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Abstract. A periodic marketing network that operates between two ecological zones and entails a movement of goods over long distances is examined. This network includes as special cases the dumbbell and dendritic marketing networks. Moreover, its analysis can be easily extended to the hourglass network, multilevel systems, and networks wherein the traders at the lowest level follow marketing rings. The model assumes that there is perfect competition in each market of the network. This leads to a set of nonlinear difference equations from which time series in all prices and commodity flows can be computed recursively. It is shown that, for a given set of time-invariant supply and demand functions in the lowest-level markets, the network has a unique equilibrium state. Some qualitative conclusions concerning the propagation of disturbances and the significance of trader profit margins are then drawn. The model can be modified to allow at least some of the traders to be monopsonists or monopolists in their respective markets.

1. Introduction. This work is a continuation of a study initiated in [5] of the dynamic economic behavior of the marketing systems occurring in third-world countries. The papers [5] and [6] were devoted to one kind of marketing network that W.O.Jones [2] refers to as the two-level system. It consists of a number of central markets at the upper level that supply

and are in turn supplied by rural markets at the lower level. Trade flows between the two levels but not exclusively along either level. Thus, the graph of this marketing network is bipartite. Others, such as C.A. Smith [4] refer to such structures as solar systems. There is some difference between these two concepts in that for solar systems each lower-level rural market is coupled by trade primarily to only one higher-level central market, whereas in the two-level system a rural market may interact appreciably with two or more central markets.

Another kind of system was examined in [7] and [8]. It too consists of a number of central markets supplying and supplied by many rural markets, but now traders follow various marketing rings through the rural markets before returning periodically to the central markets.

In both of these systems each trader engages in comparatively short-range trade, and a good moves a short distance between a rural market and a central market or along a single marketing ring. In the present work we study a system where at least some traders move goods large distances perhaps between different ecological zones. This system, which we refer to as an interzonal marketing network, is illustrated in Figure 1. We assume that an agricultural commodity is produced in one ecological zone, throughout which the rural markets ϕ_m are located. The commodity is brought by producers to the ϕ_m where it is sold to small-scale traders, who bulk and transport it to the central markets Ψ_j of that zone. Large-scale traders buy the commodity in the Ψ_j , bulk it still further, and ship it over long distances to the central markets Ψ_k of another ecological zone that does not produce the

commodity. In the γ_k the commodity is sold to many small-scale traders who transport it to the rural markets ω_r of the second ecological zone, where it is retailed to consumers. This system occurs, for example, in Nigeria; cowpeas are produced in the north, gathered in rural markets and shipped through such northern cities as Kano and Zaria to Ibadan, Lagos, Enugu, and other southern cities, where the cowpeas are then redistributed to lower-level markets [2; pp. 118-119], [3].

It should be pointed out that the subnetwork consisting only of the ϕ to Ψ branches and the γ to ω branches is assumed to be a collection of solar systems. The only place we allow loops (in the graph-theoretic sense) to occur is among the Ψ to γ branches. It should also be pointed out that our analysis extends readily to networks having even more levels at the lower end, that is, to networks having trees (again in the graph-theoretic sense) appended to the ϕ and ω markets.

A special case of this network arises when there is only one Ψ market and one γ market and therefore just one long-distance transportation branch between the two ecological zones. This is the network discussed by W.O.Jones [2; pp. 118-119] as the basic redistribution system. E.P.Scott refers to it as the dumbbell model [3].

Another special case - at least from a network-theoretic point of view - is the dendritic marketing system [1; pp. 83-92], [3], [4; pp. 177-179]. Its economic function is much different from that of the interzonal model, for it serves as the marketing channel through which raw materials are extracted from a colonial or neocolonial region and shipped through a port city to overseas

markets. We can make the network of Figure 1 represent a dendritic system by restricting it as follows. The network has only one ω market, which serves as the port city. Also, it has one γ market, which functions as a major inland assembly and wholesale market. On the other hand, it has several Ψ markets and many ϕ markets, which serve respectively as lower-level assembly markets and lowest-level rural markets. Finally, there are no loops among the Ψ to γ branches (or anywhere else) so that the graph of the overall network is a tree. Here too, we can extend the model by allowing additional markets at still lower levels beneath the ϕ markets.

Another variation of these marketing networks is the hourglass model. It occurs when an isolated region having two ecological zones is dominated by one central city. In this case, all the Ψ and γ markets are replaced by one highest-level central market Ψ . Goods are shipped from the ϕ markets to Ψ where they are sold and redistributed as shipments to the ω markets. This is similar to a solar system, but now the examined commodity flows toward Ψ along some of the branches and flows away from Ψ along the remaining branches. Strictly speaking, this is not a special case of Figure 1 because we have to coalesce all the Ψ and γ markets and the branches between them into one market in order to obtain the hourglass configuration. Nevertheless, the analysis given below can be directly applied to the hourglass model.

The primary objective of this paper is to construct a dynamic economic model of the interzonal marketing network

(Figure 1) from which time series in every price and commodity flow can be computed once an appropriate set of initial conditions is chosen. As with our prior works the model is a set of nonlinear difference equations derived from the excess-supply functions in the ϕ markets, the demand functions in the ω markets, and the transfer-supply functions of the traders under the assumptions that there is perfect competition in every market.

In addition, we prove that our model has a unique equilibrium state whenever the excess-supply functions in the ϕ_m and the demand functions in the ω_r are fixed with respect to time. We end this paper with four short sections. In the first two of these some qualitative conclusions are drawn concerning the propagation of disturbances in the marketing network and the significance of small or large profit margins for the traders. The penultimate section briefly describes the extension of the present work to interzonal marketing networks wherein the traders in the lowest-level rural markets follow rings of markets. In the last section, we indicate how our model can be modified to allow some of the traders to act as monopsonists in the ϕ markets and monopolists in the ω markets.

2. Assumptions. The notations used herein follow for the most part that of our prior works on periodic markets. The indices of the markets ϕ_m , Ψ_j , Υ_k , and ω_r follow a consecutive numbering; that is, $m = 1, \dots, M$; $j = M+1, \dots, J$; $k = J+1, \dots, K$; and $r = K+1, \dots, R$. We shall assume that each market in the system is periodic and opens regularly on only one and the same day of each marketing week, but the market days for the various markets are staggered throughout the week. (The market

week need not be seven days long. Marketing weeks as short as two days and as long as 10 or 12 days can be found in various parts of the world.) The time variable $\nu = \dots, -1, 0, 1, \dots$ numbers the marketing weeks. We assume that each market clears at a single price on each of its market days and therefore the clearance price and quantity exchanged can be written as functions of the market-week index ν .

We also assume that each trader restricts his trading activity to just one branch of Figure 1. The market where he buys (sells) the commodity will be called his initial (respectively, final) market. A trader that operates between one of the ϕ markets and one of the ψ markets will be called a $\phi:\psi$ trader, and a subscript will be appended to either ϕ or ψ or both if we wish to specify the particular market or markets the trader is using. A similar terminology is used for other pairs of initial and final markets.

The $\phi:\psi$ traders and the $\gamma:\omega$ traders complete each cycle of their trade within one week's time. For example, consider all the $\phi:\psi_j$ traders as they buy goods in their respective ϕ markets in week ν (but in general in different days of the week). After shipping their goods to ψ_j , they sell them there either in week ν or in week $\nu+1$. In both cases, they then return to their ϕ markets where they buy more goods in week $\nu+1$. Thus, when we use the week numbers as the discrete time variable, the time delay between the buying and selling of a shipment from ϕ_m to ψ_j , which we denote by τ_{mj} , equals either 0 or 1. The corresponding time delay between γ_k and ω_r is denoted by τ_{kr} , which also equals either 0 or 1. As for the long-distance shipments between ψ_j

and γ_k , the corresponding time delay is τ_{jk} , which we allow to be any nonnegative integer. Thus, a trader buying in Ψ_j and selling in γ_k may have several shipments in transit during any one week. This may require that the trader operates in one or both of his markets through agents who keep in touch with him by telephone.

Actually, our notation allows the case where all the markets meet daily, that is the marketing week is one day long. This simply requires that a trader operating between, say, ϕ_m and Ψ_j buys, transports, and sells goods every day.

We turn now to the supply and demand functions in the various markets. Any variable such as a price or quantity exchanged that belongs to a particular market is given the same index as that market. Thus, by referring to the index of such a variable, one can ascertain to which kind of market that variable belongs. As usual, p denotes price and q denotes quantity demanded, supplied, or stored; p is restricted to the range $0 \leq p < \infty$.

The excess-supply function in ϕ_m is denoted by $S_m(p, \nu)$. For fixed ν , it is a function of p alone, but it can change as ν varies. We let $S_m(p, \nu)$ be negative for sufficiently small p in order to allow in ϕ_m buyers other than the traders. This is illustrated in Figure 2. We think of $S_m(p, \nu)$ as being negative only for the very lowest values of p but do not need to impose this condition. What we do impose are

Conditions 1. For fixed ν , $S_m(p, \nu)$ is a continuous, strictly increasing function of p such that $S_m(0, \nu) = \sigma_m^-(\nu) \leq 0$ and, as $p \rightarrow \infty$, $S_m(p, \nu) \rightarrow \sigma_m^+(\nu) > 0$.

The demand function in ω_r is denoted by $D_r(p, \nu)$ and is assumed

to satisfy Conditions 2. A typical $D_r(p, \nu)$ is illustrated in Figure 3.

Conditions 2. For fixed ν , $D_r(p, \nu)$ is a continuous, nonnegative, decreasing function of p such that, as $p \rightarrow 0+$, $D_r(p, \nu) \rightarrow \infty$. It is strictly decreasing for $0 < p \leq \pi_r(\nu)$, where $\pi_r(\nu) > 0$, and is equal to zero for $\pi_r(\nu) \leq p < \infty$.

The economic behavior of the traders can be derived by treating each trader as a profit-maximizing firm that supplies the service of transferring the ownership of goods over space and time. This derivation has been presented in some detail in [6]. Therefore, we shall merely state herein the results of that derivation.

We number the traders with the index i . Any variable that belongs exclusively to the i th trader is denoted with a superscript i . Let α_a and β_b denote respectively the initial and final markets of the i th trader; a and b are the indices for those markets. Within α_a the i th trader is a buyer, and his demand function there is characterized by a function V_{ab}^i which is derived from that trader's behavior as a firm. V_{ab}^i satisfies the following.

Conditions 3. V_{ab}^i is a Lipschitz continuous, nonnegative function on the real line such that $V_{ab}^i(x) = 0$ for $x \leq 0$, $V_{ab}^i(x)$ is strictly increasing for $0 \leq x < \infty$, and $V_{ab}^i(x)$ tends to a finite limit as $x \rightarrow \infty$.

(In this work we invoke merely the continuity of V_{ab}^i , not its Lipschitzian character. However, we also use Theorem 2 of [5] whose hypothesis required that V_{ab}^i satisfy a Lipschitz condition.)

An examination of the trader's cost functions indicates that V_{ab}^i should have a step-like shape, rising rapidly for small positive values of x and then leveling off for larger values of x . However, except for Section 6, we do not need to impose this shape as a requirement on V_{ab}^i .

The demand function of the i th trader in his initial market α_a is

$$V_{ab}^i [E_{ab}^i(\nu) - T^i - p] \quad (2.1)$$

where p denotes the price in α_a and $E_{ab}^i(\nu)$ is the price he expects to receive in his final market β_b at the time the goods he is presently buying in α_a reaches his final market. Let $P_a(\nu)$ denote the price in α_a at ν . Then, T^i is the smallest value of $E_{ab}^i(\nu) - P_a(\nu)$ above which the trader is willing to transfer goods. We assume that $T^i > 0$. This demand function is illustrated in Figure 4. $E_{ab}^i(\nu)$ varies in general with ν , thereby causing the trader's demand function to shift vertically as ν changes.

Furthermore, we assume that $E_{ab}^i(\nu)$ is determined by the i th trader through some memory function M_{ab}^i of prior prices in β_b . If within a single week the market day for β_b precedes the market day for α_a ,

$$E_{ab}^i(\nu) = M_{ab}^i [P_b(\nu), P_b(\nu-1), P_b(\nu-2), \dots]. \quad (2.2a)$$

Otherwise,

$$E_{ab}^i(\nu) = M_{ab}^i [P_b(\nu-1), P_b(\nu-2), P_b(\nu-3), \dots]. \quad (2.2b)$$

These equations mean that, while trading in α_a , the trader prognosticates about the next price in β_b by extrapolating from prior prices in β_b .

Conditions 4. M_{ab}^i is a monotonic increasing function of each of its arguments, and a history of a constant price P_b in β_b yields the same price P_b for $E_{ab}^i(\nu)$. N_b^i , which is defined below also satisfies these conditions.

We can assign a more sophisticated prognosticating ability to the trader by making M_{ab}^i depend not only on prices in β_b but on prices in other markets as well. We can in fact admit market news into our model by including the prices in markets other than α_a and β_b . This will alter (2.2) but not any other equations in our dynamic model.

The i th trader is a seller within β_b , his economic behavior now being characterized by a supply function. We shall analyze two cases, one where the trader does not store goods and therefore sells the goods he brings into β_b at whatever price he can get. In this case, his supply function is perfectly inelastic. When this is so for all the traders, we refer to our model as the no-storage model. The second case, which we call the storage model, allows the trader to store goods in β_b if he feels that the price in β_b will improve with time. We assume that, in order to determine how much to store, the trader compares the per-unit gain he expects from the price improvement to the per-unit cost of storage, and, thinking marginally, chooses that quantity for storage for which the gain and the cost are equal. This requires the i th trader to know the per-unit cost of storing goods from ν to $\nu+1$ as a function Z_b^i of the amount q stored. We impose upon each Z_b^i the following.

Conditions 5. Z_b^i is a continuous, strictly increasing function of q for $0 \leq q \leq B_b^i$, and $Z_b^i(q) = \infty$ for $B_b^i < q < \infty$.

B_b^i is the i th trader's storage capacity. Since it is infinitely costly for him to store more than B_b^i , he will sell the goods he holds in excess of B_b^i for whatever price he can get. We set $I_b^i = Z_b^i(B_b^i)$.

Moreover, we assume that the trader, while operating in β_b at ν , estimates an expected price $F_b^i(\nu)$ in β_b at $\nu+1$ by means of a memory function N_b^i of past prices in β_b .

$$F_b^i(\nu) = N_b^i[P_b(\nu-1), P_b(\nu-2), P_b(\nu-3) \dots] \quad (2.3)$$

N_b^i is also assumed to satisfy Conditions 4. Again, a more sophisticated prognosis can be had by introducing prices from other markets among the arguments of N_b^i . For example, if the trader while operating in β_b has access to news about prior prices in a market δ_d that supplies goods (perhaps indirectly through other markets) to β_b and if it takes τ weeks for goods to be shipped from δ_d to β_b , then a shortfall in δ_d , signalled by a sharp rise in price there, will be felt in β_b τ weeks later on. So, the trader might predict a rise in price in β_b at $\nu+1$; this ability can be formulated by including the price $P_d(\nu-\tau+1)$ among the arguments of N_b^i .

The total amount of goods the trader has on hand in β_b at ν is denoted by

$$G_b^i(\nu) = A_b^i(\nu-1) + U_{ab}^i(\nu-\tau_{ab}). \quad (2.4)$$

Here, $A_b^i(\nu-1)$ is the amount the i th trader has stored in β_b from $\nu-1$ to ν . τ_{ab} is the time in weeks it takes to ship goods from α_a to β_b (assumed to be the same for all traders operating between α_a and β_b). $U_{ab}^i(\nu-\tau_{ab})$ is the amount the i th trader brings

into β_b at ν from α_a . Thus, $U_{ab}^i(\nu - \tau_{ab})$ is the amount the i th trader buys in α_a at $\nu - \tau_{ab}$.

$$U_{ab}^i(\nu - \tau_{ab}) = V_{ab}^i [E_{ab}^i(\nu - \tau_{ab}) - T^i - P_a(\nu - \tau_{ab})] \quad (2.5)$$

When (2.5) is zero, we say that the i th trader cuts off. When all the traders along one branch of Figure 1 cut off, we say that the branch has cut off.

From these assumptions one can conclude [6; Section 4] that in the storage model the i th trader's supply function in his final market β_b in week ν is

$$S_b^i(p, \nu) = \begin{cases} G_b^i(\nu) & \text{for } p \geq F_b^i(\nu) \\ \max \left\{ 0, G_b^i(\nu) - W_b^i[F_b^i(\nu) - p] \right\} & \text{for } F_b^i(\nu) - I_b^i \leq p \leq F_b^i(\nu) \\ \max \left\{ 0, G_b^i(\nu) - B_b^i \right\} & \text{for } p \leq F_b^i(\nu) - I_b^i \end{cases} \quad (2.6)$$

where it is understood that p is restricted to positive values. Also, W_b^i is the function-inverse of Z_b^i . $S_b^i(p, \nu)$ is sketched in Figure 5 for the case where $F_b^i(\nu) - I_b^i$ and $G_b^i(\nu) - B_b^i$ are both positive. On the other hand, $S_b^i(p, \nu)$ may intersect the ordinate or the abscissa on its curved part. When it intersects the ordinate, $S_b^i(p, \nu)$ is identical to the ordinate below the intersection point. In Figure 5, $P_b(\nu)$ is the clearance price in β_b , and $Q_b^i(\nu)$ is the amount the i th trader sells in β_b ; thus, the amount he stores from ν to $\nu+1$ is

$$A_b^i(\nu) = G_b^i(\nu) - Q_b^i(\nu) = G_b^i(\nu) - S_b^i[P_b(\nu), \nu].$$

Henceforth, we tacitly assume that all the assumptions and conditions stated in this section hold - except when alternatives are explicitly stated.

3. Dynamic equations. The price $p = P_m(\nu)$ at which the market ϕ_m clears at ν is obtained by solving the following relation that equates excess supply to aggregate trader demand in ϕ_m

$$S_m(p, \nu) = \sum_i V_{mj}^i [E_{mj}^i(\nu) - T^i - p] \quad (3.1)$$

Here, the summation is understood to be over the indices i of all the $\phi_m:\Psi$ traders, that is, over the indices for all traders having ϕ_m as their initial market. $E_{mj}^i(\nu)$ is determined in accordance with (2.2). With $p = P_m(\nu)$, the right-hand side of (3.1) becomes the amount $U_{mj}(\nu)$ of goods carried out of ϕ_m by the traders in week ν .

Let us now digress for a word of explanation about our symbolism. It is not apparent from the right-hand side of (3.1) alone just which indices i should be included in the summation. We specified that in the sentence following (3.1). (An alternative would be to use a more cumbersome notation, such as appending $\phi_m:\Psi$ as a subscript on the summation symbol.) It should also be noted that in (3.1) j changes in general as i varies, and should therefore be denoted by $j(i)$; we prefer the simpler notation j since the sentence following (3.1) removes any uncertainty concerning our notation. We shall follow this procedure in our subsequent equations involving aggregations over subsets of traders.

Next, consider market ψ_j . The clearance price $p = P_j(\nu)$

in ψ_j at ν is determined by solving

$$\sum_i S_j^i(p, \nu) = \sum_i V_{jk}^i [E_{jk}^i(\nu) - T^i - p]. \quad (3.2)$$

Now, the left-hand side is the aggregate of all the supply functions for the $\phi:\psi_j$ traders, and the right-hand side is the aggregate of all the demand functions (2.1) for the $\psi_j:\gamma$ traders. In the no-storage model, each $S_j^i(p, \nu)$ is constant with respect to p and indeed is equal to $U_{mj}^i(\nu - \tau_{mj})$ as given by (2.5); m is the index for the initial market for the i th $\phi:\psi_j$ trader, and τ_{mj} is the delay in shipping goods from ϕ_m to ψ_j . In the storage model, $S_j^i(p, \nu)$ is given by (2.6). The location of the upper corner point of $S_j^i(p, \nu)$ - and thereby the location of $S_j^i(p, \nu)$ itself - in the (p, q) plane is established by the values of (2.3) and (2.4), variables that can be determined from the prior behavior of the marketing system.

Similarly, the clearance price $p = P_k(\nu)$ in γ_k at ν is the solution of

$$\sum_i S_k^i(p, \nu) = \sum_i V_{kr}^i [E_{kr}^i(\nu) - T^i - p], \quad (3.3)$$

where the summation on the left-hand side is over the indices of all the $\psi:\gamma_k$ traders and the summation on the right-hand side is over the indices of all the $\gamma_k:\omega$ traders. Here too, $S_k^i(p, \nu)$ is given in the same two ways indicated in the preceding paragraph, one way for the no-storage model and the other way for the storage model.

Finally, the clearance price $p = P_r(\nu)$ in ω_r at ν is that price at which aggregate trader supply equals demand in ω_r :

$$\sum_i S_r^i(p, \nu) = D_r(p, \nu) \quad (3.4)$$

The summation herein is over the indices of all the $\gamma:\omega_r$ traders. Again, $S_r^i(p, \nu)$ is determined in either of two ways.

Equations (3.1) through (3.4) can be used recursively to compute all the prices and quantity flows in our zonal model, once all the $S_m(p, \nu)$ and $D_r(p, \nu)$ are specified for $\nu = 1, 2, 3, \dots$ and an appropriate set of initial conditions has been assumed. For example, if we wish to determine prices and quantity flows for $\nu = 1, 2, 3, \dots$, we must specify certain prices including those in the arguments of the memory functions (2.2) and (2.3) for certain values of $\nu \leq 0$. Indeed, when $\tau_{ab} \geq 1$, the amount (2.5) of goods brought into β_b at $\nu = 1, 2, 3, \dots$ by an $\alpha_a:\beta_b$ trader depends upon the prices

$$P_a(1-\tau_{ab}), P_a(2-\tau_{ab}), \dots, P_a(0) \quad (3.5)$$

and upon the trader's estimates $E_{ab}^i(1-\tau_{ab}), E_{ab}^i(2-\tau_{ab}), \dots$. This requires that all the prices (3.5) and any additional prices that appear as arguments in the memory functions (2.2) for $\nu = 1-\tau_{ab}, \dots, 0$ and in the arguments of (2.3) for $\nu = 0$ be specified as well. In addition, when dealing with the storage model, we have to specify the quantities $A_b^i(0)$ the traders store from $\nu = 0$ to $\nu = 1$. (If instead of assuming only price values as the initial conditions we also assume the flow values $U_{ab}^i(\nu)$ for appropriate ν , then fewer initial prices need be specified.)

Once the needed initial conditions are known, the recursive computation of all prices and quantity flows is straightforward.

The equations (3.1) to (3.4) are solved first for those markets that meet on the first day of week $\nu = 1$, then for those markets that meet on the second day of week $\nu = 1$, and so forth. More specifically, consider markets α_a and β_b an $\alpha_a:\beta_b$ trader. Once $E_{ab}(1)$ is known, every trader's demand curve (Figure 4) in α_a is located in the (p, q) plane for $\nu = 1$. Also, $E_{ab}^i(1-\tau_{ab})$ and $P_a(1-\tau_{ab})$ are also known and therefore so too is $U_{ab}^i(1-\tau_{ab})$. The last quantity is the value of the perfectly inelastic supply of the i th trader in β_b at $\nu = 1$ in the case of the no-storage model. For the storage model, $U_{ab}^i(1-\tau_{ab})$ in conjunction with the initial condition $A_b^i(0)$ on the i th trader's storage determines $G_b^i(1)$. Moreover, the initial price conditions determine through (2.3) the value $F_b^i(1)$. The last two values locate the upper corner of $S_b^i(p, \nu)$ and thereby the i th trader's supply function in β_b at $\nu = 1$ in the storage model. In this way, the aggregate supply and demand function in all the markets can be determined and thereby all the clearance prices. These prices also determine how much each trader sells in his final market at $\nu = 1$ and, if he has a storage facility, how much he stores from $\nu = 1$ to $\nu = 2$. It also determines how much he buys in his initial market at $\nu = 1$. We can therefore repeat this computation to determine all prices and flows for $\nu = 2$, then for $\nu = 3$, and so forth.

4. The equilibrium state. An interzonal marketing network is said to be in an equilibrium state or simply in equilibrium if for all m and r the excess-supply functions $S_m(p, \nu)$ and the demand functions $D_r(p, \nu)$ do not vary with ν and if all the prices also do not vary with ν . As a consequence, it will follow that all the commodity flows are also invariant with respect to ν .

In this section, we discuss the existence and uniqueness of the equilibrium states. Since all variables are time-invariant in an equilibrium state, we can and will simplify our notation by dropping the argument ν . For example, we will replace the notation $P_m(\nu)$ by P_m ; that is, instead of having P_m denote a mapping of ν into a price, P_m will now represent a price.

We first establish the existence and uniqueness of the equilibrium states for the no-storage model. This result requires a fairly long argument, but, once it is obtained, its extension to the storage model turns out to be a brief matter.

In view of Conditions 4, when an equilibrium state is in force, the memory functions (2.2) and (2.3) yield the following simple formula for the expected prices.

$$E_{ab}^i = F_b^i = P_b$$

Moreover, for the no-storage model the dynamic equations (3.1) through (3.4) simplify into (4.1) through (4.4) respectively when in addition the equilibrium prices are substituted for the variable p . In ϕ_m ,

$$S_m(P_m) = \sum_i V_{mj}^i (P_j - T^i - P_m), \quad (4.1)$$

the summation being for all $\phi_m: \Psi$ traders. In Ψ_j ,

$$\sum_i V_{mj}^i (P_j - T^i - P_m) = \sum_i V_{jk}^i (P_k - T^i - P_j), \quad (4.2)$$

the summation on the left being for all $\phi: \Psi_j$ traders and the summation on the right being for all $\Psi_j: \gamma$ traders. In γ_k ,

$$\sum_i V_{jk}^i (P_k - T^i - P_j) = \sum_i V_{kr}^i (P_r - T^i - P_k), \quad (4.3)$$

the left-hand summation being for all $\Psi:\gamma_k$ traders and the right-hand summation being for all $\gamma_k:\omega$ traders. Finally, in ω_r ,

$$\sum_i V_{kr}^i (P_r - T^i - P_k) = D_r(P_r), \quad (4.4)$$

the summation being for all $\gamma:\omega_r$ traders.

Theorem 1. The no-storage model for Figure 1 has a unique equilibrium state for each set of time-invariant excess-supply functions $S_m(p)$ and time-invariant demand functions $D_r(p)$.

Proof. The proof of this theorem is divided into three steps. The second step is by far the longest. Were it not for the existence and uniqueness theorems of [6], which we invoke, the third step would also be quite long.

Step 1. Consider the subnetwork induced by Ψ_j and its adjacent ϕ markets. We first show that each $P_j > 0$ determines a unique inflow C_j to Ψ_j from all the ϕ_m adjacent to Ψ_j . Indeed, for one such ϕ_m , a given P_j determines the demand function $V_{mj}^i (P_j - T^i - p)$ for the i th $\phi_m:\Psi_j$ trader and thereby the aggregate demand function in ϕ_m for all the $\phi_m:\Psi_j$ traders. The intersection of that aggregate demand function with the given supply function $S_m(p)$ determines a unique price P_m in ϕ_m and the total ϕ_m to Ψ_j flow U_{mj} . Summing over all the ϕ_m adjacent to Ψ_j , we obtain a unique inflow $C_j = \sum U_{mj}$ to Ψ_j .

Moreover, by the monotonicities of the S_m and the V_{mj}^i , as P_j increases, each $\phi_m:\Psi_j$ trader increases his contribution U_{mj}^i to U_{mj} , except possibly when U_{mj}^i remains zero (i.e., when the i th trader remains cut off). This means that C_j is monotonic increasing

with respect to P_j . Furthermore, the continuities of the S_m and V_{mj}^i imply that these variations are continuous. Altogether, C_j is a continuous, monotonically increasing function of $P_j > 0$, as illustrated in Figure 6. We denote the function $P_j \mapsto C_j$ by S_j^e and call it the derived-equilibrium-supply function in Ψ_j . For sufficiently small P_j , all the $\phi_m: \Psi_j$ traders cut off, and therefore there is a unique price $P_j^- > 0$ such that S_j^e is strictly increasing for $p \geq P_j^-$ and is zero for $0 < p \leq P_j^-$. This argument shows that, when C_j and P_j are related by $C_j = S_j^e(P_j)$, there exists for each $P_j > 0$ a unique set of prices P_m in the ϕ_m adjacent to Ψ_j for which equilibrium holds in the subnetwork induced by Ψ_j and those ϕ_m .

Step 2. Our next objective is to show that, under equilibrium conditions, each γ_k has a certain demand function D_k^e derived from the demand functions in the ω_r .

Choose any γ_k and restrict r to the indices of those ω markets that are adjacent to γ_k . Then, choose any set of $P_r > 0$ and thereby the $Q_r = D_r(P_r)$ such that $\sum Q_r < Q_L$, where $Q_L = \sum V_{kr}^i(P_r - T^i)$; Q_L is the abscissa intercept of the aggregate of the $V_{kr}^i(P_r - T^i - p)$ functions of p for all the $\gamma_k: \omega$ traders. Such a choice of the P_r can be made because, as the P_r increase, the Q_r decrease to zero while Q_L increases toward larger positive values. Now, let $Q_k = \sum Q_r$ and equate Q_k to γ_k 's aggregate demand function

$$\sum_i V_{kr}^i(P_r - T^i - p) \quad (4.5)$$

of all the $\gamma_k: \omega$ traders. The resulting equation has a unique

positive solution for p , which we denote by P_k . For each $\gamma_k : \omega_r$ trader, let

$$U_{kr}^i = V_{kr}^i (P_r - T^i - P_k).$$

This is the amount the i th trader transports from γ_k to ω_r under equilibrium conditions. Then, let U_{kr} denote the amount $\sum U_{kr}^i$ transported by all the $\gamma_k : \omega_r$ traders. Thus, we have

$$\sum_r U_{kr} = Q_k = \sum_r Q_r \quad (4.6)$$

where on both sides the summation is over the indices r for the ω markets adjacent to γ_k .

In other words, U_{kr} is the amount transported into and supplied in ω_r , and Q_r is the amount demanded in ω_r . If $U_{kr} = Q_r$ for every admissible r , we have found a set of prices for γ_k and its adjacent ω_r such that the subnetwork induced by these markets is in equilibrium. If however $U_{kr} \neq Q_r$ for at least one admissible r , we will hold P_k fixed and will adjust the P_r in such a fashion that (4.7) will be satisfied as well as (4.6) after all the adjustments are made.

So, examine in turn each of the $U_{kr} - Q_r$. Let $r = \rho$ be the first index for which $U_{k\rho} \neq Q_\rho$. If $U_{k\rho} > Q_\rho$, start decreasing P_ρ but keep P_k and all the other P_r fixed. This decreases $U_{k\rho}$ and increases Q_ρ . Eventually, $U_{k\rho} - Q_\rho$ will become negative because, as P_ρ approaches zero, $U_{k\rho}$ will reach zero whereas Q_ρ will increase toward larger positive values. These variations are continuous and monotonic with respect to changes in P_ρ . Therefore, by the intermediate-value theorem, there exists a unique P_ρ such that $U_{k\rho} = Q_\rho$ for the fixed P_k . (However, (4.6)

will no longer hold if we vary only one of the P_r .)

On the other hand, if $U_{k\rho} < Q_\rho$, start increasing P_ρ but keep P_k and all the other P_r fixed. This decreases Q_ρ and eventually increases $U_{k\rho}$. Eventually, $U_{k\rho} - Q_\rho$ will become positive because, as $P_\rho \rightarrow \infty$, $U_{k\rho}$ increases toward larger positive values while Q_ρ decreases and reaches zero. As before, we can conclude again that there is a unique P_ρ such that $U_{k\rho} = Q_\rho$ at the fixed P_k .

Keep doing this until all the P_k for which $U_{kr} \neq Q_r$ have been adjusted in turn to achieve $U_{kr} = Q_r$, keeping P_k fixed throughout. Although (4.6) will be violated during some of this process, it will hold again at the end. This yields then the unique equilibrium state for the subnetwork induced by γ_k and its adjacent ω_r for which P_k is the price in γ_k .

The process can be repeated for different values of P_k to obtain a function $D_k^e: P_k \mapsto Q_k$, which we shall refer to as the derived-equilibrium-demand function $\wedge \gamma_k$. We now show that D_k^e is continuous and monotonic decreasing, as illustrated in Figure 7.

While keeping each P_r constant, decrease P_k from the value it had above by some amount such that P_k remains positive. This increases each U_{kr} but keeps each Q_r fixed, except possibly when Q_r is zero and U_{kr} remains \wedge at the new lower value of P_k . At the new value of P_k , we will have $U_{kr} \geq Q_r$, with equality occurring only in the exceptional case. For each r such that $U_{kr} > Q_r$, we can restore equality at the new value of P_k by decreasing P_r an appropriate amount; indeed, by virtue of the argument used in establishing the existence of D_k^e , this increases Q_r or keeps it fixed at zero in the exceptional case. (It also decreases U_{kr} .) When equality is restored for all admissible r , $Q_k = \sum Q_r$ will

have increased - or stayed fixed at zero. Clearly, Q_k will not become larger than the maximum aggregate capacity $\sum V_{kr}^i(\infty)$ of all the $\gamma_k:\omega$ traders, no matter how much P_k is initially decreased. Again, these variations are continuous with respect to the changes in P_k and the P_r .

The opposite variations occur when P_k is initially increased and the P_r are then increased to restore the equalities $U_{kr} = Q_r$. Note that, since $D_r(p) = 0$ for $p \geq \pi_r$, under equilibrium conditions, $Q_k = 0$ for all sufficiently large P_k .

We can conclude that D_k^e is defined for all positive values of P_k and is a continuous, monotonically decreasing function of P_k . As $P_k \rightarrow 0+$, Q_k approaches a finite value. At those values of P_k for which $D_k^e(P_k)$ is positive, D_k^e is strictly decreasing. Also, $D_k^e(P_k) = 0$ for all sufficiently large P_k .

Step 3. Finally, we assert that the subnetwork of Figure 1 induced by all the Ψ and γ markets also has a unique equilibrium when S_j^e is the supply function in Ψ_j and D_k^e is the demand function in γ_k ; that is, there exists a unique set of prices in the Ψ_j and γ_k for which equilibrium holds in the stated subnetwork of Ψ to γ branches. This fact follows immediately from Theorems 2 and 3 of [6] because our S_j^e satisfy the conditions that the $L_j + S_j$ of that work satisfy and our D_k^e satisfy the conditions that the G_k^{-1} of that work satisfy. (There is one unessential difference between the conditions of [6] and the conditions on D_k^e . In [6] the demand functions tend to ∞ as $p \rightarrow 0+$, whereas D_k^e tends to a finite limit. The proofs of [6] extend immediately to the present case.)

We can now piece together all the subnetworks. We start with the unique equilibrium state for the Ψ to γ subnetwork with the found S_j^e and D_k^e functions. The unique price in each Ψ_j then determines a unique equilibrium state in each ϕ to Ψ_j subnetwork (Step 1) and the unique price in each γ_k determines a unique equilibrium state in each γ_k to ω subnetwork (Step 2). This completes the proof.

It is worth pointing out that this theorem can be extended immediately to marketing networks having more levels than those of Figure 1 so long as the extensions of the network of Figure 1 consist of trees appended to the ϕ and ω markets. Such an extension at a ϕ market is shown in Figure 8. Given a set of time-invariant supply functions for the λ_n , we need merely repeat Step 1 to obtain derived-equilibrium-supply functions for the θ_s . From these we get through Step 1 again derived-equilibrium-supply functions in the ϕ_m . A similar procedure can be used when marketing-network trees are appended to the ω markets; we now keep repeating Step 2 working back from the lowest-level markets, wherein time-invariant demand functions are given, to obtain derived-equilibrium-demand functions in the ω markets and the markets beyond ω above the lowest level. Finally, as in Step 3, the equilibrium state for Figure 1 can be extended to the appended trees to establish an overall unique equilibrium state.

As another extension of Theorem 1 we have the following assertion about our storage model. It too can be extended to multilevel marketing networks that extend Figure 1 as indicated in the preceding paragraph.

Theorem 2. The storage model for Figure 1 has a unique equilibrium state for each set of time-invariant excess-supply functions $S_m(p)$ and time-invariant demand functions $D_r(p)$.

Proof. Given a storage model, we can construct a unique no-storage model by replacing every trader's storage-model supply function S_b^i in his final market β_b by a vertical line (i.e., by an infinitely inelastic supply function) that coincides with the upper vertical portion of S_b^i . We call the result the associated no-storage model.

If a storage model has an equilibrium state, all prices are time-invariant, and every expected price $F_b^i(\nu)$ for a market β_b coincides with the past history of a constant price P_b in β_b ; that is, $F_b^i(\nu) = P_b$ for all ν and i . Consequently, in an equilibrium state, every trader operates at the upper-corner point of his supply function in his final market, and therefore no trader stores goods. Thus, the associated no-storage model has the same equilibrium state. Since the associated no-storage model can have only one equilibrium state according to Theorem 1, the same is true for the storage model. Conversely, if the associated no-storage model has an equilibrium state, then the storage model has the same equilibrium state with the upper-corner price of every trader's final-market supply function S_b^i coinciding with the clearance price in that market. Thus, we need merely invoke Theorem 1 to complete the proof.

A corollary of Theorem 2 is that Equations (4.1) through (4.4) also hold for the equilibrium state of the storage model.

5. The propagation of disturbances. When traders have price information only about the markets in which they trade, disturbances in market conditions are propagated only through the trading activity. As a consequence, a disturbance travels from market to market, proceeding no further in any one week than one branch of the network. This was pointed out by W.O.Jones [2]. Our prior models of periodic markets exhibit this phenomenon, and so too does our present interzonal model.

For example, assume that the network is in equilibrium and then a shortfall occurs at $\nu = 1$ in one of the ϕ markets, say, in ϕ_1 , this shortfall exhibiting itself as a sudden shift to the left of the excess-supply function. That is, $S_1(p, 1)$ lies substantially to the left of $S_1(p, 0)$. Assume also that all other $S_m(p, \nu)$ and $D_r(p, \nu)$ remain fixed with respect to ν . The shift in S_1 raises the price in ϕ_1 but does not affect the prices in other markets at $\nu = 1$. It also results in fewer goods being sent to the Ψ market adjacent to ϕ_1 , say Ψ_j . At $\nu = 1 + \tau_{1j}$, the shortfall reaches Ψ_j and raises the price there. This induces the other $\phi_m : \Psi_j$ traders ($m \neq 1$) to raise their expected prices and thereby their demands in the ϕ_m at $\nu = 2 + \tau_{1j}$. It also curtails the amount sent to the γ markets adjacent to Ψ_j . The process repeats in those markets, resulting in raised prices and fewer goods being transported into the ω markets adjacent to those γ markets. In fact, the disturbance propagates throughout the entire network in this step-by-step fashion. Precise arguments can be constructed to trace out how the disturbance first reaches the various markets in the network and to determine the times at which it does so. The arguments are quite the same as those given in [5] and [6] and won't be repeated here. Once the initial

disturbance has passed through a market, that market will be subjected in general to oscillating prices as the disturbance not only propagates on to newer markets but also reflects back and forth between the markets it has already reached. This can be demonstrated by computer simulations of our model.

Similarly, a disturbance in demand in one of the ω markets, say, a sudden shift to the right of $D_\rho(p, \nu)$ at $\nu = 1$ raises the price in ω_ρ and induces the $\gamma_k:\omega_\rho$ traders to raise their expected prices at $\nu = 2$ in γ_k . This raises demand and thereby price in γ_k and causes more goods to flow toward ω_ρ . It also causes fewer goods to flow toward the other ω_r markets ($r \neq \rho$) adjacent to γ_k because the corresponding traders have not raised their expected prices and demand functions in γ_k . This process too can be precisely traced to see how it propagates throughout the network.

Note that a disturbance passes from an initial market α_a to a final market β_b along one branch in τ_{ab} weeks, that is, no faster than the speed at which a good travels from α_a to β_b . On the other hand, a disturbance travels from β_b to α_a at the speed at which information passes back from β_b to α_a and can be acted upon in α_a , a possibly faster process. Actually, had we assumed that our memory functions (2.2) and (2.3) were also sensitive to prices in markets other than the indicated final market, then the disturbance could travel between markets faster than the trading activity would by itself allow. Here again is an argument for improved market news.

6. Profit margins. An examination of the operating costs for a trader [6] indicates that his V_{ab}^i curve has step-like shape; it rises sharply at small positive values of its argument and then levels off at higher argument values. Consequently, his demand curve in his initial market has the form illustrated in Figure 4. It is elastic when he operates at small expected profits, that is, at prices just below $E_{ab}^i(\nu) - T^i$ and is inelastic when he operates at large expected profits, that is, at substantially lower prices. (The large $\Psi:\gamma$ traders, who are shipping over long distances - perhaps by train, may not be as restricted by capacity constraints, and so the inelastic portion of their demand curves may not be so steep.) As a result, when all traders are operating at large \wedge ^{expected} profit margins, a disturbance in supply in one of the ϕ markets propagates toward the ω markets with relatively large changes in price but relatively small changes in the commodity flows. If on the other hand all the traders are operating at small \wedge ^{expected} profit margins, the opposite holds true.

The same thing can be said for a disturbance in demand at an ω market, with the following exception. In the no-storage model the initial transmittal of the disturbance back through the ω , γ , and Ψ markets will not result in any change in the total amount sold regardless of whether the traders are operating at large or small \wedge ^{expected} profit margins. The disturbance merely makes itself felt through price variations. This is because the supply functions in those markets are perfectly inelastic. Subsequent and lateral propagations will however involve quantity variations.

7. Ring-terminated markets. A not unusual itinerary for a small-scale trader operating in the lowest-level rural markets is one wherein the trader follows a ring of markets. For example, after leaving his Ψ_j , the trader follows a sequence of ϕ markets buying up goods in each one and then returns to Ψ_j after one week. A similar route may be followed by a trader on the ω side of the network; he buys goods in his γ_k and then proceeds through a sequence of ω markets selling those goods and returning to γ_k after one week's time to repeat the process. Such a one-week's itinerary is called a ring. The dynamic behavior and the equilibrium states of periodic marketing networks consisting solely of rings was examined in [7] and [8].

We can extend the present interzonal model to the case where some or all of the $\phi:\Psi$ traders or $\gamma:\omega$ traders follow rings of markets, so long as the rings are mutually disjoint except at the Ψ and γ markets. This can be done by combining the analyses of [7] and [8] with that of the present work. In addition to dynamic equations, which can be used recursively to compute time series in the prices and commodity flows, we can also establish the existence of a unique equilibrium state when the supply functions in the ϕ markets and the demand functions in the ω markets are fixed with respect to time. The proof of the latter follows the scheme presented in Section 4, but it requires some significant changes to incorporate the arguments of [7] and [8].

8. Monopsonist and monopolist traders. We have assumed so far that there is perfect competition in every market. This means among other things that there are many traders competing with each other. Quite often however traders are few in number in one or more of the markets and collude. What model of oligopsony or oligopoly is most appropriate for the behavior of the traders in this case and how such a model can be incorporated into our interzonal marketing network is presently uncertain - at least to this author. However, if we go to the extreme case where a single trader acts as a monopsonist or monopolist, then the standard theory of such an agent can easily be incorporated into our interzonal model.

For example, assume that in a ϕ market there is but one trader buying goods as a monopsonist from many suppliers, and refer to Figure 9. $S^{-1}(q)$ denotes the supply schedule written as a function of q and with its dependence on the time-variable suppressed. The marginal cost to the monopsonist of buying the amount q is $C'(q) = p + qdS^{-1}/dq$. Since S^{-1} is monotonic increasing, the C' curve lies above the S^{-1} curve. On the other hand, the trader's expected marginal revenue product is his demand curve H (also written as a function of q). Standard theory states that the monopsonist buys that quantity Q at which $C'(q) = H(q)$ and does so at the price $P = S^{-1}(Q)$. Thus, we need merely replace (3.1) by the present relationships in order to allow a monopsonist in one or more of the ϕ markets.

Similarly, the standard theory of a monopoly can be used to incorporate a monopolist trader selling the commodity in an ω market to many consumers. In this case, the demand curve

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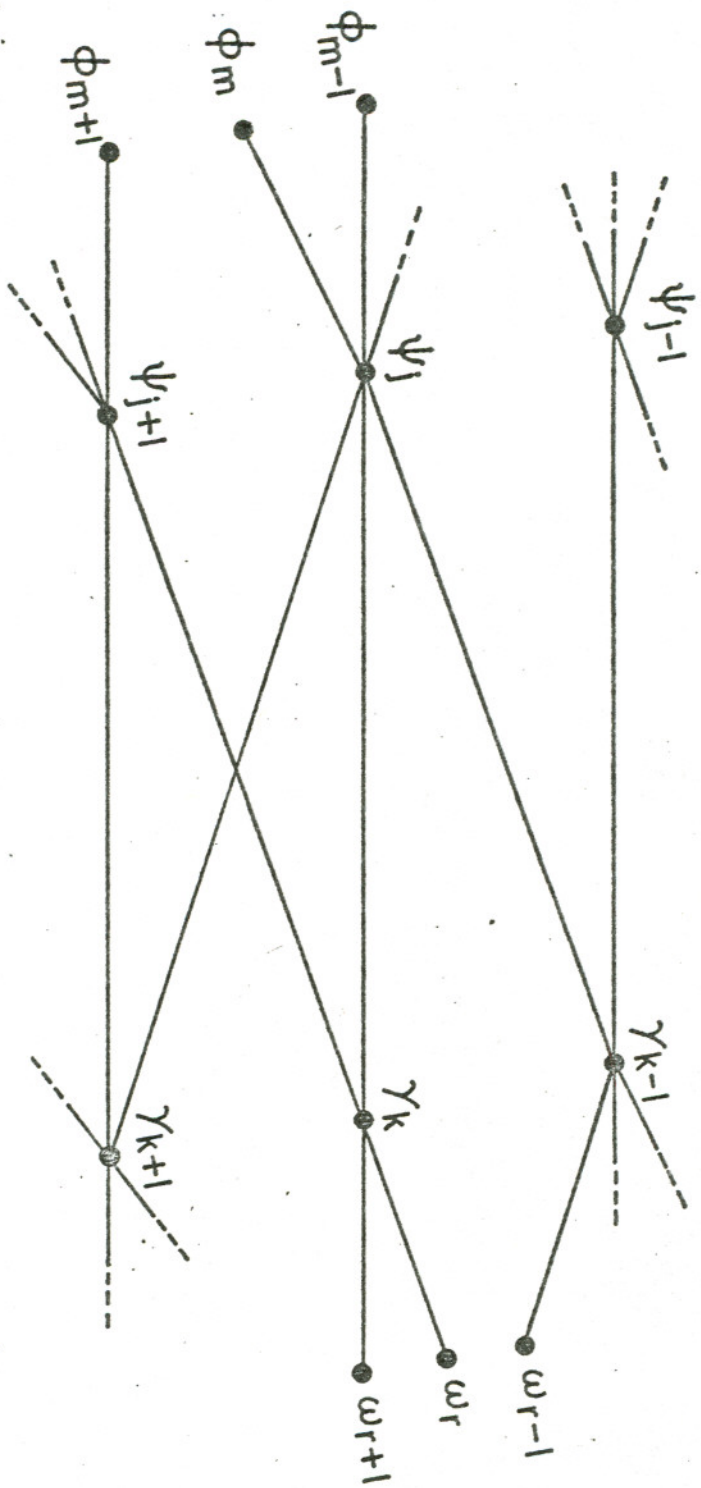


FIGURE 1

FIGURE 3

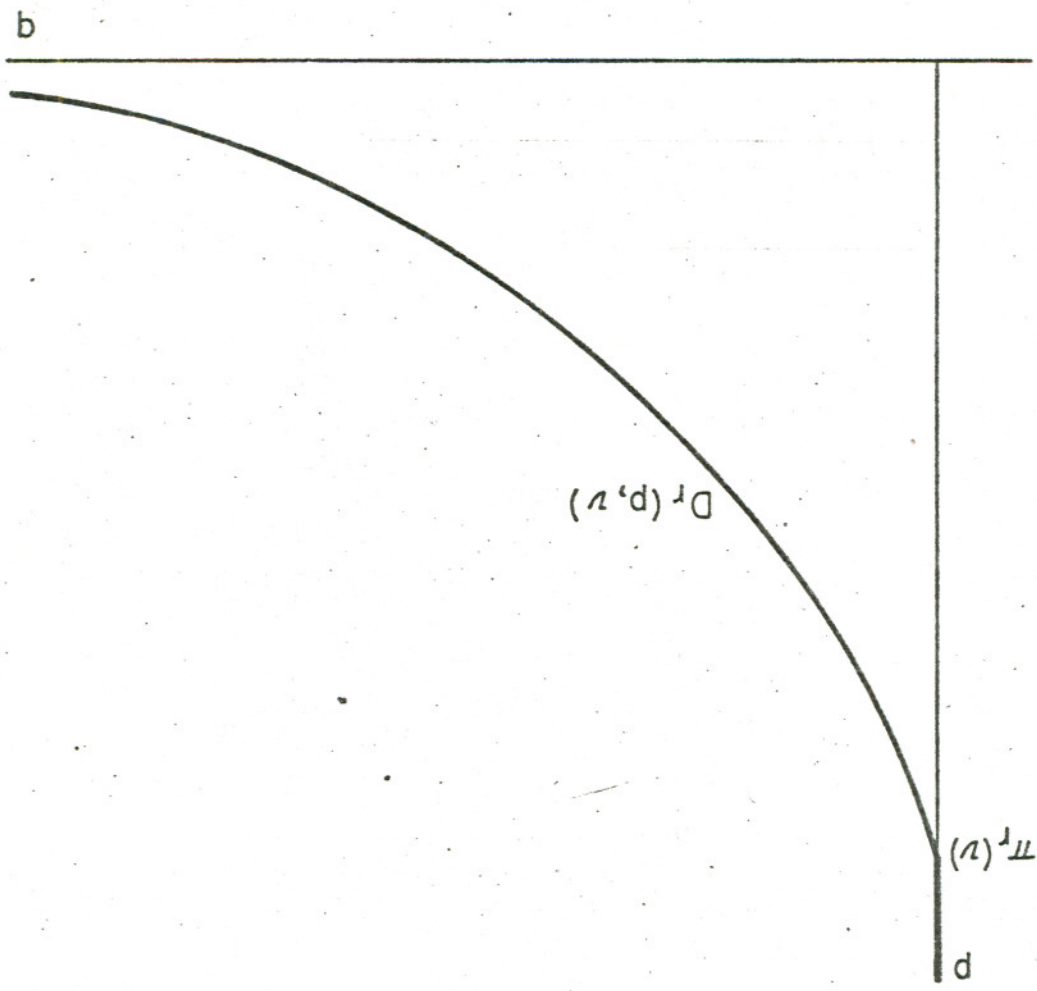


FIGURE 2

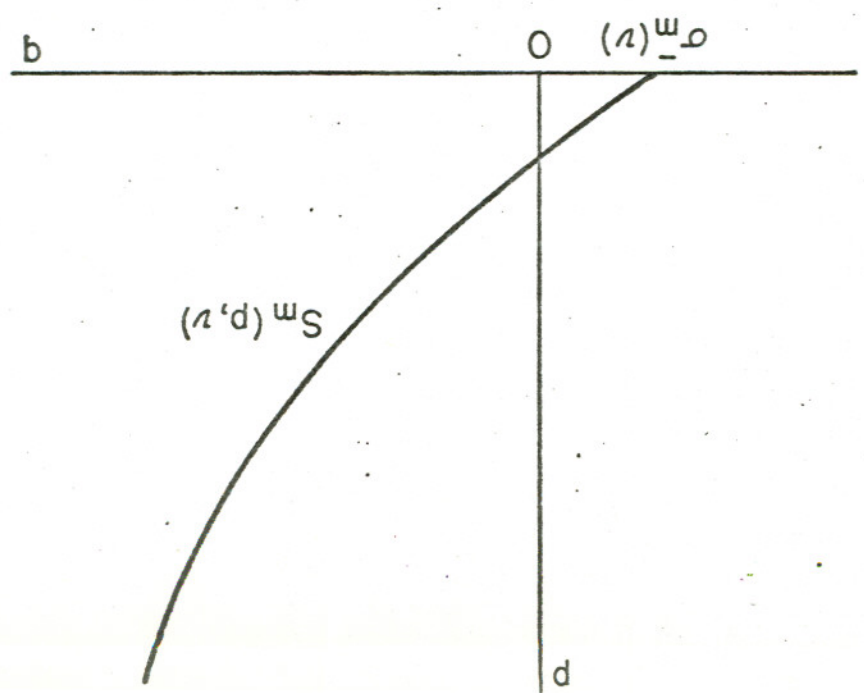


FIGURE 5

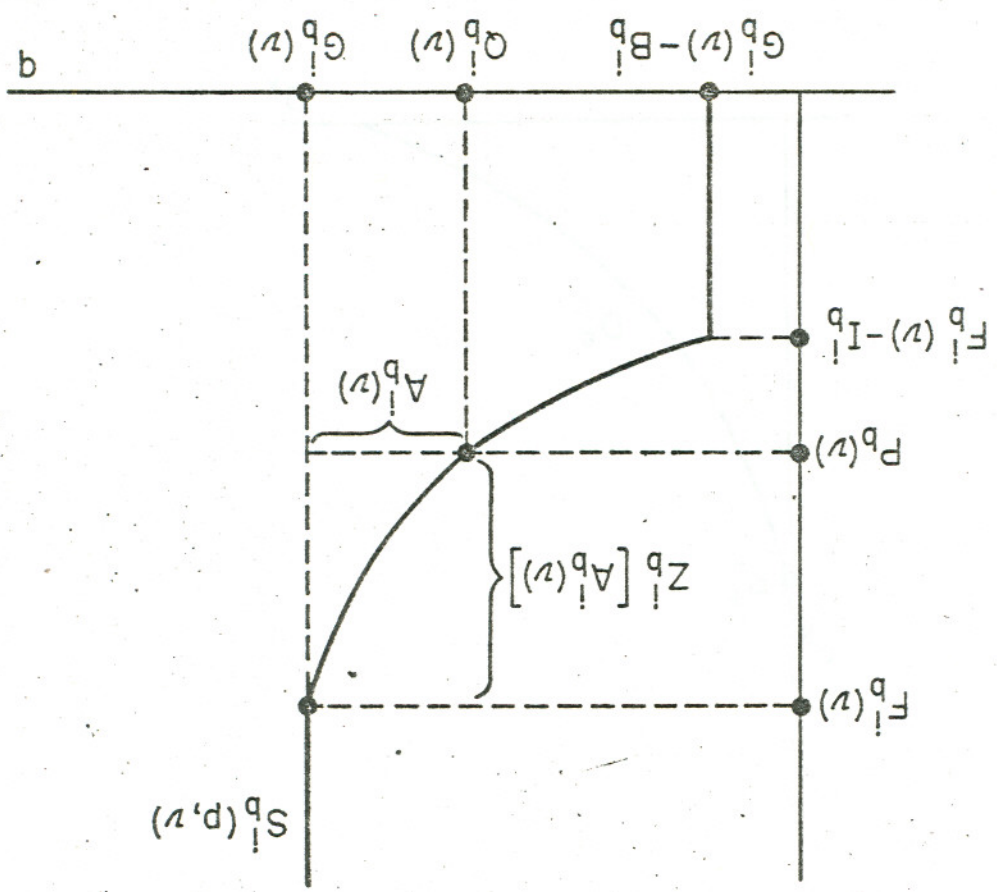


FIGURE 4

