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OF
PERIODIC MARKETING RINGS

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Abstract. A periodic marketing network operating on an n-day market week is examined. Traders are assumed to stock up on the first day of the week in a number of restocking centers and then to follow various marketing rings during the rest of the week selling goods in rural markets. These rings are allowed to intersect and diverge in any fashion. An economic model for this is devised which allows the computation of time series in the various prices and quantity flows. The key step is a modelling of each trader's demand curve in his restocking center and of his supply curves in the other markets of his ring. It is shown that a shortfall in supplies in one restocking center can lead - under suitable circumstances - to a fall (rather than a rise) in price in at least one of the rural markets. This yields one possible explanation of the apparently erratic behavior of periodic markets that has been reported in the literature. We also show that a single isolated marketing ring always has a unique equilibrium state.

1. Introduction. Virtually all of the theoretical literature on the spacio-temporal structure of periodic markets has been concerned with such questions as the following. How did periodic markets originate? Where are they located? What is their hierarchal structure? How do their time schedules synchronize? Which firms are mobile and which are fixed? What kinds of routes do mobile traders follow? (See, for example, R.H.T.Smith's survey paper [8] or the many references in R.J.Bromley's bibliographies [1], [2].) However, there has been hardly any theorizing about the dynamics of price and commodity-flow variations over space and time. In 1968, W.O.Jones [6] explicitly posed this problem for two-level periodic marketing networks and described a phenomenon that apparently arises when there is very little market news: namely, price information propagates from market to market primarily by means of the trading activity, and therefore the marketing system as a whole responds only sluggishly to variations in supply and demand. More recently, two alternative mathematical models of two-level periodic markets have been proposed [9], [12], [13]; they reproduce Jones' phenomenon and establish other results as well, such as the existence, uniqueness, and stability of equilibrium states.

The purpose of the present work is to mathematically model another common type of periodic marketing system: one wherein the markets open on a time-staggered schedule and the traders follow rings of market-places. We hasten to add that the rings we consider herein are defined only in terms of the individual traders. We do not assume any aggregate shifting of entire markets from place to place. Instead, we allow the individual trader rings to intersect and diverge in any fashion. In general, the union of the individual trader rings yields no more than a system of spatially distributed market-places which open on a staggered and periodic - but otherwise unstructured - time schedule. (In this regard, see the discussions of [3] and [8].)

The model we develop is an economic one; it relates price variations and commodity flows in the marketing system to the supply and demand fanctions in our network of markets. In our analysis the traders are the key agents. Each trader is assumed to follow some particular ring and to possess a transfer-supply schedule. Moreover, we take it that each trader stocks up in a restocking center on the first day of the market week and then proceeds to sell goods in the subsequent markets of his ring until his stock is either exhausted or he reaches the end of his ring on the last day of the market week. This process keeps repeating.

The principal result of our work is a model for ring-type periodic marketing networks which allows the prediction of prices and commodity flows - at least in principle. This we believe has not been attained before. To be sure, the use of our model to represent actual periodic markets would require the measurement of a large number of factors, such as supply and demand functions, a virtually impossible task we expect. Nevertheless, our model can be used to draw qualitative conclusions concerning the behavior of periodic markets. This we do.

As is to be expected from a model based on dynamic difference equations, our model exhibits W.O.Jones' market-by-market propagation of price disturbances. Another result is also of importance, we feel. It has been indicated in the literature that price swings in periodic markets appear at times to be erratic and unpredictable; see [5; pp. 23-25] and [7; pp. 21-22]. Our model indicates how at least some of these apparently erratic price swings can be explained. As was mentioned above, the key to our analysis is our assumed output behavior of the traders. We propose a new model for this. In this regard, let us quote D.W.Jones [5; pp. 24-25]. "Previously available models of periodic marketing ... offer only simplistic relationships between prices and the endogeneous variables. Several decades of experience in being surprised by peasant output responses should warn us that we are studying complex, general equilibrium systems in which indirect (or at least unsuspected) relationships attain some importance." We hope that our present model will help to dispel some of the mystery.

2. A trader's excess-supply function. Assume there is a spatial distribution of market-places, each of which opens as a periodic market on a particular day of the market week and is closed on all other days of the market week. The market week is taken to have n days, where $n \geq 2$; $s = 1, \ldots, n$ will be the index for those days, numbered chronologically. We let $t = \ldots, -1, 0, 1, \ldots$ be the index for all market days (non-market days are ignored) and $\nu = \ldots, -1, 0, 1, \ldots$ be the index for the market week. Therefore, $t = \nu n + s$ and

ν: "NU" lower case

$$\nu = \left[\frac{t - 1}{n}\right]$$

where [x] denotes the largest integer that does not exceed x.

Each trader is assumed to traverse periodically a ring of markets $\ldots, \Phi_1, \Phi_2, \ldots, \Phi_n, \Phi_1, \Phi_2, \ldots, \Phi_n, \Phi_1, \ldots,$ where Φ_s denotes a market that opens on the sth day of the market week. Different traders will follow in general different rings. In this work we confine our attention to the one-commodity case. We assume that Φ_1 is a restocking center and that $\Phi_2, \ldots, \Phi_n$ are rural markets. We also assume the following pattern of buying and selling for a trader: Usually, he buys a week's supply of stock in Φ_1 and then proceeds along $\Phi_2, \ldots, \Phi_n$ selling that commodity. However, if he exhausts his stock before reaching Φ_n, he returns home and waits for the beginning of the next market week to resume his trading cycle. The model we shall develop will allow the trader to buy goods in Φ_s, where $2 \leq s \leq n$, if the price there is sufficiently low and to sell goods in Φ_1 if the price there is sufficiently high.

Φ: "PHI" Capital.

As was mentioned above, the keystone of our approach is the assumed behavior of the traders. To quote Bromley [3], "Of the various groups of market participants, the most complex and least-understood group is probably that of 'traders'." We shall simply slice through the complexity by assuming herein that the traders are rational economic agents, namely, profit-maximizing firms that supply the service of transferring goods and ownership over space and time; that is, traders buy goods in Φ_1, bulk them, ship them, and sell them to new owners. A trader's supply function for that service can be derived from the standard theory of production costs. This was done in a prior work [13], but, since it is crucial to our present effort, we briefly present the argument here as well.

Throughout this paper, we adopt the following notational convention. When s = n, s + 1 will mean 1; when s = 1, s - 1 will mean n. (We have addition and subtraction modulo n except that* our integers range from 1 to n rather than from 0 to n - 1.)

Figure 1 shows the cost functions accruing to a trader as he transfers a quantity q from one market to another. AFC means average fixed cost, AVC means average variable cost, and MC means marginal cost. The heavy line is the trader's supply curve for the transfer of goods from ϕ_s to ϕ_{s+1}. (We measure the service he supplies by the amount q of goods he transfers.) The curve coincides with the p-axis for low values of p, then jumps to the minimum point on the AVC curve, and finally follows the MC curve for large values of p [4; pp. 73-74]. Reasons for the shapes we assign to these curves are given in [13; Section 2]. Let us merely say at this point that the sharp rise in the marginal-cost curve and therefore in the trader's transfer-supply curve for larger values of q is due to the fact that the per-unit cost remains relatively low until the capacity of his transportation equipment is approached, at which point it rises rapidly; in other words, we assume that it is very costly to the trader to overload his equipment appreciably.

The actual amount $C_s(t)$ of goods the trader transports from ϕ_s to ϕ_{s+1} is determined by his transfer-supply curve and the price he expects to receive for that service. That price is the difference between the clearance price $E_s(t)$ he expects in ϕ_{s+1} at t+1 while operating in ϕ_s at t and the clearance price $P_s(t)$ he finds in ϕ_s at t. $E_s(t)$ is determined from some memory function of past prices in ϕ_{s+1} and possibly of past prices in the other ϕ_k as well. That memory function should be a monotonically increasing function of each past price, more recent past prices should have a greater effect in determining $E_s(t)$, and a history of a constant price in ϕ_{s+1} should result in the same value for $E_s(t)$. The simplest reasonable rule that satisfies these conditions is obtained by setting $E_s(t)$ equal to the last price in ϕ_{s+1} at time $t + 1 - n$.

In accordance with our arguments in [13], we replace the horizontal jump at $p = T$ by a sharp but continuous transition just above the value $p = T$. This yields the trader's transfer-supply curve in ϕ_s at t shown in Figure 2; it has the shape we shall assume throughout this paper.

In Figure 2, p represents the per-unit price the trader expects to receive for transferring q units from ϕ_s to ϕ_{s+1}. In other words, a knowledge of p allows us to obtain from Figure 2 the amount q the trader wishes to have while traveling from ϕ_s to ϕ_{s+1}. This implies a demand curve for the trader in ϕ_s, which we can plot by altering the meaning of p. We now let p be the price of the commodity in ϕ_s, and we treat $E_s(t)$ as a parameter. This is indicated on the ordinate axis of Figure 3, where the actual per-unit price the trader expects for his service is the distance from $p = P_s(t)$ to the value $E_s(t)$. Thus, to obtain that demand curve we simply flip the curve of Figure 2 upside-down and shift it vertically. The result is shown in Figure 3; it is the demand curve used in [12] and [13].

Now, however, there is another complication we must take into account. The trader brings into ϕ_s an amount $C_{s-1}(t-1)$ of goods, which is in general greater or less than the amount $q = C_s(t)$ he would demand in ϕ_s had he no goods at all. Therefore, if $C_{s-1}(t-1) > C_s(t)$, he will sell $C_{s-1}(t-1) - C_s(t)$, and, if $C_s(t) > C_{s-1}(t-1)$, he will buy $C_s(t) - C_{s-1}(t-1)$. How should he value the goods he possesses as he operates in ϕ_s? By the price $P_s(t)$ they command in ϕ_s, we claim. That is, goods on hand and goods on sale have the same value for him, we assume. All this implies the existence of an excess-supply function for the trader as he operates in ϕ_s. It is shown in Figure 4 and is obtained by flipping the curve of Figure 3 through a right-to-left reversal and then shifting the result to the right by the amount $C_{s-1}(t-1)$.

In summary, when the price $p = P_s(t)$ of the commodity in ϕ_s is above $E_s(t) - T$, the trader sells all of his stock $C_{s-1}(t-1)$ because he does not wish to transport any goods to ϕ_{s-1} according to Figure 2. When $P_s^{\dagger} < P_s(t) < E_s(t) - T$, the trader sells the amount $Q_s(t)$ and transports the remainder $C_s(t)$ to ϕ_{s+1}; this case is illustrated in Figure 4. When $P_s(t) < P_s^{\dagger}$, the trader buys the amount $-Q_s(t)$ (now, $Q_s(t) < 0$) and transports $C_s(t) = C_{s-1}(t-1) + |Q_s(t)|$ to ϕ_{s+1}. All this applies whether the trader is in his restocking center ϕ_1 or in one of his rural markets ϕ_s, $2 \leq s \leq n$.

For a given trader, how will his transfer-supply curve (Figure 2) vary from one market to another? The main change we feel will be in the value of T, the minimum value on the AVC curve. To be sure, the variable costs will depend upon which two markets are at hand, and this will effect not only T but also the MC curve.

But, the MC curve contributes to the transfer-supply curve only where MC rises rapidly due to the full-loading or overloading of the trader's transport equipment, and this loading effect should be about the same for the various markets. So, we might simplify our analysis by assuming that the curves of Figures 1 and 2 maintain exactly the same shape for all markets and merely change in the value of T as the market changes. However, we will not impose this assumption, even though we draw our curves as though we had.

3. A single marketing ring. As a first application of our model of a trader, we consider a single isolated marketing ring $\{\phi_1, \phi_2, \dots, \phi_n\}$ which is cyclically traversed by a number of traders. In the next section we discuss the more pertinent case of many intersecting but distinct marketing rings, each ring being defined by the one or more traders who traverse it.

We indicate in Figure 5 how the various excess-supply and excess-demand functions determine the prices in the markets and the quantity flows between them. For that illustration we have assumed $n = 4$. The traders are represented by excess-supply functions whose aggregates are indicated in Figure 5 by the increasing functions. All the other agents in the ϕ_s are represented by excess-demand functions whose aggregates are the decreasing functions in Figure 5. The restocking center ϕ_1 is distinguished from the other markets by the fact that the agents other than the traders are represented therein by an aggregate supply function, which is indicated in Figure 5(a) as a negative excess-demand function $D_1(p, t)$, $t = 1$.

For the sake of clearer illustrations, we have assumed three further conditions for Figure 5. (i) The traders possess the same memory functions of past prices. (ii) Their transfer-supply functions (Figure 2) do not change shape from one market to another; only the T values may change. (iii) If the T value for one trader does change from one market to another market, those values for all traders change by exactly the same amount. As a result, the aggregate excess-supply function for the traders does not change shape from one market to another; it merely shifts its position. None of these three assumptions are essential to our arguments, and we do not impose them in our discussion.

In Figure 5, T_s is the minimum of the T values for all the traders in the market ϕ_s. Also, our illustrations are for the single cycle corresponding to the time values t = 1, 2, 3, 4. The curves of Figure 5 have been drawn to illustrate the following circumstances:

The traders exhaust all of their supplies during the preceding cycle and return to ϕ_1 empty-handed at t = 1; that is, the aggregate amount $C_4(0)$ they bring into ϕ_1 at t = 1 is equal to zero. (The same is true at t = 5.)

Furthermore, in the restocking center ϕ_1 the excess-demand function $D_1(p, 1)$ (which is negative and therefore in actuality a supply function) is large enough to yield a relatively low market price $P_1(1)$. In fact, $D_1(p, 1)$ intersects the aggregate excess-supply function below the nearly horizontal portion of the latter curve, which implies that all or almost all of the traders stock up in ϕ_1 and carry full loads to ϕ_2, the sum of those loads being $C_1(1)$.

In ϕ_2 the expected price $E_2(2)$ is somewhat larger than $E_1(1)$, and $D_2(p, 2)$ intersects the aggregate excess-supply function on its nearly horizontal portion to yield the market price $P_2(2)$. That price determines through each individual trader's excess-supply function the amount each trader will sell and the amount he will carry forward to ϕ_3 (see Figure 4). Aggregating those quantities over all traders, we obtain the total amount $Q_2(2)$ sold and the total amount $C_2(2)$ transported to ϕ_3. A similar process occurs in ϕ_3, where the amount $Q_3(3)$ is sold and the amount $C_3(3)$ is carried on to ϕ_4.

In ϕ_4 the traders expect in ϕ_1 at t = 5 the price $E_4(4)$, a usually low value - as shown. This causes their aggregate excess-supply function in ϕ_4 to be shifted downward substantially. Thus, the demand function $D_4(p, 4)$ intersects that supply function on its strictly vertical portion, which implies that the traders sell all their remaining goods in ϕ_4; that is, $C_4(4) = 0$.

The process now shifts back to ϕ_1 at t = 5. An illustration much like Figure 5(a) could now be drawn, but $C_4(0)$ should be replaced by $C_4(4)$, and $C_1(1)$, $D_1(p, 1)$, $E_1(1)$, and $P_1(1)$ should be replaced by $C_1(5)$, $D_1(p, 5)$, $E_1(5)$, and $P_1(5)$. Similar notational alterations in the subsequent illustrations would continue this graphical discussion.

Of course, individual traders may completely sell out before reaching ϕ_4; in fact, they will all do so if $D_2(p, 2)$ or $D_3(p, 3)$ is so large or $E_2(2)$ or $E_3(3)$ is so low that the intersection point in Figure 5(b) or 5(c) occurs on the strictly vertical portion of the aggregate excess-supply function. On the other hand, $D_4(p, 4)$ may be so low or $E_4(4)$ so high that the intersection point in Figure 5(d) is on the nearly horizontal portion of the supply curve; in this case the amount $C_4(4) > 0$ is transported back to ϕ_1 and the aggregate excess-supply curve in ϕ_1 at $t = 5$ is shifted to the right through the distance $C_4(4)$. We have not illustrated these possibilities.

The important point is that this graphical analysis can be carried out to determine all prices and flows for $t = 1, 2, 3, \ldots$ once the following exogeneous parameters are specified: (1) $D_s(p, t)$ for all $s = 1, \ldots, n$ and all $t = 1, 2, 3, \ldots$. (2) Each trader's transfer-supply curve (Figure 2). (3) Each trader's memory function that determines the price he expects in the next market from prior prices (these memory functions may vary from trader to trader and from market to market). (4) Initial conditions which specify the various amounts the individual traders bring in to ϕ_1 at $t = 1$ from ϕ_4 and also specify enough prior prices to allow the determination of all needed expected prices from the individual memory functions.

We could at this point write down explicit equations by which this recursive process could be carried out algebraically. However, we will postpone doing so until the more common periodic marketing network consisting of distinct but intersecting marketing rings has been introduced. This is our next objective.

4. *The multiring model.* In this case one or more days of the marketing week have more than one open market. Thus, we have to alter our notation. We assume that each market meets only once a week. (This is no restriction on the rural markets, for we can view a rural market that meets r times per week as r different

markets that happen to meet in the same market-place.) We index the markets that open on the sth day of the market week by $j = 1, \dots, m_s$ and denote them by ϕ_{sj}. Thus, there are m_s open markets on the sth market day, and the total number of markets is $m = m_1 + \cdots + m_n$.

We can symbolically represent the entire marketing network as in Figure 6. All the markets opening on a given market day are gathered together. Each arrow represents the possible movement of traders and goods from one market to another between two consecutive market days. An arrow that points from a market ϕ_{sj} to a market $\phi_{s+1,\,k}$ is conventionally called an arc or a ϕ_{sj}-to-$\phi_{s+1,k}$ arc. It can happen at times that all the traders that ordinarily follow the ϕ_{sj}-to-$\phi_{s+1,k}$ arc sell all their goods before they leave ϕ_{sj}, in which case no goods traverse the ϕ_{sj}-to-$\phi_{s+1,k}$ arc. At this particular time the arc is said to be cut off.

The graph-theoretic infrastructure of this ring-type periodic marketing network is a digraph that is the union of a number of distinct but possibly intersecting cycles or rings of length n. (No. 91) ← We use the words "cycle" and "ring" synonomously to mean a directed cycle that is viewed by at least one trader as his marketing itinerary.

As before, the ϕ_{1j} represent restocking centers, and the supply and demand curves for one of the ϕ_{1j} are like those of Figure 5(a). The ϕ_{sj}, where $2 \leq s \leq n$, are the rural markets and have supply and demand curves like those of Figure 5(b), 5(c), and 5(d). As in the preceding section, for any market ϕ_{sj} the agents other than the traders are represented by aggregate excess-demand functions, which are negative for $s = 1$ and positive for $2 \leq s \leq n$. On the other hand, the traders in ϕ_{sj} are represented by aggregate excess-supply functions obtained by adding horizontally their individual excess-supply functions shown in Figure 4. The latter are determined once each trader's expected price $E_s(t)$ and incoming stock $C_{s-1}(t-1)$ are specified. Now, however, since the traders passing through a particular market will in general be following different rings, the shapes of the aggregate excess-supply functions may vary substantially from market to market along any particular ring.

Let us write down a set of nonlinear difference equations from which all the prices and commodity flows of our marketing network can be recursively computed from given initial conditions. To do so, we will have to define a variety of symbols.

We have already defined p, q, t, s, ν, n, j, and ϕ_{sj}. $D_{sj}(p, t)$ denotes the excess-demand functions for all the agents in ϕ_{sj} at time t toher than the traders. We impose the following conditions on $D_{sj}(p, t)$.

<u>Conditions A_1</u>. For every fixed $t = \nu n + 1$, where $\nu = 0, 1, 2, \ldots$ and for every j in the index set of the restocking centers ϕ_{1j}, $D_{1j}(p, t)$ is a negative, continuous, strictly decreasing function of p for $0 < p < \infty$ such that $D_{1j}(p, t) \rightarrow 0-$ as $p \rightarrow 0+$ and $D_{1j}(p, t) \rightarrow -\infty$ as $p \rightarrow \infty$. Also, for $p = 0$, $D_{1j}(0, t)$ denotes the nonnegative q axis. For each fixed $t = \nu n + s$, where $\nu = 0, 1, 2, \ldots$ and $s = 2, \ldots, n$, and for every j in the index sets of the rural markets ϕ_{sj}, $D_{sj}(p, t)$ is a positive, continuous, strictly decreasing function of p for $0 < p < \infty$ such that $D_{sj}(p, t) \rightarrow \infty$ as $p \rightarrow 0+$ and $D_{sj}(p, t) \rightarrow 0+$ as $p \rightarrow \infty$. Also, we

use the convention that $D_{sj}(p, t) = 0$ if and only if $p = \infty$.

(One might wish to allow these demand functions to be identically equal to zero for some ranges of p. But this can be effectively encompassed by taking the functions to be exceedingly small but positive on those ranges. Similarly, we could replace the limits at ∞ by sufficiently large finite limits. We have imposed our stricter requirements to avoid some unessential complications in our subsequent arguments.)

Next, we number all the traders in our marketing network by the index $i = 1, 2, \ldots, I$, where I is the total number of traders. Corresponding to each i there is a sequence of n index pairs (s, j) determined by the ith trader's itinerary, namely, the subscripts of the ϕ_{sj} in that trader's ring. In the following, various parameters pertaining to the ith trader will be so designated by a superscript i.

$C^i_{sj}(t)$ designates the amount of goods the ith trader carries forward from his market ϕ_{sj} to the next market in his ring between the market days $t = \nu n + s$ and $t + 1$.

The transfer-supply curve for the ith trader in ϕ_{sj} (Figure 2) will be denoted by $V^i_{sj}(p - T^i_{sj})$. We shall assume that the costs of transferring goods from ϕ_{sj} to the ith trader's next market remains the same from week to week and therefore that the function V^i_{sj} does not vary from week to week. However, V^i_{sj} does change in general as (s, j) varies through the market indices of the ith trader's ring. We also assume the next conditions.

<u>Conditions A_2</u>. For each i, s, and j, V^i_{sj} is a continuous nonnegative function on the real line such that $V^i_{sj}(x) = 0$ for $x \leq 0$, $V^i_{sj}(x)$ is strictly increasing for $0 \leq x < \infty$, and $V^i_{sj}(x)$

tends to a finite limit as $x \to \infty$.

T^i_{sj} is the value T indicated in Figure 2 that pertains to the ith trader in ϕ_{sj}; that is, it is the minimum value on his AVC curve for the transferring of goods from ϕ_{sj} to his next market. We assume that T^i_{sj} too does not change from week to week but that it can vary as (s, j) changes.

Still another assumption is that the price $E^i_{sj}(t)$ the ith trader expects to receive in his next market $\phi_{s+1,k(i)}$ at time t + 1 while operating in ϕ_{sj} at time $t = \nu n + s$ is determined by a memory function M^i_{sj} specific to him and possibly dependent on the values of s and j.

$$(4.1) \quad E^i_{sj}(t) = M^i_{sj}[P_{s+1,k(i)}(t+1-n),\ P_{s+1,k(i)}(t+1-2n),\ P_{s+1,k(i)}(t+1-3n),\ \ldots]$$

Here, the arguments of M^i_{sj} denote the prices in $\phi_{s+1,k(i)}$ prior to t. We could generalize this still further by allowing M^i_{sj} to depend on prior prices in other markets of the ith trader's ring - and even on other markets outside his ring assuming that some market news of other markets outside his ring exists. In any case, and as we stated before, the M^i_{sj} should be a monotonic increasing function of each price among its arguments, more recent prices should be more effective in determining $E^i_{sj}(t)$ than earlier prices, and a history of constant prices in ϕ_{sj} should yield the same price for $E^i_{sj}(t)$. We could, for example, satisfy all of these conditions with the simple rule:

$$(4.2) \qquad E^i_{sj}(t) = P_{s+1,k(i)}(t+1-n)$$

Next, we let $F^i_{sj}(t) = E^i_{sj}(t) - T^i_{sj}$. This is the price at the corner point of the ith trader's excess-supply curve for ϕ_{sj} at time t. Then, that excess-supply curve (see Figure 4) can be written as

$$(4.3) \qquad C^i_{s-1,h(i)}(t-1) - V^i_{sj}\left[F^i_{sj}(t) - p\right]$$

where $C^i_{s-1,h(i)}(t-1)$ is the amount of goods the ith trader brings into ϕ_{sj} at time t from the preceding market $\phi_{s-1,h(i)}$ of his ring. Finally, let $\sum_{sj}$ denote a summation over the indices i of those traders that possess ϕ_{sj} as a market in their rings. Then, the aggregate of all trader excess-supply functions for ϕ_{sj} is

$$(4.4) \qquad S_{sj}(p, t) = \sum_{i\,sj}\left\{C^i_{s-1,h(i)}(t-1) - V^i_{sj}\left[F^i_{sj}(t) - p\right]\right\}$$

We can now determine the market-clearance price in ϕ_{sj} at t by equating the aggregate excess-demand function to the aggregate excess-supply function:

$$(4.5) \qquad D_{sj}(p, t) = S_{sj}(p, t)$$

and then seeking the solution $p = P_{sj}(t)$. That there is one and only one such solution for given s, j, and t follows immediately from Conditions A_1 and A_2.

In summary, our model for the multiring periodic marketing system consists of Equations (4.1), (4.4), and (4.5) coupled with all the assumptions stated in this section. In addition, we should state explicitly that this construction assumes that there is perfect competition within every market on each of its market days, but not for the entire marketing system as a whole.

As in the single-ring model, these equations allow us to determine all the prices and commodity flows through recursive computations starting with an appropriate set of initial conditions. For example, assume we are given a multiring marketing network, a specification of every trader's ring, and the V^i_{sj}, T^i_{sj}, M^i_{sj}, and $D_{sj}(\cdot, t)$ for every i, s, j, and t. Assume furthermore that initial conditions are given as specifications of all the $C^i_{nj}(0)$ and of enough prior prices in every Φ_{sj} to determine through (4.1) every trader's expected prices for t = 1, 2, ... , n. Then, (4.1) and (4.5) together can be used to determine sequentially every price $P_{sj}(t)$, as well as the quantity sold $Q^i_{sj}(t)$ and forward flow $C^i_{sj}(t)$ for each trader, as t progresses through 1, 2, 3, ... This yields thereby a complete determination of the dynamic behavior of our marketing network under the assumptions stated in this section.

One can conceive of modelling actual marketing networks in just this way, but this would require the estimation of a very large number of parameters, a discouraging prospect. We feel that the main value of our model is that it provides for the first time a means of explicitly examining the prices and quantity flows of an idealized multiring periodic marketing network. This enables us to draw qualitative conclusions concerning dynamic behavior. Such conclusions may then be compared to known properties of actual periodic marketing networks, or alternatively they may serve as conjectured properties worthy perhaps of empirical examination. It is to considerations of this sort that we now turn.

5. Apparently erratic price behavior. One peculiarity of periodic marketing networks that has been commented on in the literature is their irregular behavior. For one such comment, refer again to the statement of D.W.Jones concerning surprising peasant output responses quoted in the Introduction. With regard to observed price variations in certain markets in Nigeria, W.O. Jones was led "to suggest serious malfunctioning of the market in response to short-term changes in market information" and to the interpretation that "the market was responding erratically and unpredictably to new information" [7; pp. 21-22].

In a prior work concerning two-level periodic marketing networks, we showed how apparently contradictory price signals can be generated by a single disturbance in supply. In particular, a sudden oversupply at one market can lead to an initial fall in price propagating in one direction of the network and an initial rise in price propagating in another direction. This can occur when some arcs of the marketing network cut off (that is, when no traders carry goods along those arcs because of unfavorable prices at their terminal markets). We shall now indicate how another apparently contradictory price variation can arise in our multiring network. More interestingly, this event can occur even without any arcs cutting off.

Consider the marketing network of Figure 7, where ϕ_{11} and ϕ_{12} are restocking centers on the first day of the four-day marketing week and ϕ_2, ϕ_3, and ϕ_4 are rural markets meeting on the next three days. We shall show how a sudden shortfall in ϕ_{11} can appear in ϕ_3 two days later as either a rise or a fall in price depending on the demand in ϕ_3. We will argue graphically.

We assume that one group of traders operate out of ϕ_{11} and proceed along the upper ring of Figure 7 and that another group of traders operate out of ϕ_{12} and proceed along the lower ring. We let W_1 and W_2 denote the aggregate excess-supply curves in the various markets for the first and second groups respectively. (Again, we draw our illustrations as though the W_1 and W_2 curves have the same shape and do not change from market to market; they merely shift their positions.)

Figure 8 shows one possible sequence of positions for W_1 and W_2. D_{11}, D_{12}, and D_2 denote the excess-demand functions in ϕ_{11}, ϕ_{12}, and ϕ_2 respectively. (We have let $D_{11} = D_{12}$, but this too is not an essential assumption.) We view the positions of Figure 8 as a normal situation, just for the sake of argument. In this case, both groups of traders acquire the same amount in ϕ_{11} and ϕ_{12} and sell most of it in ϕ_2 to arrive with their aggregate excess-supply function $W_1 + W_2$ in ϕ_3, as shown in Figure 8(d).

Now, consider Figure 9. We assume that a sudden short-fall in supply occurs in ϕ_{11} on the first day of the market week. This shifts the excess-demand function D_{11} to the steeper position shown in Figure 9(a). In ϕ_{12}, D_{12} remains in its normal position, as shown in Figure 9(b). We wish to examine how this shift in D_{11} affects the price in ϕ_3. We shall assume still further that the expected prices depend upon prior prices not only in the market to which the trader will next go but also in other markets as well. For example, if the price in ϕ_{11} rises at t = 1, the trader will raise the price he expects in ϕ_3 while operating in

Φ_2 at t = 2. (We could also assume that this price rise in Φ_{11} at t = 1 induces a simultaneous rise in his current expected price in Φ_2 while operating in Φ_{11} at t = 1. This would merely alter somewhat the changes we describe below. We won't make this assumption.)

Figure 9(c) shows the excess-supply curve for both groups of traders simultaneously, and Figure 9(d) shows their aggregate. With the excess-demand function D_2 positioned as shown in Figure 9(d) (the same as in Figure 8(c)), the traders working out of Φ_{12} sell all their goods in Φ_2, whereas those working out of Φ_{11} sell in Φ_2 only a small part of their stocks. This yields the W_1 and W_2 of Figure 9(e) and the aggregate $W_1 + W_2$ in Figure 9(f).

For Φ_3 we now wish to compare the $W_1 + W_2$ curve in the normal case (Figure 8(d)) with the $W_1 + W_2$ curve in the shortfall case (Figure 9(f)). We do this by drawing them in Figure 10 on the ssme set of axes. Consider the comparatively low demand function D_L in Figure 10. The resulting price in the normal case is lower than the resulting price in the shortfall case, as is to be expected. But, for the relatively high demand function D_H, the situation is reversed. The price in the normal case is higher than the price in the shortfall case. Without the theory provided by our model, one might view the latter occurrence as inexplicable.

We can summarize this phenomenon as follows. When Φ_{11} has a shortfall and Φ_{12} stays normal, both groups of traders acquire about the same amount in Φ_{11} and Φ_{12}, but the price in Φ_{11} is much higher than the price in Φ_{12}. In Φ_2, the traders out of

ϕ_{11} sell very little because they need a high price to make a profit, but the traders out of ϕ_{12} sell out. Thus, in ϕ_3, the traders out of ϕ_{12} have nothing to sell, but the traders out of ϕ_{11} still have most of their stock, amounting to more than what both trader groups would have in aggregate under normal conditions. This means that the shortfall price in ϕ_3 can be lower than the normal price and the shortfall sales can be larger than the normal sales if demand in ϕ_3 is high enough to cause the traders out of ϕ_{11} to sell most of their stock.

Finally, it should be stated that this phenomenon need not always happen. Its occurrence depends on the relative positions of the supply and demand functions and on the memory functions of past prices.

6. *Equlibrium in a single ring.* Once a dynamic model becomes available, a natural question to ask as whether it has an equilibrium state. This question does not appear to have been asked about periodic marketing systems evidently because dynamic models had not been available for them. The first results along these lines appear in [12] and [13]. (For a less realistic model of periodic markets, similar results appear in [9], [10], and [11].)

The importance of addressing this question lies in the following facts. If it can be shown that no equilibrium state exists, then we can conclude that the periodic marketing system is condemned to perpetual price variations even when the exogeneous supply and demand functions are fixed with respect to time. One might then ask what regularities those variations might have. For example, do limit cycles exist, and, if so, are they stable? On the other hand, if an equilibrium state exists, then we have the possibility of steady and predictable prices, upon which rational economic planning can be based. In the latter case, the questions of uniqueness and stability for the equilibrium state should be investigated.

With regard to our general model of Section 4, we have not been able to establish or to disprove the existence of an equilibrium state. However, we have been able to show that the single-ring model of Section 3 does have one and only one equilibrium state. This is what we now present.

As before, let $R = \{\phi_1, \phi_2, \dots, \phi_n\}$, where $n \geq 2$, be a single isolated marketing ring traversed cyclically by m traders. We use the notation of Section 3 but now delete the time symbol t because* we're interested in equilibria in time. There being but one open market on each day, we also drop the market index j. In addition, we set

$$C_s = \sum_i C_s^i = \sum_i V_s^i(E_s - T_s^i - P_s),$$

and

$$D_s(P_s) = Q_s = \sum_i Q_s^i.$$

R with fixed excess-demand functions $D_s(p)$, $s = 1, \dots, n$, is said to be in equilibrium if none of its variables change with time. The corresponding set of all prices and quantity flows is called an equilibrium state.

Assume an equilibrium state exists. In view of our assumed properties of the memory functions and the fact that prices remain constant, every trader's expected price E_s^i is equal to the price P_{s+1}. We let $E_s = P_{s+1}$ denote this common value of all the E_s^i. This means that the aggregate excess-supply functions S_s line up with each other as is indicated in Figure 11, where we have assumed that $n = 4$. (In general, the shapes of the S_s may change from market to market, because, for one reason, for different i the T_s^i may change differently as s varies. We have nevertheless indicated an unvarying shape in Figure 11.)

It is important to note that, if in an equilibrium state a positive amount of goods is carried from ϕ_s to ϕ_{s+1}, we must have that $P_s < E_s - \min_i T_s^i$, and therefore $E_{s-1} < E_s$ since $E_{s-1} = P_s$. Thus, if a positive amount of goods reaches the last market ϕ_n, we will have $E_n < E_1 < \cdots < E_{n-1}$. In this case, we cannot have a positive amount of goods being carried from ϕ_n to ϕ_1 in an equilibrium state, for such would imply that $E_{n-1} < E_n$, which coupled with the last conclusion would yield $E_{n-1} < E_{n-1}$, an impossibility. Thus, $C_n = 0$.

Next, we show that in an equilibrium state a positive amount of goods must reach ϕ_n, and therefore we will always have

$$(6.1) \quad E_n = P_1 < E_1 = P_2 < E_2 = P_3 < \cdots < E_{n-1} = P_n.$$

Indeed, assume that no goods reach ϕ_n. Then there will be some smallest $s > 1$ for which no goods reach ϕ_s; that is, $C_{s-1} = 0$.

If $s \geq 3$, then $C_{s-2} > 0$. This means that in ϕ_{s-1} the intersection between D_{s-1} and the excess-supply function S_{s-1} will be on the strictly vertical part of S_{s-1} located above $q = C_{s-2} > 0$. (See Figure 12.) Hence, $E_{s-1} < \infty$, and therefore $P_s = E_{s-1} < \infty$. However, since $0 \leq D(P_s) = Q_s \leq C_{s-1} = 0$, we have from Conditions A_1 that $P_s = \infty$, a contradiction.

If $s = 2$, our assumption that no goods reach ϕ_2 requires that E_1 be so low that S_1 intersects D_1 at the origin. Again we conclude that $P_2 = E_1 < \infty$. But, by the last sentence of the preceding paragraph, $P_2 = \infty$, again a contradiction.

Thus, we have proven that $C_s > 0$ for $s = 1, \ldots, n-1$, that $C_n = 0$, and that (6.1) holds. Hence, $P_n < \infty$. Consequently, every price P_s is positive and finite, and some goods are sold in every market (i.e., $Q_s = C_{s-1} - C_s > 0$). The following relations subsume these facts and the condition that all variables are independent of time.

$$(6.2) \qquad 0 > D_1(P_1) = -C_1,\ C_n = 0,\ E_1 = P_2 > P_1 + \min_i T_1^i$$

$$(6.3) \qquad 0 < D_s(P_s) = C_{s-1} - C_s < C_{s-1},\ E_s = P_{s+1} > P_s + \min_i T_s^i$$

$$s = 2, \ldots, n-1$$

$$(6.4) \qquad 0 < D_n(P_n) = C_{n-1},\ 0 < E_n = P_1 < P_n < \infty$$

We have shown that these relations are necessary for the occurrence of an equilibrium state. They are sufficient as well, for their fulfillment during the marketing weeks prior to, say, $t = 1$ insures their fulfillment for all subsequent marketing weeks. This can be seen by applying the recursive analysis described in Sections 3 and 4.

We now argue that, given all the D_s and every trader's transfer-supply curve, there always exists an equilibrium state and it is unique. To show this, we steadily increase E_1 from the value 0. For some low E_1 we will have the situation indicated in Figure 13. Setting $P_2 = E_1$, as must be the case in an equilibrium state, we obtain a point A_2 on D_2 and the corresponding amount

Q_2^* sold in ϕ_2. But, for this low value of E_1, the amount C_1 brought into ϕ_2 is less than Q_2, an impossibility. Thus, for this E_1, no equilibrium state is possible.

Let B_2 be the point on D_2 corresponding to C_1. As E_1 increases, A_2 moves upward and B_2 moves downward along D_2 until for some unique E_1 we get $A_2 = B_2$ and an S_2 positioned as is S_4 in the ϕ_4 diagram of Figure 11. At this stage, the horizontal positioning of S_2 is determined by the fact that S_2's strictly vertical part is located at $q = C_1$. However, S_2's vertical positioning is not uniquely determined. It merely satisfies the condition that D_2 intersects S_2 somewhere on S_2's stricltly vertical part. Hence, $C_2 = 0$.

Next, assume a further increase in E_1. This dictates a higher point A_2 on D_2. Now, as is illustrated in Figure 14, S_2's position is uniquely determined because not only must its strictly vertical part lie at $q = C_1$, which has increased, but it must also pass through A_2, which has shifted leftward and upward. Such a unique position for S_2 exists by virtue of our assumed continuity and monotonicity properties for D_2 and for each trader's transfer-supply function.

On the other hand, for the situation shown in Figure 14, we have again an A_3 on D_3 dictated by $E_2 = P_2$ and a B_3 on D_3 dictated by C_2. Since $A_3 \neq B_2$, no equlibrium state can exist. However, as E_1 keeps increasing still further, so too do C_1, C_2, and E_2 with S_2 shifting its position in a unique fashion. Moreover, A_3 approaches B_3 until the critical case is acheived in ϕ_3 when A_3 first equals B_3 and D_3 intersects S_3 on the latter's strictly vertical part. At this point. $C_3 = 0$. Any further increase in E_1 generates a situation in ϕ_3 similar to that shown in the ϕ_2 diagram of Figure 14; in particular, C_3 becomes positive.

We continue this process of increasing E_1 and generating a movement of goods further along the sequence of markets $\phi_1, \phi_2, \ldots, \phi_n$. Eventually, we will get the situation where the critical case occurs in ϕ_n (i.e., $C_1 > 0, \ldots, C_{n-1} > 0, C_n = 0$) with the S_s $(s = 1, 2, \ldots, n-1)$ uniquely positioned. The last step is to insert an S_n which intersects D_n at the quantity $q = C_{n-1}$ on S_n's strictly vertical part and which is positioned vertically to satisfy $E_n = P_1$. These conditions uniquely locate S_n.

At this point, we have fulfilled the relations (6.2), (6.3), and (6.4). Thus, an equilibrium state has been found. That equilibrium state is unique, for any further increase in E_1 will dictate a $C_n > 0$, a violation of the second relation in (6.2).

We summarize this section with the following conclusions: Let there be given a single periodic marketing ring with time-invariant excess-demand functions D_s and time-invariant traders' transfer-supply curves. Let the dynamic behavior of this ring be dictated by the assumptions and equations of Section 4. Then, there exists one and only one equilibrium state. Necessary and sufficient conditions for the occurrence of that equilibrium state is the fulfillment of (6.2), (6.3), and (6.4).

7. Propagation of disturbances. W.O.Jones [6] described another characteristic of periodic markets, one that occurs when there is very little market news and traders do not alter their marketing routes. Price disturbances propagate essentially

through the trading activity and therefore in a step-by-step fashion, progressing in general only along one arc between two consecutive market days. Thus, if a price disturbance occurs in one market, say, Φ on day $t = 1$, the earliest time at which the effects of that disturbance can reach a distant market, say, Ψ is $d+1$ where d is the length (measured by the number of arcs) of the shortest directed path from Φ to Ψ.

However, it can happen that the time it takes for the disturbance to reach Ψ may be even longer. This may occur when at least one arc of every shortest Φ-to-Ψ directed path does not have traders operating on it because of some temporary price conditions, that is, when those arcs cut off just before the disturbance reaches them and remain cut off for a while. Thus, the disturbance may have to traverse a longer directed path in order to reach Ψ, if it does so at all.

We showed in [12] and [13] that two-level periodic marketing networks transmit the initial swing of a price disturbance properly so long as cutoff does not occur. That is, if a disturbance reaches a distant market in the minimum possible time, then a shortfall will be detected as an initial rise in price and a sudden oversupply as an initial fall in price. However, subsequent variations may oscillate in both directions. In fact, in [13] we showed how cutoff may block out the initial price increment and then induce the transmittal of an opposite price swing, yielding thereby a confusion in price signalling.

In our present model the situation can be even worse. As we saw in Section 5, the initial price swing itself can be misleading even when no cutoff occurs. For example, under certain circumstances, that initial swing can be a fall in price when a shortfall in supply has occurred in another market sometime earlier. Thus, we see again that periodic markets can exhibit contrary behavior. The basic reason seems to be that we're now dealing with a cobweb-type of phenomenon, but now the cobweb variations are distributed over and transmitted through a network.

All this argues for the development of better market-news services and improved transportation facilities. If traders are made aware of market conditions outside their respective rings and if good transportation routes are available, traders may respond to price-arbitraging opportunities by altering their routes. This should allow the overall marketing network to respond to disturbances with more of its resources and thereby to mitigate the effects of those disturbances.

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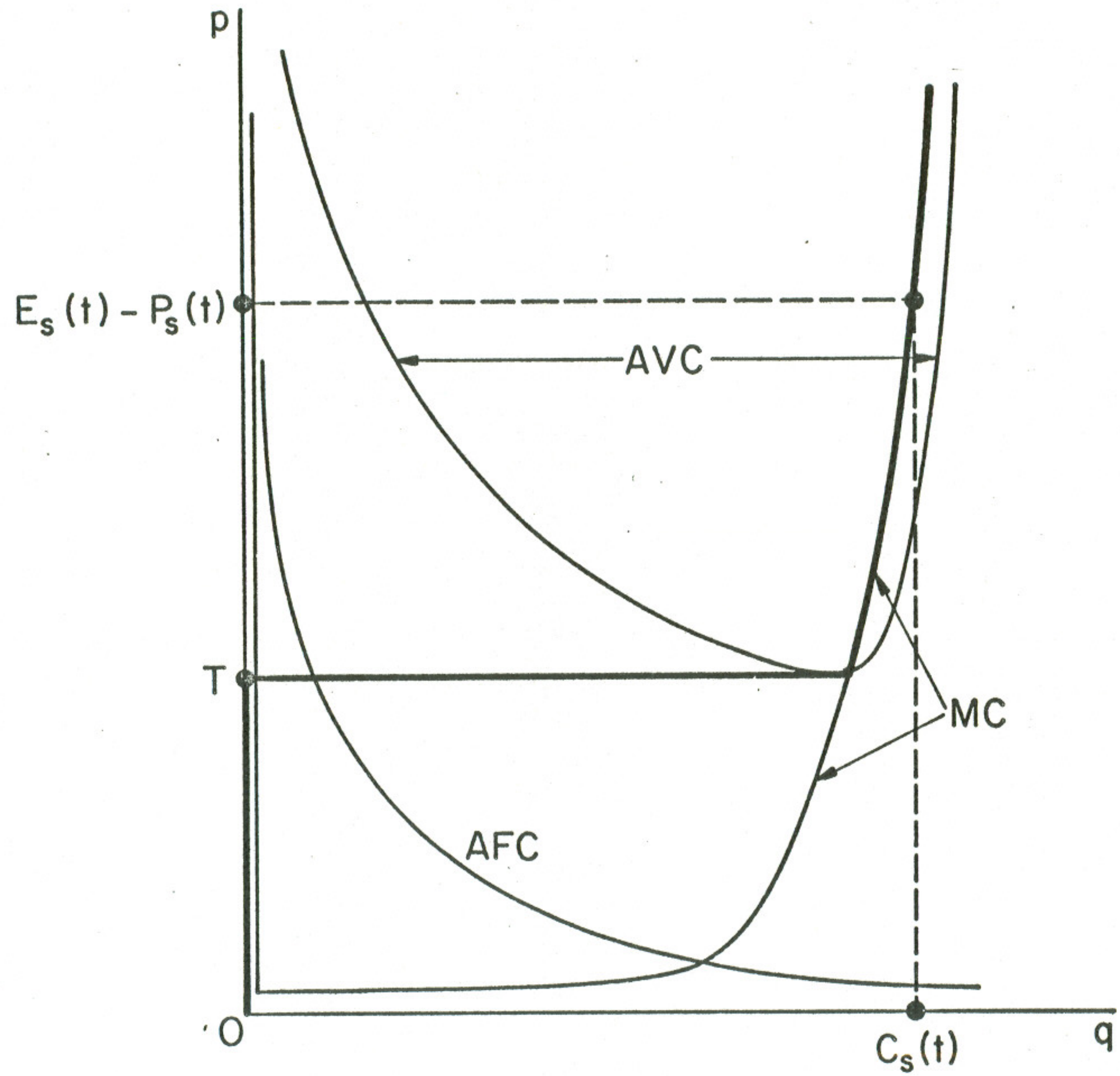


Figure I.

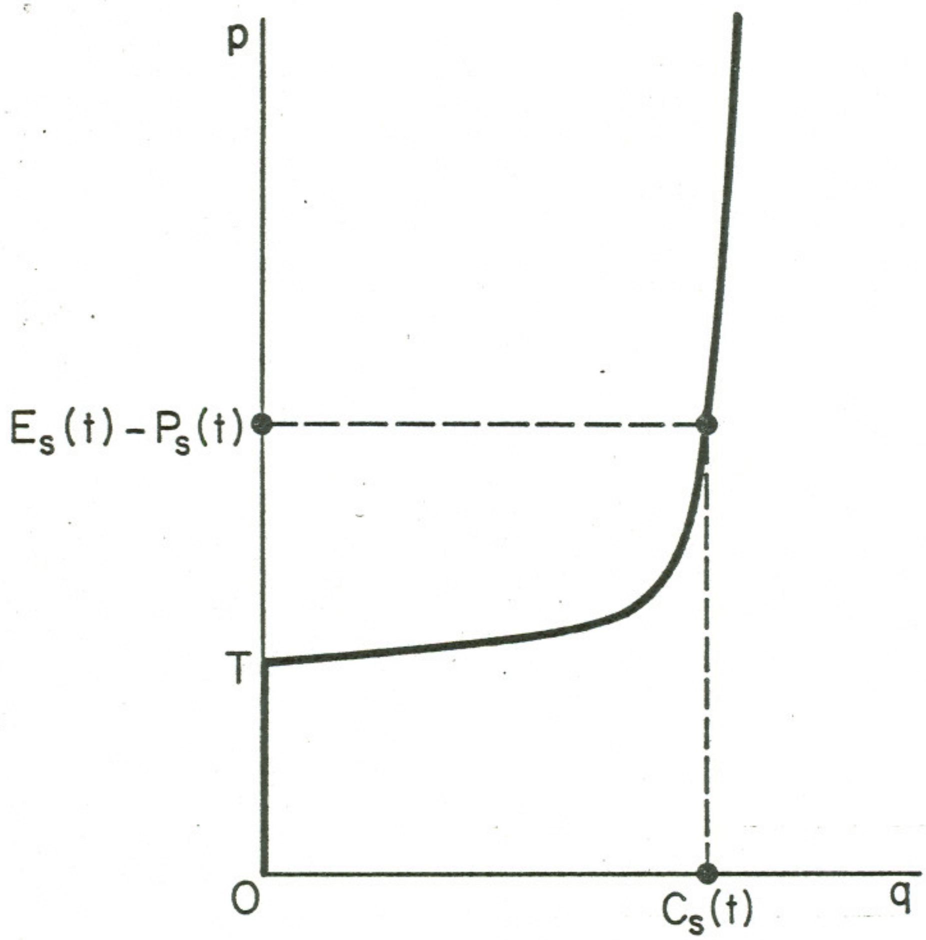
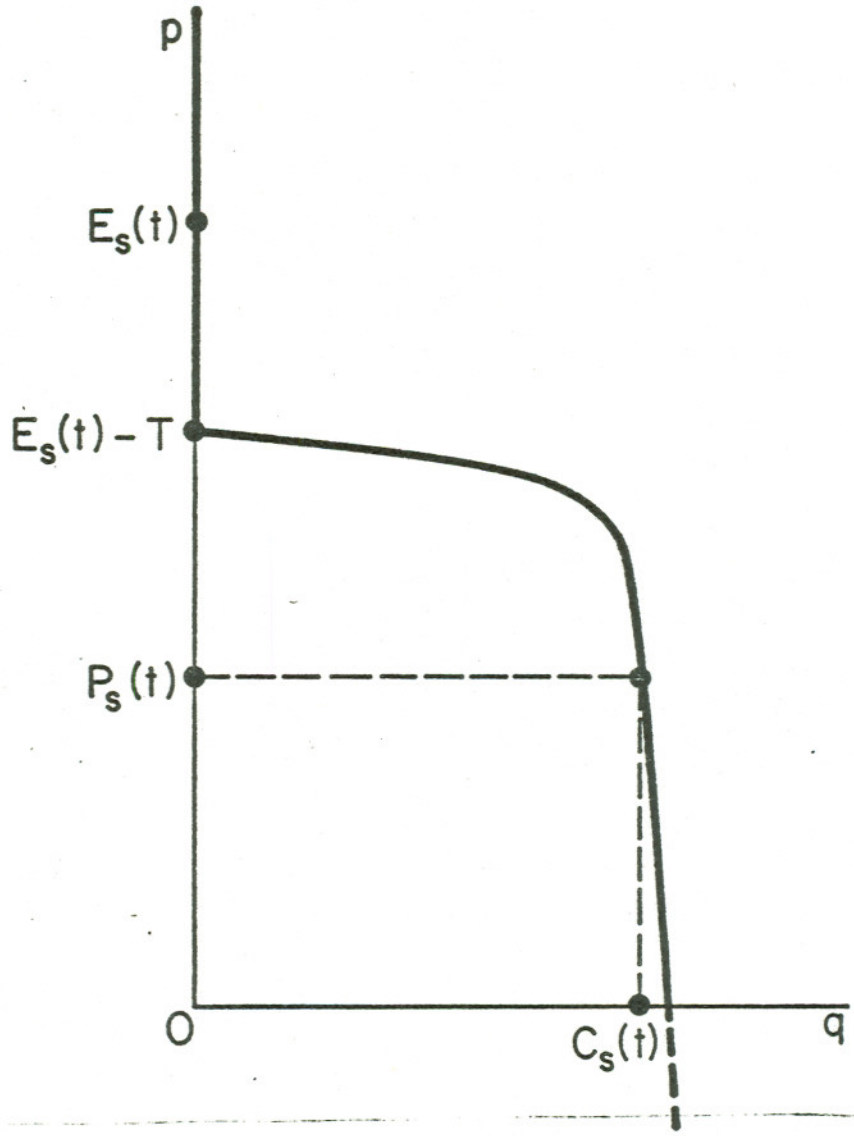


Figure 2.



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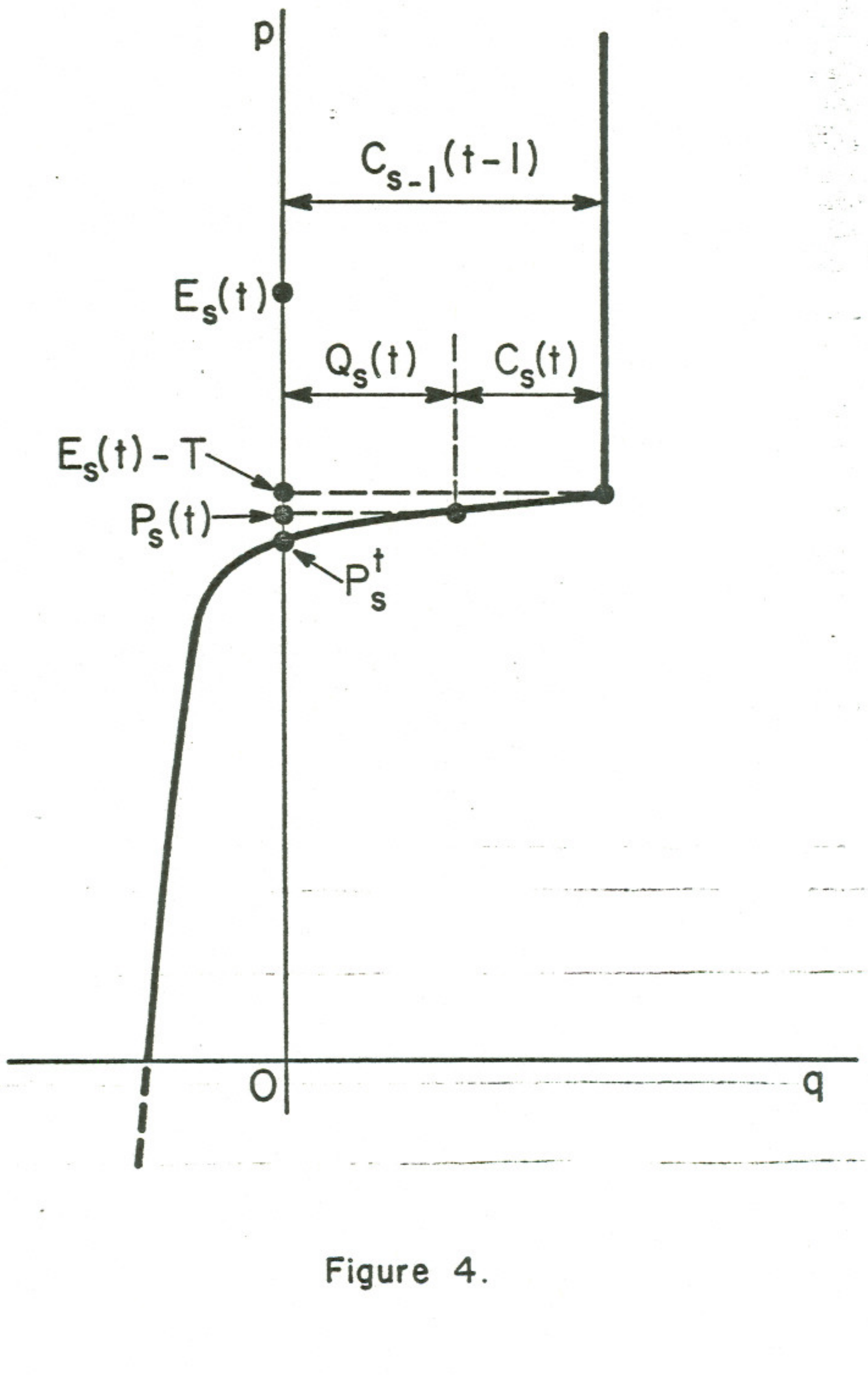

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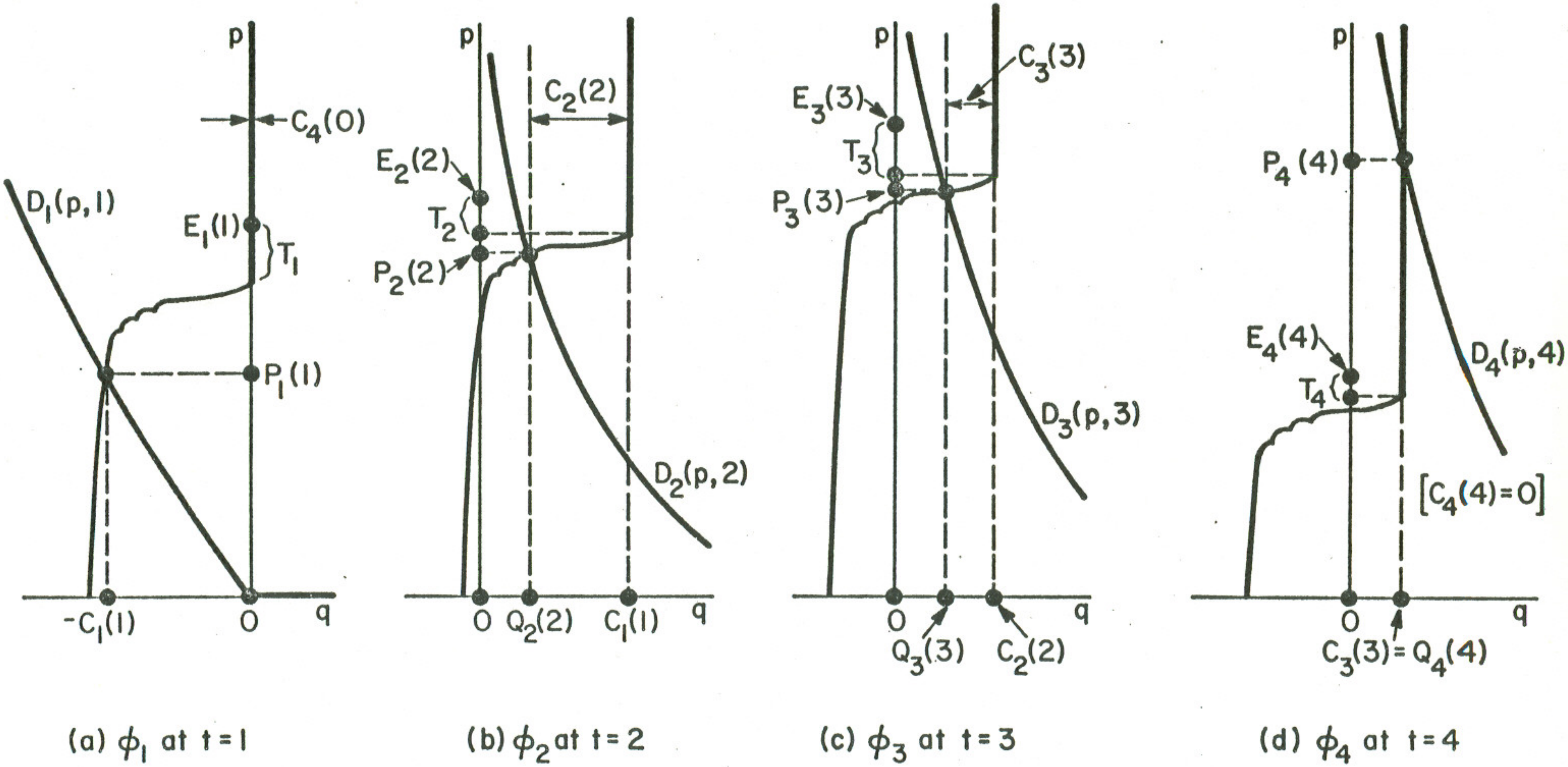


p
$C_4(0)$
$D_1(p,1)$
$E_1(1)$
T_1
$P_1(1)$
$-C_1(1)$
0
q
(a) ϕ_1 at t=1
p
$C_2(2)$
$E_2(2)$
T_2
$P_2(2)$
$D_2(p,2)$
0
$Q_2(2)$
$C_1(1)$
q
(b) ϕ_2 at t=2
p
$C_3(3)$
$E_3(3)$
T_3
$P_3(3)$
$D_3(p,3)$
0
$Q_3(3)$
$C_2(2)$
q
(c) ϕ_3 at t=3
p
$P_4(4)$
$E_4(4)$
T_4
$D_4(p,4)$
$[C_4(4)=0]$
0
$C_3(3)=Q_4(4)$
q
(d) ϕ_4 at t=4

Figure 5.

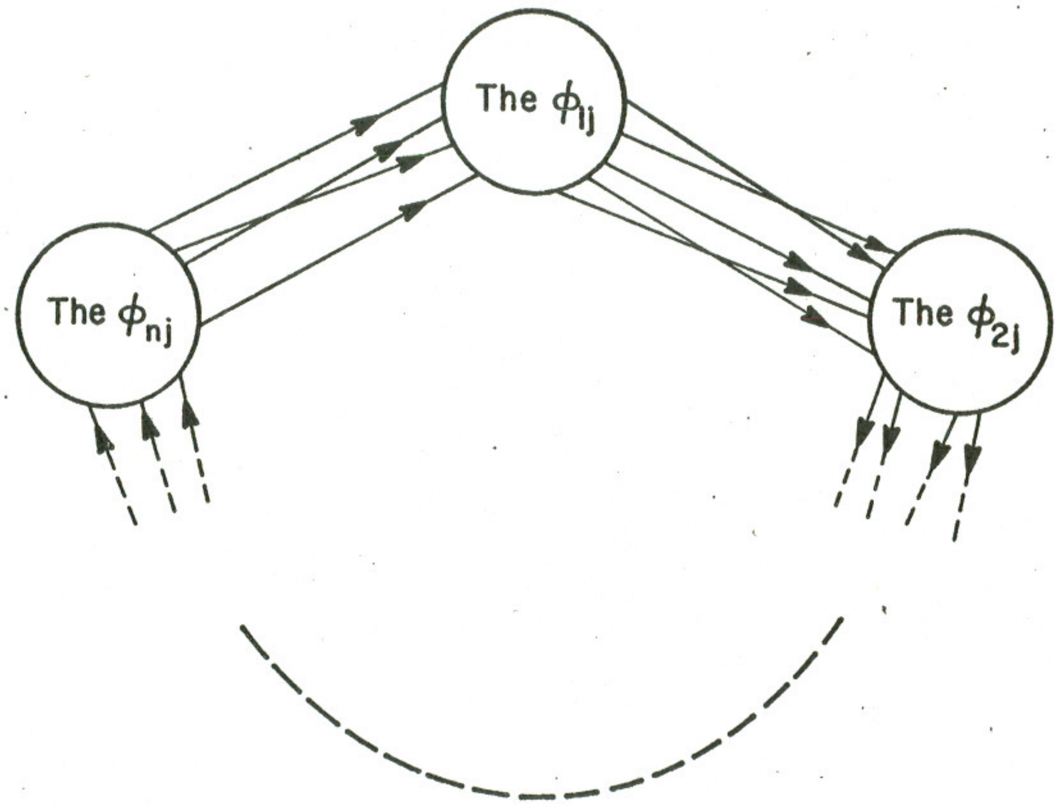


Figure 6.

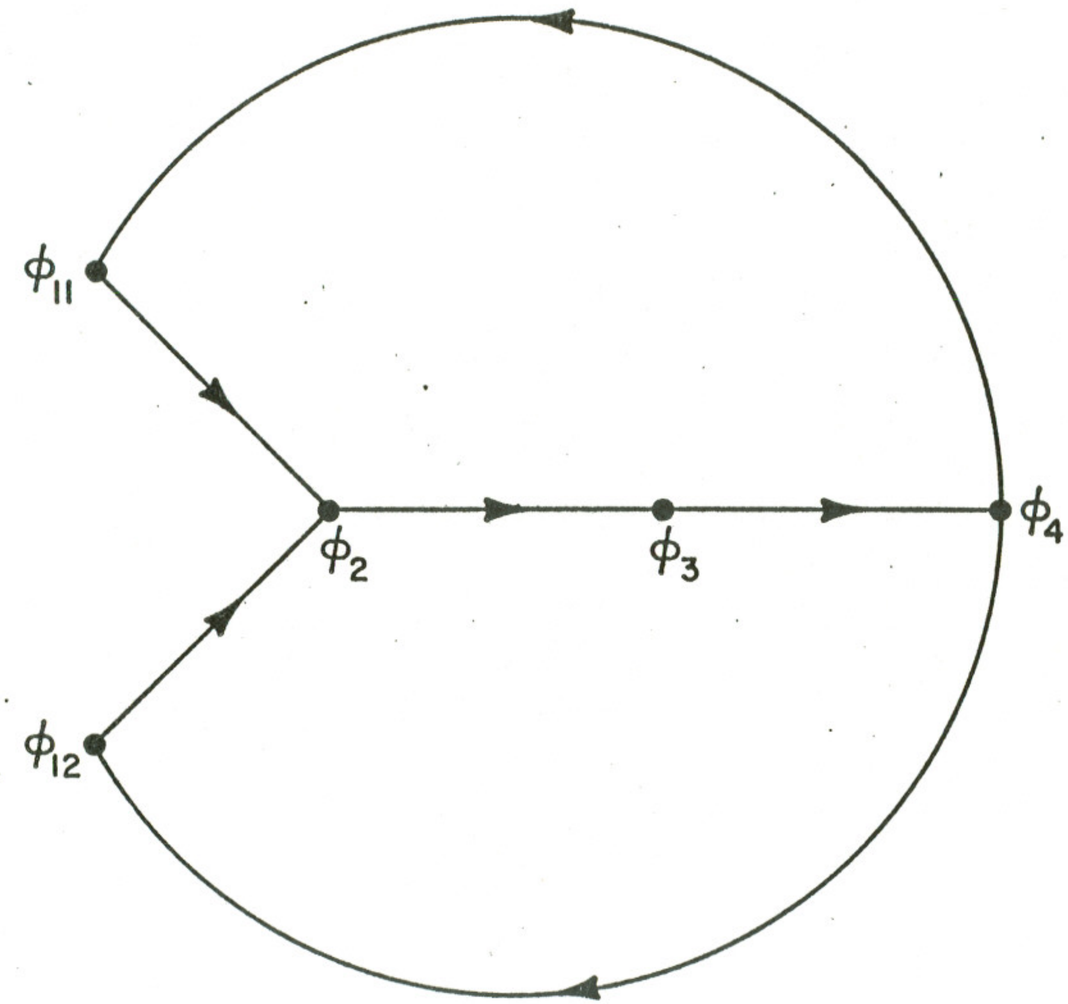
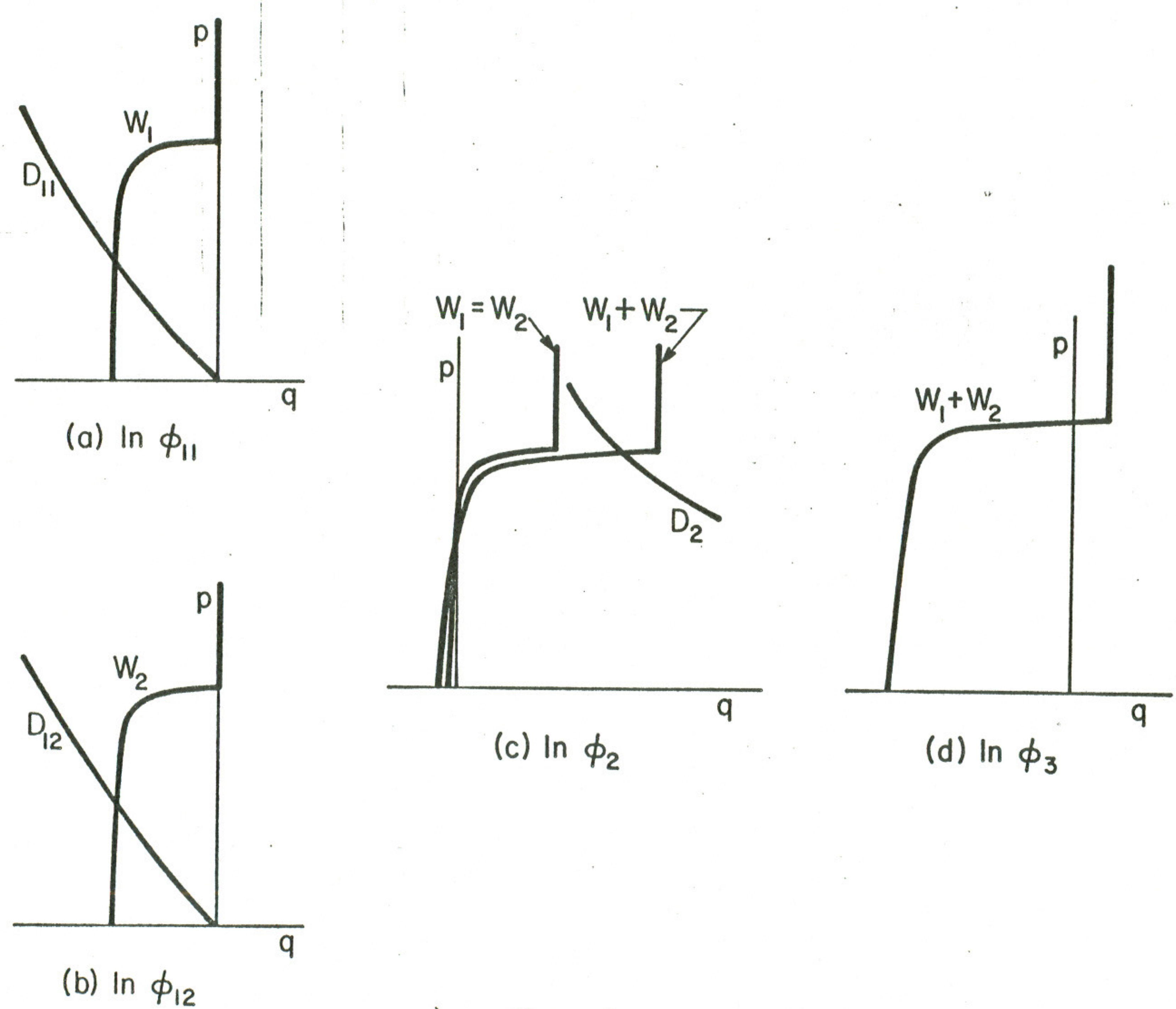
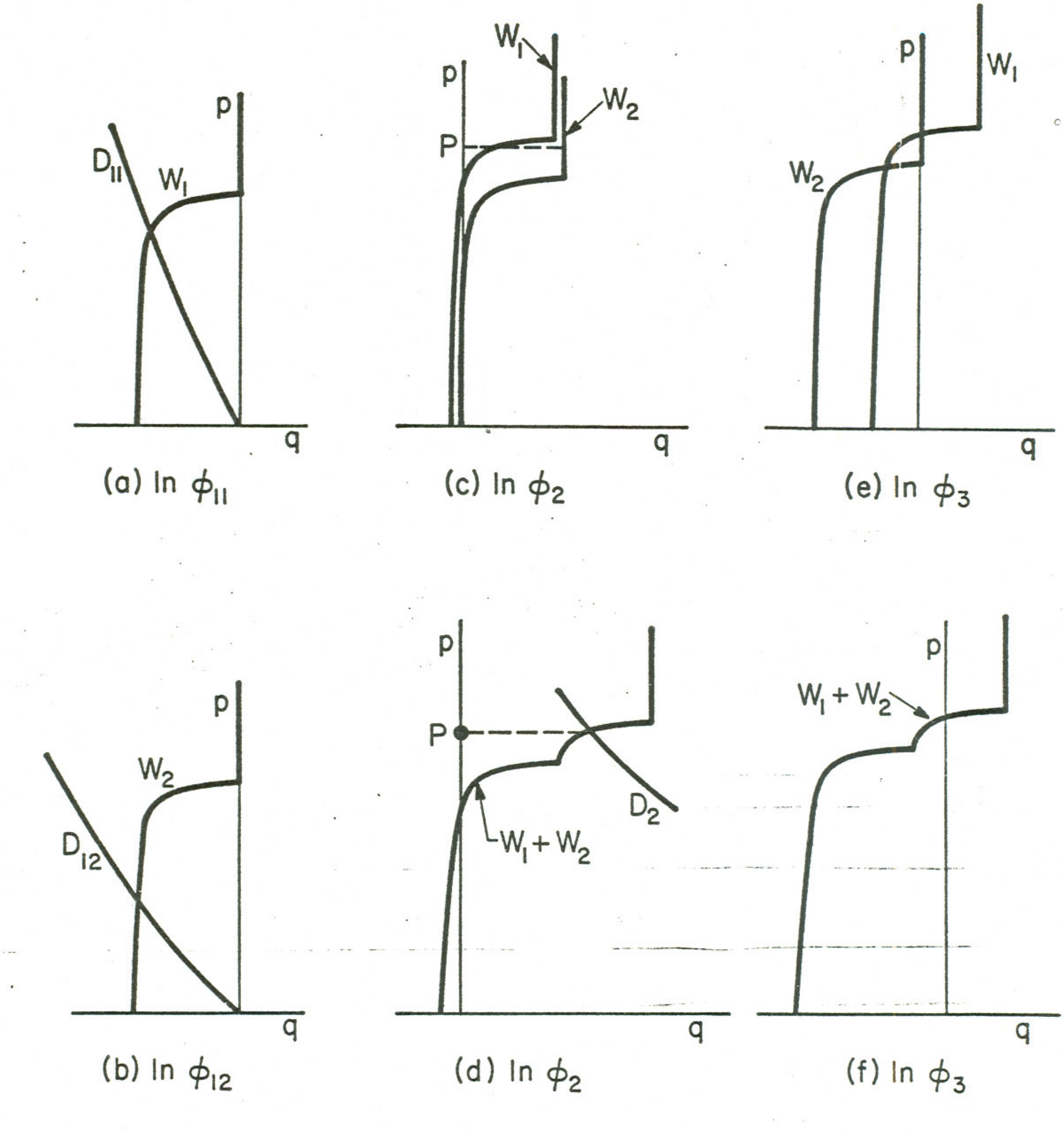


Figure 7.



p
W1
D11
q
(a) ln φ11
p
W2
D12
q
(b) ln φ12
W1 = W2
W1 + W2
p
D2
q
(c) ln φ2
p
W1 + W2
q
(d) ln φ3

Figure 8.



p
D11
W1
q
(a) ln φ11
p
W1
W2
P
q
(c) ln φ2
p
W1
W2
q
(e) ln φ3
p
W2
D12
q
(b) ln φ12
p
P
W1 + W2
D2
q
(d) ln φ2
p
W1 + W2
q
(f) ln φ3

Figure 9.

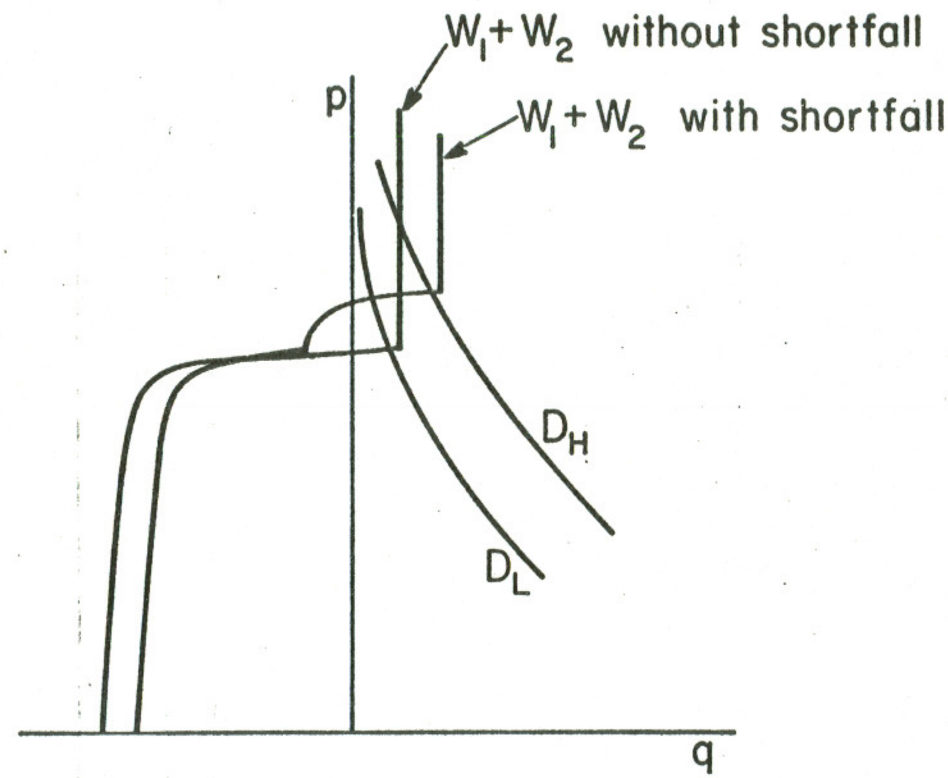
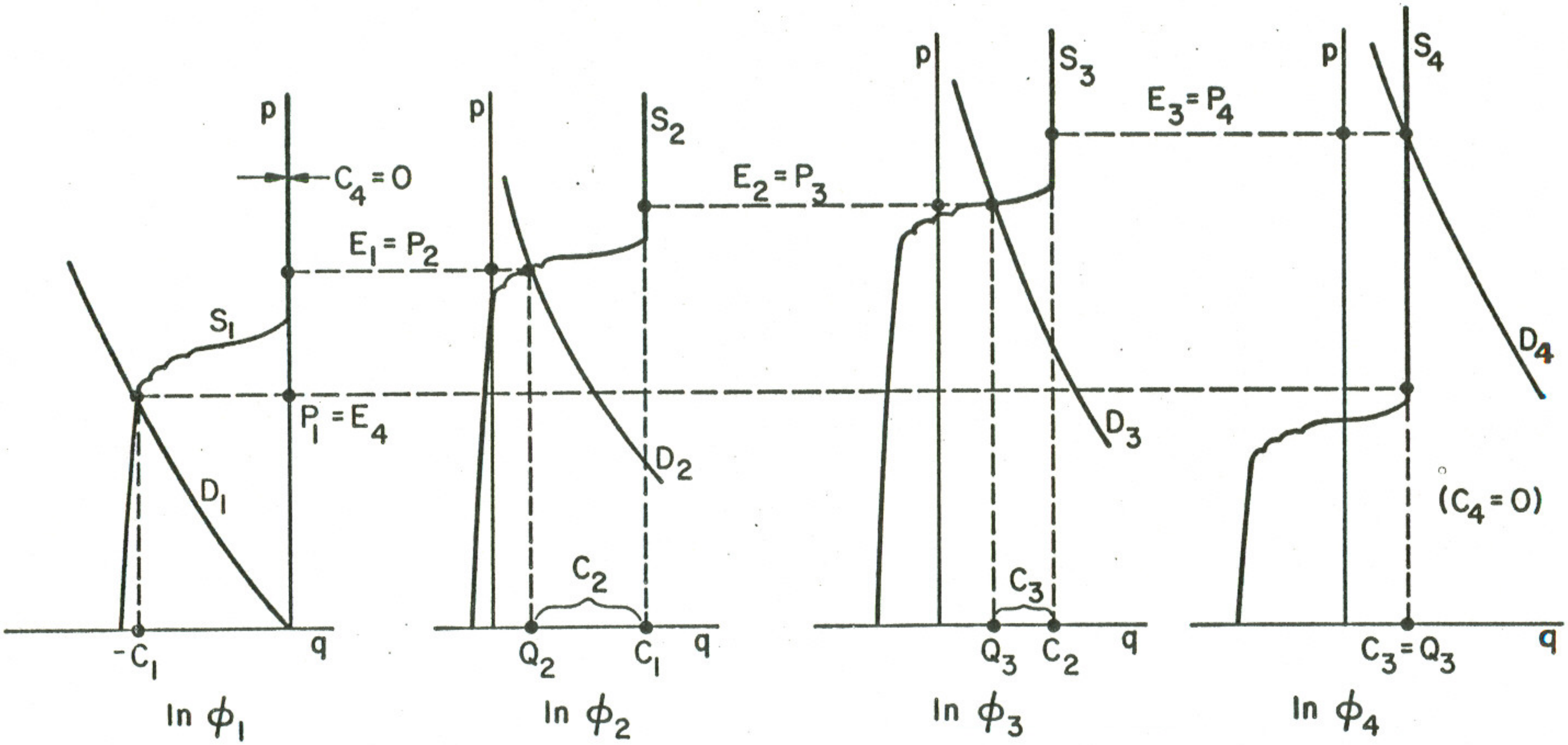


Figure 10.



p
$C_4 = 0$
S_1
$P_1 = E_4$
D_1
$-C_1$
q
In ϕ_1
p
$E_1 = P_2$
S_2
D_2
C_2
Q_2
C_1
q
In ϕ_2
$E_2 = P_3$
p
S_3
D_3
C_3
Q_3
C_2
q
In ϕ_3
$E_3 = P_4$
p
S_4
D_4
$(C_4 = 0)$
$C_3 = Q_3$
q
In ϕ_4

Figure 11.

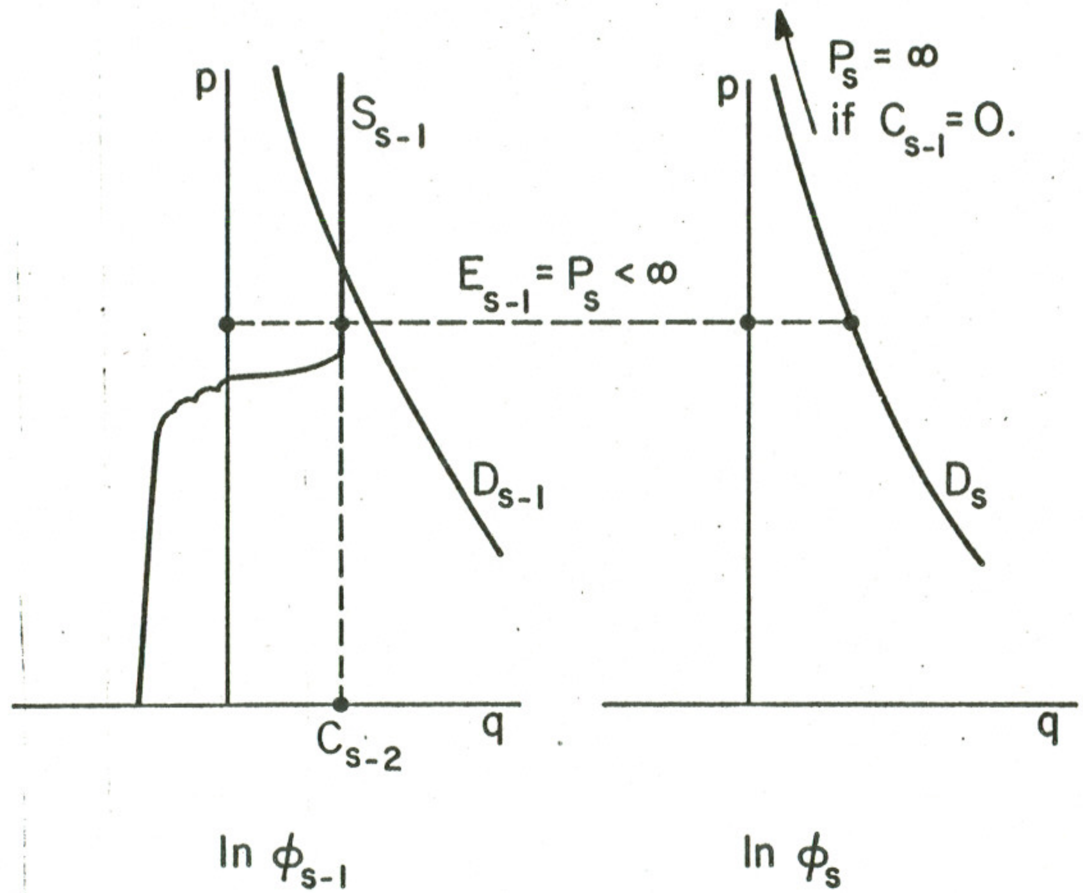

p
S_{s-1}
$E_{s-1} = P_s < \infty$
D_{s-1}
C_{s-2}
q
$\ln \phi_{s-1}$
p
$P_s = \infty$
if $C_{s-1} = 0$.
D_s
q
$\ln \phi_s$

Figure 12.

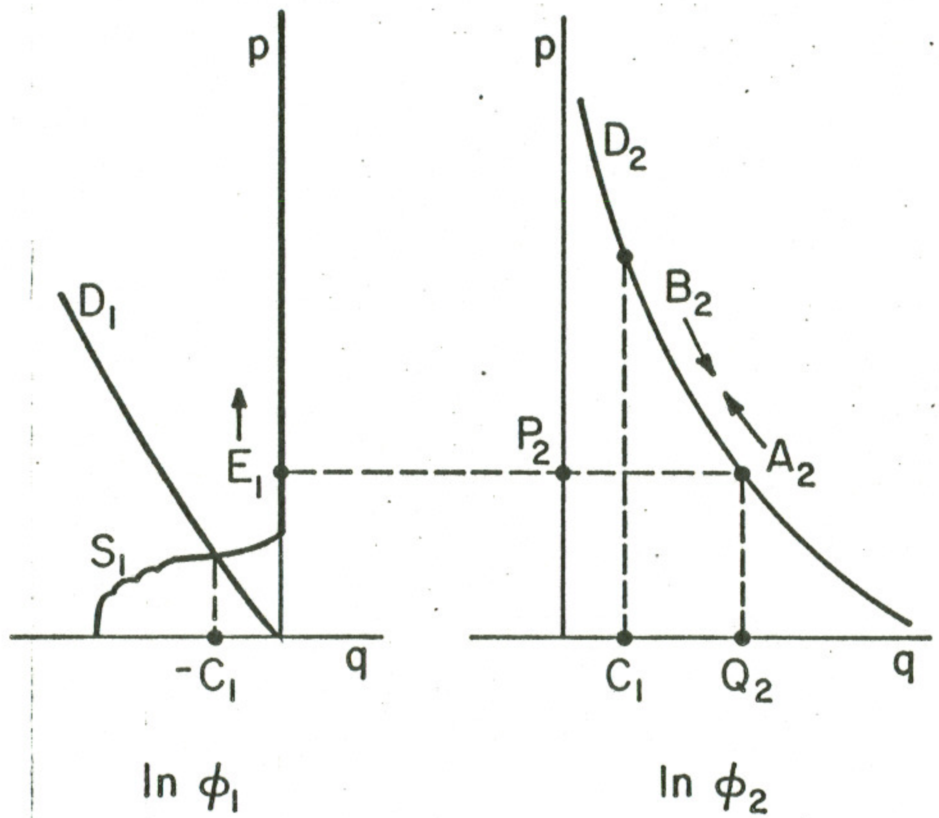


Figure 13.

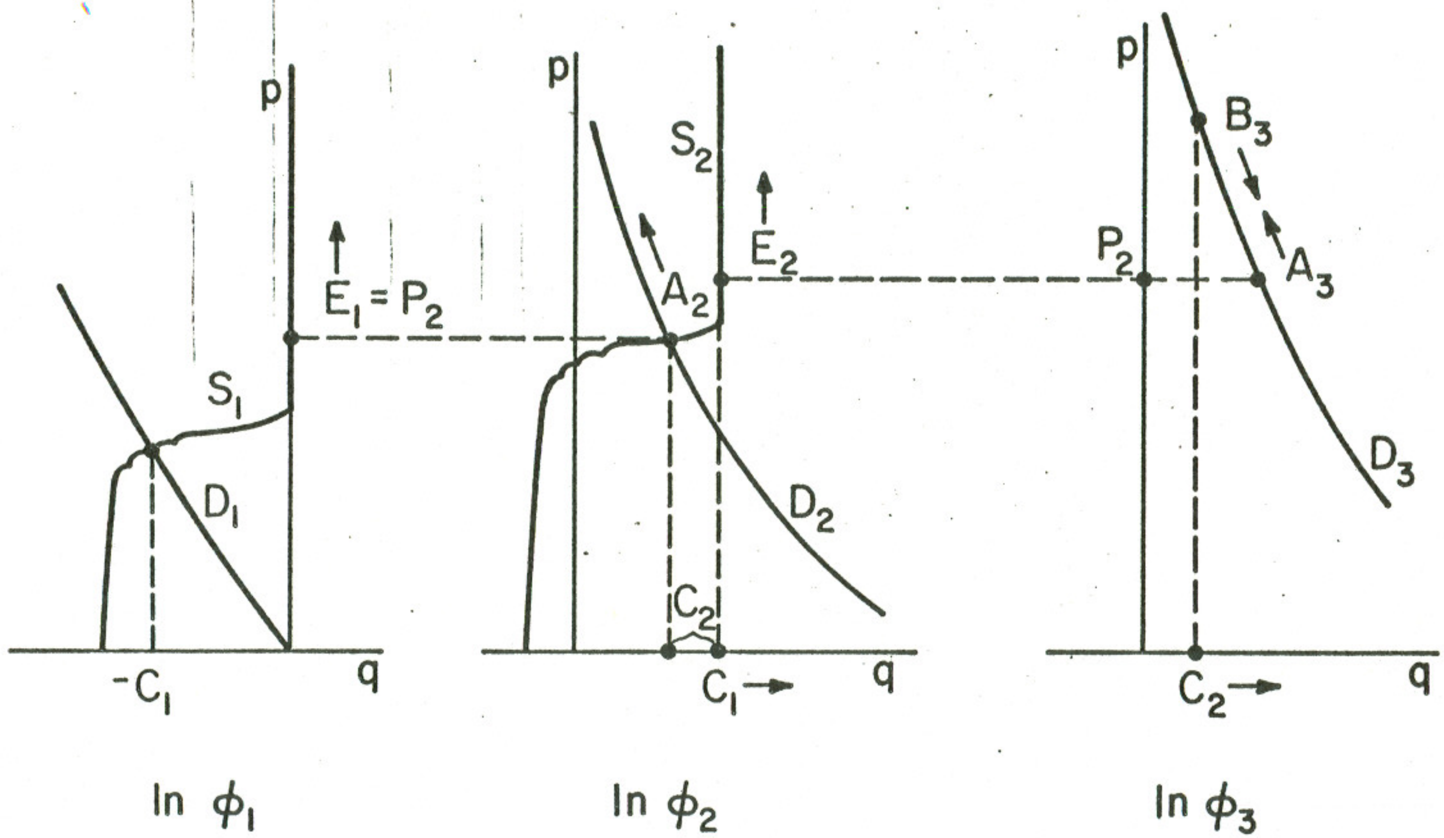

p
q
-C1
S1
D1
E1 = P2
ln φ1
p
S2
A2
E2
D2
C2
C1
q
ln φ2
p
B3
P2
A3
D3
C2
q
ln φ3

Figure 14.