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THE CHARACTERISTIC - RESISTANCE METHOD
FOR GROUNDED SEMI-INFINITE GRIDS

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The Characteristic-Resistance Method
for Grounded Semi-infinite Grids*

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Abstract. This work analyzes a uniform half-plane rectangular grid of positive resistances each node of which is connected to ground through another positive resistance. Current sources are imposed only on the boundary of the grid. It is shown that, if those current-source values are quadratically summable, then there is a unique current flow in the grid under Kirchhoff's node and loop laws, Ohm's law, and the condition of finite power dissipation in the network. The paper provides a method for actually computing what those currents are. It accomplishes this by extending the characteristic-impedance concept to lumped transmission lines of operator-valued parameters. The method extends to more general types of uniform grids having more complicated graphs or positive-real branch impedances. In the latter case, a method is developed for calculating the transient responses in the grid for given time-dependent current sources. Grids with positive-real branch impedances arise in the discretization of the partial-differential equations of a variety of physical phenomena. Finally, an approximate computation of the currents in a grid of nonlinear monotone resistances can also be achieved by means of a linearization technique.

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1. Introduction. Existence and uniqueness theorems for the current flows in infinite electrical networks have been the subject of rigorous analysis for almost a decade, starting with the seminal work of Flanders [4] and continuing with the efforts of Dolezal [2] and the author [18], [20]. However, except for some highly particular cases such as the lumped transmission line, there have been very few results concerning the actual computation of the current flows in an infinite network in the case where the total power dissipation or stored energy is required to be finite. A notable accomplishment in this regard is Flanders' justification [5] of the various analyses of the constant-resistance square grid. If the assumption of finite power or finite energy is dropped, a method [19], [20] becomes available, but it requires the a priori assignments of the currents in certain branches called "joints". How to choose the joint currents to ensure finite power or finite energy remains an open question for general infinite networks.

This paper attacks that problem in the case of the grounded semi-infinite grid of Figure 1, wherein Y_a and Y_c are positive conductances and Z_b is a positive resistance. Such a network arises, for example, as the discrete approximation of the following heat-flow problem. Let there be a very thin uniform half-plane layer of heat-conductive material M_1 coating a somewhat thicker medium M_2 of considerably lower heat conductivity, which in turn coats a medium of infinite heat conductivity M_3 . This is shown in cross-section in Figure 2. The materials are understood to extend infinitely in both perpendicular directions to the indicated cross section and infinitely to the right as well. Ambient temperature is taken to be zero, which is the temperature throughout M_3 . A small rod-shaped source of heat H is applied to the left-hand edge of M_1 . Unlike the media M_1 , M_2 , and M_3 , H is not infinite in extent in the perpendicular directions; H supplies only a finite amount of heat flow to

to yield finite power, and a limb analysis [19] then becomes feasible. Alternatively, all the node voltages and thereby all the currents can be determined by our computations on the unit circle.

Our method works for networks with more complicated graphs than the one shown in Figure 1. We can allow each node of that figure to be adjacent to more than two nodes in the same horizontal row in which the considered node appears. We need only require that each node have a finite degree and that the grid remain uniform. (See Figure 5 of [15].) Moreover, the element values and graphs can vary from horizontal row to horizontal row and even in the values of Z_b so long as some periodicity in these variables occur along the vertical direction.

Actually, these grounded-grid configurations arise from the discretization of the partial-differential equations of a variety of physical phenomena. Now, however, the branch parameters, rather than being resistances, are in general positive-real impedances [8], [9], [15]. This raises the question of how the transient responses within such grids can be computed. An answer is developed based upon Papoulis' inversion method for the Laplace transformation. Briefly stated, we apply our analysis for purely resistive grids along a sequence of points on the real positive axis of the complex plane and apply Papoulis' method.

Finally, some results of Dolezal [3] allow us to use our characteristic-resistance method to compute approximately the currents in a grid of nonlinear monotone resistances. If the nonlinearities are not too severe, the grid can be approximated by a grid of linear resistances and a bound on the ensuing error between the current responses of the nonlinear grid and its linearization can be stated.

2. An existence and uniqueness theorem. We will need an extension of the existence and uniqueness theorem of [16] to the case where the currents and voltages are Hilbert-space-valued and the branch conductances are positive

The inner product of two coboundaries \underline{v}' and \underline{w}' in \mathcal{V} is defined to be $\sum (v_j, g_j w_j)$. Thus, the norm of \underline{v}' is

$$\|\underline{v}'\| = [\sum (v_j, g_j v_j)]^{1/2}$$

Finally, we define the conductance operator G of N to be the mapping of any 1-cochain $\underline{v}' = \sum v_j B_j'$ into the 1-chain $G\underline{v}' = \sum g_j v_j B_j$.

All the arguments of [16] carry over to this case of an operator network. They establish the following existence and uniqueness theorem.

Theorem 2.1. Let N satisfy Conditions A, and let its branch parameters satisfy

$$(2.2) \quad \sum (g_j^{-1} h_j, h_j) < \infty .$$

Then, there exists a unique $\underline{v}' \in \mathcal{V}$ such that

$$(2.3) \quad \langle \underline{w}', \underline{h} - G \underline{v}' \rangle = 0$$

for all $\underline{w}' \in \mathcal{V}$.

This theorem states in effect that four conditions determine a unique set of branch voltages: Kirchhoff's loop law (\underline{v}' is a coboundary), the finite-power dissipation condition (2.1), the finite-power-available condition (2.2), and a generalized form of Tellegen's theorem (2.3), which encompasses Kirchhoff's node law and Ohm's law as consequences.

Our infinite grids possess two more properties that will be worth exploiting.

Conditions B. (i) N is locally finite except for a single ground node which is adjacent to every other node.

(ii) There are only a finite number of different operators among all the g_j .

We can now identify \mathcal{V} with a certain subspace of all coboundaries as follows.

Lemma 2.1. Condition B(ii) and the positivity and invertibility of each g_j imply that $\underline{v}' = \sum v_j B_j' \in \mathcal{V}$ if and only if the v_j satisfy Kirchhoff's loop law and $\sum \|v_j\|^2 < \infty$.

Proof. Kirchhoff's loop law is equivalent to the assertion that \underline{v}' is a

connecting ground to that node by $B_{\nu, j}$ whose voltage with respect to $w' - w'_y$ is not zero. For each index j occurring in Σ_1 , there is a unique index $i = i(j)$. Moreover, as $\nu \rightarrow \infty$, the minimum of all the indices j in Σ_1 and Σ_2 and of all their corresponding indices $i(j)$ tends to infinity.

The absolute value of (2.5) is bounded by

$$\begin{aligned} & \Sigma_1 |(x_i, h_j)| + \Sigma_1 |(x_i, g_j v_j)| + \Sigma_2 |(w_j, h_j)| + \Sigma_2 |(w_j, g_j v_j)| \\ = & \Sigma_1 |(g_j^{\frac{1}{2}} x_i, g_j^{-\frac{1}{2}} h_j)| + \Sigma_1 |(g_j^{\frac{1}{2}} x_i, g_j^{\frac{1}{2}} v_j)| \\ & + \Sigma_2 |(g_j^{\frac{1}{2}} w_j, g_j^{-\frac{1}{2}} h_j)| + \Sigma_2 |(g_j^{\frac{1}{2}} w_j, g_j^{\frac{1}{2}} v_j)| \end{aligned}$$

(Remember that $i = i(j)$; thus, the sum Σ_1 on j also sums over the indices $i = i(j)$.) By Schwarz's inequality, this quantity is bounded by

$$\begin{aligned} & \Sigma_1 \|g_j^{\frac{1}{2}} x_i\| \|g_j^{-\frac{1}{2}} h_j\| + \Sigma_1 \|g_j^{\frac{1}{2}} x_i\| \|g_j^{\frac{1}{2}} v_j\| \\ & + \Sigma_2 \|g_j^{\frac{1}{2}} w_j\| \|g_j^{-\frac{1}{2}} h_j\| + \Sigma_2 \|g_j^{\frac{1}{2}} w_j\| \|g_j^{\frac{1}{2}} v_j\| \\ \leq & [\Sigma_1 \|g_j^{\frac{1}{2}} g_i^{-\frac{1}{2}} g_i^{\frac{1}{2}} x_i\|^2 \Sigma_1 (g_j^{-1} h_j, h_j)]^{\frac{1}{2}} \\ & + [\Sigma_1 \|g_j^{\frac{1}{2}} g_i^{-\frac{1}{2}} g_i^{\frac{1}{2}} x_i\|^2 \Sigma_1 (v_j, g_j v_j)]^{\frac{1}{2}} \\ & + [\Sigma_2 (w_j, g_j w_j) \Sigma_2 (g_j^{-1} h_j, h_j)]^{\frac{1}{2}} \\ & + [\Sigma_2 (w_j, g_j w_j) \Sigma_2 (v_j, g_j v_j)]^{\frac{1}{2}} \end{aligned}$$

Since there are only a finite number of different operators among all of the g_j , there is a constant M such that $\|g_j^{\frac{1}{2}} g_i^{-\frac{1}{2}}\|^2 < M$ for all i and j . Thus,

$$\Sigma_1 \|g_j^{\frac{1}{2}} g_i^{-\frac{1}{2}} g_i^{\frac{1}{2}} x_i\|^2 \leq M \Sigma_1 (x_i, g_i x_i)$$

We have already noted that the minimum of all the indices j in Σ_1 and Σ_2 and of the corresponding $i(j)$ tends to infinity as $\nu \rightarrow \infty$. Therefore, these estimates coupled with (2.1) and (2.2) imply that (2.4) tends to zero as $\nu \rightarrow \infty$, which is what we had to show.

node and loop laws and Ohm's law are satisfied. This branch-voltage vector determines and is determined by a unique node-voltage vector in ℓ_{2r} .

3. A ladder network of Hilbert ports. Our objective is to derive a set of equations from which the currents and voltages in the infinite grid of Figure 1 can be numerically computed. This will be accomplished by replacing the infinite grid of Figure 1 by a ladder network each of whose branches is a Hilbert port [14] with respect to ℓ_{2r} obtained from a subnetwork of the grid. We need in fact only two different Hilbert ports, those shown in Figures 4(b) and 4(c). Figure 4(a) is a vector-valued current source whose value is given by (2.6). As we shall see, the Hilbert port admittance Y of Figure 4(b) and the Hilbert-port impedance Z of Figure 4(c) are continuous linear mappings of ℓ_{2r} into ℓ_{2r} given by certain Laurent matrices. A Laurent matrix is an infinite matrix of the form $A = [A_{jk}]$, where $j, k = \dots, -1, 0, 1, \dots$, such that $A_{j,k} = A_{j+1, k+1}$ for all j, k [6; p. 135]. Thus, all its rows are the same except for horizontal shifts. To specify a Laurent matrix, it suffices to specify its row for $j=0$; we call that row the principal row and denote it by

$$A^\dagger = [\dots, A_{0,-1}, (A_{0,0}), A_{0,1} \dots]$$

where the parentheses are used to identify the $j=0, k=0$ entry in A .

Because of the disconnected form of the Hilbert port of Figure 4(c), only one current vector i can respond when a voltage-source vector in ℓ_{2r} is connected to that Hilbert port. Moreover, i will also be in ℓ_{2r} because of the constant value of Z_b of the impedance therein. Thus, the impedance Z of that Hilbert port is truly a Laurent matrix, in fact, a diagonal matrix with the principal row:

$$(3.1) \quad Z^\dagger = [\dots, 0, 0, (Z_b), 0, 0 \dots]$$

On the other hand, many different responding (node) voltage vectors are possible when a current-source vector i in ℓ_{2r} is impressed upon the Hilbert port of Figure 4(b). However, since Figure 4(b) with that current source satisfies

voltages and currents of Figure 5 based upon the characteristic-resistance approach to periodic structures.

4. The characteristic-resistance method. We wish to determine the ℓ_{2r} -valued currents and voltages in the ladder network of Figure 5, where H is a given ℓ_{2r} -valued current source. We shall use the standard characteristic-resistance method for periodic transmission lines, which now requires our extension of it to the case where the line's parameters are operators—rather than scalars.

We will show later on that the characteristic-resistance operator Z_0 indicated in Figure 5 is invertible. Assume this to be true for the moment and set $Y_0 = Z_0^{-1}$. Then, by the usual argument that the characteristic resistance is not altered if a single Y and a single Z are removed from the beginning of the ladder structure, we have

$$(4.1) \quad (Y_0 - Y)(Z_0 + Z) = 1.$$

Here, 1 denotes the identity operator on ℓ_{2r} .

This equation has more than one solution Z_0 . Let us argue heuristically for the moment to see which solution we should seek. Z_0 should describe the behavior of the network of Figure 1. By the symmetry of that network, a shift of the set of current sources H_k to the right or left should result in the same shift in the responding node voltages. That is, Z_0 should commute with the bilateral shift. This means that Z_0 should be a Laurent matrix [6; p. 135]. Moreover, the solution dictated by Theorem 2.1 disallows any energy being injected into the network from infinity. Moreover, the ladder network is passive. This suggests that Z_0 should be a positive operator.

To find such a Z_0 , we shall make use of the natural isomorphism between ℓ_{2r} and the space $L_2(0, 2\pi)$ of (equivalence classes of) quadratically integrable

In view of the remarks of the preceding paragraph, we seek the positive solution $\tilde{Z}_0(x)$ of (4.2) where $\tilde{Y}_0(x) = \tilde{Z}_0(x)^{-1}$. It is

$$(4.3) \quad \tilde{Z}_0(x) = \frac{-\beta(\alpha - 2\cos x) + [\beta^2(\alpha - 2\cos x)^2 + 4\beta(\alpha - 2\cos x)]^{\frac{1}{2}}}{2Y_a(\alpha - 2\cos x)}$$

where $\beta = Y_a Z_b$. Since $\alpha > 2$, $\tilde{Z}_0(x)$ is clearly positive, even, continuous, bounded, and bounded away from zero for all x . (The other root of (4.2) is negative for all x). Moreover, $\tilde{Y}_0(x)$ is given by

$$(4.4) \quad \tilde{Y}_0(x) = \frac{\beta(\alpha - 2\cos x) + [\beta^2(\alpha - 2\cos x)^2 + 4\beta(\alpha - 2\cos x)]^{\frac{1}{2}}}{2Z_b}$$

and has the same properties. Thus, Z_0 and Y_0 are the Laurent matrices whose principal rows are the Fourier coefficients of the functions $\tilde{Z}_0(x)$ and $\tilde{Y}_0(x)$ respectively. Therefore,

$$(4.5) \quad Z_0 = (2Y)^{-1} [-YZ + (Y^2 Z^2 + 4YZ)^{\frac{1}{2}}]$$

and

$$(4.6) \quad Y_0 = (2Z)^{-1} [YZ + (Y^2 Z^2 + 4YZ)^{\frac{1}{2}}].$$

By the aforementioned isomorphism, Z_0 and Y_0 are positive invertible operators on ℓ_{2r} .

Laurent matrices commute. Also, the square root and the inverse of a positive invertible operator A commute with every operator that commutes with A . These facts allow us to manipulate Y and Z in much the same way as one might manipulate scalars. It follows from Figure 5 that

$$(4.7) \quad V_1 = Z_0 H$$

Thus,
$$I_2 = H - I_1 = H - Y Z_0 H = (1 - Y Z_0) H$$

Alternatively,

$$I_2 = (Z_0 + Z)^{-1} V_1 = (Z_0 + Z)^{-1} Z_0 H = (1 + Y_0 Z)^{-1} H.$$

This lemma allows us to prove

Theorem 4.1. The solution for the network of Figure 5 given by (4.5) through (4.10) is precisely the solution dictated by Theorem 2.2.

Proof. We shall show that the solution given by (4.5) through (4.10) satisfies the conditions of Theorem 2.2. First note that the network of Figure 5 satisfies Conditions A and B. Number and orient the branches in accordance with the currents in Figure 5 and let v_j be the branch voltages (i.e., voltage drops). Since the V_{2n+1} are node voltages, Kirchhoff's loop law is automatically satisfied by the corresponding v_j . The branch currents, as determined by Ohm's law in its operator form, satisfy Kirchhoff's node law. Indeed, for every n ,

$$\begin{aligned} I_{2n+2} + I_{2n+1} &= \theta^{n+1} H + Y V_{2n+1} = \theta^{n+1} H + Y \theta^n V_1 \\ &= \theta^n (\theta H + Y Z_0 H) = \theta^n (\theta + Y Z_0) H. \end{aligned}$$

By (4.8), $\theta + Y Z_0 = 1$. Hence,

$$I_{2n+2} + I_{2n+1} = \theta^n H = I_{2n}$$

which fulfills Kirchhoff's node law.

Finally, we have to show that the vector of branch voltages is in $\ell_2(H_r)$, where $H_r = \ell_{2r}$. For the vertical branches in Figure 5, we may invoke Lemma 4.1 to write

$$\sum_{n=0}^{\infty} \|v_{2n+1}\|^2 \leq \sum_{n=0}^{\infty} \|\theta^n v_1\|^2 \leq \|v_1\|^2 \sum_{n=0}^{\infty} \delta^{2n}$$

The right-hand side is a finite quantity because $0 < \delta < 1$. Similarly, for the horizontal branches of Figure 5, we have

$$\sum_{n=1}^{\infty} \|Z I_{2n}\|^2 \leq \|Z\|^2 \sum_{n=1}^{\infty} \|\theta^n H\|^2 \leq \|Z\|^2 \|H\|^2 \sum_{n=1}^{\infty} \delta^{2n} < \infty$$

This completes the proof.

Furthermore, the components of the vector voltages V_{2n+1} are the node voltages of Figure 1. Therefore, the branch voltages of Figure 1 determined from these node voltages satisfy Kirchhoff's loop law.

Finally, the sum of the squares of all the branch voltages within any Hilbert port of impedance Z is equal to $\|Z I_{2n}\|^2$, where I_{2n} is its port current vector. For any Hilbert port of admittance Y with the port voltage vector V_{2n+1} , the sum of the squares of the voltages across all the admittances Y_c is equal to $\|V_{2n+1}\|^2$. On the other hand, the voltage across any admittance Y_a in that Hilbert port is the difference between the two node voltages on Y_a . Since $(x - y)^2 \leq 2x^2 + 2y^2$, it follows that the sum of the squares of the voltages across all the Y_a in that Hilbert port is no larger than $4 \|V_{2n+1}\|^2$. Thus, the sum of the squares of all the branch voltages in the grid—as determined by our two-step procedure—is no larger than

$$\sum_{n=1}^{\infty} \|Z I_{2n}\|^2 + 5 \sum_{n=0}^{\infty} \|V_{2n+1}\|^2$$

In the last paragraph of the proof of Theorem 4.1, this quantity was shown to be finite. Hence, the branch-voltage vector as determined by our two-step procedure is a member of ℓ_{2r} . This completes the proof.

6. The computation of the voltages and currents in the grid. We can use the Fourier-series representation of the members of ℓ_{2r} to convert the equations of Section 4 into a form that is convenient for numerical computation. We have already noted that the periodic function \tilde{Z}_0 corresponding to the Laurent matrix Z_0 is given by (4.3). Similarly, we can use (4.8) to compute the periodic function $\tilde{\theta}$, which is the multiplier corresponding to the Laurent matrix θ . We obtain

$$(6.1) \quad \tilde{\theta}(x) = 1 + \frac{1}{2} \beta (\alpha - 2 \cos x) - \frac{1}{2} [\beta^2 (\alpha - 2 \cos x)^2 + 4 \beta (\alpha - 2 \cos x)]^{\frac{1}{2}}$$

limb analysis [19]. Our present analysis has shown how those joint voltages must be assigned in order to obtain the finite-power-dissipation condition, a heretofore open problem.

Here are some numerical examples regarding the grid of Figure 1. As before, V_{2n+1} denotes the vector of node voltages in the $(n+1)$ st horizontal row of nodes for $n = 0, 1, 2, \dots$. We display the 0th-indexed entry in each vector with parentheses. Because each of these vectors have even symmetry around the 0th-indexed entry, we need merely give their entries for nonnegative indices.

Example 6.1. Let $Y_a = 1$, $Y_c = 1$, and $Z_b = 1$. Also, let $H = [\dots, 0, 0, (1), 0, 0, \dots]^T$. Therefore, $\tilde{H}(x) = 1$. Moreover, $\alpha=3$, $\beta=1$. Then, $Z_0(x)$ is given by (4.3), and $\tilde{\Theta}(x)$ by (6.1). Upon computing the Fourier coefficients of (6.2) and (6.3) we obtain all the V_{2n+1} for $n = 0, 1, 2, \dots$. Ohm's law then yields all the branch currents. Some results for the V_{2n+1} are:

$$V_1 = [\dots, (.3216), .0996, .0322, .0108, .0037, .0013, \dots]^T$$

$$V_3 = [\dots, (.0873), .0445, .0185, .0072, .0027, .0010, \dots]^T$$

$$V_5 = [\dots, (.0259), .0170, .0086, .0038, .0016, .0006, \dots]^T$$

$$V_7 = [\dots, (.0082), .0062, .0036, .0018, .0008, .0004, \dots]^T$$

$$V_9 = [\dots, (.0027), .0022, .0014, .0008, .0004, .0002, \dots]^T$$

$$V_{11} = [\dots, (.0009), .0007, .0005, .0003, .0002, .0001, \dots]^T$$

Example 6.2 Now set $Y_a = 1$, $Y_c = 2$, $Z_b = 10$ and $\tilde{H}(x) = 1$. Since the conductances Y_c to ground and the resistances Z_b are now larger, we should expect a more rapid decay in the node voltages as we progress into the grid. This is substantiated by the following results.

$$V_1 = [\dots, (.2797), .0730, .0190, .0050, .0013, .0003, \dots]^T$$

$$V_3 = [\dots, (.0087), .0042, .0016, .0005, .0002, .0001, \dots]^T$$

$$V_5 = [\dots, (.0003), .0002, .0001, .0000, .0000, \dots]^T$$

$$V_7 = [\dots, (.0000), .0000, \dots]^T$$

There is still another way of extending our method to more complicated grids, and that is to introduce three-dimensional ones. Figure 6 illustrates the three-dimensional half-space analogue to Figure 1. Let ξ , η and ζ represent the three coordinates of three-dimensional Euclidean space. Restrict ξ and η to the integers and ζ to the positive integers. Then, all such triplets (ξ, η, ζ) will be the locations of our grid's nodes, except for an additional ground node. For a fixed ζ and variable ξ and η , we have the ζ th horizontal plane of nodes in Figure 6. The positive ζ direction is the downward vertical direction. Every pair of nodes at a distance of one unit apart is connected by a branch. Every horizontal branch has a positive conductance Y_a , and every vertical branch has a positive resistance Z_b . Moreover, every node is connected to ground through a positive conductance Y_c . Finally, every node of the form $(\xi, \eta, 1)$ has a current source $H_{\xi, \eta}$ connected to it from ground. $H_{\xi, \eta}$ is a real number, which we will also denote by H_k , where $k = (\xi, \eta) = (k_1, k_2)$ is a doublet of integers k_1 and k_2 .

To analyze this configuration, it is convenient to alter our definition of the Hilbert space ℓ_{2r} in a nonessential way. First of all, we let D denote the space of all ordered doublets whose entries are integers. Thus, $k \in D$. An element of ℓ_{2r} is now taken to be a two-dimensional array $\{a_k : k \in D\}$ of real numbers a_k such that

$$\sum_{k \in D} a_k^2 = \sum_{k_1 = -\infty}^{\infty} \sum_{k_2 = -\infty}^{\infty} a_{(k_1, k_2)}^2 < \infty.$$

The inner product of two elements $a = \{a_k\}$ and $b = \{b_k\}$ in ℓ_{2r} is now the double infinite series:

$$(a, b) = \sum_{k \in D} a_k b_k$$

A nodal analysis shows that the admittance operator Y for the Hilbert port corresponding to any horizontal plane is given by $Y = [Y_{jk}]$, where

$$Y_{jk} = \begin{cases} Y_c + 4Y_a & \text{for } j = k \\ -Y_a & \text{for } |j_1 - k_1| + |j_2 - k_2| = 1 \\ 0 & \text{otherwise} \end{cases}$$

This is a four-dimensional analogue of a row-finite Laurent matrix.

Similarly, the impedance operator $Z = [Z_{jk}]$ of the Hilbert port of resistances Z_b is given by

$$Z_{jk} = \begin{cases} Z_b & \text{for } j = k \\ 0 & \text{for } j \neq k \end{cases}$$

We now have the analogue of a diagonal matrix with equal main-diagonal terms.

Next, we exploit the isomorphism between our present space l_{2r} and the space of double Fourier series; that is, $a = [a_k: k \in D]$ corresponds to

$$\begin{aligned} \tilde{a}(x) &= \sum_{k \in D} a_k e^{i(k,x)} \\ &= \sum_{k_1 = -\infty}^{\infty} \sum_{k_2 = -\infty}^{\infty} a_{(k_1, k_2)} e^{i k_1 x_1 + i k_2 x_2} \end{aligned}$$

where $k = (k_1, k_2)$, $x = (x_1, x_2)$, and $(k, x) = k_1 x_1 + k_2 x_2$. Under this correspondence, the operator Y transforms into multiplication by the function

$$\tilde{Y}(x) = Y_a [\gamma - 2 \cos x_1 - 2 \cos x_2]$$

where

$$\gamma = \frac{Y_c}{Y_a} + 4 > 4.$$

The operator Z corresponds to multiplication by the constant function $\tilde{Z}(x) = Z_b$.

Finally, the given current-source array H corresponds to

$$\tilde{H}(x) = \sum_{k \in D} H_k e^{i(k,x)}$$

Figure 6 for any n . From these, any voltage or current can be obtained once again (subject to the usual caveat about computer time).

What we have just described is a formal method of computing the voltages and currents in Figure 6. Of equal importance is the fact that the existence and uniqueness discussions of the prior sections carry directly over to our present three-dimensional grid. Indeed, all the theorems, and in particular Theorems 2.2 and 5.1, hold once again for the grid of Figure 6. Only a few minor modifications, primarily those of notation, need be made.

Finally, we point out once again that we can augment Figure 6 by adding other horizontal branches to pairs of nodes further apart than one unit, so long as the uniformity of each horizontal plane is maintained. This merely alters the function $\tilde{Y}(x)$ by adding additional terms of the form $b_{\nu\mu} \cos \nu x_1 \cos \mu x_2$ where ν and μ are integers. Furthermore, variations in the graphs and element values in the vertical direction can also be allowed so long as periodicity in the vertical direction is maintained; this leads to the problem of solving more complicated equations for $\tilde{Z}_0(x)$.

Example 7.1. As an example, we have computed the node voltages for the grid of Figure 6 for the case when $Y_a = Z_b = Y_c = 1$, $H_{0,0} = 1$, and $H_{\xi,\eta} = 0$ if $\xi \neq 0$ or $\eta \neq 0$. For the node voltages, we use the indexing system explained at the beginning of this section. The values of the node voltages for the first five horizontal planes of nodes (i.e., for $\zeta = 1, \dots, 5$) are given below. Because of the symmetry of our grid and of the applied current sources, we need merely display the node voltages for $\xi = 1, 2, 3, \dots$ and $\eta = 1, \dots, \xi$. The other node values are obtained by interchanging ξ with η and by using even symmetry for negative indices. We terminate the tables at $\xi = 4$.

8. Grids of impedances. So far, we have restricted our attention to purely resistive grids. We now wish to allow capacitors, inductors, transformers, etc. as grid parameters. More specifically, we shall now generalize the grid of Figure 1 by assuming that Y_a , Z_b , and Y_c are (scalar) positive-real functions. Our objective is to show that, if the current sources in Figure 1 are given as suitably restricted Laplace-transformable distributions on the time axis, then there exists a unique set of Laplace-transformable voltage and current distributions that satisfy Kirchhoff's and Ohm's laws and a certain form of the finite-power-dissipation condition. They will be determinable from our operator version of the characteristic-impedance method.

To accomplish this, we shall make use of the results and notations of [17]. C_+ will denote the open right half of the complex plane C :

$$C_+ = \{s \in C: \operatorname{Re} s > 0\}$$

For $s \in C_+$, Ω_s is the closed cone:

$$\Omega_s = \{Z \in C: |\arg z| \leq |\arg s|\},$$

where it is understood that the origin is a member of Ω_s . ℓ_2 denotes the complexification of ℓ_{2r} . By an "operator", we henceforth mean a continuous linear mapping of ℓ_2 into ℓ_2 . For any operator F , $W[F]$ is the numerical range of F :

$$W[F] = \{(Wa, a): a \in \ell_2, \|a\| = 1\}$$

P is the set of all analytic operator-valued functions F on C_+ such that, for every $s \in C_+$, $W[F(s)] \subset \Omega_s$. Thus, if $F \in P$, $F(\sigma)$ is a positive operator for each $\sigma > 0$. P_i is the set of all $F \in P$ such that, for every fixed $s \in C_+$, $W[F(s)]$ is bounded away from the origin; that is, there exists a $\delta > 0$ such that $\operatorname{Re}(F(s)a, a) \geq \delta \|a\|^2$ for all $a \in \ell_2$. Thus, for each $s \in C_+$, and $F \in P_i$, $F(s)$ is an invertible operator. It was shown in [17] that, if $F \in P_i$ and $G \in P$, then $F + G \in P_i$; also, if $F \in P_i$ and if F^{-1} denotes the function $s \rightarrow [F(s)]^{-1}$, then $F^{-1} \in P_i$.

Now, a standard property of a scalar positive-real function F is that $|\arg F(s)| \leq |\arg s|$ for $s \in C_+$ [12; Theorem 5]. Since Y_a and Y_c are scalar positive-real functions, this fact coupled with (8.2) implies that $Y \in P$. Since Y_c is not identically equal to zero, the minimax theorem for harmonic functions shows that, given any compact subset $E \subset C_+$, there is a $\delta > 0$ for which

$$\operatorname{Re}(Y(s)a, a) \geq \operatorname{Re} Y_c(a) \|a\|^2 \geq \delta \|a\|^2$$

for all $s \in E$. Thus, $Y \in P_i$, and in addition Y satisfies Hypothesis (ii) of Theorem 8.1 when we set $F_k(s) = Y(s)$.

A simpler argument shows that, if Z_b is a scalar positive-real function, then the Laurent matrix Z , as defined by (3.1), also is in P_i and satisfies Hypothesis (ii) of Theorem 8.1 when $F_k(s) = Z(s)$. Moreover, row-finite Laurent matrices commute, and therefore so too do $Y(s)$ and $Z(s)$. Finally, upon setting $F_k(s) = Y(s)$ for k odd and $F_k(s) = Z(s)$ for k even and noting that the series and shunt elements of Figure 5 remain invariant, we see that Hypothesis (iii) of Theorem 8.1 is also satisfied.

Thus, we may invoke Theorem 8.1 to conclude that, as $n \rightarrow \infty$, the Z_n , defined by (8.1), converge to a member of P_i . But, the Z_n are the driving-point impedance operators of truncations of the ladder network of Figure 5. Hence, their limit is the characteristic impedance Z_0 of that infinite ladder network. Thus, we have established

Theorem 8.2. Let Z be defined by (3.1) and Y by (3.2), where Z_b , Y_a , and Y_c are scalar positive-real functions. Then, Z and Y are members of P_i , and so too is the characteristic-impedance operator Z_0 of the infinite ladder network of Figure 5.

P_i is a subset of the class of all positive* operator-valued functions [14; p. 178]. Moreover, an operator-valued positive* function is the Laplace transform of a right-sided operator-valued distribution, where C_+ is understood

$H(s) \in \mathcal{L}_2$. Let τ be the infimum of the support of h , and let $\sigma = \operatorname{Re} s > 0$. Then,

$$|H_k(s)| = \left| \int_{\tau}^{\infty} h_k(t) e^{-st} dt \right| \leq \int_{\tau}^{\infty} |h_k(t)| e^{-\sigma t} dt.$$

By Schwarz's inequality

$$|H_k(s)|^2 \leq \frac{e^{-2\sigma\tau}}{2\sigma} \int_{\tau}^{\infty} |h_k(t)|^2 dt$$

Therefore

$$(8.4) \quad \sum_{k=-\infty}^{\infty} |H_k(s)|^2 \leq \frac{e^{-2\sigma\tau}}{2\sigma} \sum_{k=-\infty}^{\infty} \int_{\tau}^{\infty} |h_k(t)|^2 dt$$

Now, $\{\sum_{k=-r}^r |h_k(t)|^2 : r = 1, 2, 3, \dots\}$ is a monotonic sequence of integrable functions which converge to $\sum_{k=-\infty}^{\infty} |h_k(t)|^2$ for almost all t . So, by Levi's theorem, we can interchange the summation with the integration to write

$$\sum_{k=-\infty}^{\infty} |H_k(s)|^2 \leq \frac{e^{-2\sigma\tau}}{2\sigma} \int_{\tau}^{\infty} \sum_{k=-\infty}^{\infty} |h_k(t)|^2 dt.$$

The integral on the right-hand side is the square of the norm of $h \in L_2(\mathbb{R}, \mathcal{L}_{2r})$.

This proves the first conclusion. The second conclusion now follows immediately from the reality of h .

Under the hypothesis of Theorem 8.2 and Lemma 8.1, $V_1(s)$, $V_{2n+1}(s)$, and $I_{2n}(s)$, as given by (4.7), (4.9), and (4.10), are all \mathcal{L}_2 -valued Laplace transforms with C_+ contained in their regions of definition [14; Theorem 6.3-2]. Moreover, their respective inverse Laplace transforms $v_1(t)$, $v_{2n+1}(t)$, and $i_{2n}(t)$ will be \mathcal{L}_{2r} -valued distributions with supports bounded on the left at the origin. The reality of these distributions is a consequence of [14; Theorem 8.13-1] and the exchange formula [14; Theorem 6.3-2]. The various components of these distributions yield the transient voltages and currents of our infinite grid under the finite-power condition on the real positive axis of the s -domain.

9. The computation of transient responses. We can couple the last conclusion of Theorem 8.3 with a method of Papoulis to compute the transient behavior of our electrical grids when their branch impedances are scalar positive-real functions. Let's illustrate this with a particular example.

Consider the grid of Figure 1. Let us assume that in the time domain the 0th-indexed current source is the unit-step function and that all the other sources are zero. Therefore, the Laplace transform of the current-source vector is

$$(9.1) \quad H(s) = [\dots, 0, 0, (\frac{1}{s}), 0, 0, \dots]^T, \quad s \in C_+.$$

Let us also assume that Y_c is the admittance of a parallel combination of a resistor and capacitor, Y_a is a conductor, and Z_b is a resistor. In particular, we take

$$Y_c(s) = s + 1, \quad Y_a = 1, \quad Z_b = 1.$$

Then, α becomes a function of s :

$$\alpha(s) = \frac{Y_c(s)}{Y_a} + 2 = s + 3, \quad s \in C_+.$$

$\beta = Y_a Z_b = 1$ as before. Upon substituting $\alpha(s)$ and β in (4.3) and (6.1), we obtain \tilde{Z}_0 and $\tilde{\theta}$ as functions of both x and s . Thus, the Fourier coefficients of (6.2) and (6.3) can be obtained as functions of s ; these Fourier coefficients are the Laplace transforms of the time-dependent node voltages in our grid.

Let's compute the transient voltage v at the node to which the 0th-indexed current source is connected. Its Laplace transform V is the constant term in the Fourier-series expansion of $\tilde{Z}_0(x)\tilde{H}(x)$. Here, $\tilde{Z}_0(x)$ is a function of s as well as x by virtue of its dependence on the parameter $\alpha(s)$. By (9.1), $\tilde{H}(x) = s^{-1}$; $\tilde{H}(x)$ is independent of x . Thus,

$$(9.2) \quad V(s) = \frac{1}{s\pi} \int_0^\pi \tilde{Z}_0(x) dx.$$

replacing r by a linear resistor r_0 : $i \mapsto r_0 i$, analyzing the resulting linear grid by our characteristic-resistance method, and then determining a bound on the error between the current responses of the nonlinear grid and its linear approximation.

More specifically, Dolezal's results are as follows. Let Y_a^{-1} , Z_b , and Y_c^{-1} all be the same nonlinear resistor r . Assume that $r(0) = 0$ and that the slope of r is bounded as follows:

$$\alpha(p-q)^2 \leq [r(p) - r(q)] (p-q) \leq \beta (p-q)^2$$

for all p and q on the real line. Here, α and β are constants satisfying

$$0 < \alpha \leq \beta < 3\alpha.$$

Furthermore, let us assume that each current source H_k and its parallel resistance Y_c^{-1} has been replaced by a voltage source e_k in series with r . Let

$$e = [\dots, e_{-1}, e_0, e_1, \dots]^T \in \ell_2$$

A theorem of Dolezal [3; Theorem 3] asserts that, under these conditions, there exists a unique vector $i \in \ell_2$ of branch currents in this nonlinear network.

To obtain some estimation of what i is, Dolezal replaces every r by a linear resistance r_0 chosen as follows:

Let A be either a real positive number or ∞ . Set

$$r_0 = \frac{1}{2} (S_A + J_A)$$

where

$$S_A = \sup_{p \in K_A} p^{-1} r(p), \quad J_A = \inf_{p \in K_A} p^{-1} r(p),$$

$$K_A = [-A, A] - \{0\}.$$

Now, we let j denote the vector of branch currents in this linear approximation to the given nonlinear grid with the aforementioned voltage sources e_k . $j \in \ell_2$ can be computed by our characteristic-resistance method. (A change of voltage sources into current sources presents no difficulties). Finally, Dolezal

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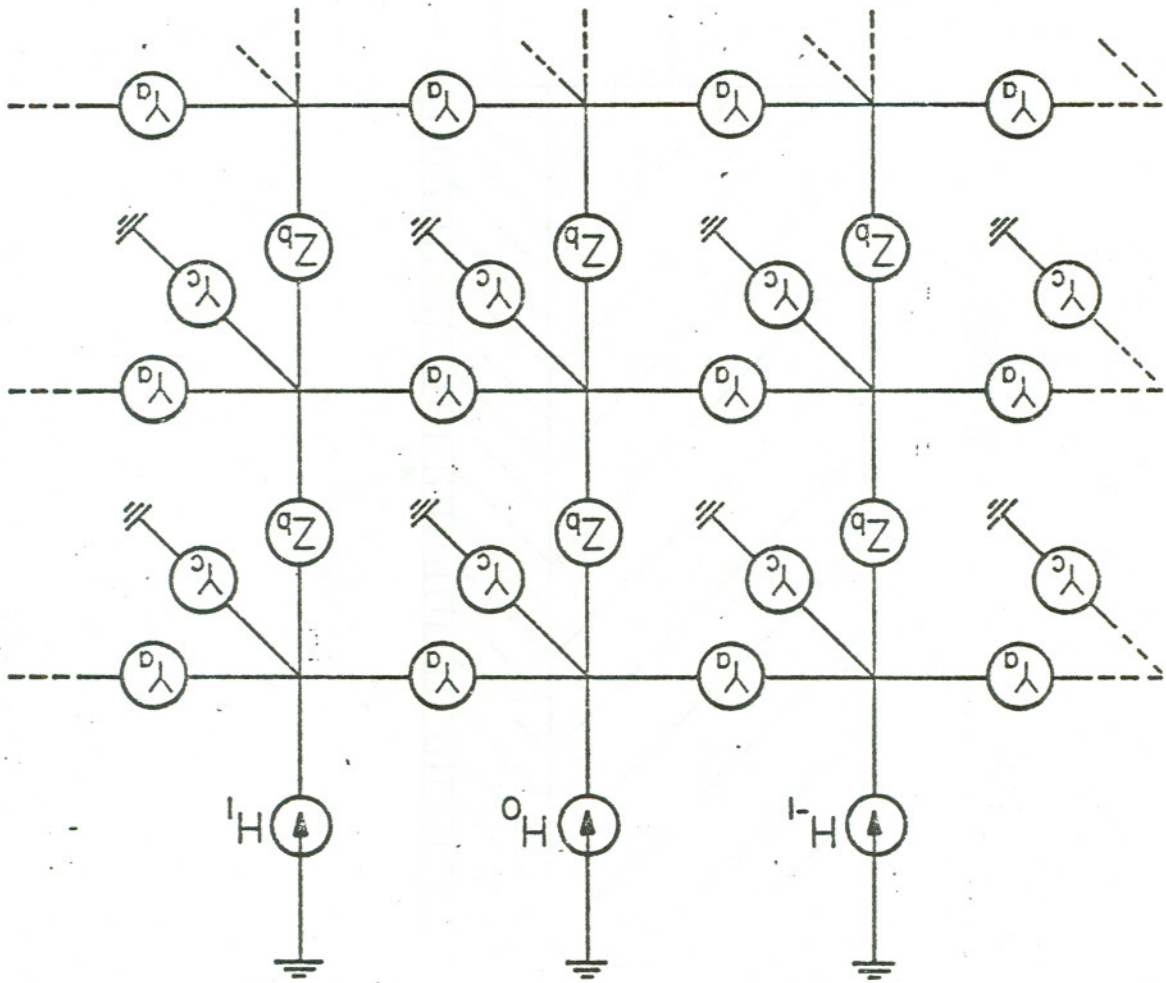
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Figure 1.



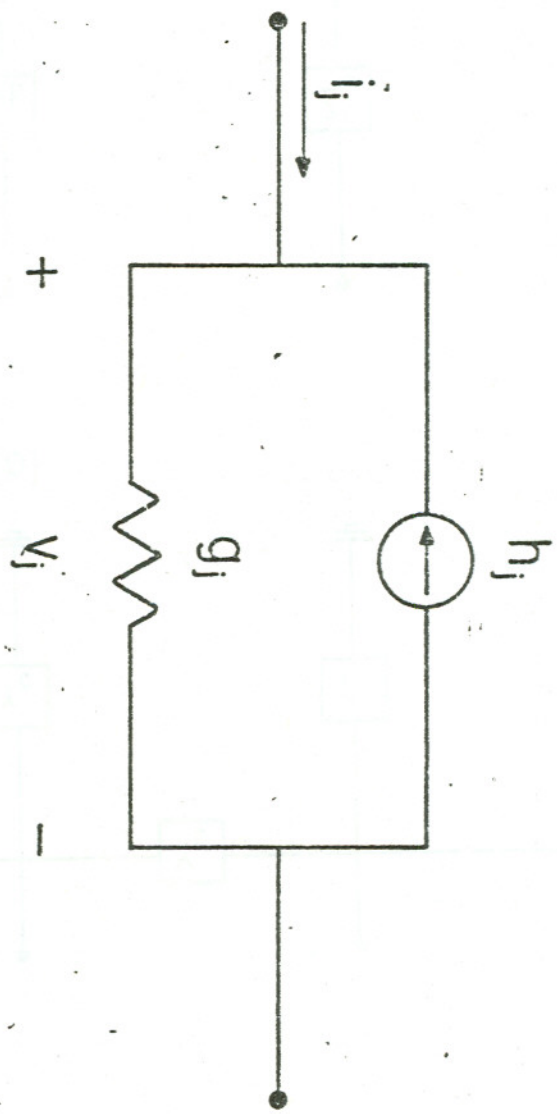


Figure 3.

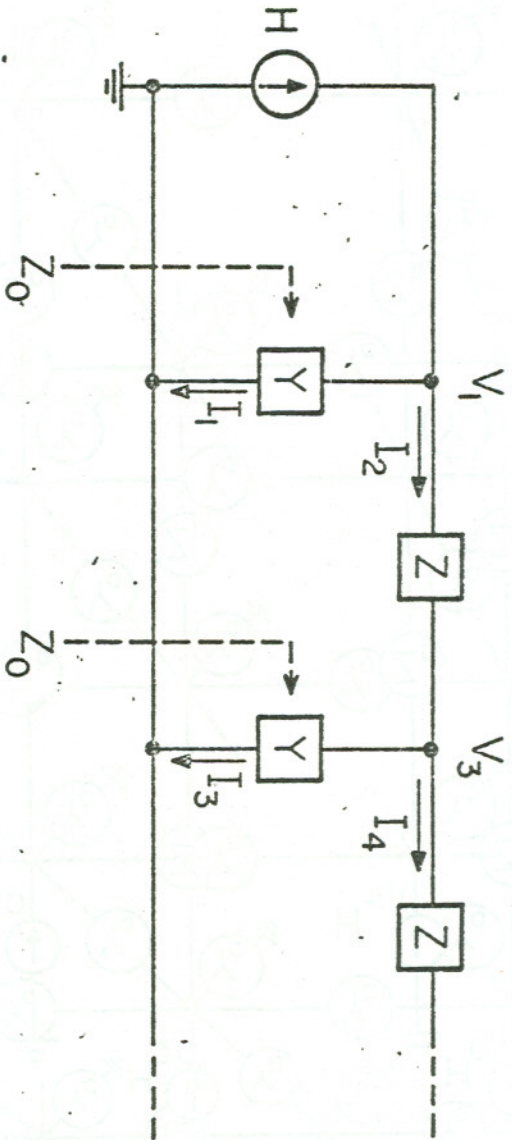


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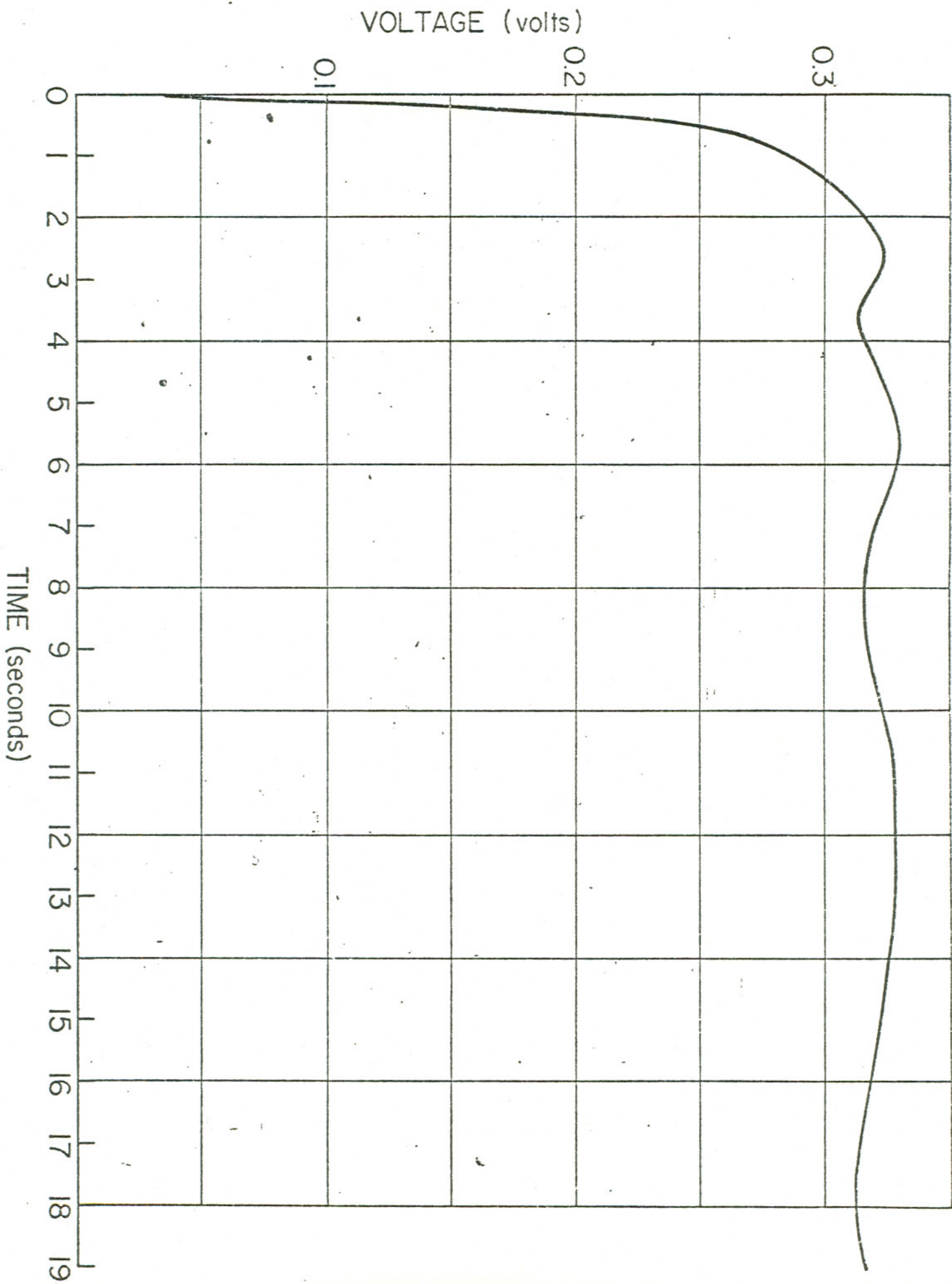


Figure 7.