THE LIMB ANALYSIS OF COUNTABLY INFINITE ELECTRICAL NETWORKS

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1. Introduction. Unlike their finite counterparts, infinite electrical networks with a given set of branch emf's can in general have an infinity of different sets of branch currents satisfying Ohm's law and Kirchhoff's node and loop laws. This is so even when all the branch emf's are zero. This phenomena is reflected in the fact that the customary mesh and cutset analyses fail when they are applied to infinite networks. In two prior papers [5], [6], a method for computing a network's response under no more than the aforementioned laws was developed that was successful for a variety of infinite networks. The objective of the present work is to investigate its applicability to all countably infinite electrical networks. We shall refer to this method as "limb analysis".

Limb analysis has two basic parts, a graph-theoretic analysis wherein a certain spanning forest in the network and a partition of the network into finite subnetworks are constructed, and an analytic analysis wherein Ohm's law and Kirchhoff's node and loop laws are applied. We shall prove in this work that there exists a solution to the graph-theoretic part for every connected countably infinite network. More specifically, we shall show that every such network possesses the kind of spanning forest and associated partition demanded by limb analysis. Also, we shall establish a sufficient condition on the branch-impedance values which insures that the analytic part of limb analysis can also be carried out.

2. Some Definitions and Terminology. An electrical network is a graph with an assignment of electrical parameters, such as emf sources and impedances, to the branches. For conciseness of terminology, we will not maintain a distinction between an electrical network and its graph and will apply graph-theoretic terminology directly to the network.

Every branch in the network is required to have two distinct nodes: just one is not allowed. However, many (even an infinity of) branches are allowed to have the same pair of nodes. A countably infinite network or simply a countable network is one having a countably infinite set of branches and either a finite or countably infinite set of nodes. Throughout this work, the electrical network N will always be assumed to be connected and countably infinite. A node is said to be finite (infinite) if its degree is finite (respectively, countably infinite). A network is called locally finite if all its nodes are finite. A node of a subnetwork of N will be called N-finite or N-infinite if it is respectively finite or infinite as a node of N; it can happen of course that an N-infinite node appears as a finite node in the subnetwork.

A node-induced subnetwork H of N is a given subset of the node set of N together with every branch that has both of its nodes in that subset of nodes. We say that H is induced by its nodes. On the other hand, a branch-induced subnetwork K of N is simply the network consisting of the branches in a given subset of the branch set of N together with all their nodes. Now, we say that K is induced by its branches.

We will at times treat a single branch b as a subnetwork of N; it is then understood that we are really dealing with the subnetwork induced by b, that is, b in conjunction with its two nodes. The union $H \cup K$ of two networks H and K is the network whose node set (branch set) is the union of the node sets (respectively, branch sets) of the two given networks. The intersection $H \cap K$ of two subnetworks H and K of N is the subnetwork of those nodes and branches that are in both H and K. When $H \cap K$ is not void (i.e., has at least one node), we say that H and K intersect or meet. H - K denotes the subnetwork of N consisting of all nodes in H that are not in K in conjunction with all branches in H that are not incident to any nodes in K.

A partition $\{M_p\}$ of a subnetwork M of N is a collection of branch induced subnetworks M_p of M such that every branch of M appears in one and only on M_p and $M = \cup M_p$. A branch b is said to be adjacent to a subnetwork K if b is not in the branch set of K and at least one node of b is in the node set of K. On the other hand, a node n_0 and a subnetwork M are called adjacent if n_0 is not in the node set of M and there exists a branch joining n_0 to a node of M. A branch is said to join two subnetworks of N if those subnetworks do not meet and b has a node in each of the subnetworks.

A path is an alternating sequence on nodes and branches such that no node appears more than once and every branch in the sequence is incident at the nodes immediately preceding and succeeding it in the sequence. Such a sequence is allowed to have a beginning or an end, but the initial and final terms in the sequence must be nodes. A path is called <u>endless</u> if the corresponding sequence has neither a beginning nor an end, <u>one-ended</u> if that sequence has either a beginning or an end but not both, and <u>finite</u> if it has both a beginning and an end. A network is said to be <u>connected</u> if each pair of nodes is contained in some finite path. Konig's lemma [4; p. 40] states that, in a connected, infinite, locally finite graph, every node has at least one one-ended path beginning at that node. A <u>loop</u> is defined as is a finite path except that the beginning and ending nodes are the same. It follows from Konig's lemma that any network that is connected and countably infinite has at least one infinite node or one one-ended path. Two paths are said to be <u>node-distinct</u> if they share no nodes in common.

A <u>forest</u> T is a network containing no loops; T is called a <u>tree</u> if it is also connected. When T is a subnetwork of N, it is said to be <u>spanning in</u> N if it contains every node in N.

3. The Chainlike Structure. R.Halin [1] has introduced the concept of an "m-times chainlike" infinite graph. It is useful in the investigation of the maximum number of node-distinct one-ended paths in the given graph. For our purposes we will need a variation of his idea; to this end, we introduce the following definition.

A network G will be called <u>chainlike</u> if it is locally finite and can be partitioned in accordance with

$$G = \bigcup_{p=1}^{\infty} G_p, \tag{1}$$

$$\left(\bigcup_{i=1}^{p} G_i\right) \cap G_{p+1} = V_{p+1}, \quad p = 1, 2, 3, \ldots, \tag{2}$$

where the following conditions are satisfied.

1. Each G_p is a finite network.

2. Each subnetwork V_{p+1} consists of m_{p+1} isolated nodes where $m_{p+1} < \infty$.

3. The sequence $\{m_p\}_{p=2}^{\infty}$ is monotonic increasing (but not necessarily strictly so).

4. $V_{p+1} \subseteq \left(\bigcup_{i=1}^{p} G_i\right) - \left(\bigcup_{i=1}^{p-1} G_i\right)$ for every p; that is, V_{p+1} shares no nodes in common with $\bigcup_{i=1}^{p-1} G_i$.

5. In each G_{p+1} there are m_{p+1} node-distinct finite paths from the nodes in V_{p+1} to m_{p+1} of the m_{p+2} nodes in V_{p+2}.

If the m_p remain constant at a finite value m for all p sufficiently large, we will call G <u>m-times chainlike</u>. On the other hand, if $m_p \to \infty$ as $p \to \infty$, we will call G <u>divergently chainlike</u>.

A modification of the proof of Theorem 1 in [1] coupled with the paragraph following Theorem 3' in that paper establishes the following [2].

<u>Proposition 3.1</u>. Every locally finite network is chainlike. It has m, but not m+1, node-distinct one-ended paths when and only when it is m-times chainlike. It has an infinity of node-distinct one-ended paths when and only when it is divergently chainlike.

4. A Partition of N and a Certain Spanning Forest in N.

Let N be a given countable connected network. Let G be the network obtained from N by deleting all the infinite nodes of N: that is, G is the set of all finite nodes in N in conjunction with all branches of N that are not incident to infinite nodes. By Proposition 3.1, G is chainlike. In the following we use the notation of the preceding section for the chainlike structure of G. We can partition the nodes of G into two sets, the first consisting of those nodes that do not lie on one-ended paths and the second consisting of those that do. Let K and M be respectively the subnetworks of G induced by the nodes in the first and second sets. We allow either K or M to be void.By Konig's lemma, all the components of K are finite, whereas those of M are infinite.

Furthermore, M, if it exists, must also be chainlike with the same sets V_p as those of G. Indeed, by Condition 5 of the definition of a chainlike network, every node in any V_p lies on a one-ended path and hence must belong to M. So, by (2), each component of K must lie within a single G_p and possess no vertices in common with V_p, V_{p+1}, or the node-distinct finite paths described in Condtion 5. (Moreover, by Condition 1, each G_p can contain no more than a finite number of the components of K.) Thus, the removal of K from G will yield once again the same chainlike structure for the remaining network M.

Now, consider the union of all the finite paths mentioned in Condition 5 of the preceding section. This union consists

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of m (or an infinity of) vertex-distinct one-ended paths in M if M is m-times (or, respectively, divergently) chainlike. By a <u>spine</u> we will mean either one of these one-ended paths or one of the infinite nodes of N.

We shall now construct in N a spanning forest and a partition of N which satisfy conditions similar to those given in the hypothesis of Theorem 5.1 of [5]. This corresponds to the graph-theoretic part of limb analysis mentioned in the Introduction. The following procedure for constructing the the spanning forest and the partition of N involves a number of steps in which certain nodes, branches, or components are designated. It can happen that some of these entities do not exist in N. When this occurs, it is understood that the corresponding step is simply skipped.

Let M_p be that subnetwork of G_p obtained by removing from G_p all components of K that are contained in G_p. We also let q_k, where k = 1, 2, ... , denote the infinite nodes of N, and we assign the indices of the q_k as follows. First of all, we arrange the infinite nodes that are not adjacent to any of the M_p into a sequence X. Then, we let $q_1, q_2, \ldots, q_a$ be the infinite nodes that are adjacent to M_1 and let q_{a+1} be the first node in X; we also set $Q_1 = \{q_1, \ldots, q_{a+1}\}$. Next, we let $q_{a+2}, q_{a+3}, \ldots, q_b$ be the infinite nodes that are adjacent to M_2 but not adjacent to M_1, and let q_{b+1} be the second node in X; also, $Q_2 = \{q_{a+2}, \ldots, q_{b+1}\}$. Then, we let $q_{b+2}, q_{b+3}, \ldots, q_c$ be the infinite nodes that are adjacent to M_3 but not adjacent to $M_1 \cup M_2$, and let q_{c+1} be the third node of X; also, $Q_3 = \{q_{b+2}, \ldots, q_{c+1}\}$. We continue this numbering procedure until all infinite nodes are labelled; simultaneously, the set of all infinite nodes is partitioned into the family $\{Q_s\}$. Note that, since each M_p is locally finite and has a finite number of nodes, it has only a finite number of adjacent infinite nodes. Consequently, this numbering procedure is feasible.

Observe that each component of K must be adjacent to at least one q_k but not to any nodes in M because N is connected and these components arise through the deletion of the q_k from N. We label the components of K as $K_{k,j}$ in the following way. $K_{k,1}$, $K_{k,2}$, $K_{k,3}$, ... will denote those components of K that are adjacent to q_k and are not adjacent to those q_i for which $i < k$. When $i > k$, $K_{k,j}$ may be adjacent to q_i. For any fixed k, there may be a finite or infinite number of $K_{k,j}$.

Finally, for each pair of infinite nodes q_i and q_j, we label the branches joining those two nodes by $b_{i,j,1}$, $b_{i,j,2}$, $b_{i,j,3}$, ...

Now, let K_1 be the union of all $K_{k,1}$ for which $q_k \in Q_1$. K_1 is a finite network since each $K_{k,1}$ is a finite network and Q_1 is a finite set. Let P_1 be the union of all $b_{i,j,1}$ for which $q_i \in Q_1$ and $q_j \in Q_1$. Clearly, P_1 is a finite network. Let R_1 be the union of P_1, all branches joining M_1 to Q_1, and all branches joining K_1 to any of the q_k (not necessarily in Q_1). Since M_1 and K_1 are finite networks, R_1 is too. Finally, we let N_1 be the finite subnetwork of N induced by all the branches in M_1, K_1, and R_1. We inductively continue this process of constructing finite subnetworks N_p of N as follows. Let p be an integer greater than one. Let K_p be the union of all $K_{k,p-s+1}$ for which $q_k \in Q_s$, where $s = 1, 2, \ldots, p$. As was true for K_1, K_p is also a finite subnetwork. Next, let P_p be the finite network induced by all $b_{i,j,p-s+1}$ for which q_i and q_j both lie in $\bigcup_{r=1}^{s} Q_r$ and at least one of them lie in Q_s, where $s = 1, 2, \ldots, p$. Then, let R_p be the union of P_p, all branches joining M_p to any of the q_k, and all branches joining K_p to any of the q_k. It follows from the finiteness of P_p, M_p, and K_p that R_p is a finite network too. Finally, set N_p equal to the union of M_p, K_p, and R_p. Thus, N_p is a finite subnetwork of N.

From the way we have constructed the N_p it can be seen that every branch of N lies in one and only one of the N_p. That is, $\{N_p\}_{p=1}^{\infty}$ is a partition of N into finite subnetworks. Moreover, $N_p \cap N_{p+1} = V_{p+1} \cup W_{p+1}$, where V_{p+1} is the finite set (possibly void) of N-finite nodes specified in (2) for the chainlike structure of G and W_{p+1} is a finite set (possibly void) of N-infinite nodes. Furthermore, $N_p \cap N_m$ is just a finite set (possibly void) of N-infinite nodes if $|p - m| > 1$.

We recall that a spine in N is defined to be any infinite node or any one of the one-ended paths arising from the union of the finite paths mentioned in Condition 5 above. Our next step is to construct in each N_p a spanning forest F_p such that each component of F_p meets one and only one spine and contains all the nodes and branches of that spine that lie in N_p.

To this end, let A_1 be the union of all nodes and branches in N_p that lie in spines. Choose any branch b_1 in N_p that is adjacent to A_1. If b_1 has one node that is not in A_1, set $A_2 = A_1 \cup b_1$; otherwise, set $A_2 = A_1$. Choose another branch b_2 in N_p that is adjacent to A_2 and set $A_3 = A_2 \cup b_2$ if b_2 has one node that is not in A_2; otherwise, set $A_3 = A_2$. Continue this procedure until all branches in N_p that are not in spines have been considered. This yields a spanning forest F_p in N_p with the desired properties.

We now set $F = \bigcup_{p=1}^{\infty} F_p$.

<u>Lemma 4.1.</u> F is a spanning forest in N such that each component of F meets one and only one spine in N and contains all the nodes and branches of that spine.

<u>Proof</u>. If $p \neq m$, a component of F_p and a component of F_m can meet only if they meet the same spine, and, if they do meet, their intersection lies in that spine and is either an N-infinite node or, if $m = p+1$, possibly a node in V_m. Otherwise, either (2) or the disconnectedness of some component of K from the rest of $N - q_1 - q_2 - q_3 - \cdots$ would be violated. It follows that F contains no loops and is therefore a forest. Moreover, F is spanning because it is spanning in each N_p. The rest of the conclusion follows directly from the way each F_p was constructed in N_p.

We shall call each component of F a <u>limb</u>. The set of all limbs will be called a <u>full set of limbs</u>. Since F is spanning, every node of N belongs to a unique limb.

<u>Lemma 4.2</u>. If n_0 is a node in N, then there exists a

unique finite path P contained in the limb L that contains n_0 such that n_0 is one terminal node of P, the other terminal node n_1 of P belongs to the spine of L, and P lies entirely outside the spine of L except for n_1. (If n_0 lies in a spine, then $n_0 = n_1$ and P is the degenerate path consisting of n_0 alone.)

Proof. Choose any node n_2 in the spine S of L. (When S is an N-infinite node, we choose S itself.) Since L is a tree, there exists a unique finite path from n_0 to n_2 lying in L. In tracing that path from n_0 to n_2, we will find a first node n_1 that lies in S, and the traced path from n_0 to n_1 is the path P we seek. Again, P must be unique since L is a tree.

Lemma 4.3. The path P of Lemma 4.2 lies entirely within a single N_p. When P is not degenerate, its nodes other than n_1 are not common to two or more of the N_p.

Proof. This follows from the facts that L is the union of its intersections with each N_p and two such intersections meet only at a single node in the spine of L if they meet at all.

Lemma 4.4. (i) If the spine of L is a one-ended path, then L is an infinite tree containing neither N-infinite nodes now endless paths. Moreover, given any node n_0 of L, there exists a unique one-ended path starting at n_0 and contained entirely in $\bigcup_{s=p}^{\infty}(L \cap N_p)$, where N_p is the subnetwork containing n_0.

(ii) If the spine of L is an N-infinite node, then L may be either a finite or infinite tree, but in either case it contains exactly one N-infinite node, the spine itself, and does not contain any one-ended or endless path.

Proof. (i) Since L is a component of the forest F and contains a one-ended path, it is an infinite tree. By Lemma 4.1, L contains exactly one spine. Since the N-infinite nodes are all spines, L cannot contain an N-infinite node. Furthermore, $L = \bigcup_{p=1}^{\infty} (L \cap N_p)$. Each $L \cap N_p$ is a finite tree. Also, for $|p - m| > 1$, $L \cap N_p$ does not meet $L \cap N_m$, whereas $L \cap N_p$ meets $L \cap N_{p+1}$ at and only at the unique spine node residing in both N_p and N_{p+1}. It follows that L cannot contain an endless path.

It also follows from this structure for L that there is exactly one one-ended path in $\bigcup_{s=p}^{\infty} (L \cap N_p)$ starting from a given node n_0 of $L \cap N_p$. That path lies entirely in L's spine if n_0 belongs to the spine. Otherwise, that path is the union of the n_0 to n_1 path P of Lemma 4.2 and the one-ended path in the spine starting at n_1.

(ii) Now, assume that L's spine is the N-infinte node q_k. As in (i), L is a tree and cannot contain any other N-infinite node. $L \cap N_p$ is a finite tree for each p. For $p \neq m$, $L \cap N_p$ and $L \cap N_m$ meet at and only at q_k. That is, q_k is a cut-node for L, and its removal from L leaves only finite components. Consequently, L cannot contain any one-ended or endless path. Finally, in the process of constructing all the F_p, either a finite or infinite number of branches may have been added to q_k to produce L.

Lemma 4.5. If b is any branch of a limb L, then L - b has exactly two components. One of them contains all of L's spine except possibly a finite portion of that spine. The other component is a finite tree all of whose nodes are N-finite.

Proof. Since L is a tree, the removal of b must yield exactly two components. If b is not a part of L's spine, all of that spine will appear in one of those components. The only other possibility is that the spine is a one-ended path and b is in that path. the removal of b will then split the spine into a finite path and a one-ended path, with the latter appearing in one of the aforementioned components. Finally, if the last conclusion were not true, either L would contain two N-infinite nodes, or (by Konig's lemma) L would contain an N-infinite node and a one-ended path, or L would contain an endless path, all of which are impossible according to Lemma 4.4.

A branch of N that is not in the forest F will be called a tie. Thus, a branch is or is not a tie depending on the choice of F. Let d be any tie. Two possibilities arise:

(i) Both nodes of d lie in the same limb L: Since L is a tree, $L \cup d$ contains exactly one loop, namely, the union of b and the unique path in L connecting the nodes of d. We shall refer to that loop as the d-orb or tie orb.

(ii) The two nodes of d lie in different limbs, say, L_1 and L_2: In this case we define the d-orb or tie orb as the unique (finite or infinite) path $d \cup P_1 \cup P_2 \cup H_1 \cup H_2$, where P_1 and P_2 are the two finite paths in, respectively, L_1 and L_2 as specified in Lemma 4.2. Also, if the spine of L_1 is an N-infinite node, H_1 is that node; if the spine of L_1 is a one-ended path, H_1 is the one-ended path lying in the spine and starting at the node where P_1 terminates. H_2 lies in the spine of L_2 and is defined similarly. Thus, we see that this d-orb lies in $L_1 \cup d \cup L_2$.

<u>Lemma 4.6</u>. Let N_p be the subnetwork that contains a given tie d. Then, the d-orb is contained entirely within $\bigcup_{s=p}^{\infty} N_s$. Moreover, the d-orb is contained entirely within N_p alone under either one of the following conditions.

(i) Both nodes of d lie in the same limb.

(ii) The two nodes of d lie in different limbs, both of which have spines that are N-infinite nodes.

<u>Proof</u>. Since $\{N_p\}$ is a partition of N, d must lie in a single N_p. For any limb L, $L \cap N_p$ is a tree. Hence, under condition (i), the d-orb will lie in N_p. On the other hand, if d's nodes lie in different limbs, the d-orb is equal to $d \cup P_1 \cup P_2 \cup H_1 \cup H_2$. By Lemma 4.3, P_1 and P_2 also lie in N_p. When the spines of both limbs are N-infinite nodes, H_1 and H_2 are those nodes and therefore also lie in N_p, which implies that the d-orb lies in N_p. The only other possibility is that one or both of H_1 and H_2 are one-ended paths, but in any case H_1 and H_2 will lie in $\bigcup_{s=p}^{\infty} N_s$ according to Lemma 4.4(i). Therefore, so too will the d-orb.

<u>Lemma 4.7</u>. Let b be any branch in F. Then, there are only a finite number of tie orbs that contain b.

<u>Proof</u>. Let L be the limb that contains b. Let H be that finite component of L - b whose nodes are all N-finite (Lemma 4.5). Let d be any tie. The following results follow

directly from the definition of the d-orb. If d is not adjacent to H, then the d-orb does not contain b. If one node of d is in H and the other node of d is not in H, then the d-orb contains b. Finally, if both nodes of d lie in H, then the d-orb does not contain b. Thus, the only tie orbs that contain b are those whose ties have one and only one node in H. Since H is finite and all its nodes are N-finite, there can be only a finite number of such ties.

5. Current Flows Satisfying Kirchhoff's Node Law. Henceforth we assume that every branch in N has an orientation. Thus, if branch b and node n are incident, then b is either incident away from or incident toward n. The current in a branch is a complex number measured with respect to the branch's orientation. Also, if d is a tie, we assign to the d-orb that orientation which agrees with the orientation of d (i.e., while tracing the d-orb in the direction of the d-orb's orientation, we pass through d in the direction of d's orientation).

Kirchhoff's node law. For each N-finite node n, $\Sigma \pm i_{k_j} = 0$ where the summation is over all branches b_{k_j} incident to n, i_{k_j} is the current in b_{k_j}, and the plus (minus) sign is used if the corresponding branch is incident away from (respectively, toward) n. For each N-infinite node, no restriction is imposed. (Thus, Kirchhoff's node law is a condition concerning only the N-finite nodes.)

Lemma 5.1. The specification of the currents in all the ties and the imposition of Kirchhoff's node law uniquely determines the current in each branch of F.

<u>Proof</u>. After numbering all the branches of F in any fashion, we let b_1, b_2, b_3, ... denote those branches. Every limb, and therefore F as well, must contain at least one end node that is N-fininte. This is because each limb, being a tree, has at least two end nodes and at most one N-infinite node. We shall call a branch of F an end branch if and only if it is incident to an end node that is N-finite. Among all the end branches of F, choose the one, say, b_{k_1} having the least index k_1 and let n_1 be its N-finite end node. Thus, all other branches in N that are incident to n_1 are finite in number and have specified currents. Kirchhoff's node law therefore uniquely determines the current in b_{k_1}.

Inductively, assume that the currents have already been determined in the branches b_{k_1}, ... , b_{k_j} of F. Among all the end branches of F - b_{k_1} - ... - b_{k_j}, choose that end branch $b_{k_{j+1}}$ having the lowest index and let n_2 be its N-finite end node. All other branches in N that are incident to n_2 will be finite in number and their currents will be known. So, Kirchhoff's node law determines the current in $b_{k_{j+1}}$.

Note that at no step will F - b_{k_1} - ... - b_{k_j} have a component that does not contain at least part of a spine. In the event that F contains a finite limb L and all but one, say, b_{k_i} of the branches of L have been treated, it follows that b_{k_i} will have one N-finite node n_r and one N-infinite node q_s. The procedure then applies Kircchoff's node law to n_r to determine the current in b_{k_i}. No contradiction can arise at q_r because Kirchhoff's node law places no restriction at the N-infinte nodes.

Let b_k be any branch of F and let L be the limb that contains b_k. This procedure will eventually assign a current to b_k. Indeed, by Lemma 4.5, one of the two components, say, H of L - b_k is a finite tree all of whose nodes are N-finite. Thus, the procedure will eventually assign a current to every branch of H and then to b_k as well. Moreover, the current in b_k will depend only on the ties adjacent to H and therefore will be independent of the way the branches in F were numbered. That is, the current in b_k is uniquely determined by the tie currents. This completes the proof.

Under the notation defined in Lemma 4.7 and its proof, b lies in a given d-orb if and only if one but not both of the nodes of d lies in H. Now, assume that the current in d is i and the currents in all other ties are zero. Then, a repetition of the proof of Lemma 5.1 shows that Kirchhoff's node law requires that all branches not in the d-orb have zero current whereas all branches b in the d-orb have the current $\pm$ i; here again, the plus (minus) sign is used if b's orientation agrees (disagrees) with the d-orb's orientation. We shall say that the tie d <u>induces</u> the current $\pm$ i (or zero) in a branch if that branch is (is not) in d's orb. (Thus, d induces i in itself.)

Now assume that arbitrary currents are assigned to all the ties in N. By virtue of Lemma 4.7, only a finite number of ties induce nonzero currents in any given branch. Therefore, we may apply superposition to conclude that the currents induced in all branches of N by all the ties are finite and satisfy Kirchhoff's node law. In view of the uniqueness assertion of Lemma 5.1, we can conclude with the following.

Lemma 5.2. Let there be given a current flow in G such that Kirchhoff's node law is satisfied. Then, the current in any branch is equal to the finite sum of the currents induced in that branch by the ties.

6. Joints and Chords. In order to make use of Kirchhoff's loop law, which we will state later on, we construct a spanning tree in N by adding certain ties to the limbs.

In particular, add to F_1 as many ties in N_1 as possible without forming any loops in the resulting subnetwork. Continue the procedure considering in turn F_1, F_2, F_3, ... as follows. Let $j_1, \ldots, j_m$ be the ties that have been added to $F_1, F_2, \ldots, F_{p-1}$, where $p \geq 2$. Then, add to F_p as many ties in N_p as possible without forming any loops in the union of those ties in N_p with $F_1 \cup \cdots \cup F_{p-1} \cup j_1 \cup \cdots \cup j_m$. After completing this procedure, let J be the set of all the added ties and let J' be the subnetwork of N induced by those ties. The members of J will be called joints, and J will be called a full set of joints.

Lemma 6.1. $F \cup J'$ is a spanning tree in N.

Proof. Since F spans N, so too does $F \cup J'$. Also, $F \cup J'$ will not contain any loops because no loops were allowed in the process of constructing J. Finally, suppose that $F \cup J'$ is disconnected. Let n_1 and n_2 be two nodes appearing in different components of $F \cup J'$. Since N is connected, there exists a path P in N joining n_1 and n_2. In tracing P we will find at least one tie, say, d joining two different components of $F \cup J'$. But, this is a contradiction; for, in the process of constructing J, d would have been chosen as a joint, thereby connecting those components.

Those ties that are not joints will be called chords. Set $T = F \cup J'$. Since T is a spanning tree, each chord a generates in conjunction with T a unique loop, which we will call either the $a \cup T$ loop or the chord-tree loop. It is the union of a with the unique path in T connecting the nodes of a. Assume that a lies in N_p. If both nodes of a lie in the same limb of F, then the $a \cup T$ loop is identical with the a-orb and lies entirely in N_p. However, if the two nodes of a lie in different limbs, say, L_1 and L_2, then the $a \cup T$ loop is different from the a-orb (since the latter is now a path) and lies in $\bigcup_{s=1}^{p} N_s$. Indeed, if there were no path in $T \cap (\bigcup_{s=1}^{p} N_s)$ joining the nodes of a, then a would have been chosen as a joint, in contradiction to the assumption that a is a chord. Thus, such a path does exist, and its union with a yields the $a \cup T$ loop lying in $\bigcup_{s=1}^{p} N_s$. This result coupled with Lemma 4.6 yields the following.

Lemma 6.2. Let N be any connected countable network. Choose the partition $\{N_s\}_{s=1}^{\infty}$ of N and the spanning forest F as stated in Section 4. Also, choose the full set J of joints as stated in this section. Set $T = F \cup J'$. Let a be any chord in N_p. Then, the a-orb lies in $\bigcup_{s=p}^{\infty} N_s$ and the $a \cup T$ loop lies in $\bigcup_{s=1}^{p} N_s$.

We note in passing that this lemma states that condition (i) of the hypothesis of Theorem 5.1 of [5], the main theorem of that work, is satisfied by every connected countable network.

7. The Space C^∞ of One-sided Sequences of Complex Numbers. C^∞ will denote the set of all infinite vectors of the form $\underset{\sim}{x} = [x_1, x_2, x_3, \ldots]^T$, where the components x_k are complex numbers. (The superscript T denotes matrix transpose.) No restriction is placed on the growth of the x_k as $k \to \infty$. Multiplication by a complex number and addition are defined componentwise, and this makes C^∞ a linear space.

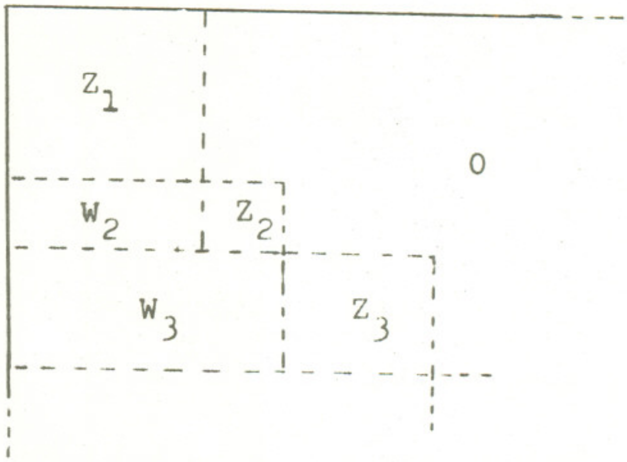
Now, consider an infinite matrix of the form

$$Z = \begin{bmatrix} Z_{11} & Z_{12} & Z_{13} & \cdots \\ Z_{21} & Z_{22} & Z_{23} & \cdots \\ Z_{31} & Z_{32} & Z_{33} & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{bmatrix}$$

where each Z_{jk} is a complex number. Z defines a mapping $\underset{\sim}{x} \mapsto Z\underset{\sim}{x}$ of C^∞ into C^∞ by means of the customary definition of the matrix product Zx if and only if every row of Z has no more than a finite number of nonzero entries. In this case Z is said to be <u>row-finite</u>, and the mapping $\underset{\sim}{x} \mapsto Z\underset{\sim}{x}$ is linear.

Asuume in addition that Z has the partitioned form

(3) $Z =$



where each Z_p is a square finite $k_p \times k_p$ matrix and all the entries to the right of these Z_p are zero. If every Z_p is nonsingular, then the equation

$$Z\underset{\sim}{x} = \underset{\sim}{y},$$

where $\underset{\sim}{y}$ is a given vector in C^∞, has a unique solution $\underset{\sim}{x} \in C^\infty$. It can be obtained by first solving

$$Z_1 [x_1, \ldots, x_{k_1}]^T = [y_1, \ldots, y_{k_1}]^T$$

for the first k_1 components of x. Then, the next k_2 components of $\underset{\sim}{x}$ can be determined by solving

$$Z_2 [x_{k_1+1}, \ldots, x_{k_2}]^T = [y_{k_1+1}, \ldots, y_{k_2}]^T - W_2 [x_1, \ldots, x_{k_1}]^T.$$

Continuing in this way, we can determine all the conponents of $\underset{\sim}{x}$. When Z has the form of (3) wherein each Z_p is nonsingular, we shall say that Z is <u>invertible in blocks</u>.

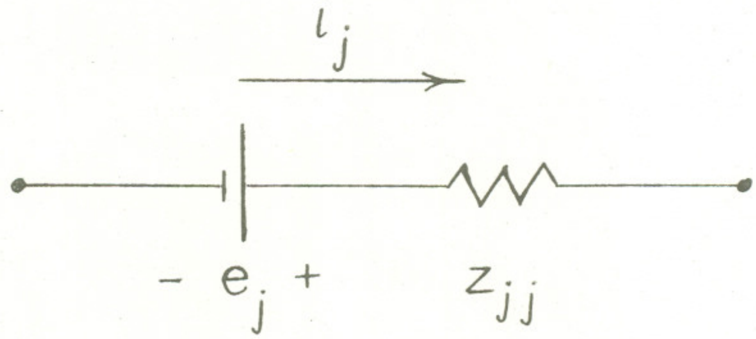


Figure 1.

8. The Network Equations. So far, we have only invoked Kirchhoff's node law. Another law we shall exploit is Kirchhoff's loop law; it concerns the branch voltage drops around the loops in N. As with branch currents, a branch voltage drop is a complex number measured with respect to the branch's orientation. Also, an oriented loop in N is a loop to which a direction of traversal is assigned. Henceforth, it is understood that every loop in N has an orientation assigned to it.

Kirchhoff's Loop Law. Around every oriented loop in N, $\sum \pm v_{k_j} = 0$, where the sum is over all branches b_{k_j} in the loop, v_{k_j} is the voltage drop in b_{k_j}, and the plus (minus) sign is used if the orientation of b_{k_j} agrees (disagrees) with the orientation of the loop.

We shall assume that each branch b_j of our electrical network N has the structure shown in Figure 1, where the arrow designates b_j's orientation and currents and voltages are measured with respect to that orientation. Here, e_j, i_j, and z_{jj} are complex numbers representing respectively b_j's emf, current, and self-impedance; the flow of i_j through z_{jj} produces the voltage drop $z_{jj}i_j$. In addition, it is assumed that, for $k \neq j$, a current i_k in branch b_k produces a voltage drop $z_{jk}i_k$ in branch b_j; here, z_{jk} is called the mutual impedance coupling the current in b_k to a voltage drop in b_j. The total voltage drop v_j in b_j measured with respect to b_j's orientation is

$$(4) \qquad v_j = -e_j + \sum_k z_{jk} i_k$$

where the summation is over all branch indices including $k = j$. Henceforth, whenever we say that N satisfies Kirchhoff's node and loop laws, it is tacitly understood that the branch currents and voltages are related in accordance with (4).

Kirchhoff's node and loop laws, coupled with (4), are not in general enough to force a unique current flow in N, as is shown by example in [5]. This is reflected in the fact that the customary mesh analysis of finite networks fails in general for infinite networks; for one thing, given a fundamental system of mesh currents with respect to some spanning tree, it can happen that an infinity of such currents flow through a single branch, which can lead in turn to divergent series in the analysis.

On the other hand, because of Lemma 6.2 we can apply limb analysis as follows: First of all, assume that, if b_j is a branch in N_p, then $z_{jk} \neq 0$ only if $b_k \in N_s$ where $s \leq p$. This means that voltage drops are induced in b_j through mutual coupling only by a finite number of branch currents, namely, the currents on some or all of the branches in $\bigcup_{s=1}^{p} N_s$. Next, number all the joints consecutively using the positive integers. Let $\underset{\sim}{j} = [j_1, j_2, \ldots]^T$ be the (finite or infinite) vector of all joint currents where j_k is the current in the kth joint. We will assign the values of the j_k arbitrarily.

Furthermore, consecutively number the chords 1, 2, 3, ... starting first with the chords in N_1, then proceeding to the chords in N_2, then proceeding to the chords in N_3, and so forth. Let $\underset{\sim}{c} = [c_1, c_2, \ldots]^T$ be the vector of chord currents where c_k is the current in the kth chord.

If Kirchhoff's node law is satisfied, the current in each limb can be written as the finite sum of the currents induced in that limb branch by the chords and joints. (See Lemma 5.2.) Moreover, if Kirchhoff's loop law is satisfied, we can write a sequence of Kirchhoff's loop law equations, one for each chord-tree loop, in the order of the chord indices; in doing so, each chord-tree loop is assigned the orientation that agrees with its chord's orientation. Upon transposing the terms invoking joint currents to the right-hand side, we obtain the matrix equation

$$Z\underset{\sim}{c} = \underset{\sim}{g}, \tag{5}$$

where $Z = [Z_{jk}]$ is a matrix, whose entries are linear combinations of the self and mutual impedances in N. Also, g is a vector whose entries are linear combinations of the branch emf's and the joint currents where the coefficients of the joint currents are self and mutual impedances. We assume that the emf's and impedances are given and we assign the values of the joint currents arbitrarily. Our prior hypothesis on mutual coupling in conjunction with Lemma 6.2 shows that Z has the partitioned form of (3). If each Z_p therein is nonsingular then Z is invertible in blocks. This allows us to solve for $\underset{\sim}{c}$, after which we can compute all the branch currents by using Lemma 5.2.

Before discussing conditions on the branch impedances which insure that Z is invertible in blocks, let us take note of how this limb analysis avoids the aforementioned pitfalls that render the customary mesh analysis inoperative for infinite networks. First of all, it identifies a set of branches namely, the joints to which one is free to assign currents arbitrarily, leading thereby to unique currents in the remaining branches. Moreover, by Lemma 5.2 only a finite number of chord and joint currents are induced in any branch. (Contrast this to the application of mesh analysis to infinite networks wherein an infinity of mesh currents will in general pass through a given tree branch.) This in turn allows us to apply Kirchhoff's loop law around the chord-tree loops to get network equations represented by (5) wherein Z is row-finite. Actually, our numbering procedure, Lemma 6.2, and our hypothesis on mutual coupling forces Z to have the partitioned form of (3).

9. Chord Dominance. The kth equation in the expansion of (5) corresponds to Kirchhoff's loop law written for the $a_k \cup T$ loop, where a_k is the kth chord. But, the only chord that induces a nonzero current in a_k is a_k itself, and the only chord contained in the $a_k \cup T$ loop is again a_k. These facts imply that the self impedance of z_{a_k} of a_k appears only as

an added term in the kth main-diagonal entry Z_{kk} of Z and nowhere else. That is,

$$Z_{kk} = z_{a_k} + z^{kk}$$

where z^{kk} is independent of z_{a_k}; also, Z_{ms} is independent of z_{a_k} if either one or both of m and s are not equal to k. Therefore, by varying z_{a_k}, we vary only Z_{kk} and no other entry of Z. In fact, if all the chord impedances $|z_{a_k}|$ are chosen sufficiently large, then all the blocks Z_p in (3) will become dominated by their diagonal elements and thereby nonsingular [3; p. 32].

An explicit condition of this nature can be obtained if we examine how the various branch impedances appear in the entries of the block Z_p in (3). We have

$$Z_{kk} = z_{a_k} + \sum_j \pm z_j^{kk}$$

and, for $s \neq k$,

$$Z_{ks} = \sum_j \pm z_j^{ks}.$$

Here, both summations are finite. Also, $\sum_j \pm z_j^{kk}$ contains the self impedances of all the branches other than a_k in the intersection of the $a_k \cup T$ loop with the a_k-orb as well as those mutual impedances that couple currents in the branches of the a_k-orb to voltage drops in branches of the $a_k \cup T$ loop. Finally, $\sum_j \pm z_j^{ks}$, where $k \neq s$, contains the self impedances of all branches in the intersection of the $a_k \cup T$ loop with the a_s-orb as well as those mutual impedances that couple currents in the branches of the a_s-orb to voltage drops in branches of the $a_k \cup T$ loop. The plus (minus) sign in front of z_j^{kk} or z_j^{ks} is used if a positive current in chord a_k or respectively chord a_s produces via this impedance a positive (negative) voltage drop in the the $a_k \cup T$ loop when z_j^{kk} or z_j^{ks} is taken to be one ohm.

Now, assume that the chord a_k lies in N_p and let $\underline{1}$, $\underline{1}+1$, ... , m be the indices of all the chords in N_p. Thus, $\underline{1} \leq k \leq m$. If

$$(6) \qquad |z_{a_k}| > \left|\sum_j \pm z_j^{kk}\right| + \sum_{\substack{s=\underline{1} \\ s\neq k}}^{m} \left|\sum_j \pm z_j^{ks}\right|,$$

then

$$|Z_{kk}| \geq |z_{a_k}| - \left|\sum_j \pm z_j^{kk}\right| > \sum_{\substack{s=\underline{1} \\ s\neq k}}^{m} \left|\sum_j \pm z_j^{ks}\right| = \sum_{\substack{s=\underline{1} \\ s\neq k}}^{m} |Z_{ks}|.$$

Hence, if (6) holds for every chord a_k in N, then each Z_p will truly be dominated along its rows by its diagonal elements. Similarly, if

$$|z_{a_k}| > \left|\sum_j \pm z_j^{kk}\right| + \sum_{\substack{s=\underline{1} \\ s\neq k}}^{m} \left|\sum_j \pm z_j^{sk}\right|,$$

then, as above,

$$|Z_{kk}| > \sum_{\substack{s=\underline{1} \\ s\neq k}}^{m} |Z_{sk}|.$$

So, if (7) holds for every chord a_k in N, then each Z_p will be dominated along its columns by its diagonal elements. Finally, if either (6) holds for every chord in N or (7) holds for every chord in N, we shall say that N is chord dominant with respect to T and J, where as always T is the chosen tree $F \cup J'$ and J is the chosen full set of joints.

Theorem. Let N be a connected countable network. Then, there exists in N a spanning forest F, a full set J of joints, and a partition $\{N_s\}_{s=1}^{\infty}$ of N into finite subnetworks N_s such that, for $T = F \cup J'$ and for each chord a in N_p, the corresponding $a \cup T$ loop lies in $\bigcup_{s=1}^{p} N_s$ and the corresponding a-orb lies in $\bigcup_{s=p}^{\infty} N_s$. Assume that all branches have parameters of the form shown in Figure 1 and assume that mutual coupling is only of the type where a current in branch b_k produces a voltage drop in branch b_j with the additional restriction that the voltage drop is zero whenever $b_k \in N_s$, $b_j \in N_p$, and $s > p$. Also, assume that all emf's and self and mutual branch impedances are given. Arbitrarily assign values to all the joint currents. Number the chords as stated in Section 8. Upon writing Kirchhoff's loop law around each chord-tree loop and invoking Kirchhoff's node law to express each branch current as the finite sum of the chord and joint currents induced in that branch (Lemma 5.2), we obtain a system of equations which have the matrix form (5) where $\underset{\sim}{c}$ is the unknown vector of chord currents and $\underset{\sim}{g}$ is a known vector depending on the branch emf's, the branch self and mutual impedances, and the joint currents; Moreover, Z has the partitioned form of (3). Z is invertible on C^{∞} whenever each Z_p in its partitioned form is nonsingular. A sufficient condition for this to be so is that N be chord dominant with respect to T and J. When Z is invertible on C^{∞}, $\underset{\sim}{c}$ will be uniquely determined, and, according to Lemma 5.2, so too will be all the branch currents. Moreover, when Z is invertible on C^{∞}, any set of branch currents that satisfy Kirchhoff's node and loop laws will correspond in this way to a particular choice of joint currents.

Proof. Everything has been established by our foregoing arguments except for the last sentence. Under any given set of branch currents, the joint currents will be specified. Because Kirchhoff's node and loop laws are satisfied, (5) holds. The invertibility of Z implies that the chord currents, as determined by (5), must coincide with the given chord currents. The limb-branch currents, as determined by Lemma 5.2, must also coincide with the given ones by virtue of the uniqueness assertion of Lemma 5.1.

10. Some Closing Remarks. We note in passing that the analysis of [5; Section 6] can now be applied to determine the dimension dim $\mathcal{H}$ of the linear space $\mathcal{H}$ of all homogeneous current flows in N. By a "homogeneous current flow" we mean a vector of all the branch currents in N, where Kirchhoff's node and loop laws are satisfied as well as (4) with all $e_j = 0$. When N satisfies the hypothesis of the Theorem, we have dim $\mathcal{H}$ = $|J|$, where $|J|$ is the cardinality of the full set of joints. Since N is connected, $|J| = x - 1$ where x is the cardinality of the set of spines in N.

It is also worth noting that our graph-theoretic results allow us to make a cutset analysis that is dual to the loop analysis of this section; that is, chord orbs are replaced by cutsets each of which contain exactly one limb branch, chord-tree loops are replaced by incidence cutsets, impedances are replaced by admittances, and the network equations are now generated by Kirchhoff's node law, rather than by Kirchhoff's loop law. In fact, this cutset analysis extends to countable networks the discussion in [5; Section 8], which was restricted to locally finite networks. However, we have now established that hypothesis (i) of Theorem 8.1 of [5] will always be satisfied by a countable connected network when T and J are chosen as indicated above.

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