|}{n+1}\right)^{\alpha}.$$

Setting $C = (n+1)/\nu$ and invoking the fact that $\tilde{\phi}$ is a continuous function of rapid descent, we may dominate the first summation on the right-hand side of (11) by

$$\frac{1}{\nu}\sum_{k=-n}^{n}\left[1-\left(\frac{n+1-|k|}{n+1}\right)^{\alpha}\right]\left|\tilde{\phi}\left(\frac{2\pi k}{\nu}\right)\right|$$

$$\leq \frac{1}{\nu C^{\alpha}}\sum_{k=-n}^{n}\left[C^{\alpha}-\left(C-\left|\frac{k}{\nu}\right|\right)^{\alpha}\right]\frac{M}{1+\left|\frac{k}{\nu}\right|^{\alpha+2}} \tag{12}$$

$$\leq \frac{1}{\nu}\sum_{p=1}^{\alpha}\binom{\alpha}{p}C^{-p}\sum_{k=-n}^{n}\left|\frac{k}{\nu}\right|^{p}\frac{M}{1+\left|\frac{k}{\nu}\right|^{\alpha+2}}$$

Here, M is a sufficiently large constant. Given any $\varepsilon > 0$, we can choose an N such that for $\nu \geq N$ and for $p = 1, 2, \ldots, \alpha$ the difference between

$$\frac{1}{\nu}\sum_{k=-n}^{n}\left|\frac{k}{\nu}\right|^{p}\frac{M}{1+\left|\frac{k}{\nu}\right|^{\alpha+2}},$$

and

$$\int_{-\infty}^{\infty}\frac{M|x|^{p}}{1+|x|^{\alpha+2}}\,dx \tag{13}$$

is bounded by ε. Note that (13) is a finite quantity. Moreover, as $\nu\to\infty$, $C\to\infty$. It follows that the right-hand side of (12) converges to zero as $\nu\to\infty$. This proves our original assertion.

The functions,

$$g_{\nu\rho}^{[\alpha]}(t) = e^{-\rho t}g_{\nu}^{[\alpha]}(t) \qquad (\rho \geq 0),$$

will be used to generate approximations that are realizable by dissipative networks. Fig. 5 shows some examples of it for

$\alpha = 1$, $\nu = 3, 5, 7$, $p = 2$, and $\rho = 1/2$. Since $g_\nu^{[\alpha]}$ converges in $\mathcal{D}'$ to δ as $\nu \to \infty$, it immediately follows that $g_{\nu\rho}^{[\alpha]}$ does also.

Now, if f is once again a distribution whose support is bounded and contained within the positive t-axis, then an approximation to f is found, as before, to be

$$f_{\nu\rho}^{[\alpha]}(t) = g_{\nu\rho}^{[\alpha]} * f$$
$$= \frac{1}{\nu} \sum_{k=-n}^{n} \frac{(n+1-|k|)\cdots(n+\alpha-|k|)}{(n+1)\cdots(n+\alpha)} F\left(-\rho + \frac{i2\pi k}{\nu}\right) \exp\left(-\rho t + \frac{i2\pi k t}{\nu}\right) \tag{14}$$
$$(n = \nu^p;\ p > 1;\ \alpha = 1, 2, 3, \ldots;\ \nu = 1, 2, 3, \ldots).$$

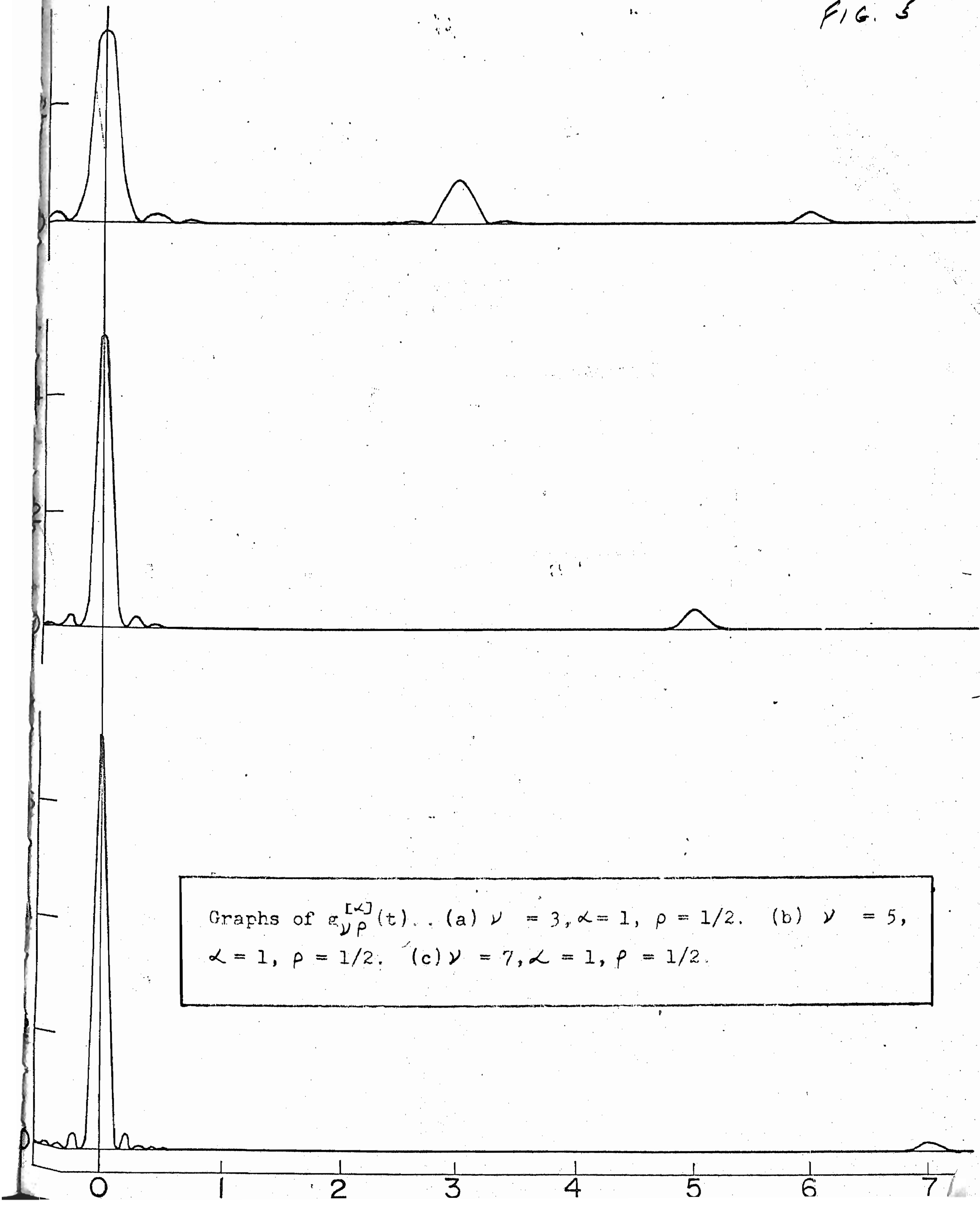
For $\rho \geq 0$, a realizable approximation is

$$f_{\nu\rho+}^{[\alpha]}(t) = f_{\nu\rho}^{[\alpha]}(t) 1_+(t) \tag{15}$$

By the continuity of convolution and the fact that the support of f is bounded and in $\varepsilon < t < \infty$ ($\varepsilon > 0$), it follows that (14) and (15) converge in $\mathcal{D}'$ to f. We may again relax the restrictions on the support of f in the same way as before.

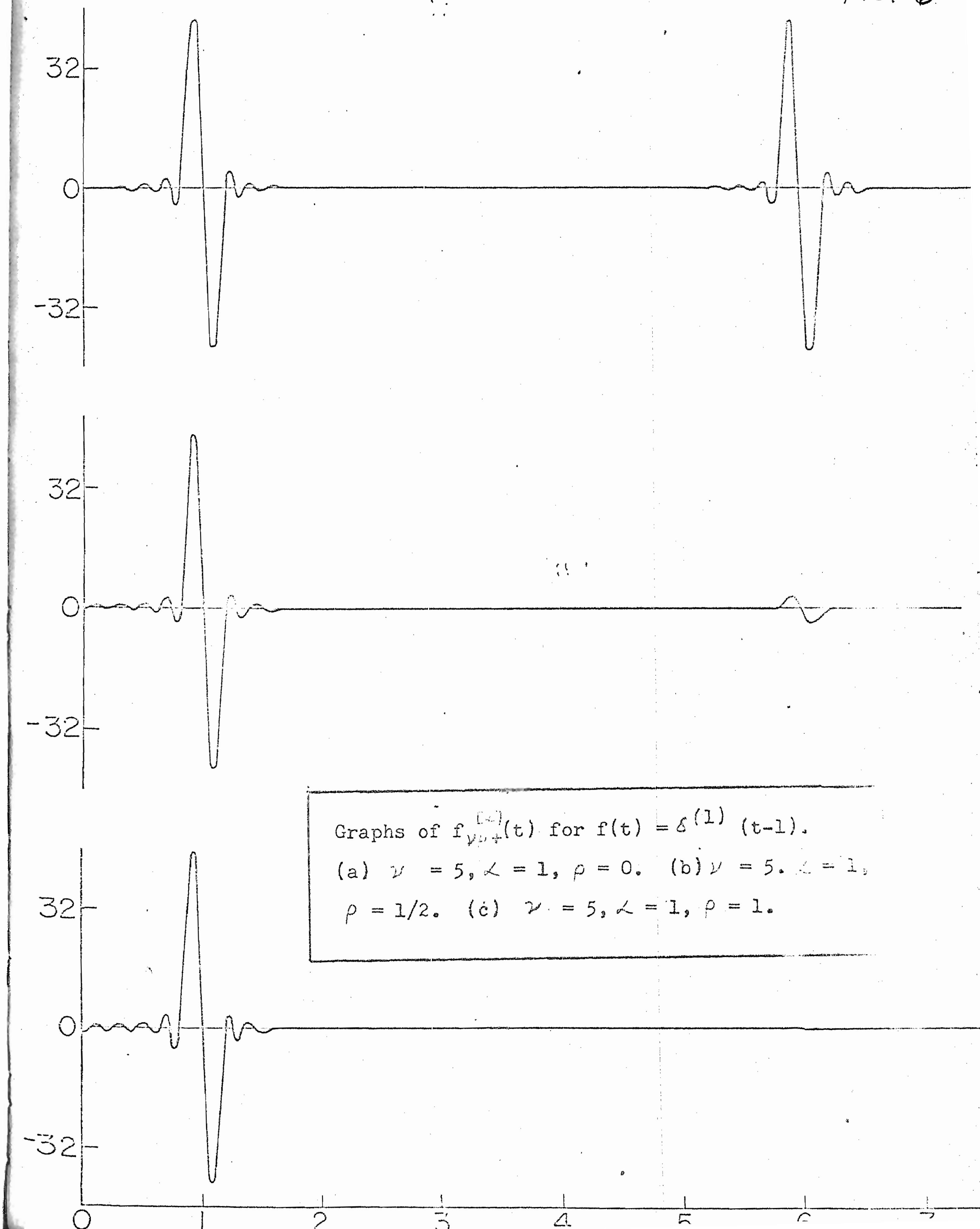
As was mentioned above, when one proceeds toward larger α, the approximations become smoother but slower. Another illustration of this is provided by Figs. 2, 6, and 7.

FIG. 5



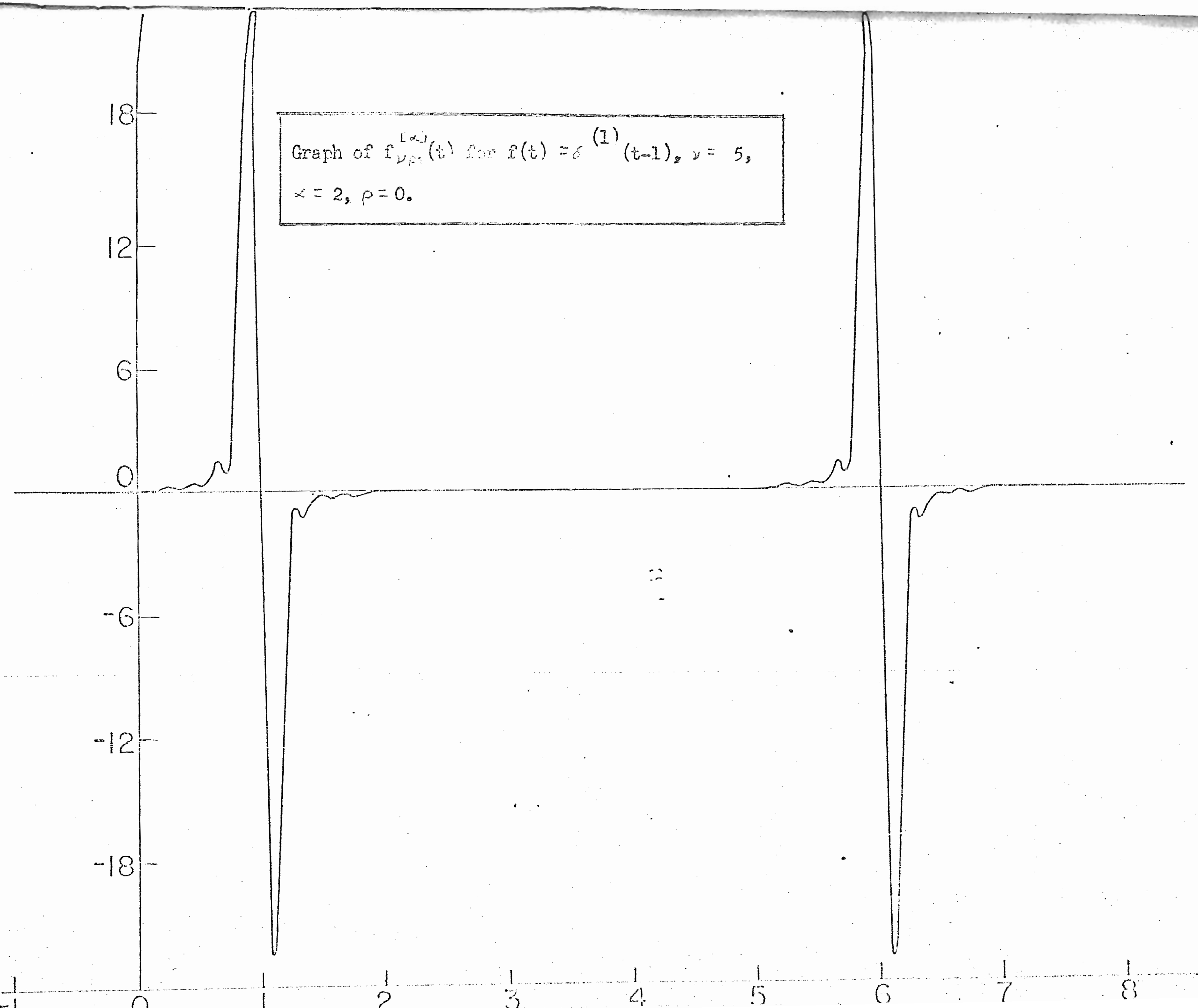
Graphs of $g_{\nu\rho}^{[\alpha]}(t)$. (a) $\nu = 3, \alpha = 1, \rho = 1/2$. (b) $\nu = 5$, $\alpha = 1$, $\rho = 1/2$. (c) $\nu = 7, \alpha = 1, \rho = 1/2$.

FIG. 6

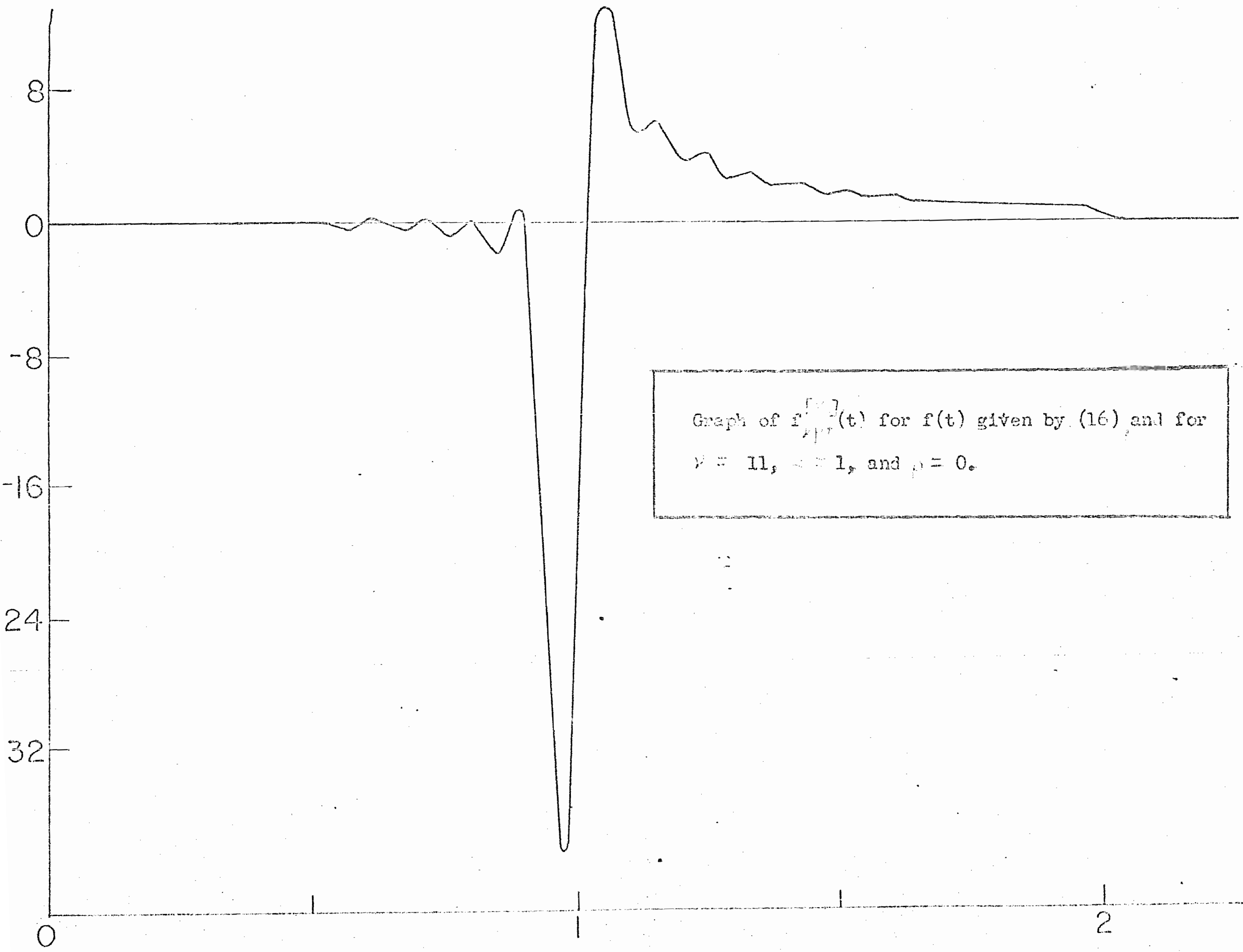


Graphs of $f^{(\alpha)}_{\nu\rho+}(t)$ for $f(t) = \delta^{(1)}(t-1)$.
(a) $\nu = 5$, $\alpha = 1$, $\rho = 0$. (b) $\nu = 5$, $\alpha = 1$, $\rho = 1/2$. (c) $\nu = 5$, $\alpha = 1$, $\rho = 1$.

FIG. 7



Graph of $f_{\nu\rho}^{[\alpha]}(t)$ for $f(t) = \delta^{(1)}(t-1)$, $\nu = 5$, $\alpha = 2$, $\rho = 0$.



Graph of $f^{[\nu]}_{\lambda,\mu,\rho}(t)$ for $f(t)$ given by (16) and for $\nu = 11$, $\lambda = 1$, and $\rho = 0$.

FIG. 8

THE APPROXIMATION OF PSEUDOFUNCTIONS

So far, we have only approximated the delta functional and its derivatives. Let us now turn to another class of distributions, the pseudofunctions. (See Secs. 1-4 and 2-5 of [55].) These distributions are based on Hadamard's concept of the finite part of a divergent integral.

Let

$$f(t) = \mathrm{Fp}\ \frac{1_+(t-1)1_+(2-t)}{t-1} . \qquad (16)$$

Applying our approximating procedure and the first Cesaro mean, we obtain for $\rho = 0$, $\alpha = 1$, and $p = 2$

$$f^{[1]}_{\nu\rho +}(t) = 1_+(t)\frac{2}{\nu^3+\nu} \sum_{k=1}^{\nu^2} (\nu^2+1-k)\left[A(k)\cos\frac{2\pi k(t-1)}{\nu} + B(k)\sin\frac{2\pi k(t-1)}{\nu}\right].$$

Here,

$$A(k) = \int_0^{\frac{2\pi k}{\nu}} \frac{(\cos x) - 1}{x}\, dx,$$

$$B(k) = \int_0^{\frac{2\pi k}{\nu}} \frac{\sin x}{x}\, dx.$$

This approximation is plotted in Fig. 8 for $\nu = 11$.

Note that over the interval, $0 < t < 2$, our approximation follows the function $1_+(t)/t$ fairly well except in the vicinity of $t = 1$, where it has a sharp negative pulse. The need for this pulse can be appreciated in a heuristic way as follows.

For the sake of an analogy that we shall point out in a moment, consider an approximation $h(t)$ to the unit step function $1_+(t)$. We take $h(t)$ to be differentiable for all t and identically equal to $1_+(t)$ except in the vicinity of $t = 0$, where it rises monotonically but continuously. The sharper we make this rise the better will $h(t)$ approximate $1_+(t)$. Clearly, $h^{(1)}(t)$ is a sharp positive pulse of unit area around $t = 0$. This is a classical approximation to the delta functional and it reflects the fact that the distributional derivative of $1_+(t)$ is $\delta(t)$.

Now, it is also a fact that in the distributional sense

$$\text{Fp} \frac{1_+(t)}{t} = \frac{d}{dt} \left[1_+(t) \log t \right].$$

In analogy to the discussion of the preceding paragraph, we might expect to get an approximation to $\text{Fp}\, 1_+(t)/t$ by differentiating a differentiable approximation to $1_+(t) \log t$. We could take such an approximation $g(t)$ to be identical to $1_+(t) \log t$ except in the vicinity of $t = 0$, where it decreases monotonically and sharply from the value, zero, to the value, $\log \varepsilon$, where ε is a small positive number $(0 < \varepsilon < 1)$. Then,

$g^{(1)}(t)$ will have a sharp pulse around $t = 0$, whose area is $\log \varepsilon$, a negative number. Moreover, as $\varepsilon \to 0$, this negative area increases indefinitely in magnitude. This explains the shape of the plot of Fig. 8 in the vicinity of $t = 1$.

As a more substantial example, consider the pseudofunction,

$$f(t) = \mathrm{Fp}\, \frac{1_+(t-1)1_+(2-t)}{(t-1)^3} \qquad (17)$$

For $p = 0$, we obtain in this case the following Cesaro approximating means.

$$F^{[\alpha]}_{\nu p+}(t) = 1_+(t) \left\{ -\frac{1}{2\nu} + \frac{2}{\nu} \sum_{k=1}^{\nu^p} C_{k\nu\alpha} \left[(A_{k\nu} - \tfrac{1}{2}) \cos (at-t) + (B_{k\nu} - a) \sin (at-t) \right] \right\}$$

Here,

$$a = \frac{2\pi k}{\nu},$$

$$A_{k\nu} = \int_0^1 \frac{1}{x^3} \left[(\cos ax) - 1 + \frac{a^2x^2}{2} \right] dx,$$

$$B_{k\nu} = \int_0^1 \frac{1}{x^3} \left[(\sin ax) - ax \right] dx,$$

$$C_{k\nu\alpha} = \frac{(\nu^p+1-k) \cdots (\nu^p+\alpha-k)}{(\nu^p+1) \cdots (\nu^p+\alpha)} .$$

This is plotted in Fig. 9 for $\alpha = 1, 2, 3$ and $\nu = 5$. Note again the smoothing effect as we proceed to the higher order Cesaro means. These graphs provide convenient visualizations of the distribution in the vicinity of its singular point.

THE CONVERGENCE BEHAVIOR

Our approximation technique possesses certain converge properties that make it useful not only for distributions b also for ordinary functions. In stating these properties w shall need the concept of the order of a distribution. distribution f is said to be of finite order if there exist nonnegative integer q such that f(t) is the qth distributio derivative for all t of some continuous function. In this c the smallest nonnegative integer r that q can be is called t order of f. It is a fact that every distribution of bounded support is of finite order. (See corollary 3-4-2a of [55].) One of the convergence properties is that uniform convergenc will be obtained over the continuous portions of the given distribution f so long as we choose in our approximation tec nique a Cesaro mean whose order is greater than the order of More precisely, we may state

Theorem 1: If the distribution f(t) is of bounded supp and of order r, if it is a continuous function over the fini interval, $a < t < b$, and if $\alpha > r$, then $f_{\nu\rho}^{[\alpha]}(t)$ converges unif ly to f(t) over $a_1 \leq t \leq b_1$, where $a < a_1 < b_1 < b$.

In order to establish this theorem, we shall need two lemmas.

Lemma 1: Let $g_{\nu}^{[\alpha]}(t)$ be defined as in (9), where $n = \nu$
$p > 1$, and α is a positive integer. Then, for every integer such that $0 \leq q \leq \alpha - 1$,

$$\frac{d^q}{dt^q} g_{\nu}^{[\alpha]}(t)$$

converges to zero as $\nu \to \infty$ uniformly over the interval, $a \leq t \leq \nu/2$, where $a > 0$.

Proof: Let K_n^{α} be defined by

$$g_{\nu}^{[\alpha]}(t) = \frac{2}{\nu} K_n^{\alpha}\left(\frac{2\pi t}{\nu}\right). \qquad (1$$

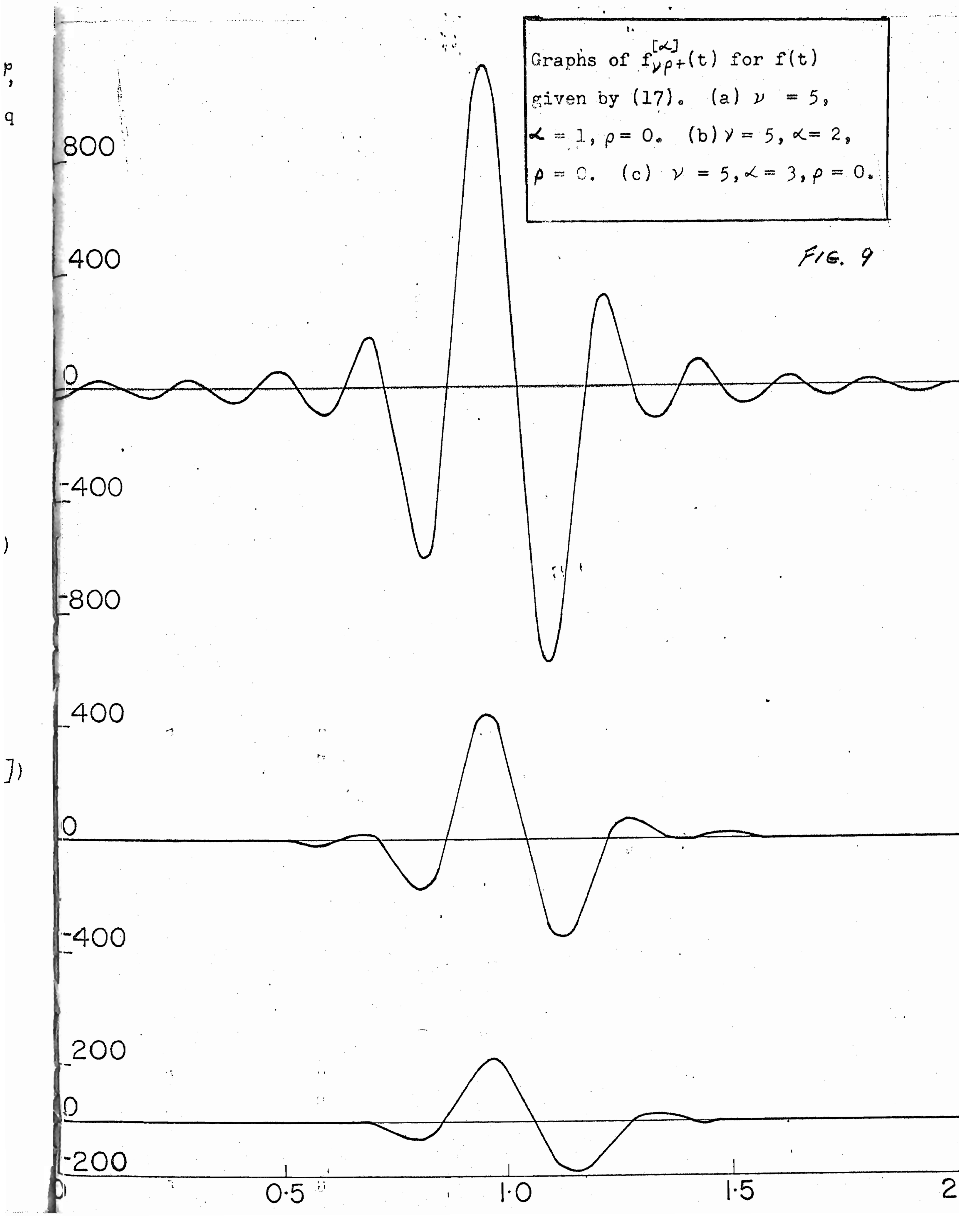
Therefore,

$$\left| \frac{d^q}{dt^q} g_{\nu}^{[\alpha]}(t) \right| = \frac{2}{\nu} \quad \frac{d^q}{dt^q} K_n^{\alpha}\left(\frac{2\pi t}{\nu}\right) \qquad (1$$

and, by a known result (see eqn. (1.10), p. 60, Vol. II of [
the right-hand side is dominated by

$$\frac{M}{t^{\alpha+1}} \nu^{(q-\alpha)(p-1)} \qquad (2$$

over the interval, $\nu^{(1-p)} \leq t \leq \nu/2$. Here, M is some constant. Lemma 1 now follows immediately.



Graphs of $f^{[\alpha]}_{\nu\rho+}(t)$ for f(t) given by (17). (a) $\nu = 5$, $\alpha = 1$, $\rho = 0$. (b) $\nu = 5$, $\alpha = 2$, $\rho = 0$. (c) $\nu = 5$, $\alpha = 3$, $\rho = 0$.
FIG. 9
800
400
0
-400
-800
400
0
-400
200
0
-200
0
0·5
1·0
1·5
2

Lemma 2: Let $g_\nu^{[\alpha]}(t)$ be defined as before with α being a positive integer. If η is some small positive constant, then, as $\nu \to \infty$, the following limits hold.

$$\int_\eta^{\nu/2} \left| g_\nu^{[\alpha]}(x) \right| dx \longrightarrow 0 \tag{21}$$

$$\int_{-\nu/2}^{-\eta} \left| g_\nu^{[\alpha]}(x) \right| dx \longrightarrow 0 \tag{22}$$

$$\int_{-\eta}^{\eta} g_\nu^{[\alpha]}(x)\, dx \longrightarrow 1 \tag{23}$$

Proof: Using the same estimate as in the previous proof (see eqn. (1.10), p. 60, Vol. II of [68]), we have

$$\int_\eta^{\nu/2} \left| g_\nu^{[\alpha]}(x) \right| dx = \frac{2}{\nu} \int_\eta^{\nu/2} \left| K_n^\alpha\left(\frac{2\pi x}{\nu}\right) \right| dx$$

$$\leq \frac{1}{\pi} \int_{\frac{2\pi\eta}{\nu}}^{\pi} \frac{M}{\nu^{p\alpha}\, y^{\alpha+1}}\, dy < \frac{M}{\alpha\pi(2\pi\eta)^\alpha \nu^{(p-1)\alpha}} \longrightarrow 0$$

as $\nu \to \infty$. The limit (22) can be shown in the same way.

Finally, note that

$$\int_{-\nu/2}^{\nu/2} g_\nu^{[\alpha]}(x)\, dx = 1,$$

as is clear from (9). The limit (23) now follows from this result and from (21) and (22). Q. E. D.

Proof of theorem 1: For the sake of clarity, we sketch $f(\tau)$ in Fig. 10, where it is understood that $f(t)$ is continuous over $a < \tau < b$ but may be a singular distribution over $x < \tau < a$ and $b < \tau < y$. We also illustrate there $g_{\nu\rho}^{[\alpha]}(t-\tau)$ as well as other values of τ that we shall use. It is assumed throughout that

$$t - \frac{\nu}{2} < x_1 < x < a < c_1 < c < t - \eta < t < t + \eta < d < d_1 < b < y < y_1 < t + \frac{\nu}{2}$$

and that the support of $f(\tau)$ is contained in $x \leq \tau \leq y$.

Let $\mu(\tau)$ and $\lambda(\tau)$ be infinitely differentiable functions that satisfy the following conditions: $\mu(\tau) = 1$ over neighborhoods of $[x,a]$ and $[b,y]$ and the support of $\mu(\tau)$ is contained in $[x_1, c_1]$ and $[d_1, y_1]$. $\lambda(\tau) = 1$ over a neighborhood of $[c_1, d_1]$ and the support of $\lambda(\tau)$ is contained in $[c,d]$. (Here, the brackets denote closed intervals.) Moreover, $\mu(\tau) + \lambda(\tau) = 1$ over a neighborhood of $[x,y]$.

Now,

$$f_{\nu\rho}^{[\alpha]}(t) = f(t) * g_{\nu\rho}^{[\alpha]}(t)$$
$$= \langle f(\tau), \mu(\tau) e^{-\rho(t-\tau)} g_{\nu}^{[\alpha]}(t-\tau)\rangle + \langle f(\tau), \lambda(\tau) e^{-\rho(t-\tau)} g_{\nu}^{[\alpha]}(t-\tau)\rangle. \qquad (24)$$

Let P be a bound on $e^{-\rho t}$ for $a < t < b$. Since f is of order r, ~~there exists a constant C such that (see Secs. 3-3 and 3-4 of [55])~~

$$\left|\langle f(\tau), \mu(\tau) e^{-\rho(t-\tau)} g_{\nu}^{[\alpha]}(t-\tau)\rangle\right| \leq C \sup_{\substack{x_1 \leq \tau \leq c_1 \\ d_1 \leq \tau \leq y_1}} \left| \frac{d^r}{d\tau^r}\left[\mu(\tau) e^{-\rho(t-\tau)} g_{\nu}^{[\alpha]}(t-\tau)\right]\right|$$

$$\leq CP \sup_{\substack{x_1 \leq \tau \leq c_1 \\ d_1 \leq \tau \leq y_1}} \left| \sum_{q=0}^{r} \binom{r}{q} \frac{d^{r-q}}{d\tau^{r-q}}\left[\mu(\tau) e^{\rho\tau}\right] \frac{d^q}{d\tau^q} g_{\nu}^{[\alpha]}(t-\tau)\right|$$

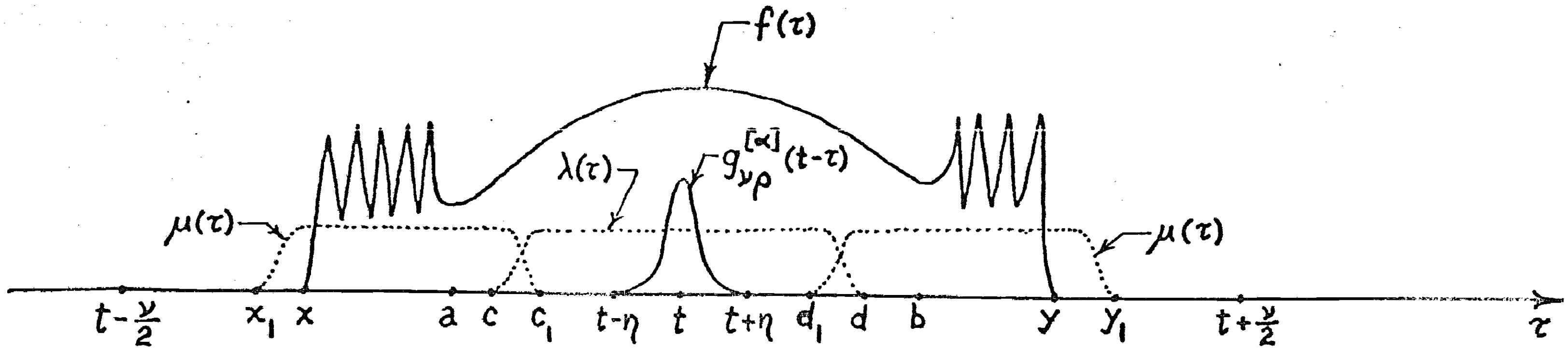

f(τ)
λ(τ)
$g_{\nu\rho}^{[\alpha]}(t-\tau)$
μ(τ)
μ(τ)
$t-\frac{\nu}{2}$ x_1 x a c c_1 t-η t t+η d_1 d b y y_1 $t+\frac{\nu}{2}$ τ

Fig. 10

By lemma 1 and the fact that $g_\nu^{[\alpha]}$ is an even function, we see that the right-hand side of the last inequality converges to zero as $\nu \to \infty$ uniformly for $c_1 + \eta < t < d_1 - \eta$. Since we are free to choose c and c_1 arbitrarily close to a, d and d_1 arbitrarily close to b, and η arbitrarily small, the same comment holds for any closed interval $[a_1, b_1]$ that is interior to the open interval (a,b).

Next, consider

$$\langle f(\tau), \lambda(\tau) e^{-\rho(t-\tau)} g_\nu^{[\alpha]}(t-\tau)\rangle = \int_c^d f(\tau)\lambda(\tau) e^{-\rho(t-\tau)} g_\nu^{[\alpha]}(t-\tau)\, d\tau.$$

We shall estimate this integral by considering in turn the intervals, $[c, t-\eta]$, $[t+\eta, d]$, and $[t-\eta, t+\eta]$. First of all,

$$\left|\int_c^{t-\eta} f(\tau)\lambda(\tau) e^{-\rho(t-\tau)} g_\nu^{[\alpha]}(t-\tau)\, d\tau\right| \leq P\left[\sup_\tau |f(\tau)\lambda(\tau) e^{\rho\tau}|\right] \int_{t-\frac{\nu}{2}}^{t-\eta} |g_\nu^{[\alpha]}(t-\tau)|\, d\tau$$
$$= P\left[\sup_\tau |f(\tau)\lambda(\tau)\, e^{\rho\tau}|\right] \int_\eta^{\nu/2} |g_\nu^{[\alpha]}(x)|\, dx$$

The last expression, which is independent of t, converges to zero as $\nu \to \infty$ according to lemma 2. As before, this conclusion holds for all t in the ~~cloud~~ CLOSED interval $[a_1, b_1]$ that lies inside the open interval (a,b).

A similar argument can be constructed for the interval $[t+\eta, d]$.

Finally, noting that $\lambda(\tau) = 1$ over $[t-\eta, t+\eta]$, we may write

$$\int_{t-\eta}^{t+\eta} f(\tau)\lambda(\tau) e^{-\rho(t-\tau)} g_\nu^{[\alpha]}(t-\tau)\, d\tau$$
$$= \int_{t-\eta}^{t+\eta} [f(t) + \varepsilon_1(t,\tau)]\,[1+\varepsilon_2(\tau)]\, g_\nu^{[\alpha]}(t-\tau)\, d\tau \qquad (25)$$

Here, $\varepsilon_1(t,\tau)$ is the error term for $f(\tau)$ and $\varepsilon_2(\tau)$ is the error term for $e^{-\rho(t-\tau)}$. Moreover, $\varepsilon_1(t,\tau)$ is uniformly continuous for $c < t < d$ and $t-\eta < \tau < t+\eta$ so that $\varepsilon_1(t,\tau) \to 0$ as $\eta \to 0$ uniformly for $c < t < d$. Also, $\varepsilon_2(\tau) \to 0$ as $\eta \to 0$ because of the continuity of the exponential function. These facts combined with (23) demonstrate that the right-hand side of (25) converges to $f(t)$ uniformly for all t in any closed interval $[a_1, b_1]$ that lies inside the open interval (a,b). This completes the proof.

We indicated before that if we replace $g_\nu^{[\alpha]}$ by $g_{\nu\rho}^{[\alpha]}$, where $\rho > 0$, we can damp out the subsequent repetitions of the approximation to the given distribution of bounded support by merely choosing ρ and/or ν sufficiently large. We shall now restate this assertion more precisely and shall then prove it.

<u>Theorem 2:</u> <u>If the distribution</u> $f(t)$ <u>has a bounded support contained in the finite interval</u>, $x \le t \le y$, <u>if the order of</u> f <u>is</u> r, <u>if</u> $\rho > 0$, <u>and if</u> $\alpha > r$, <u>then</u> $f_{\nu\rho}^{[\alpha]}(t)$ <u>converges to zero uniformly over</u> $y + \eta \le t < \infty$, <u>where</u> η <u>is any fixed positive number</u>.

<u>Proof</u>: Let

$$x_1 < x < y < y_1 < \eta + y$$

and let $\theta(t)$ be an infinitely differentiable function that equals one over a neighborhood of $[x,y]$ and is zero outside $[x_1, y_1]$. We wish to show that

$$\left| f_{\nu\rho}^{[\alpha]}(t) \right| = \left| \langle f(\tau), \theta(\tau) e^{-\rho(t-\tau)} g_\nu^{[\alpha]}(t-\tau) \rangle \right|$$

can be made less than any preassigned positive $\varepsilon > 0$ over all of the interval, $y + \eta \le t < \infty$, by choosing ν large enough. We may again write (see Secs. 3-3 and 3-4 of [55])

$$\left| f_{\nu\rho}^{[\alpha]}(t) \right| \le C \sup_{x_1 \le \tau \le y_1} \left| \frac{d^r}{d\tau^r} \left[\theta(\tau) e^{-\rho(t-\tau)} g_\nu^{[\alpha]}(t-\tau) \right] \right|$$

$$= C \sup_{x_1 \le \tau \le y_1} \sum_{q=0}^{r} \left| \frac{d^{r-q}}{d\tau^{r-q}} \left[\theta(\tau) e^{-\rho(t-\tau)} \right] \frac{d^q}{d\tau^q} g_\nu^{[\alpha]}(t-\tau) \right| \qquad (26)$$

Firstly, consider the case where $t - \nu/2 \le \tau \le t - \eta$. We have that $\left| \frac{d^{r-q}}{d\tau^{r-q}} \left[\theta(\tau) e^{-\rho(t-\tau)} \right] \right| \le B \quad (q = 0, 1, \ldots, r)$

where B is a sufficiently large constant.

Also, by lemma 1, for a given $\varepsilon_1 > 0$ and for all sufficiently large ν

$$\left| \frac{d^q}{d\tau^q} g_\nu^{[\alpha]}(t-\tau) \right| < \varepsilon_1 \qquad (q = 0, 1, \ldots, \alpha - 1)$$

Therefore, for all $t \ge y + \eta$ (27)

$$\left| f_{\nu\rho}^{[\alpha]}(t) \right| \le C\, B\, r\, \varepsilon_1 = \varepsilon_2 .$$

Thus, we may preassign ε_2 rather than ε_1.

Next, let $-\infty < \tau \le t - \nu/2$. By another known result (see eqn. (1.9), p. 60, Vol. II of [68]),

$$\left| \frac{\partial^q}{\partial\tau^q} g_\nu^{[\alpha]}(t-\tau) \right| \le B\, \nu^{(q+1)(p-1)} . \qquad (28)$$

Moreover, every term inside the summation sign of (26) has an exponential damping factor. Consequently, in view of (28) and the fact that $t-\tau \geq \nu/2$, we see that each such term converges to zero as $\nu \to \infty$ uniformly for all $t \geq y + \eta$. This result combined with (27) proves the theorem.

We also indicated before that we can take into account certain distributions whose supports are bounded on the left but are unbounded on the right. The precise statement is

Theorem 3: If the distribution f(t) has a support bounded on the left and is of order r, if its Laplace transform has a region of convergence, Re $s > \sigma_1$, if it is a continuous function over the finite interval, $a < t < b$, if $\alpha > r$, and if $\sigma_1 < -\rho$, then $f_{\nu\rho}^{[\alpha]}$ converges uniformly to f(t) over $a_1 \leq t \leq b_1$, where $a < a_1 < b_1 < b$.

Proof: The proof of this theorem is quite similar to that of theorem 1. We again refer to Fig. 10 but it is now understood that $y = y_1 = \infty$ and that $\mu(\tau)$ equals one over a neighborhood of $[b,\infty)$ as well as over a neighborhood of $[x,a]$.

Now, refer to (24). Since f is Laplace-transformable, the first term on the right-hand side of (24) may be rewritten as

$$\langle f(\tau)\, e^{c\tau}, \mu(\tau) e^{-\rho t} e^{(\rho-c)\tau} g_\nu^{[\alpha]}(t-\tau)\rangle,$$

where $\sigma_1 < -c < -\rho$. Let P again be a bound on $e^{-\rho t}$ for $a < t < b$. The distribution $f(\tau)e^{c\tau}$ will have the same order as that of $f(\tau)$ and, therefore, the last expression is dominated by (see Sec. 4-4 of [55]).

$$CP \sup_{\substack{x_1 \le \tau \le c_1 \\ d_1 \le \tau < \infty}} \left| (1+\tau^2)^r \sum_{q=0}^{r} \binom{r}{q} \frac{d^{r-q}}{d\tau^{r-q}} \left[\mu(\tau) e^{(p-c)\tau}\right] \frac{d^q}{d\tau^q} g_\nu^{[\alpha]}(t-\tau) \right|$$

$$\le CP \sup_{\substack{x_1 \le \tau \le c_1 \\ d_1 \le \tau \le d_1 + \nu/2}} |\ldots| + CP \sup_{\substack{x_1 \le \tau \le c_1 \\ d_1 + \nu/2 \le \tau < \infty}} |\ldots| .$$

Since $p-c<0$, there is an exponential damping factor in every term inside the summation sign. By lemma 1 the first supremum converges to zero as $\nu \to \infty$ uniformly for $c_1 + \eta < t < d_1 - \eta$. Also, by our estimate (28) and the presence of the exponential damping factor, the second supremum also converges to zero as $\nu \to \infty$ uniformly for t in the same interval. This result again holds true for all t in any closed interval $[a_1, b_1]$ that lies inside the open interval (a,b).

The rest of the proof, which is concerned with the second term on the right-hand side of (24), is precisely the same as the corresponding part of the proof of theorem 1.

So far, we have examined convergence only over those open intervals where the distribution f is a continuous function. In general, one is interested in all of f and not just its continuous parts. The problem is, however, that the convergence criteria that one ordinarily uses for functions (such as uniform convergence or minimization of the mean square error) are inapplicable to singular distributions. Actually the topology of the space $\mathcal{D}'$ of all distributions is quite complicated (see pp. 64-77 of [54]) and there does not appear to be any way of constructing a simple convergence criterion for this space. Indeed, the statement that "an approximating element is close to a given element in some space" usually implies that some metric can be assigned to the space. $\mathcal{D}'$ does not appear to be a metrizable space.

Nevertheless, a convergence criterion can be constructed if we restrict our attention to some fixed finite closed interval I. Let $\mathcal{D}_I$ be the space of those elements of $\mathcal{D}$ whose supports are contained in I. It is a fact that every distribution ~~is a bounded functional~~ satisfies on $\mathcal{D}_I$ ~~in the sense that~~ (see Secs. 3-3 and 3-4 of [55]).

$$|\langle f, \varphi \rangle| \leq C \sup |\varphi^{(r)}(t)|$$

for every φ in $\mathcal{D}_I$, where C and r depend only on f and I and not on the choice of φ. In view of this fact, we can construct the following convergence criterion: Choose an $\varepsilon > 0$ and construct some approximation f_ν to f. Then, we can say that "f_ν approximates f sufficiently well over the interior of I" if

$$\frac{|\langle f - f_\nu, \varphi \rangle|}{\sup |\varphi^{(r)}|} \leq \varepsilon$$

for every φ in $\mathcal{D}_I$. This criterion is probably too complicated to be of practical use.

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