

STATE UNIVERSITY OF NEW YORK AT
STONY BROOK

College of Engineering Technical Report
No. 365

A DYNAMIC TRANSPORTATION MODEL

A. H. Zemanian

This work was supported by the National Science Foundation
under Grant MCS 80-20386.

May 25, 1981

A DYNAMIC TRANSPORTATION MODEL*

A.H.Zemanian

State University of New York at Stony Brook

Abstract. A new and considerably different analysis of a transportation network is presented, wherein a commodity is supplied at a number of locations to shippers who bulk and transport it to various markets and then sell it to wholesalers. Most, if not all, of the prior analyses of commodity transportation networks use mathematical programming techniques to determine prices and commodity flows under optimal conditions. In contrast to this, the present work is a fully dynamic analysis. It models the supply and demand behaviors of the shippers and then aggregates those behaviors to obtain recursive equations that determine price and commodity-flow variations under possibly dynamic disequilibrium. The shippers base their shipment decisions upon price information and may store goods if prices appear to be unfavorable. It is shown that this dynamic system has a unique equilibrium state, which is asymptotically stable under certain conditions on the slopes of the various supply, demand, and storage schedules.

*This work was supported by the National Science Foundation under Grant MCS 80-20386.

Examined herein is the day-by-day transportation and distribution of a commodity from spatially dispersed points of supply to other spatially dispersed points of demand. The initial supply points are characterized by supply functions and the terminal demand points by demand functions. The transportation between these points is the business of a variety of shippers who act as monopsonists at each demand point (one shipper for each demand point) and are able to ship the commodity to any of the demand points. The principal distinction of the present work, as compared to prior studies of such transportation problems, is that a fully dynamic and adaptive model is constructed, one wherein the shippers individually decide from day to day how much of the commodity they will store and how much they will transport. The system need never be in equilibrium even though the shippers behave as profit-maximizing firms.

It appears that virtually all of the previous works on this topic use mathematical programming, primarily linear or quadratic programming, to determine how much of the commodity is transported from each supply point to each demand point and how much is stored over successive time periods. Thus, optimization techniques are used; for example, the cost of transportation and storage might be minimized or alternatively some welfare function might be maximized. Seminal works on this subject are Enke (1951), Hitchcock (1941), Koopmans (1948), Samuelson (1952), and Tramel and Seale (1959). A more recent and thorough exposition is provided by Takayama and Judge (1971). As a result of the optimization approach, the thrust of these prior works is toward

the determination of a spatial and temporal equilibrium analysis for the overall marketing system. They provide hardly any information about how the system would behave in disequilibrium, despite the fact that the marketing system as a whole may quite possibly remain in disequilibrium as the supply and demand functions keep varying with time. Moreover, even when the supply and demand functions remain fixed with time, these optimization models do not indicate how a transportation system that is initially in disequilibrium may adjust itself toward (or perhaps away from) the equilibrium state. In other words, the mathematical programming techniques are unsuitable for stability analyses.

One category of optimization analyses that take into account the fact that the exogenous determinants of the transportation system might be varying with time is adaptive or recursive programming; see Day (1973), Chapter 19 of Takayama and Judge (1971), and the references therein. The past and expected future supply and demand schedules are assumed. An objective function is then optimized to determine prices and commodity flows for future time periods, and the computed commodity flows are then implemented for the next time period. This process is repeated at each new time period as more information about supply and demand is received. In this way, time sequences in commodity flows are planned. However, the shippers, who actually decide what the commodity flows are to be, most probably do not plan their shipments in this way, especially if there are two or more shippers. For one reason, it is unlikely that they will know what the past and present supply and demand schedules facing them are, not to

mention their future schedules or the schedules other shippers in distant markets face. Usually, the only information about market conditions that is readily available concerns past clearance prices in the various wholesale terminal markets. Moreover, the shippers do not plan in unison for an optimal allocation of commodity flows in the entire transportation system. Instead, they individually try to maximize their profits using whatever information about past prices is available to them.

These are the considerations that motivate what we believe is a radically different analysis of a transportation system. We build our model on an atomistic foundation by treating each shipper as a profit-maximizing firm. We then aggregate their individual day-by-day activities to obtain an overall dynamic analysis of the entire transportation system.

This procedure is similar to that used in our prior works on periodic marketing networks (Zemanian 1979, 1980, 1981, in press, to appear), a system that is common in the rural underdeveloped areas of the world. Our present analysis concerns a system that is compatible with a developed industrialized economy, and thus there are significant differences. The principal one is the following. In our prior works we assumed that there was virtually no market news and that each trader confined his activities to a single pair of markets or perhaps to a single ring of markets. (This is not unusual in underdeveloped economies; see Jones (1968).) We now relax these assumptions. We assume that there is considerable, if not complete, information about past prices in the terminal markets and that each shipper customarily sends

goods to several, perhaps all of the terminal markets. Indeed, as price conditions change, each shipper generally alters the proportions of the goods he allocates to his various terminal markets and is in fact free to initiate or terminate a flow of goods to any terminal market in the system.

Another difference concerns the storage behavior of the shipper. In our prior works, we assumed that each shipper stored goods only in his terminal market. This is reasonable when that shipper has only one terminal market, for then he will be ready to use immediately his storage facility in accordance with the variations in his terminal price. However, when he is shipping to many terminal markets from a single initial market, it is also advantageous to store goods in this initial market, for those goods can then be allocated to whatever terminal markets offer the best price without necessitating a reshipment from other terminal markets. In this work we allow the shipper to store goods in any or all of his markets.

Still another difference concerns competition. As in our prior works, we assume herein that the terminal markets are perfectly competitive so that the shipper is a price taker in every one of his terminal markets. However, in contrast to our prior works, we assume that the shipper is a monopsonist in his initial market. (However, we did indicate at the end of Zemanian (to appear) how that model can be modified to allow the traders at the lowest levels to act as monopsonists or monopolists.) Such monopsonistic behavior might occur, for example, when the commodity is an agricultural good. Each shipper might then establish his buying operation in a different geographic location so

widely separated from the other locations that the farmers who sell to him find it impractical to sell to any other shipper.

1. The Transportation Network

Figure 1 illustrates the transportation network we shall discuss in this work. Each ϕ_m represents a location where many competing suppliers sell their goods to a single shipper, a monopsonist. The γ_k represent the terminal markets to which the goods are sent by the shippers. The ψ_j are the locations in the vicinity of the ϕ_m at which the shippers bulk goods and prepare them either for shipment or storage. We treat the ψ_j as hypothetical markets which determine how much the shippers store in the vicinity of the ϕ_m and how much they transport to the various γ_k . The arcs between these markets indicate the possible ways good may flow from the ϕ_m through the ψ_j to the γ_k .

We use the following numbering system for the indices m , j , and k : $m = 1, 2, \dots, M$; $j = M+1, M+2, \dots, J$; $k = J+1, J+2, \dots, K$. Since each ϕ_m has only one shipper and therefore only one adjacent ψ_j , we must have that $J = 2M$. Moreover, we number the ψ_j adjacent to a ϕ_m such that $j = m+M$. The advantage of this numbering system is that we can ascertain to which kind of location a variable such as a price pertains simply by noting its index. For example, P_m denotes a price in a ϕ market whereas P_k denotes a price in a γ market. This allows us to manage with fewer symbols, an advantage since there are so many variables in this work. *How 91* Other assumptions are the following. All the markets (the ψ_j as well) meet on all the marketing days. We designate the latter by the discrete time variable $t = \dots, -1,$

0, 1, A single price $P_m(t)$, $P_j(t)$, or $P_k(t)$ is established in each of the ϕ_m , ψ_j , or γ_k respectively for every t . At each t every shipper is fully informed of the prior prices $P_k(t-1)$, $P_k(t-2)$, ... in every γ_k . It simplifies our notation still further to assume that each shipper can ship goods to every γ_k , and for this reason an arc has been drawn in Figure 1 from each ψ_j to each γ_k . In the event that the last two assumptions do not hold - that indeed because of a gap in market news or perhaps some government regulation or some physical barrier - he cannot ship to a particular γ_k , we can accommodate this situation into our model by taking that shipper's minimum average variable cost T_{jk} of shipping goods from his ψ_j to that γ_k as exceedingly large, say, ∞ .

The reason we treat the ψ_j as nodes distinct from the ϕ_m , even though each ψ_j can be identified with exactly one ϕ_m , is the following. We have to devise some way of determining how much of the goods each shipper buys in his ϕ_m , how much he stores at ψ_j , and how much he ships to each of the γ_k . Our model for this purpose is based upon the theory of a profit-maximizing firm. On the one hand, it takes into account the costs to the shipper of buying, bulking, and preparing for shipment or storage the goods he has bought in ϕ_m . The marginal cost for doing so depends upon the amount of goods he is processing (i.e., transferring from ϕ_m to ψ_j). This cost coupled with the supply function in ϕ_m leads to a derived supply function at ψ_j . On the other hand, the marginal cost in shipping goods from ψ_j to a particular γ_k depends only upon the amount being shipped along this arc, not on the total amount being sent to all of

the γ markets. At least, we assume this is the case. So, in determining how the demand in a γ_k coupled with the cost of shipping from Ψ_j to γ_k is reflected as a derived demand in Ψ_j , we have to treat each such shipment separately. To this end, we use each Ψ_j as a hypothetical market where the derived supply function from ϕ_m is equated to the aggregate of the derived demand functions from the γ_k to establish a clearance price and the quantities stored and shipped to the various γ_k . The clearance price is taken to be the f.o.b. price at the supply locations. This is the basic idea of our paper. We now turn to details.

2. A Shipper's Transfer Supply Function and Derived Demand

A shipper can be viewed as a firm that supplies the service of transferring the ownership of goods over space and time. His supply schedule can be derived in the standard way from his marginal-cost and average-variable-cost curves. This in turn leads to his demand curve in the market at which he buys the goods he will be shipping. The derivation of all this has been presented in detail in Zemanian (1980, Section 2) and therefore will not be repeated here. We will merely explain its conclusion, namely, the derived-demand curve shown in Figure 2, where as usual p denotes price and q quantity.

Consider two markets between which a shipper transports goods. We let α_a denote the market where the shipper acquires goods and call it his initial market. Similarly, β_b will denote the market to which the shipper sends those goods; β_b will be referred to as his final market. In our model there is no more than one shipper operating between any two markets, and therefore we can specify

a shipper by calling him the $\alpha_a:\beta_b$ shipper. In fact, the index pair ab identifies that shipper.

The curve in Figure 2 is the shipper's demand schedule in his initial market α_a at time t . Thus, if at time t the price in α_a is $P_a(t)$, then the quantity the shipper acquires therein is $Q_a(t)$.

While operating in α_a at time t , the shipper maintains an expectation $E_{ab}(t)$ of a price that those goods will receive in his final market β_b at time $t+\tau_{ab}$, where τ_{ab} is the time it takes for goods to be shipped from α_a to β_b . That expected price $E_{ab}(t)$ is determined by a memory function M_{ab} of prior prices in the final market β_b :

$$(2.1) \quad E_{ab}(t) = M_{ab}[P_b(t-1), P_b(t-2), P_b(t-3), \dots]$$

That is, the shipper extrapolates from past prices to estimate a future price. Actually, M_{ab} may even depend upon past prices in other markets. However, we won't allow this generalization in our discussion even though most of the results of this paper extend readily in this way. We take M_{ab} to be monotonically increasing with respect to each of its arguments; moreover, if all those arguments have the same value, we assume that M_{ab} assigns that value to $E_{ab}(t)$.

T_{ab} is that critical expected price increment from α_a to β_b below which the shipper will not transport goods and above which he will. Thus, for $p > E_{ab}(t) - T_{ab}$ the shipper demands nothing in his initial market, whereas for $p < E_{ab}(t) - T_{ab}$ he has a positive demand. This demand function is symbolized by

$$(2.2) \quad q = V_{ab}[E_{ab}(t) - T_{ab} - p]$$

where $V_{ab}(x)$ is zero for $x \leq 0$, positive for $x > 0$, and continuous everywhere. In fact, an examination of the operating costs facing the shipper indicates that V_{ab} should have the step-like shape shown in Figure 2 (Zemanian, 1980, Section 2). Note that, as $E_{ab}(t)$ changes, the curve of Figure 2 shifts vertically by the same amount.

3. A Shipper's Storage Schedule and the Corresponding Supply Function

In this paper we will be working with two different models, one in which none of the shippers store goods and another in which at least some of them do. In the former case, each shipper takes whatever price he can get in the market to which he delivers goods; thus, his supply schedule therein is perfectly inelastic. In the latter case, which we now discuss, his supply schedule has some elasticity, for the less favorable the prices, the more he will store. This storage behavior has also been discussed in detail in Zemanian (1980, Section 4), and therefore in the present section we merely define symbols and explain the pertinent relationships.

We let $Z_{ab}(q)$ denote the per-unit marginal cost of storing the amount q of goods at β_b from one day to the next. For $0 < q \leq B_{ab}$, we assume that Z_{ab} is a continuous, positive, strictly monotonically increasing function with $Z_{ab}(B_{ab}) = I_{ab}$; here, B_{ab} denotes the maximum storage capacity at β_b available to the $a:\beta_b$ shipper, and I_{ab} denotes the per-unit marginal

cost of storing at maximum capacity. For $q > B_{ab}$, we set $Z_{ab}(q) = \infty$; this insures in effect that the $\alpha_a:\beta_b$ shipper will not store goods at β_b in excess of B_{ab} . Moreover, we allow $Z_{ab}(0+)$ to be either zero or positive.

Next, we define W_{ab} on the interval $Z_{ab}(0+) < p \leq I_{ab}$ as the inverse function of Z_{ab} ; that is, $W_{ab}(p) = q$ for $Z_{ab}(0+) < p \leq I_{ab}$ if and only if $p = Z_{ab}(q)$ for $0 < q \leq B_{ab}$. We then extend W_{ab} to all real values of p by setting $W_{ab}(p) = 0$ for $p \leq Z_{ab}(0+)$ and $W_{ab}(p) = B_{ab}$ for $p \geq I_{ab}$. This makes W_{ab} a continuous function of p .

The goods the $\alpha_a:\beta_b$ shipper has in storage at β_b may or may not be offered for sale by him depending upon the current price in β_b and the price he expects in β_b one day hence. In other words, the goods in storage provide a component of that shipper's supply schedule at β_b . More specifically, let $F_{ab}(t)$ be the price the $\alpha_a:\beta_b$ shipper expects to receive in β_b at $t+1$ while operating in β_b at t . We assume that $F_{ab}(t)$ is determined by a memory function N_{ab} of past prices in β_b :

$$(3.1) \quad F_{ab}(t) = N_{ab}[P_b(t-1), P_b(t-2), P_b(t-3), \dots]$$

We assign the same monotonicity and constancy properties to N_{ab} as we did to M_{ab} . Moreover, N_{ab} may also depend upon prices in other markets, but as with M_{ab} we won't allow this generalization. It follows that, if the current price $P_b(t)$ in β_b is less than $F_{ab}(t)$, the trader will store at least some goods rather than offering them for sale. Indeed, let $A_{ab}(t-1)$ be the amount the

$\alpha_a : \beta_b$ shipper has in storage from $t-1$ to t . Of this amount he will offer only the amount

$$(3.2) \quad A_{ab}(t-1) - W_{ab}[F_{ab}(t) - p]$$

for sale if the price in β_b is p because $W_{ab}[F_{ab}(t) - p]$ is the amount he will store in β_b from t to $t+1$. Thus, (3.2) is that component of the shipper's supply schedule in β_b at t resulting from his storage behavior. This is illustrated in Figure 3.

For p above $F_{ab}(t)$, the shipper offers all of $A_{ab}(t-1)$ for sale. On the other hand, at p_1 he offers for sale only the portion q_1 from the goods in storage, and, if p_1 is truly the clearance price, he stores

$$A_{ab}(t) = A_{ab}(t-1) - q_1$$

from t to $t+1$. At a still lower price p_2 , (3.2) may take on the negative value q_2 . This simply means that the shipper wishes to add the amount q_2 to his stored goods and exhibits this as a negative supply, that is, as a positive demand.

Subsequently, we will add (3.2) to another supply component arising from the goods the shipper brings into β_b at the beginning of day t to obtain his overall supply schedule in β_b at t ,

4. Dynamics in ϕ_m and Derived Supply in Ψ_j

The amount $U_m(t)$ of goods acquired by a shipper from his ϕ_m is determined by the standard theory of the monopsonist. This is illustrated in Figure 4. $S_m(p, t)$ is the exogeneously given aggregate supply function of all the suppliers to ϕ_m at t . For p and q related by this curve, the marginal cost $MC_m(q, t)$

to the shipper of acquiring q goods in ϕ_m at t is $p + qdp/dq$; the inverse function of MC_m with respect to q is denoted by $MC_m^{-1}(p, t)$. We assume that S_m is an increasing function of p . Consequently, MC_m is situated above S_m as indicated in Figure 4. The marginal cost of acquiring the amount $U_m(t)$ is $R_m(t)$, but the price paid to the suppliers is $P_m(t)$.

In accordance with Section 2, the shipper's demand function $D_m(p, t)$ in ϕ_m at t is given by (2.2). Now, however, we replace the expected price in Ψ_j by the f.o.b. price $P_j(t)$ at which Ψ_j clears. $P_j(t)$ is determined by the dynamics in Ψ_j , a matter we discuss in the next section. Assume for the moment that $P_j(t)$ is known. This locates $D_m(p, t)$ within Figure 4. The resulting intersection of D_m with MC_m determines $U_m(t)$ and $R_m(t)$, and the vertical projection onto S_m determines $P_m(t)$. In symbols, we have

$$(4.1) \quad D_m(p, t) = V_{mj}[P_j(t) - T_{mj} - p]$$

so that

$$(4.2) \quad U_m(t) = MC_m^{-1}[R_m(t), t] = V_{mj}[P_j(t) - T_{mj} - R_m(t)]$$

and

$$(4.3) \quad U_m(t) = S_m[P_m(t), t].$$

With $P_j(t)$ given, (4.2) implicitly determines $R_m(t)$ and thereby $U_m(t)$. Then, (4.3) implicitly determines $P_m(t)$.

This construction determines a relationship between corresponding pairs $p = P_j(t)$ and $q = U_m(t)$, which we denote by $q = X_j(p, t)$. X_j serves as a derived supply function in Ψ_j .

In view of the assumed shapes of D_m and MC_m , X_j has the form indicated in Figure 5.

Before leaving this section, we should comment on the shipper's ability to determine the amount $U_m(t)$ he will buy for a given $P_j(t)$ in accordance with this construction. To do so, he must know what $MC_m(q, t)$ is, but this may not be the case. It appears to be more realistic to assume that the shipper simply tries to act optimally as a monopsonist. In this case, we should replace MC_m by what the shipper believes it to be. $U_m(t)$ is then determined from the given $P_j(t)$ by the intersection of D_m with the shipper's estimate of MC_m . He then pays the price dictated by (4.3) in order to acquire his desired $U_m(t)$. This may alter the derived supply function X_j , but otherwise our analysis remains the same.

5. Dynamics in Ψ_j

In addition to the derived supply function X_j , we assign to the shipper the supply function arising from his storage behavior at Ψ_j , as was explained in Section 3. This is displayed in Figure 5 by the curve labelled

$$(5.1) \quad A_{mj}(t-1) - W_{mj}[F_{mj}(t) - p],$$

where $F_{mj}(t)$ is the f.o.b. price in Ψ_j the shipper expects to have at time $t+1$:

$$(5.1) \quad F_{mj}(t) = N_{mj}[P_j(t-1), P_j(t-2), P_j(t-3), \dots]$$

The sum of the two supply functions yields the total supply function in Ψ_j at t :

$$(5.3) \quad S_j(p, t) = X_j(p, t) + A_{mj}(t-1) - W_{mj}[F_{mj}(t) - p]$$

On the other hand, as was indicated in Section 3, the shipper has a derived demand function (see Figure 2) for each of his shipments to the γ_k . We obtain the shipper's aggregate demand function $D_j(p, t)$ in Ψ_j by summing over all k :

$$(5.4) \quad D_j(p, t) = \sum_k V_{jk}[E_{jk} - T_{jk} - p]$$

Here,

$$(5.5) \quad E_{jk}(t) = M_{jk}[P_k(t-1), P_k(t-2), P_k(t-3) \dots].$$

Upon equating (5.3) to (5.4), we get an implicit equation for the shipper's f.o.b. price $P_j(t)$:

$$(5.6) \quad S_j[P_j(t), t] = D_j[P_j(t), t]$$

Then, the substitution of $p=P_j(t)$ into the various supply and demand functions determines the various amounts of goods shipped or stored. In particular,

$$(5.7) \quad U_{jk}(t) = V_{jk}[E_{jk}(t) - T_{jk} - P_j(t)]$$

is the amount shipped out of Ψ_j at t toward γ_k , and

$$(5.8) \quad U_j(t) = \sum_k U_{jk}(t)$$

is the total amount shipped to all the γ_k . As was stated before,

$$(5.9) \quad U_m(t) = X_j[P_j(t), t]$$

is the amount processed for shipping or storage at time t .

Furthermore,

$$(5.10) \quad A_{mj}(t) = W_{mj} [F_{mj}(t) - P_j(t)]$$

is the amount stored from t to $t+1$. Also, if $A_{mj}(t-1) - A_{mj}(t)$ is positive (negative), it is the amount taken out of storage (respectively, added to storage) at t .

Let us reiterate why this determination of prices and commodity flows reflects reasonable behavior on the part of a shipper who wishes to maximize profits. We invoke standard marginal analysis, but a complication arises in our dynamic model in that the shipper cannot know what prices his goods will fetch when they reach their destinations; he can only work with expected prices. Now, the price dictated by the demand function D_j is the marginal per-unit revenue the trader expects to receive for shipping goods from his warehouse in Ψ_j to a γ_k , for it is the expected price the goods fetch in the γ_k minus the marginal cost incurred in shipping those goods from Ψ_j to the γ_k . This expected marginal revenue decreases as the quantity shipped out of Ψ_j increases. (Note also that by hypothesis the γ markets function under perfect competition with many shippers; Therefore, $E_{jk}(t)$ does not depend upon the amount of goods being shipped from Ψ_j to γ_k .) On the other hand, the price dictated by the derived supply function X_j is the marginal per-unit cost of buying goods in ϕ_m and making them available for shipment from Ψ_j . Moreover, the price dictated by the storage supply function (5.1) is the expected per-unit opportunity cost of not storing the marginal good in Ψ_j . Both of these costs increase as the quantity supplied in Ψ_j increases. By standard marginal analysis, under the indicated forms for our supply and

demand functions (we precisely define these forms in Section 7 below), profit maximization occurs where the net marginal revenue on an increment of shipment is zero. So, the shipper maximizes his expected profit from his operation in Ψ_j and ϕ_m by processing the amount of goods dictated by the clearance equation (5.6).

6. Dynamics in γ_k

We assume that in each terminal market γ_k there is perfect competition between many local buyers and a variety of shippers bringing goods into γ_k from the various Ψ_j . The aggregate demand function of all the local buyers in γ_k at time t is denoted by $D_k(p, t)$ as in Figure 6. In our model it is assumed to be given exogeneously. On the other hand, each shipper who transports goods out of Ψ_j and into γ_k is characterized by a supply function like that of Figure 3 with however the following alterations. The subscripts ab are replaced by jk . The shipper has on hand in γ_k at t not only the amount $A_{jk}(t-1)$ of goods that were in storage from $t-1$ but also the amount

$$(6.1) \quad U_{jk}(t-\tau_{jk}) = V_{jk} [E_{jk}(t-\tau_{jk}) - T_{jk} - P_j(t-\tau_{jk})]$$

that he just transported into γ_k . Hence, his total supply curve is shifted horizontally to the right by the amount (6.1) to give a supply curve like the one in Figure 3 except that $A_{ab}(t-1)$ is replaced by $A_{jk}(t-1) + U_{jk}(t-\tau_{jk})$. Furthermore, the price $F_{jk}(t)$ he expects in γ_k at $t+1$ is given by the memory function

$$(6.2) \quad F_{jk}(t) = N_{jk} [P_k(t-1), P_k(t-2), P_k(t-3), \dots].$$

In symbols, that shipper's supply function in γ_k at t is

$$(6.3) \quad S_{jk}(p, t) = A_{jk}(t-1) + V_{jk} [E_{jk}(t-\tau_{jk}) - T_{jk} - P_j(t-\tau_{jk})] \\ - W_{jk} [F_{jk}(t) - p]$$

The aggregate supply function of all the shippers in γ_k at t is therefore

$$(6.4) \quad S_k(p, t) = \sum_j S_{jk}(p, t).$$

The clearance price $p = P_k(t)$ in γ_k at t is the solution of

$$(6.5) \quad S_k(p, t) = D_k(p, t),$$

and the total amount exchanged is

$$(6.6) \quad Q_k(t) = S_k[P_k(t), t] = D_k[P_k(t), t].$$

This is illustrated in Figure 6. Finally, $S_{jk}[P_k(t), t]$ is the amount the $\Psi_j: \gamma_k$ shipper sells in γ_k at t and

$$(6.7) \quad A_{jk}(t) = W_{jk} [F_{jk}(t) - P_k(t)]$$

is the amount that shipper stores in γ_k from t to $t+1$.

It can happen that the price $P_k(t)$ is so low that, for some of the shippers with high expected prices, $P_k(t)$ is situated as is p_2 in Figure 3. Those shippers will then buy goods from shippers with lower expected prices and will add those goods to their storage from t to $t+1$.

7. Assumptions and the Model

We now gather together and explicate all the assumptions we impose on the various characteristics and functions of our model.

Conditions I. For each fixed t , $S_m(p, t)$ is an exogeneously given, continuous, nonnegative function on $0 \leq p < \infty$ such that $S_m(p, t) = 0$ for $0 \leq p \leq P_m^-(t)$ and $S_m(p, t)$ is strictly increasing for $P_m^-(t) \leq p < \infty$. Moreover, $MC_m^{-1}(p, t)$ exists, is continuous on $0 \leq p < \infty$, and is strictly increasing on $P_m^-(t) \leq p < \infty$.

(It follows that, for each $p > P_m^-(t)$, $MC_m^{-1}(p, t) < S_m(p, t)$.)

Conditions II. For each fixed t , $D_k(p, t)$ is an exogeneously given, continuous, nonnegative function on $0 < p < \infty$ such that, as $p \rightarrow 0+$, $D_k(p, t) \rightarrow \infty$. It is strictly decreasing for $0 < p \leq P_k^+(t)$ and is equal to zero for $P_k^+(t) \leq p < \infty$.

Conditions III. V_{ab} is a Lipschitz continuous, nonnegative function on the real line such that $V_{ab}(x) = 0$ for $x \leq 0$, $V_{ab}(x)$ is strictly increasing for $0 \leq x < \infty$, and $V_{ab}(x)$ tends to a finite limit as $x \rightarrow \infty$.

(The Lipschitz continuity of V_{ab} was used in the proof of Theorem 2 of Zemanian (1979), a result we invoke in Section 9 below.)

Conditions IV. W_{ab} is a continuous, nonnegative, increasing function on the real line such that $W_{ab}(x) = 0$ for $x \leq 0$ and $W_{ab}(x) = B_{ab}$ for $x \geq I_{ab} > 0$.

(W_{ab} need not be strictly increasing on $0 \leq x \leq I_{ab}$. In fact, we allow W_{ab} to be identically zero on the entire real line. This would occur when the shipper has no storage facility at his final market.)

Conditions V. The memory functions M_{ab} and N_{ab} possess the following properties. If $P_b(t)$ remains constant for all $t \leq t_0$, then M_{ab} and N_{ab} assign the same constant to their corresponding expected prices at t_0 . Moreover, M_{ab} and N_{ab} are strictly increasing functions of each of their arguments.

Conditions VI. Every T_{ab} is a nonnegative number.

Conditions VII. Every τ_{ab} is a positive integer.

We assume henceforth that all the Conditions I through VII and all the assumptions stated in Section 1 hold. Our model for a transportation system then consists of Figure 1 in conjunction with the equations of Sections 2 through 6.

8. Recursive Analysis

We can use our model to compute all the prices, commodity flows, and stored amounts for $t = 1, 2, 3, \dots$ once an appropriate set of initial conditions is assigned. This will yield a fully dynamic analysis of our transportation system.

As the initial conditions, assume that the following are specified: All the amounts $A_{mj}(0)$ and $A_{jk}(0)$ stored between $t = 0$ and $t = 1$. All the f.o.b. prices $P_j(t)$ for $t = 1 - \tau_{jk}, 2 - \tau_{jk}, \dots, 0$ and for all j and k . All the f.o.b. prices for $x \leq 0$ that appear in the arguments of the memory function N_{mj} at $t = 1, 2, 3, \dots$. All the γ_k market prices $P_k(x)$ for $x \leq 0$ that appear in the arguments of the memory functions M_{jk} at $t = 1 - \tau_{jk}, 2 - \tau_{jk}, \dots, 0, 1, 2, \dots$. All the γ_k market prices $P_k(x)$ for $x \leq 0$ that appear in the arguments of the memory functions N_{jk} at $t = 1, 2, 3, \dots$. (This listing may be redundant.)

Furthermore, assume that all the supply functions $S_m(p, t)$ and demand functions $D_k(p, t)$ are given for $t = 1, 2, 3, \dots$ and that all the T_{mj} , T_{jk} , and τ_{jk} are also given. Then, to determine all the prices and commodity flows for $t = 1$ and the amounts stored from $t = 1$ to $t = 2$, proceed as follows.

From $S_m(p, t)$ derive $MC_m^{-1}(p, t)$ and then derive $X_j(p, 1)$ for every m and j , in accordance with Section 4. This in turn determines $S_j(p, 1)$ through (5.2) and (5.3). Then, determine $D_j(p, 1)$ by means of (5.4) and (5.5). The intersection between S_j and D_j determines $P_j(t)$ according to (5.6). Equation (5.7) then yields the $U_{jk}(1)$, (5.9) gives $U_m(1)$, and (5.10) gives $A_{mj}(1)$. Also, (4.2) determines $R_m(1)$ and (4.3) determines $P_m(1)$. Next, use (6.2), (6.3), and (6.4) to determine every $S_k(p, 1)$. Then, the solution of (6.5) for p gives $P_k(1)$, and (6.6) then yields $Q_k(1)$. Finally, $A_{jk}(1)$ is given by (6.7). At this point we have updated by one unit of time all the initial conditions indicated above. So, we can repeat these computations for $t = 2$. Continuing in this way, we get a complete dynamic analysis for $t = 1, 2, 3, \dots$.

9. Equilibrium

Assume that all $S_m(p, t)$ and all $D_k(p, t)$ are fixed with respect to time t . We say that our transportation system is in an equilibrium state when every price within it does not vary with t . It follows that all commodity flows will be constant too. As we shall show later on, the amounts in storage will all be zero.

We alter our notation now by dropping the arguments in t . Thus, for example, P_j will now denote a price rather than a mapping $t \mapsto P_j(t)$ of the time axis into the price axis.

It is a fact that our transportation system has a unique equilibrium state. To show this, we first consider the special case where the shippers do not have storage facilities, namely

the no-storage model. In this case, all the supply functions arising from goods in storage (see Figure 3) are absent.

In particular, (5.3) has only the X_j term on its right-hand side, and (6.3) only the V_{jk} term. Moreover, by Conditions V the memory function M_{jk} assumes the constant price of its arguments, and so $E_{jk} = P_k$. Consequently, the equations governing our no-storage model in an equilibrium state are the following.

In ϕ_m ,

$$(9.1) \quad U_m = MC_m^{-1}(R_m) = V_{mj}(P_j - T_{mj} - R_m) = S_m(P_m).$$

In Ψ_j ,

$$(9.2) \quad U_m = X_j(P_j) = \sum_k V_{jk}(P_k - T_{jk} - P_j).$$

In Υ_k ,

$$(9.3) \quad \sum_j V_{jk}(P_k - T_{jk} - P_j) = D_k(P_k).$$

Note that, according to these equations, the interactions in the ϕ_m make themselves felt in the Ψ_j and thereby in the Υ_k only through the derived supply functions X_j .

Now, the subnetwork of Figure 1 consisting of all the Ψ_j and Υ_k and the arcs between them is the same as the two-level system considered in Zemanian (1979). Moreover, (9.2) and (9.3) are the same as those equations in Zemanian (1979) (namely, Equations (13) and (14) with $L_j \equiv 0$) governing the equilibrium state discussed in that work. Because of Conditions I and III and the way X_j is constructed from the curves S_m , MC_m , and V_{mj} , X_j is a continuous nonnegative function on $0 \leq p < \infty$,

equal to zero on $0 \leq p \leq P_m^- + T_{mj}$, and strictly increasing on $P_m^- + T_{mj} \leq p < \infty$. These are precisely the conditions imposed on the S_j functions of Zemanian (1979) except for the requirement that $\lim_{p \rightarrow \infty} S_j(p) = \infty$, which is now neither needed nor imposed. Furthermore, V_{jk} and D_k also satisfy the conditions imposed in Zemanian (1979). So, we can invoke Theorems 2 and 3 of that paper to conclude that the stated subnetwork has a unique equilibrium state. But, once the P_j are specified and fixed with respect to t , (9.1) and (9.2) uniquely determine U_m , R_m , and P_m as fixed values. So, we can also conclude that our entire no-storage transportation system has a unique equilibrium state.

We turn now to our storage model, wherein at least some of the shippers store goods under appropriate conditions. Observe that in an equilibrium state, if it exists, all prices are constant, and therefore $F_{mj} = P_j$ and $F_{jk} = P_k$. This means that each trader operates at the upper corner point of Figure 3 and hence does not store goods. To put it another way, if any trader stores goods, the transportation system cannot be in an equilibrium state. Thus, we need merely seek an equilibrium state in which storage does not happen. But, this corresponds to the no-storage model, which, as we have already concluded, has a unique equilibrium state. We have established the following.

Theorem. When all the supply functions S_m and demand functions D_k are fixed with respect to time, our transportation system has a unique equilibrium state, whether or not the shippers are able to store goods. In that equilibrium state, no storage occurs.

Another consequence of our arguments is that Equations (9.1) through (9.3) also hold for the equilibrium state of our storage model.

10. Asymptotic Stability

Continue to assume that all the S_m and D_k are independent of time t . We shall now show that the equilibrium state is asymptotically stable if certain qualitative conditions on the slopes of the various supply and demand functions are satisfied. First note that, according to the dynamic equations of Sections 4 through 6, conditions in the ϕ_m make themselves felt in the ψ_j and thereby in the γ_k only through the derived supply functions X_j , which are also fixed with respect to time in this analysis of asymptotic stability. Because of this, we can again approach our problem by first examining the subnetwork consisting of the ψ_j , the γ_k , and the arcs between them. If we can show that, after a small displacement from equilibrium values, the P_j and P_k converge back to their equilibrium values, it will follow from (4.2) and (4.3) (see also Figure 4) that the P_m will do the same.

We base our examination of the incremental behavior of the transportation system around its equilibrium state on the total differentials of the clearance equations for the various markets. We shall tacitly assume that the supply, demand, and memory functions occurring in this section have first-order Taylor's expansions whose remainders tend toward zero faster than the increments in their arguments under, say, the euclidean norm, this being true in neighborhoods of the equilibrium values. Let X_j' be the derivative of X_j evaluated at the equilibrium value of its

argument P_j ; a similar meaning is assigned to $W_{mj}^!$, $W_{jk}^!$, $V_{jk}^!$, and $D_k^!$. On the other hand, let $M_{jk}^{(\nu)}$ be the partial derivative of (5.5) with respect to its ν th argument evaluated at the equilibrium-state value of P_k (that is, the equilibrium value of P_k is substituted for every one of the arguments $P_k(t-1)$, $P_k(t-2)$, $P_k(t-3)$, ...). (We will not need any notation for the partial derivatives of N_{mj} and N_{jk} .) We assume henceforth that the shippers prognosticate about future prices from only a finite number of past prices. In particular, only the prices $P_k(t-\nu)$, where $\nu = 1, \dots, r$, are taken to be the arguments of M_{jk} .

We now adopt still another quite reasonable assumption concerning storage costs: The per-unit cost of storing any amount, no matter how small, is no less than a certain positive amount. Since we are treating the amount q in storage as a continuous variable and since Z_{ab} is monotonic increasing, this condition can be expressed by the requirement that $Z_{ab}(0+) > 0$. In terms of W_{ab} , it can also be expressed by the following condition, which we impose throughout this section and in Section 12 below.

Condition IVa. W_{ab} is identically zero in a neighborhood of the origin.

This implies that all $W_{mj}^!$ and $W_{jk}^!$ are zero. It also implies that small variations in the prices from their equilibrium values will not induce nonzero storage amounts, as can be seen from (5.10) and (6.7). To put this another way, since the amounts stored depend only upon prices and not upon prior stored amounts, the amounts in storage remain zero for small perturbations around the equilibrium state.

To proceed, consider the clearance equation (5.6) for a Ψ_j . Substitute (5.2) through (5.5) into (5.6), set the $A_{mj}(t-1)$ equal to zero, take total differentials treating prices as the independent variables, and finally set all W'_{mj} equal to zero in accordance with Condition IVa. This yields

$$(10.1) \quad dP_j(t) = \left[X'_j + \sum_k V'_{jk} \right]^{-1} \sum_k V'_{jk} \left[M_{jk}^{(1)} dP_k(t-1) + \dots + M_{jk}^{(r)} dP_k(t-r) \right]$$

Next, we turn to a γ_k market and derive similar equations in total differentials. Substitute (5.5), (6.2), (6.3), and (6.4) into (6.5) with $p = P_k(t)$, set all $A_{jk}(t-1)$ equal to zero, take total differentials, and then set the W'_{jk} equal to zero. Solve the result for $dP_k(t)$:

$$(10.2) \quad \begin{aligned} dP_k(t) = & (D'_k)^{-1} \sum_j V'_{jk} \left[M_{jk}^{(1)} dP_k(t-\tau_{jk}-1) + \dots \right. \\ & \left. \dots + M_{jk}^{(r)} dP_k(t-\tau_{jk}-r) - dP_j(t-\tau_{jk}) \right] \end{aligned}$$

As the next step, let us define a vector $w(t)$ whose components are the differentials of the prices in the Ψ_j and γ_k :

$$w(t) = \left[dP_{M+1}(t), \dots, dP_J(t), dP_{J+1}(t), \dots, dP_K(t) \right]^T$$

The superscript T denotes matrix transpose. In terms of this notation, (10.1) and (10.2) can be rewritten together in the form

$$(10.3) \quad w(t) = \Gamma_1 w(t-1) + \Gamma_2 w(t-2) + \dots + \Gamma_s w(t-s)$$

where s is the maximum of all the $\tau_{jk} + r$ occurring in (10.2). Also, the Γ_μ , where $\mu = 1, \dots, s$, are square matrices of the same order as the vector $w(t)$. We have hereby linearized our

nonlinear dynamic equations around the equilibrium state. As before, we are tacitly assuming that our system is sufficiently well-behaved to allow such an incremental analysis.

Clearly, if all the entries of all the Γ_μ are sufficiently small, then from any initial choice of the vectors $w(0)$, $w(-1)$, \dots , $w(1-s)$, the sequence generated by (10.3) will tend to zero under any vector norm. This means that, so long as the coefficients of the price differentials occurring in the right-hand sides of (10.1) and (10.2) are sufficiently small, the equilibrium state of our transportation network will be asymptotically stable.

By examining all these coefficients, we can easily deduce qualitative conditions which lead to suitably small coefficients. They are the following: In absolute values, all $X_j^!$ and $D_k^!$ are to be large enough and all $V_{jk}^!$ and $M_{jk}^{(\nu)}$ are to be small enough. In terms of slopes measured with respect to the q axis, this means that the X_j and D_k should be flat enough and the $V_{jk}^!$ should be steep enough. Moreover, each of the M_{jk} should be sufficiently insensitive to variations in its arguments. The latter means that shippers do not overreact to variations in past prices.

That $V_{jk}^!$ is small means that the $\Psi_j : \gamma_k$ shipper either is operating at a large profit margin because of highly favorable prices or has cut off his operation between Ψ_j and γ_k because of unfavorable prices. This is because of the assumed step shape of the V_{jk} curve. This conclusion seems reasonable since shippers who are operating at small profit margins may radically

alter the amounts they ship as prices vary around the critical values separating profitable from losing operations.

The flatness of D_k coincides with one of the conditions for stability in the classical cobweb (Ezekial, 1938): the demand function should be elastic. On the other hand, the flatness of X_j contradicts the cobweb's other condition, namely, that the supply function should be inelastic. The explanation for this apparent discrepancy with a classical result is the following. If there were only one Ψ_j and one γ_k and no storage facilities, if $\tau_{jk} = 1$, if the memory function (5.5) was replaced by $E_{jk}(t) = P_b(t-1)$ as in Ezekial's cobweb model, and if we were only interested in the variations in $P_k(t)$ but not in those of $P_j(t)$, then our transportation network would be equivalent to a single market because Ψ_j could be incorporated into γ_k by eliminating $P_j(t)$ from the dynamic equations. In fact, the radically simplified forms of (10.1) and (10.2) could now be combined to give

$$(10.4) \quad dP_k(t) = \frac{(D_k^!)^{-1}}{(X_j^!)^{-1} + (V_{jk}^!)^{-1}} dP_k(t-2)$$

In this simple case, asymptotic stability is assumed if $D_k^!$ is large enough and $X_j^!$ and $V_{jk}^!$ are small enough in absolute values. This now agrees with the classical cobweb conditions so far as $D_k^!$ and $X_j^!$ are concerned.

The reason the latter condition on $X_j^!$ is not appropriate for our more complicated transportation network having many Ψ_j and γ_k markets is the following. First of all, in addition to

the variations in $P_k(t)$, we are indeed interested in the variations in $P_j(t)$. Moreover, a variation in the price at one of the terminal markets induces variations in the prices $P_j(t)$ in the Ψ_j markets, as is indicated by (10.1). The latter in turn induce price disturbances in other terminal markets according to (10.2). Now, by (10.1) the variations in $P_j(t)$ can be moderated by making $X_j^!$ large, which will in turn dampen the lateral transmission of price disturbances. In short, it is these two effects that lead our present analysis to a conclusion regarding $X_j^!$ different from that of the classical cobweb. Note also that asymptotic stability is still assured for the simple system of (10.4) when $X_j^!$ is large so long as $D_k^!$ remains large and $V_{jk}^!$ remains small. Thus, our two conclusions do not contradict each other.

The last note also applies to our multimarket transportation system. That is, not all of the aforementioned conditions need hold simultaneously. If some of them hold sufficiently strongly, then the entries of the Γ_μ matrices may remain sufficiently small even when the remaining conditions are relaxed to some extent. For example, $V_{jk}^!$ will not be small for an equilibrium price P_j close to and somewhat below $P_k - T_{jk}$, and the latter may very well be the case in equilibrium for at least some j and k . Nevertheless, the remaining conditions may hold so strongly that the Γ_μ entries remain adequately small to insure asymptotic stability.

Since X_j is only a derived supply curve, we should also investigate how the size of $X_j^!$ is affected by conditions in ϕ_m . Upon eliminating $R_m(t)$ in the two equations of (4.2) while noting

that MC_m does not depend on t in the present analysis, then taking total differentials (as before, a prime will denote a first derivative evaluated at an equilibrium-state value), and finally solving for $dU_m(t)$ in terms of $dP_j(t)$, we find the coefficient of $dP_j(t)$ to be

$$X_j^m = \left[(V_{mj}')^{-1} + MC_m' \right]^{-1}.$$

We would like X_j^m to be large. This will be the case if V_{mj}' is large and MC_m' is small. That MC_m' is small implies that the supply functions S_m in the ϕ_m are elastic. That V_{mj}' is large implies that V_{mj} is elastic too, but the latter will not be the case at large profit margins if V_{mj} also has a step shape. However, the step form of our V curves comes from the limited capacity of the shipper to handle goods. Once that capacity is approached, the costs of handling goods increases rapidly. This phenomenon may be more pronounced for the V_{jk} than for the V_{mj} . For instance, if the shipper transports goods by truck from Ψ_j to the γ_k and if he has only a limited number of trucks, then his marginal transporting costs will increase rapidly once his transport capacity is reached. On the other hand, such a capacity restriction may be less stringent for his processing of goods from ϕ_m to Ψ_j since no transport limitation is now involved. Hence, V_{mj} may very well have less of a step shape and may indeed be flat well beyond the shipper's transport capacity.

11. The Propagation of Disturbances

As compared to a periodic marketing system in which there is very little market news and poor connectivity between the markets, modern transportation networks can respond much more

rapidly to disturbances in supply and demand. This will especially be the case if every shipper is informed about all market prices and can ship to every terminal market. However, it may happen that because of some impediment, such as a physical barrier or government intervention or inadequate market news, some of the traders may not be able to ship goods directly to some of the terminal markets. This would be represented in Figure 1 by the deletion of some of the arcs between the ψ_j and the γ_k . Our model would still apply and the analysis given above would carry directly over to this case. We could, for example, simply set T_{jk} equal to ∞ for each ψ_j, γ_k pair that is not connected by an arc. Then, all our other notations would apply unaltered. In this case, Jones' (1968) step-by-step

propagation of a disturbance, which is characteristic of periodic marketing networks, would now arise once again. In particular, for two markets ψ_j and γ_k not connected by an arc, a disturbance would propagate from ψ_j to γ_k only by means of the trading activities along paths through other markets. In such a situation, the transportation network would react more sluggishly to a disturbance than it would were it completely connected. An analysis of Jones' step-by-step propagation of disturbances has been given in Zemanian (1979) and will not be repeated here.

12. Multiperiod Storage

Up to now, we have assumed that the shipper plans his storage by estimating the price in each of his markets one day hence and then compares his expected marginal opportunity cost of not storing a good to his expected marginal opportunity cost of not

shipping the good and then chooses that action whose negation yields the largest opportunity cost. However, the shipper may have information from which he can estimate expected prices for two or more future market days. In this case, it would be reasonable for the shipper to plan his storage for several days into the future. He can do so by computing the expected marginal opportunity costs of not storing goods for each of the storage time spans under consideration and then choosing that action whose negation has the largest opportunity cost.

To be specific, consider the case where the $\alpha_a : \beta_b$ shipper expects the price in β_b n days hence to be $F_{ab}(n, t)$, n being a positive integer. Assume that he also knows what is the per-unit marginal cost $Z_{ab}(n, \rho)$ of storing the amount ρ for n days. Let $W_{ab}(n, x)$ be the inverse function of $Z_{ab}(n, \rho)$ with respect to ρ [i.e., $\wedge_{0 < \rho \leq B_{ab}}, x = Z_{ab}(n, \rho)$ if and only if $\rho = W_{ab}(n, x)$]; for the sake of simplicity, let us assume that neither Z_{ab} nor W_{ab} have discontinuities when $0 < \rho \leq B_{ab}$]. Extend the definition of W_{ab} by setting $W_{ab}(n, x) = 0$ for $x \leq Z_{ab}(n, 0+)$ and $W_{ab}(n, x) = I_{ab}(n) = Z_{ab}(n, B_{ab})$ for $x \geq I_{ab}(n)$. Finally, assume that, for each fixed n , $W_{ab}(n, x)$ satisfies Conditions IV and IVa, and, for each fixed x , $W_{ab}(n, x)$ is a (not necessarily strictly) decreasing function of n .

When the amount in storage is ρ , the per-unit opportunity cost of not storing the marginal good is $F_{ab}(n, t) - Z_{ab}(n, \rho)$. The shipper should choose that n (or one of the optimal values of n) for which this opportunity cost is a maximum. That marginal good should then be stored for n days in the event that storage is more profitable than shipping or selling.

Instead of keeping p fixed, we could reason in terms of a fixed price p . Now, because Z_{ab} and W_{ab} are monotone increasing, the shipper should choose that n for which

$$A_{ab}(t-1) - W_{ab}[n, F_{ab}(n, t) - p]$$

is a minimum. This implies that the shipper's storage supply function in β_b at t should be

$$(12.1) \quad A_{ab}(t-1) - \max_n W_{ab}[n, F_{ab}(n, t) - p].$$

This is the expression that should replace (3.2) when multiperiod storage is allowed. Under this generalization, our analysis continues as before. The theorem on the existence and uniqueness of the equilibrium state holds once again. So too does our analysis of asymptotic stability. Indeed, at equilibrium all the $F_{ab}(n, t)$ take on the same value P_b . Moreover, if the $W_{ab}(n, x)$ are strictly decreasing functions of n for every sufficiently small value of x , then we need merely examine the derivatives of the memory functions for the $F_{ab}(1, t)$, and the analysis of Section 10 remains virtually unchanged.

13. Discounting Future Costs and Income

Another extension of our model concerns the fact that up to now we have not discounted future costs and income. This is justifiable from the fact that we are dealing with very short transport and storage periods, a matter of days in fact. However, it is easy to introduce discounting. If the daily interest rate is i , then any cost or income accruing τ days hence should be multiplied by the discount factor $(1 + i)^{-\tau}$. Thus, for example, in (2.2) $E_{ab}(t)$ should be multiplied by $(1 + i)^{-\tau_{ab}}$

and in (12.1) $F_{ab}(n, t)$ should be multiplied by $(1 + i)^{-n}$. Similarly, T_{ab} , V_{ab}^{-1} , and $Z_{ab}(n, \cdot)$ should be discounted costs or cost functions.

REFERENCES

- Day, Richard H. 1973, "Recursive programming models: a brief introduction," in Studies in Economic Planning over Space and Time, G.G.Judge and T.Takayama (Editors), North-Holland Publishing Co., Amsterdam, pp. 329-344.
- Enke, S. 1951, "Equilibrium among spatially separated markets: solution by electric analogue," Econometrica, vol. 19, pp. 40-47.
- Ezekial, M. 1938, "The cobweb theorem," Quarterly Journal of Economics, vol. 52, pp. 255-280.
- Hitchcock, F.L. 1941, "Distribution of a product from several sources to numerous localities," Journal of Mathematics and Physics, vol. 20, pp. 224-230.
- Jones, W.O. 1968, "The structure of staple food marketing in Nigeria as revealed by price analysis," Food Research Institute Studies, Stanford University, vol.VIII, pp. 95-123.
- Koopmans, T.C. 1948, "Optimal utilization of the transport system," Econometrica, vol. 16, pp. 66-68.
- Samuelson, P.A. 1952, "Spatial price equilibrium and linear programming," American Economic Review, vol. 42, pp. 283-303.
- Takayama, T., and G.G.Judge 1971, Spatial and Temporal Price and Allocation Models, North-Holland Publishing Co., Amsterdam.

Tramel, T.E., and A.D.Seale, Jr. 1959, "Reactive programming of supply and demand relations - applications to fresh vegetables," Journal of Farm Economics, vol. 41, pp. 1012-1022.

Zemanian, A.H. 1979, "Two-level periodic marketing networks without market news," Journal of Mathematical Analysis and Applications, vol. 68, pp. 509-525.

Zemanian, A.H. 1980, "Two-level periodic marketing networks wherein traders store goods," Geographical Analysis, vol. 12, pp. 353-372.

Zemanian, A.H. 1981, "A dynamic economic model of periodic marketing rings," Geographical Analysis, vol. 13, pp. 165-180.

Zemanian, A.H. (in press), "Equilibrium and stability in a periodic marketing ring," SIAM Journal on Algebraic and Discrete Methods.

Zemanian, A.H. (to appear), "An interzonal marketing network."

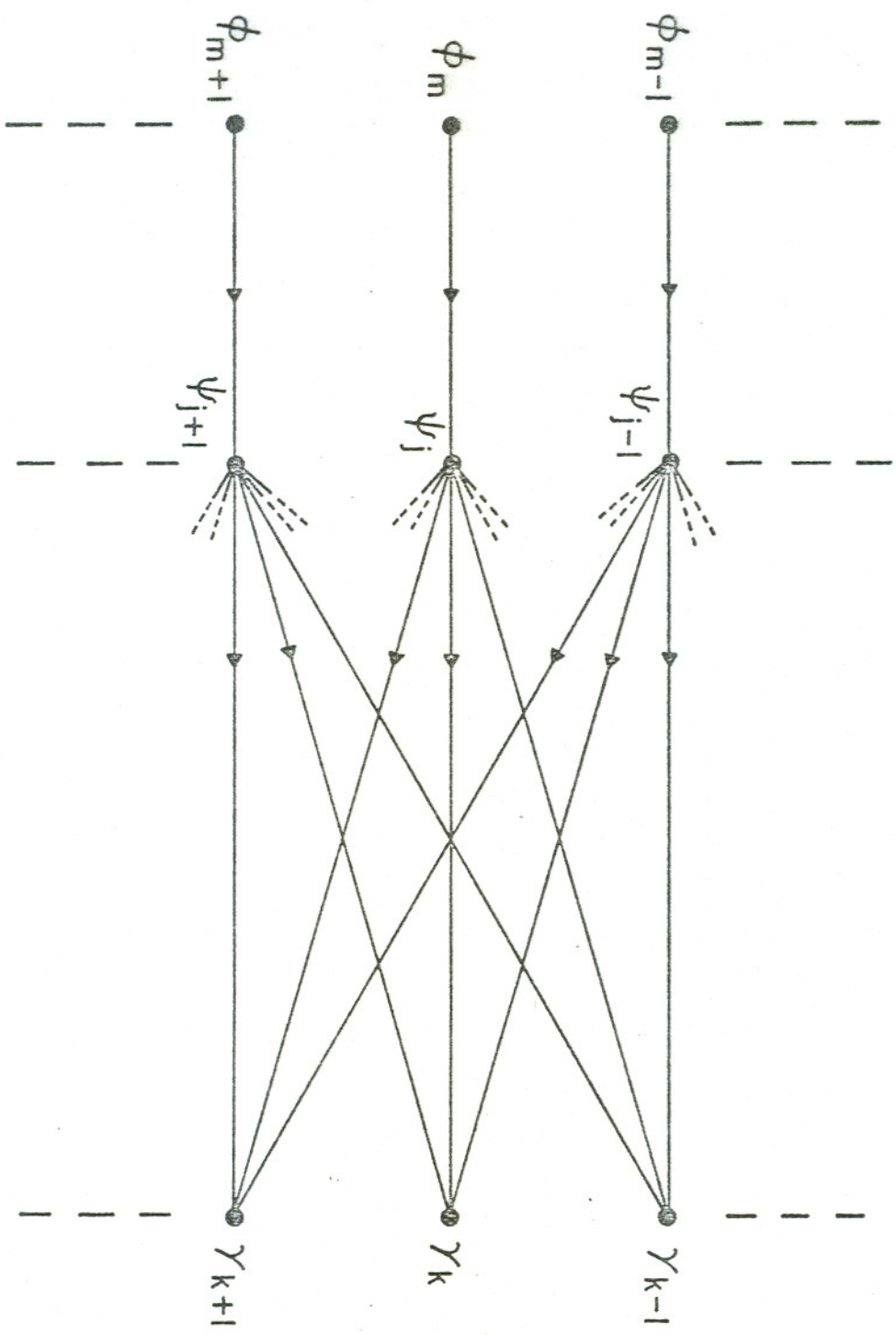


Figure 1

Figure 1

Figure 2

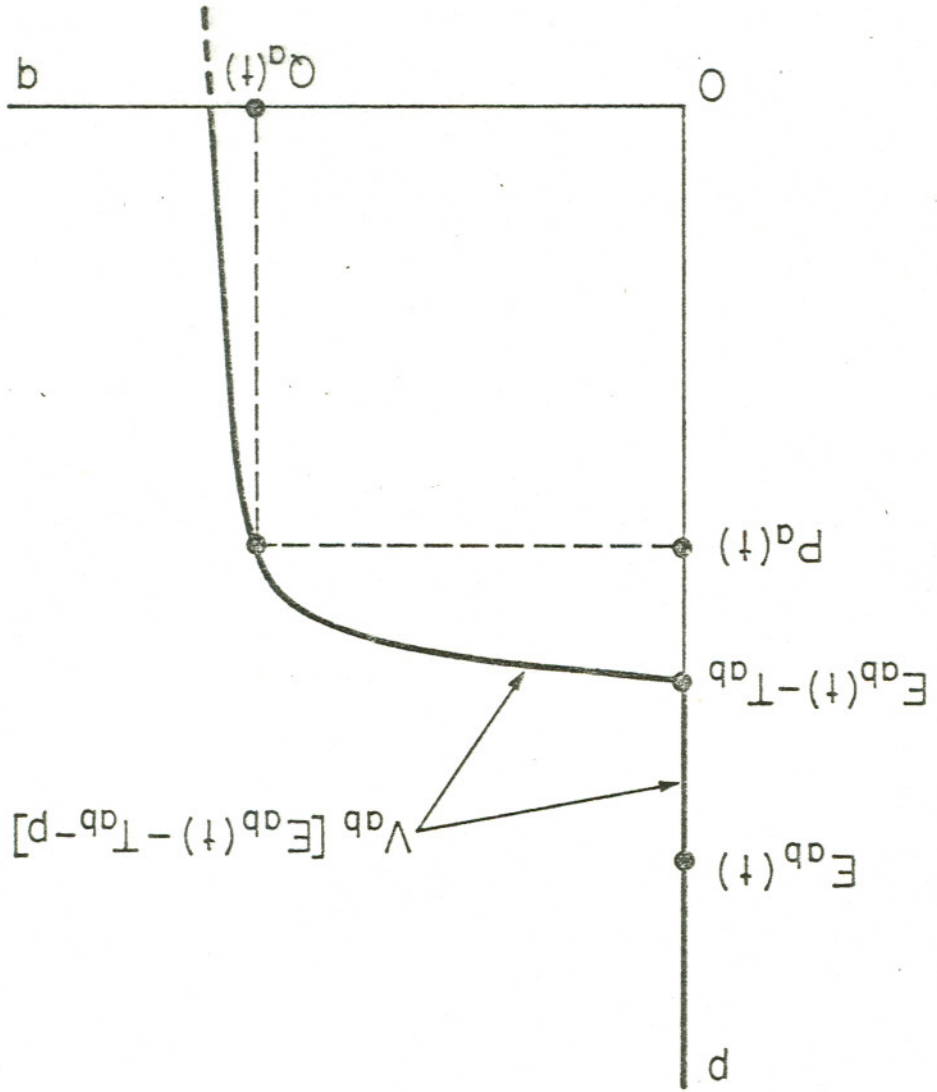
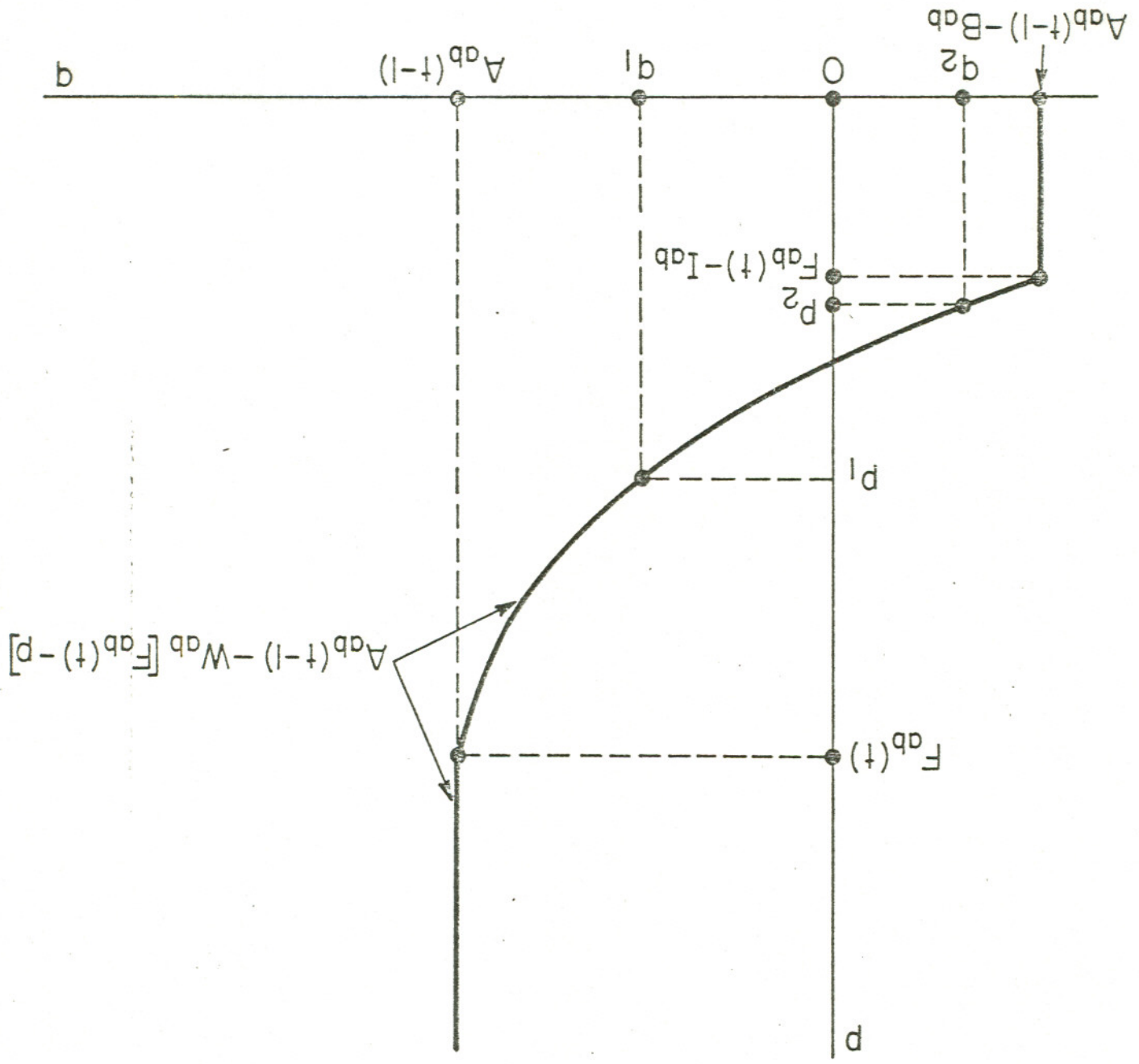


Figure 3



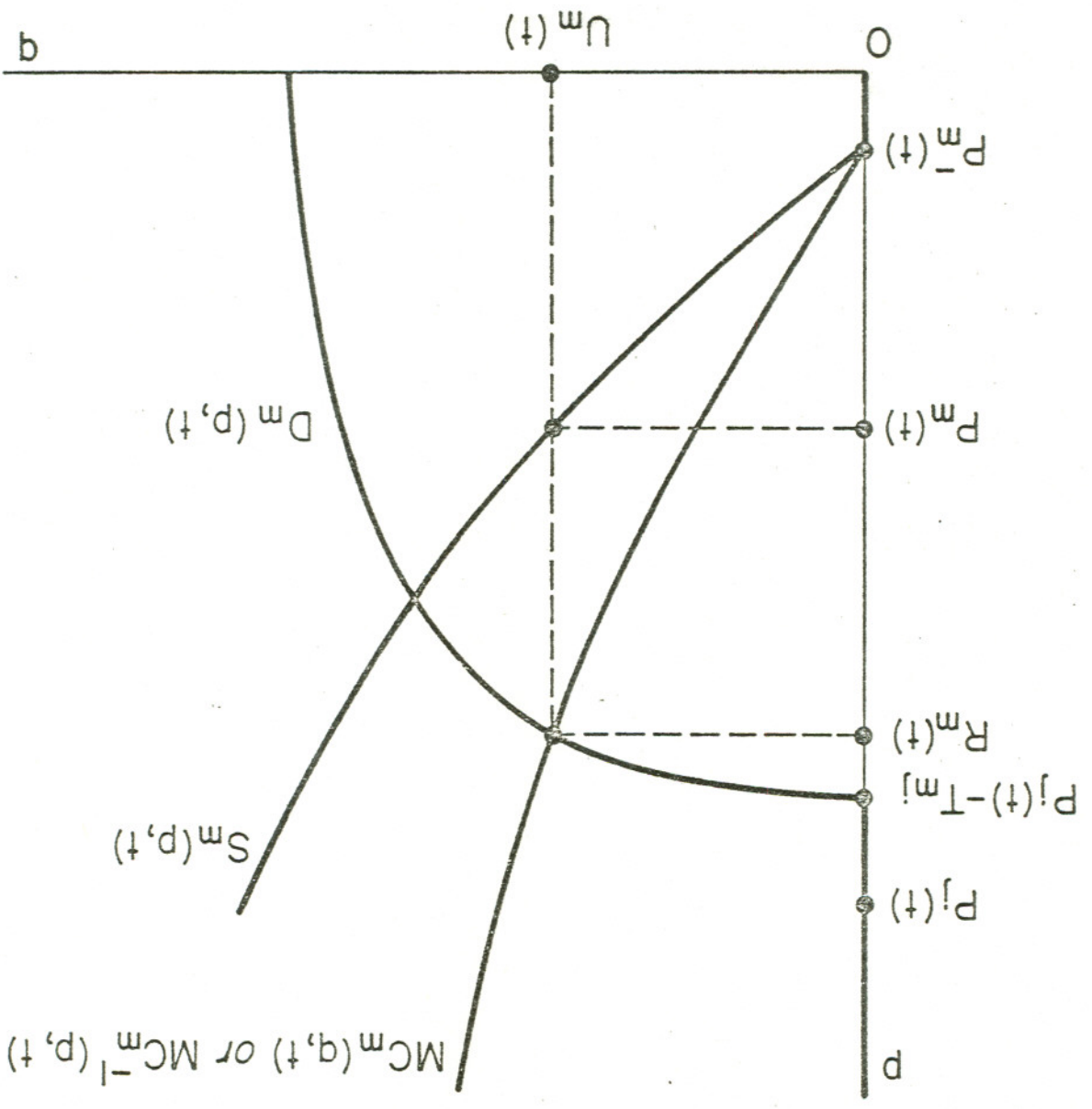


Figure 4

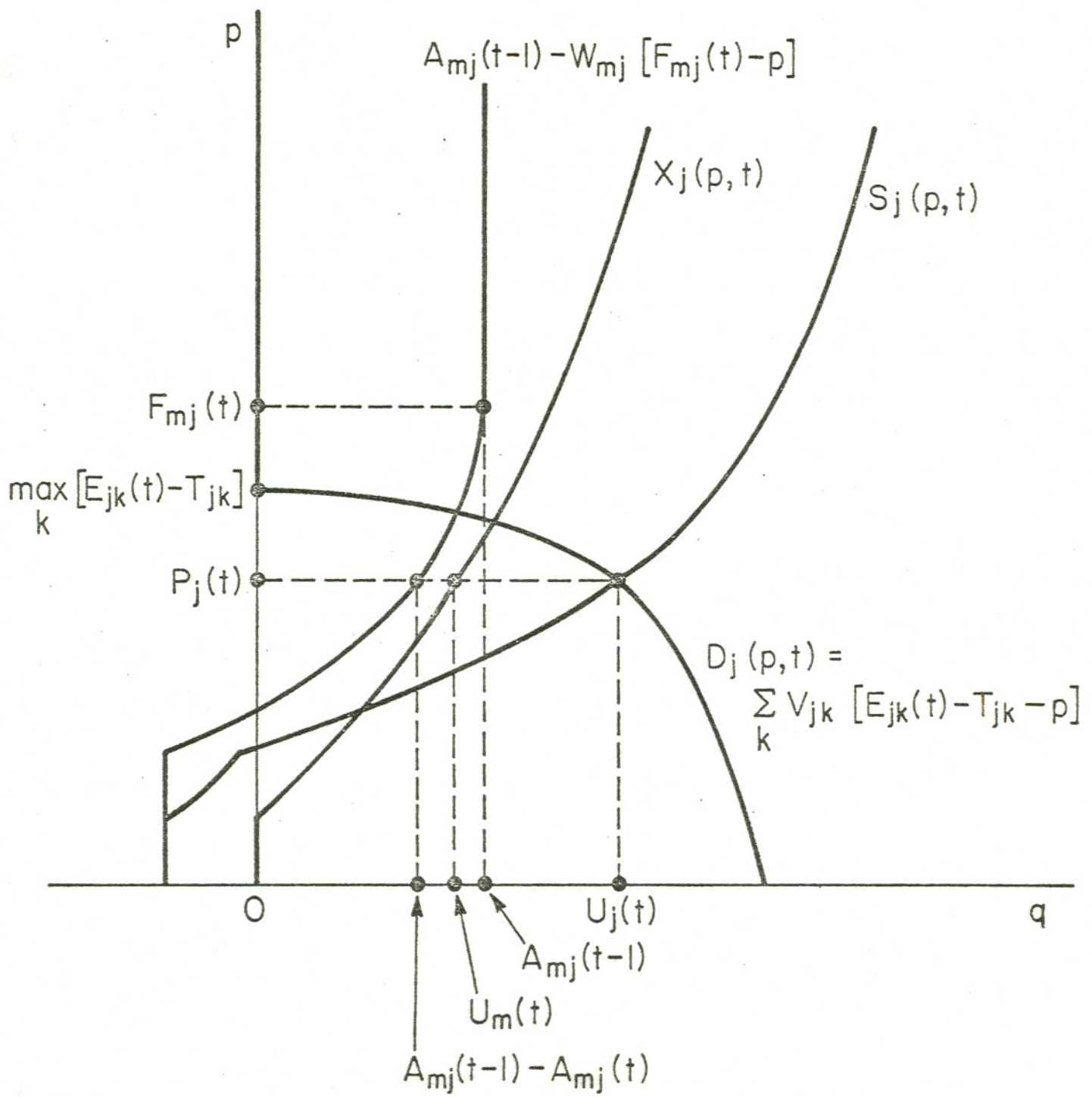


Figure 5

Figure 6

