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THE DISTRIBUTIONAL HANKEL TRANSFORMATION

by

A. H. Zemanian

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Prepared for

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ABSTRACT

The Hankel transformation is generalized in a distributional way, something that apparently has not been done before. Two different procedures are used to accomplish this. In the first procedure a topological linear space of testing functions is constructed for which the μ^{th} order Hankel transformation is a topological automorphism. The dual space consists of the μ^{th} order Hankel-transformable distributions. The distributional Hankel transformation is then defined by generalizing a variation of Parseval's formula. It turns out that the distributions to which this transformation may be applied must be of slow growth.

The second procedure yields a more general result in that there is no restriction on the rate of growth of the distributions that are to be transformed. Here again, Parseval's formula is used to define the generalized Hankel transformation, but in contrast to the previous case the testing functions for the distributions under consideration are required to have bounded supports. The Hankel transforms then turn out to be continuous linear functionals on certain classes of analytic functions.

Several applications to differential equations containing Bessel-type differential operators are also given.

LIST OF PREVIOUS PUBLICATIONS PRODUCED UNDER CONTRACT AF19(629)-2981

SCIENTIFIC REPORTS:

1. A. H. Zemanian, " A Time-Domain Characterization of Rational Positive-Real Matrices." First Scientific Report, AFCRL-63-390, College of Engineering Tech. Rep. 12, State University of New York at Stony Brook; August 5, 1963.

2. A. H. Zemanian, "A Time-Domain Characterization of Positive-Real Matrices." Second Scientific Report, AFCRL-63-391, College of Engineering Tech. Rep. 13, State University of New York at Stony Brook; August 16, 1963.

3. A. H. Zemanian, "The Time-Domain Synthesis of Distributions." Third Scienfific Report, AFCRL-64-191, College of Engineering Tech. Rep. 19, State University of New York at Stony Brook; February 1, 1964.

4. A. H. Zemanian, "The Distributional Laplace and Mellin Transformations." Fourth Scientific Report, AFCRL-64-685, College of Engineering Tech. Rep. 26, State University of New York at Stony Brook; August 15, 1964.

5. A. H. Zemanian, "Orthonormal Series Expansion of Certain Distributions and Distributional Transform Calculus." Fifth Scientific Report, AFCRL-64-995, College of Engineering Tech. Rep. 22, State University of New York at Stony Brook; November 15, 1964.

6. T. Laughlin, "A Table of Distributional Mellin Transforms." AFCRL-65-645, College of Engineering Tech. Rep. 40, State University of New York at Stony Brook; June 15, 1965.

FINAL REPORT:

A. H. Zemanian, "Application of Generalized Function Theory to Network Realizability Theory and Time Domain Synthesis of Positive-Real Functions," AFCRL-65-316, College of Engineering Tech. Rep. 41, State University of New York at Stony Brook; May 31, 1965.

PAPERS:

1. A. H. Zemanian, "The Time-Domain Synthesis of Distributions." Proceedings of the First Allerton Conference on Circuit Theory, University of Illinois; 1963.

2. A. H. Zemanian, "The Approximation of Distributions by the Impulse Responses of RLC Two-ports." Proceedings of the International Conference of Microwaves, Circuit Theory, and Information Theory; Tokyo; September, 1964.

3. A. H. Zemanian, "The Time-Domain Synthesis of Distributions." IEEE Transactions on Circuit Theory, Vol. CT-11, pp. 487-493; December, 1964.

4. A. H. Zemanian, "A Characterization of the Inverse Laplace Transforms of Rational Positive-Real Functions." J. Soc. Indust. Appl. Math., accepted for publication.

5. A. H. Zemanian, "Some Convergence Properties of Exponential Series Expansions of Distributions." J. Math. Anal. Appl., accepted for publication.

6. H. Konig and A. H. Zemanian, "Necessary and Sufficient Conditions for a Matrix Distribution to Have a Positive-Real Laplace Transform." J. Soc. Indust. Appl. Math., accepted for publication.

7. A. H. Zemanian, "The Distributional Laplace and Mellin Transformations." J. Soc. Indust. Appl. Math., accepted for publication.

8. A. H. Zemanian, "Inversion Formulas for the Distributional Laplace Transformation." J. Soc. Indust. Appl. Math., accepted for publication.

9. A. H. Zemanian, "Orthonormal Series Expansion of Certain Distributions and Distributional Transform Calculus," J. Math. Anal. Appl., accepted for publication.

1. Introduction.

In this work we extend the Hankel transformation to distributions, something that apparently has not been done before. On the other hand, it should be noted that Ditkin and Prudnikov [1], [2] have developed a Mikusinski-type operational calculus that is suitable for the operator. This report is divided into two parts. In the first part our method is in the spirit of L. Schwartz's extension of the Fourier transformation to tempered distributions [3; Vol.II], whereas is the second part we obtain a more general result by a procedure that is similar to the Ehrenpreis [4] and Gelford-Shilov [5] extension of the Fourier transformation to all distributions.

We shall make use of the following classical results concerning the Hankel transformation h_μ. The first is an inversion theorem [6; p. 52], [7; p. 456]. As is customary, $L_p(a,b)$ denotes the space of (equivalent classes of) locally integrable functions f(t) is absolutely integrable on $a<x<b$.

<u>If</u> $f(x) \in L_1(0,\infty)$, if $f(x)$ <u>is of bounded variation in a neighborhood of the point</u> $x=x_0$, <u>if</u> $\mu \geq -\frac{1}{2}$, <u>and if</u>

$$F(y) = h_\mu f = \int_0^\infty f(x)\sqrt{xy}\; J_\mu(xy)\, dx, \tag{1-1}$$

<u>where as usual</u> J_μ <u>denotes the Bessel function of first kind and order</u> μ , <u>then</u>

$$\frac{1}{2}[f(x_0+0) + f(x_0-0)] = h_\mu^{-1} F = \int_0^\infty F(y)\sqrt{x_0 y}\; J_\mu(x_0 y)\, dy \tag{1-2}$$

Egs. (1-1) and (1-2) define respectively the (ordinary) direct and inverse μ th-order Hankel transformations, which are self-reciprocal ($h_\mu = h_\mu^{-1}$). Another useful result is Parseval's formula:

If f(x) and G(y) are in L_1 $(0,\infty)$, if $\geq -\frac{1}{2}$, and if F(y) and g(x) are respectively the direct and inverse μ th-order Hankel transforms of f(x) and G(y), then

$$\int_0^\infty f(x)\, g(x)\, dx = \int_0^\infty F(y)\, G(y)\, dy \tag{1-3}$$

This form of Parseval's formula is easy to prove. Other conditions under which (1-3) holds are given by Macaulay-Owen [8].

Finally, we have a generalization of a Paley-Wiener theorem due to Griffith [9], whose result has been extended by Raman Unni [10] and by Gergen, Burlak and Dressel [11].

Let $\eta = y+iw$ and $\mu \geq -\frac{1}{2}$. A function $\phi(\eta)$ has the representation

$$\Phi(\eta) = \int_0^b \sqrt{\eta x}\; J_\mu(\eta x)\, a(x)\, dx$$

with $b>o$ and $a(x) \in L_2$ (o,b) if and only if $\Phi(y) \in L_2$ $(0,\infty)$, $\eta^{-\mu-\frac{1}{2}}\Phi(\eta)$ is an even entire function of η, and there exists a constant C such that

$$|\phi(\eta)| < C e^{b|w|}$$

for all η.

Our procedure for generalizing the Hankel transformation in the first part of this report is as follows. We construct a topological linear space $\mathcal{H}_\mu$ of testing functions on $0<x<\infty$ for which the μ th-order Hankel transformation h_μ is a topological automorphism when $\mu \geq -\frac{1}{2}$. The dual space to $\mathcal{H}_\mu$ consists of the Hankel-transformable distributions. These distributions turn out to be of slow growth as $x \to \infty$. The μ th-order distributional Hankel transformation is then defined by generalizing Parseval's equation (1-3).

The procedure in the second part of the report is to construct a testing function space B_μ of those functions in $\mathcal{H}_\mu$ that are identically zero for all sufficiently large x. Taking the μ th-order Hankel transforms of the elements of B_μ, we obtain a certain space Y_μ of analytic functions. It is found that h_μ is a topological isomorphism from B_μ onto Y_μ. The dual space B'_μ is a space of distributions on $0<x<\infty$, which behave as do the distributions in $\mathcal{H}'_\mu$ as $x \to 0+$ but have no restriction on their behavior as $x \to \infty$. By using once again an analogue to Parseval's equation (1-3) to define the μ th-order distributional Hankel transformation on B'_μ, we find that Y'_μ is the space of such Hankel transforms.

Our generalizations of the Hankel transformation are useful in solving distributional differential equations that contain the operator

$$x^{-\mu-\frac{1}{2}} D\, x^{2\mu+1} D\, x^{-\mu-\frac{1}{2}} = D^2 - \frac{4\mu^2-1}{4x^2}$$

or the Bessel operator

$$D^2 + x^{-1}D - x^{-2}\mu^2,$$

as we shall see.

In this work, x, y, and ω shall denote real one-dimensional variables; x will always be restricted to the open interval $(0, \infty)$. This interval will be denoted by I. The parameter μ is allowed to assume any real value, However, whenever we deal with the Hankel transformation, we shall require that μ be no less than $-\frac{1}{2}$.

By a smooth function we mean a function that possesses (ordinary) derivatives of all orders at all points of its domain. We shall also use the notation $D = D_x = d/dx$. At times, we shall denote a singular distribution f by f(x) merely to indicate that the testing functions, on which f is defined, have x as their independent variable. Whenever we speak of an isomorphism or automorphism, we shall mean a topological isomorphism or topological automorphism.

At times we shall deal with multivalued functions of the complex variable $\eta = y + i\omega$ that are analytic except for branch points at $\eta = 0$ and $\eta = \infty$. We shall always restrict such functions to their principal branches by requiring that $-\pi < \arg \eta \leq \pi$.

$\mathcal{D}_I$ denotes the space of smooth functions whose supports are contained in I. We assign to it the topology that makes its dual $\mathcal{D}'_I$ the space of Schwartz distributions on I [3; Vol.I, p. 65]. The number that $f \in D'_I$ assigns to $\phi \in D_I$ is denoted by $\langle f, \phi \rangle$. Moreover, we assign to D'_I the weak topology generated by the seminorms

$$\rho_\phi(f) = |\langle f, \phi \rangle| \qquad (\phi \in D_I).$$

PART I

THE HANKEL TRANSFORMATION OF CERTAIN DISTRIBUTIONS OF SLOW GROWTH

2. The Testing Function Space $\mathcal{H}_\mu$.

For each real number μ, we define a testing function space $\mathcal{H}_\mu$ as follows. A complex-valued function $\varphi(x)$ is in $\mathcal{H}_\mu$ if and only if it is defined and smooth on $0<x<\infty$ and, for each pair of nonnegative integers m and k, the number

$$\gamma^{\mu}_{m,k}(\varphi) = \sup_{0<x<\infty} \left| x^m (x^{-1}D)^k \left[x^{-\mu-\frac{1}{2}} \varphi(x) \right] \right| \qquad (2\text{-}1)$$

exists (i.e., is finite). Here

$$(x^{-1}D)^k = x^{-1} Dx^{-1}D \ldots x^{-1}D$$

where there are k differentiations. It follows through an inductive argument that, for each k, $D^k\varphi$ is of rapid descent as $x \to \infty$; that is, $x^m D^k \varphi \to 0$ as $x \to \infty$ for every m.

$\mathcal{H}_\mu$ is a linear space over the field of complex numbers. We assign a topology to it by taking $\gamma^{\mu}_{m,k}(\varphi)$ as its seminorms. Alternatively, we could use the following seminorms, which generate the same topology in $\mathcal{H}_\mu$.

$$\rho^{\mu}_{r}(\varphi) = \max_{\substack{0\le k\le r \\ 0\le m\le r}} \gamma^{\mu}_{m,k}(\varphi) \qquad (r=0,1,2,\ldots) \qquad (2\text{-}2)$$

Note that $\gamma^{\mu}_{0,0} = \rho^{\mu}_{0}$ is a norm so that both (2-1) and (2-2) constitute a separating system of seminorms [12]. We shall say that a sequence $\{\varphi_\nu\}_{\nu=1}^{\infty}$ converges in $\mathcal{H}_\mu$ if each $\varphi_\nu \in \mathcal{H}_\mu$ and, for every m and k, $\gamma_{m,k}^{\mu}(\varphi_\nu - \varphi_\kappa) \to 0$ (or equivalently, for each r, $\rho_r^{\mu}(\varphi_\nu - \varphi_\kappa \to 0)$, as ν and κ tend to infinity independently.

Clearly, D_I is a subspace of $\mathcal{H}_\mu$ and convergence in D_I implies convergence in $\mathcal{H}_\mu$.

Lemma 2-1: <u>If q is an even positive integer, then $\mathcal{H}_{\mu+q} \subset \mathcal{H}_\mu$ and the topology of $\mathcal{H}_{\mu+q}$ is stronger than the topology induced on it by $\mathcal{H}_\mu$.</u>

$$(x^{-1}D)^k \; (x^{-\mu-\frac{1}{2}}\varphi) = 2k\,(x^{-1}D)^{k-1}\,(x^{-\mu-\frac{5}{2}}\varphi) + x^2\,(x^{-1}D)^k\,(x^{-\mu-\frac{5}{2}}\varphi)$$

This shows that

$$\gamma_{m,k}^{\mu}(\varphi) \leq 2k\,\gamma_{m,k-1}^{\mu+2}(\varphi) + \gamma_{m+2,k}^{\mu+2}(\varphi).$$

..us, our lemma follows by induction on q.

Lemma 2-2: <u>$\mathcal{H}_\mu$ is sequentially complete.</u>

Proof: Let $\{\varphi_\nu\}_{\nu=1}^{\infty}$ converge in $\mathcal{H}_\mu$. By the seminorms $\gamma_{m,0}^{\mu}$ this sequence converges uniformly on every compact subset of I. Moreover,

$$(x^{-1}D)^k\,(x^{-\mu-\frac{1}{2}}\varphi_\nu) = x^{-\mu-\frac{1}{2}}\left[a_{k0}\frac{\varphi_\nu}{x^{2k}} + a_{k,1}\frac{D\varphi_\nu}{x^{2k-1}} + \dots + a_{kk}\frac{D^k\varphi_\nu}{x^k}\right.$$

where the a's denote constants. From the seminorms $\gamma_{m,k}^{\mu}$ it follows by induction that, for each k, $\{D^k \varphi_\nu\}_{\nu=1}^{\infty}$ converges uniformly on every compact subset of I. We can conclude that there exists a smooth function φ on $0<x<\infty$ such that

$$\lim_{\nu\to\infty} x^m (x^{-1}D)^k \; x^{-\mu-\frac{1}{2}} \varphi_\nu = x^m (x^{-1}D)^k \, x^{-\mu-\frac{1}{2}} \varphi ,$$

where the convergence is uniform on $0<x<\infty$, again because of the seminorms $\gamma_{m,k}^{\mu}$. Hence, $\varphi \in \mathcal{H}_\mu$. Q. E. D.

$\mathcal{O}$ will denote a space of multipliers for $\mathcal{H}_\mu$: A function θ (x) defined on I is in $\mathcal{O}$ if and only if it is smooth and for each nonnegative integer ν there is an integer n_ν such that

$$(1+x^{n_\nu})^{-1} \; |(x^{-1}D)^\nu \theta| \tag{2-4}$$

is founded on I. Note that the product of two functions in $\mathcal{O}$ is also in $\mathcal{O}$. For every fixed μ, the operation $\varphi \to \theta\varphi$ is a continuous linear mapping of $\mathcal{H}_\mu$ into $\mathcal{H}_\mu$. Indeed, let B_k be a bound on (2-4) for all ν =0, 1, ...,k. Then,

$$(x^{-1}D)^k x^{-\mu-\frac{1}{2}} \theta\varphi = \sum_{\nu=0}^{k} \binom{k}{\nu} [(1+x^{n_\nu})^{-1}(x^{-1}D)^\nu\theta][(1+x^{n_\nu}) (x^{-1}D)^{k-\nu} x^{-\mu-\frac{1}{2}}\varphi].$$

Hence,

$$\gamma_{m,k}^{\mu}(\theta\varphi) \le \sum_{\nu=0}^{k} \binom{k}{\nu} B_k [\gamma_{m,k-\nu}^{\mu}(\varphi) + \gamma_{m+n_\nu,k-\nu}^{\mu}(\varphi)],$$

which verifies our assertion.

<u>Lemma 2-3:</u> <u>If</u> p(x) <u>and</u> Q (x) <u>are polynomials</u> and Q (x)

<u>is never zero for</u> $0 \leq x < \infty$, <u>then</u> $P(x^2) / Q(x^2)$ is in $\mathcal{O}$.

<u>Proof</u>: Clearly, $P(x^2)$ is in Θ . Therefore, we need merely show that $1/Q(x^2)$ is in Θ. First, note that

$$(x^{-1}D)^k \frac{1}{Q(x^2)} \tag{2-5}$$

is bounded on $X \leq x < \infty$ for each $X > 0$ and each k=o, 1, 2.... Furthermore, in some neighborhood Ω of the origin, $1/Q(x^2)$ is an analytic function and we may apply the operator $(x^{-1}D)^k$ to its Maclaurin's series expansion

$$\frac{1}{Q(x^2)} = a + b\,x^2 + c\,x^4 + \ldots$$

to get another Maclaurin's series expansion of the same form, which is valid in Ω . Hence, for every k, (2-5) is also bounded in some (smaller) neighborhood of the origin. This completes the proof.

<u>Lemma 2-4</u>: <u>The operation</u> $\varphi \to x\varphi$ <u>is an isomorphism from</u> $\mathcal{H}_\mu$ <u>onto</u> $\mathcal{H}_{\mu+1}$.

<u>Proof</u>: For $\varphi \in \mathcal{H}_\mu$, $\gamma^{\mu+1}_{m,k}(x\varphi) = \gamma^{\mu}_{m,k}(\varphi)$. Q. E. D.

Another operator that we shall use is

$$N_\mu = x^{\mu+\frac{1}{2}}\, D\, x^{-\mu-\frac{1}{2}} ;$$

For $\varphi \in \mathcal{H}_\mu$, $\gamma^{\mu+1}_{m,k}(N_\mu \varphi) = \gamma^{\mu}_{m,k+1}(\varphi)$ which implies

<u>Lemma 2-5</u>: <u>The operation</u> $\varphi \to N_\mu \varphi$ <u>is a continuous linear</u> mapping of $\mathcal{H}_\mu$ into $\mathcal{H}_{\mu+1}$.

Also, let

$$M = x^{-\mu-\frac{1}{2}} D x^{\mu+\frac{1}{2}}.$$

<u>Lemma 2-6</u>: <u>The operation</u> $\varphi \to M_\mu \varphi$ <u>is a continuous linear mapping of</u> $\mathcal{H}_{\mu+1}$ <u>into</u> $\mathcal{H}_\mu$.

<u>Proof</u>:

$$\gamma^{\mu}_{m,k}(M_\mu \varphi) = \sup_{0<x<\infty} \left| x^m (x^{-1}D)^k x^{-2\mu-1} D x^{2\mu+2} x^{-\mu-\frac{3}{2}} \varphi \right|$$

$$= \sup \left| 2(\mu+k+1) x^m (x^{-1}D)^k x^{-\mu-\frac{3}{2}} \varphi + x^{m+2} (x^{-1}D)^{k+1} x^{-\mu-\frac{3}{2}} \varphi \right|$$

$$\leq 2(\mu+k+1) \gamma^{\mu+1}_{m,k}(\varphi) + \gamma^{\mu+1}_{m+2,k+1}(\varphi).$$

This proves that $M_\mu \varphi \in \mathcal{H}_\mu$ whenever $\varphi \in \mathcal{H}_{\mu+1}$ and that the mapping is continuous.

The preceding two lemmas show that the operator

$$M_\mu N_\mu = x^{-\mu-\frac{1}{2}} D x^{2\mu+1} D x^{-\mu-\frac{1}{2}} = D^2 - \frac{4\mu^2-1}{4x^2}$$

is a continuous linear mapping of $\mathcal{H}_\mu$ into $\mathcal{H}_\mu$.

We now take up the behavior of $\mathcal{H}_\mu$ under the μth-order Hankel transformation h_μ . Whenever we deal with h_μ , μ will be restricted to the range $-\frac{1}{2} \leq \mu < \infty$. The μ th-order Hankel transform of an arbitrary element φ in $\mathcal{H}_\mu$ always exists because φ (x) is smooth and of rapid descent as $x \to \infty$ and $\varphi(x) = C(1)$ as $x \to 0+$. In the following we let x and y denote the independent variables of the original function and the Hankel transform respectively.

<u>Lemma 2-7</u>: <u>Let</u> $\mu \geq -\frac{1}{2}$. <u>If</u> $\mathcal{a} \in \mathcal{H}_\mu$, <u>then</u>

$$h_{\mu+1}(-x\mathcal{a}) = N_\mu h_\mu \mathcal{a}, \tag{2-6}$$

$$h_{\mu+1}(N_\mu \mathcal{a}) = -y\, h_\mu \mathcal{a}, \tag{2-7}$$

$$h_\mu(-x^2\mathcal{a}) = M_\mu N_\mu h_\mu \mathcal{a}, \tag{2-8}$$

$$h_\mu(M_\mu N_\mu \mathcal{a}) = -y^2 h_\mu \mathcal{a}. \tag{2-9}$$

<u>If</u> $\mathcal{a} \in \mathcal{H}_{\mu+1}$, <u>then</u>

$$h_\mu(x\mathcal{a}) = M_\mu h_{\mu+1}\mathcal{a}, \tag{2-10}$$

$$h_\mu(M_\mu \mathcal{a}) = y\, h_{\mu+1}\mathcal{a}. \tag{2-11}$$

<u>Proof</u>: Let $\phi(y) = h_\mu[\mathcal{a}(x)]$, where $\mathcal{a} \in \mathcal{H}_\mu$. We may differentiate under the integral sign, as follows,

$$D_y\, y^{-\mu-\frac{1}{2}}\phi = \int_0^\infty \mathcal{a}(x)\sqrt{x}\; D_y[y^{-\mu} J_\mu(xy)]\,dx$$

$$= -\int_0^\infty \mathcal{a}(x)\; x^{\mu+\frac{3}{2}}\; \frac{J_{\mu+1}(xy)}{(xy)^\mu}\,dx,$$

because $(xy)^{-\mu} J_{\mu+1}(xy)$ is a bounded smooth function on $0<xy<\infty$. Hence,

$$N_\mu \phi = \int_0^\infty (-x\mathcal{a})\sqrt{xy}\; J_{\mu+1}(xy)\,dx$$

which is the same as (2-6).

To obtain (2-7), first note that $N_\mu \mathcal{a}$ is in $\mathcal{H}_{\mu+1}$ according to lemma (2-5) so that we may take its $(\mu+1)$ th-order Hankel transform. An integration by parts yields

$$h_{\mu+1}(N_\mu \mathcal{a}) = \sqrt{y}\int_0^\infty D_x(x^{-\mu-\frac{1}{2}}\mathcal{a})\, x^{\mu+1} J_{\mu+1}(xy)\,dx$$

$$= \mathcal{a}(x)\sqrt{xy}\; J_{\mu+1}(xy)\Big|_0^\infty - y\int_0^\infty \mathcal{a}(x)\sqrt{xy}\; J_\mu(xy)\,dx$$

The limit terms are zero since $\alpha(x)$ is of rapid descent as $x \to \infty$ and, as $x \to 0+$, $\sqrt{xy}\, J_{\mu+1}(xy) = O(x)$ and $\alpha(x) = O(1)$. Thus, we have obtained (2-7).

Now assume that $\alpha \in \mathcal{H}_{\mu+1}$ and set $\phi(y) = h_{\mu+1}[\alpha(x)]$. Eq. (2-10) is derived by differentiating under the integral sign, a procedure that is valid because $y^{\mu+1} J_\mu(xy)$ is bounded for $0 < x < \infty$ and for all y in any compact subset of $0 < y < \infty$.

$$D_y y^{\mu+\frac{1}{2}} \phi = \int_0^\infty \alpha(x) \sqrt{x}\, D_y \left[y^{\mu+1} J_{\mu+1}(xy) \right] dx$$
$$= \int_0^\infty \alpha(x)\, x^{\frac{3}{2}} y^{\mu+1} J_\mu(xy)\, dx$$

Multiplication by $y^{-\mu-\frac{1}{2}}$ yields (2-10).

For (2-11) we invoke lemma 2-6 and apply h_μ. An integration by parts then yields

$$h_\mu(M_\mu \alpha) = \sqrt{y} \int_0^\infty D_x\left(x^{\mu+\frac{1}{2}} \alpha\right) x^{-\mu} J_\mu(xy)\, dx$$
$$= \alpha(x) \sqrt{xy}\, J_\mu(xy) \Big|_0^\infty + y \int_0^\infty \alpha(x) \sqrt{xy}\, J_{\mu+1} (xy)\, dx$$

In view of the seminorm $\gamma_{0,0}^{\mu+1}(\alpha)$ and the fact that $\mu \geq -\frac{1}{2}$, $\alpha(x) = O(x)$ as $x \to 0+$. Moreover, $\sqrt{xy}\, J_\mu(xy)$ is bounded for $0 < xy < \infty$ and $\alpha(x)$ is of rapid descent as $x \to \infty$. Hence, the limit terms are zero, and we have obtained (2-11).

Eqs. (2-8) and (2-9) now follow from the preceding results.

<u>Lemma 2-8</u>: <u>For $\mu \geq -\frac{1}{2}$ h_μ is an isomorphism from $\mathcal{H}_\mu$ onto $\mathcal{H}_\mu$</u>

(i.e., an automorphism).

Proof: As before, let $\phi(y) = \mathfrak{h}_\mu[\mathcal{a}(x)]$. By applying (2-6) k times and (2-7) m times and noting that

$$N_{\mu+k+m-1} \cdots N_{\mu+k+1} N_{\mu+k} (x^k \mathcal{a}) = x^k N_{\mu+m-1} \cdots N_{\mu+1} N_\mu \mathcal{a},$$

we obtain

$$(-y)^m \cdot N_{\mu+k-1} \cdots N_\mu \phi(y) = \int_0^\infty (-x)^k [N_{\mu+m-1} \cdots N_\mu \mathcal{a}(x)] \sqrt{xy}\, J_{\mu+k+m}(xy)\, dx.$$

This is the same as

$$(-1)^{m+k} y^m (y^{-1} D_y)^k y^{-\mu-\frac{1}{2}} \phi(y) = \int_0^\infty x^{2\mu+2k+m+1} [(x^{-1}D_x)^m x^{-\mu-\frac{1}{2}} \mathcal{a}(x)] \frac{J_{\mu+k+m}(xy)}{(xy)^{\mu+k}}\, dx$$

Furthermore, $J_{\mu+k+m}(z) / z^{\mu+k}$ is bounded on $0 < z < \infty$ by, say, B_{km}. The last equation therefore shows that $\phi(y)$ is smooth on $0 < y < \infty$.

If n is an integer no less than $\mu + k + \frac{1}{2}(m+1)$, then $x^{2\mu+2k+m+1} < (1+x^2)^n$ for $x>0$. Hence, the last equation also yields

$$\gamma^\mu_{m,k}(\phi) \leq \int_0^\infty (1+x^2)^{n+1} \left| (x^{-1}D_x)^m x^{-\mu-\frac{1}{2}} \mathcal{a}(x) \right| \frac{B_{km}}{1+x^2}\, dx$$

$$\leq \frac{1}{2}\pi B_{km} \sum_{\nu=0}^{n+1} \binom{n+1}{\nu} \gamma^\mu_{2\nu,m}(\mathcal{a}).$$

This inequality proves that ϕ is in $\mathcal{H}_\mu$ whenever $\mathcal{a}$ is, and that the linear mapping $\mathfrak{h}_\mu$ is also continuous from $\mathcal{H}_\mu$ into $\mathcal{H}_\mu$. In view of the inversion formula (1-2) and the fact that $\mathfrak{h}_\mu^{-1} = \mathfrak{h}_\mu$, $\mathfrak{h}_\mu$ is also a one-to-one mapping of $\mathcal{H}_\mu$ onto $\mathcal{H}_\mu$ and indeed an automorphism on $\mathcal{H}_\mu$.

3. <u>The Distribution Space</u> $\mathcal{H}_\mu'$. $\mathcal{H}_\mu'$ is the dual space of $\mathcal{H}_\mu$. The complex number that $f \in \mathcal{H}_\mu'$ assigns to $\varphi \in \mathcal{H}_\mu$ will be denoted by $\langle f, \varphi \rangle$. We assign to $\mathcal{H}_\mu'$ the weak topology generated by the seminorms

$$\sigma_\varphi(f) = |\langle f, \varphi \rangle| ,$$

where the φ are arbitrary elements of $\mathcal{H}_\mu$. Under the customary definitions of equality, addition, and multiplication-by-a-complex-number, $\mathcal{H}_\mu'$ is a linear space. The use of lemma 2-2 and a small modification of M. S. Brodskii's proof of the sequential completeness of the space of distributions [13; Sec. 2.2] shows that $\mathcal{H}_\mu'$ is sequentially complete. Obviously, the restrictions of the members of $\mathcal{H}_\mu'$ to D_I comprise subspace of D_I' and convergence in $\mathcal{H}_\mu'$ implies weak convergence in D_I'.

<u>Lemma 3-1</u>: <u>For each</u> $f \in \mathcal{H}_\mu'$, <u>there exists a nonnegative integer</u> r <u>and a positive constant</u> C <u>such that, for all</u> $\varphi \in \mathcal{H}_\mu$,

$$|\langle f, \varphi \rangle| \leq C \rho_r^\mu(\varphi). \tag{3-1}$$

This boundedness property for $f \in \mathcal{H}_\mu'$ is proven in precisely the same way as is the corresponding result for distributions [13; Sec. 3-3].

Some other readily established facts are:

(i) $\mathcal{H}_\mu'$ contains every regular distribution f that corresponds to a function which is Lebesgue integrable on $0 < x < X$ for every finite $X > 0$ and is of slow growth as $x \to \infty$. In this case we have

$$< f, \varphi > = \int_0^\infty f(x) \; \varphi(x) \; dx \qquad (\varphi \in \mathcal{H}_\mu).$$

(ii) If f is a tempered distribution whose support is contained in $X \leq x < \infty$ for some $X > 0$, then $f \in \mathcal{H}_\mu'$.

(iii) On the other hand, every $f \in \mathcal{H}_\mu'$ is a distribution of slow growth as $x \to \infty$, in the following sense. Let λ be a smooth function on $-\infty < x > \infty$ that is equal to zero for $-\infty < x \leq 1$ and is equal to 1 for $2 \leq x < \infty$. Then, λf is a tempered distribution. (Here, it is understood that λf is the zero distribution on $-\infty < x < 1$.)

(iv) From lemma 2-1 we have that, for each even positive integer q, $\mathcal{H}_\mu' \subset \mathcal{H}_{\mu+q}'$ and convergence in $\mathcal{H}_\mu'$ implies convergence in $\mathcal{H}_{\mu+q}'$.

4. Some Operations on $\mathcal{H}_\mu'$.

(i)Let $f \in \mathcal{H}_\mu'$ and $\varphi \in \mathcal{H}_\mu$. We define the operation of multiplication-by- θ , where $\theta \in \Theta$, through the equations

$$\langle \theta f, \varphi \rangle = \langle f, \theta \varphi \rangle.$$

It can readily be shown that $f \to \theta f$ is a continuous linear mapping of $\mathcal{H}_\mu'$ into $\mathcal{H}_\mu'$.

(ii) The operation $f \to xf$ is defined on $\mathcal{H}'_{\mu+1}$ by

$$\langle xf, \varphi \rangle = \langle f, x\varphi \rangle,$$

here $f \in \mathcal{H}'_{\mu+1}$ and $\varphi \in \mathcal{H}_{\mu}$. The right-hand side has a sense ecause $x\varphi \in \mathcal{H}_{\mu+1}$. As a consequence of lemma 2-4, this map-ing is an isomorphism from $\mathcal{H}'_{\mu+1}$ onto $\mathcal{H}'_{\mu}$. Thus, the inverse peration $g \to g/x$ maps $\mathcal{H}'_{\mu}$ onto $\mathcal{H}'_{\mu+1}$.

(iii) The operator N_{μ} is defined on $\mathcal{H}'_{\mu}$ as follows. If $\varphi \in \mathcal{H}_{\mu+1}$, then $M_{\mu}\varphi \in \mathcal{H}_{\mu}$. So, for $f \in \mathcal{H}'_{\mu}$ we set

$$\langle N_{\mu}f, \varphi \rangle = \langle f, -M_{\mu}\varphi \rangle.$$

his defines $N_{\mu} f$ as a functional on $\mathcal{H}_{\mu+1}$. By lemma 2-6, $N_{\mu} f$ is in $\mathcal{H}'_{\mu+1}$, and $f \to N_{\mu} f$ is a continuous linear mapping of $\mathcal{H}'_{\mu}$ into $\mathcal{H}'_{\mu+1}$.

(iv) Similarly, we define M_{μ} on $\mathcal{H}'_{\mu+1}$ by

$$\langle M_{\mu}f, \varphi \rangle = \langle f, -N_{\mu}\varphi \rangle,$$

here $f \in \mathcal{H}'_{\mu+1}$, $\varphi \in \mathcal{H}_{\mu}$, and, by lemma 2-5, $N_{\mu}\varphi \in \mathcal{H}_{\mu+1}$. emma 2-5 also implies that $f \to M_{\mu} f$ is a continuous linear mapping f $\mathcal{H}'_{\mu+1}$ into $\mathcal{H}'_{\mu}$.

(v) Combining the preceding two operators, we see that the perator $M_{\mu} N_{\mu}$ is a continuous linear mapping of $\mathcal{H}'_{\mu}$ into $\mathcal{H}'_{\mu}$, efined by

$$\langle M_{\mu}N_{\mu}f, \varphi \rangle = \langle f, M_{\mu}N_{\mu}\varphi \rangle.$$

5. <u>A Distributional Hankel Transformation</u>. Henceforth, we restrict μ to the interval $-\frac{1}{2} \leq \mu < \infty$. Following Schwartz's approach to the distributional Fourier transformation, we extend the Hankel transformation to distributions by generalizing Parseval's equation (1-3). By lemma 2-8, $\phi = \mathfrak{h}_\mu \alpha$ is in $\mathcal{H}_\mu$ whenever α is. So, we define the μ th-order Hankel transform F of $f \in \mathcal{H}_\mu'$ by

$$\langle F, \phi \rangle = \langle f, \alpha \rangle \qquad (\alpha \in \mathcal{H}_\mu). \tag{5-1}$$

This determines F as a functional on $\mathcal{H}_\mu$. That F is also a distribution in $\mathcal{H}_\mu'$ follows from lemma 2-8. Indeed, (5-1) defines $\mathfrak{h}_\mu$ as a continuous linear mapping of $\mathcal{H}_\mu'$ into $\mathcal{H}_\mu'$. On the other hand, if F is taken as the given distribution in $\mathcal{H}_\mu'$, then (5-1) also serves to define $\mathfrak{h}_\mu^{-1}$ as a continuous linear mapping of $\mathcal{H}_\mu'$ into $\mathcal{H}_\mu'$. On the other hand, if F is taken as the given distribution in $\mathcal{H}_\mu'$, then (5-1) also serves to define $\mathfrak{h}_\mu^{-1}$ as a continuous linear mapping of $\mathcal{H}_\mu'$ into $\mathcal{H}_\mu'$. Moreover, (5-1) implies the following uniqueness property for $\mathfrak{h}_\mu$.

<u>Lemma 5-1</u>: <u>If</u> $f, g \in \mathcal{H}_\mu'$, <u>if</u> $F = \mathfrak{h}_\mu f$ <u>and</u> $G = \mathfrak{h}_\mu g$, <u>and if</u> $F=G$, <u>then</u> $f=g$.

(Here, the equal signs are understood in the sense of equality in $\mathcal{H}_\mu'$.)

<u>Proof</u>: For $\alpha \in \mathcal{H}_\mu$ and $\phi = \mathfrak{h}_\mu \alpha$,

$$\langle f, \alpha \rangle = \langle F, \phi \rangle = \langle G, \phi \rangle = \langle g, \alpha \rangle. \text{ Q. E. D.}$$

Altogether, then, we have obtained

<u>Theorem 1</u>: $\mathfrak{h}_\mu$ <u>is an automorphism on</u> $\mathcal{H}_\mu'$.

The ordinary Hankel transformation of a function $f \in L_1(0,\infty)$ s a special case of our distributional transformation. For, y the distributional definition of $\mathfrak{h}_\mu f = F$, we have, for $\varphi \in \mathcal{H}_\mu$ nd $\phi = \mathfrak{h}_\mu \varphi$,

$$\langle F, \phi \rangle = \langle f, \varphi \rangle = \int_0^\infty f(x) \; \varphi(x) \; dx.$$

ince $\mathcal{H}_\mu \subset L_1(0,\infty)$ we may invoke the classical Parseval equation 1-3) to write

$$\int_0^\infty f(x)\varphi(x) \; dx = \int_0^\infty F_c(y) \; \phi(y) \; dy,$$

here $F_c(y)$ denotes the ordinary Hankel transform (1-1) of (x). Hence, the distributional Hankel transform $F(y)$ is the egular distribution on $0 < y < \infty$ corresponding to the continuous inction $F_c(y)$.

<u>Theorem 2</u>: <u>If the support of</u> $f \in \mathcal{H}_\mu'$ $(\mu \geq -\frac{1}{2})$ <u>is a compact</u> <u>ibset of</u> $(0,\infty)$, <u>then</u>

$$F(y) = (\mathfrak{h}_\mu f)(y) = \langle f(x), \sqrt{xy}\, J_\mu(xy) \rangle, \tag{5-2}$$

<u>iere</u> $F(y)$ <u>is a smooth function on</u> $0 < y < \infty$ <u>that satisfies an</u> <u>iequality of the form</u>,

$$|F(y)| \leq \begin{cases} K y^{\mu+\frac{1}{2}} & (0<y<1) \\ K_\cdot y^{p} & (1<y<\infty), \end{cases} \tag{5-3}$$

<u>and</u> p <u>being sufficiently large constants</u>.

The proof proceeds by means of two lemmas.

Lemma 5-2: Let $\phi \in \mathcal{H}_\mu$ and set

$$R_\varepsilon(x) = \int_0^\varepsilon \phi(y) \sqrt{xy}\, J_\mu(xy)\, dy.$$

Then, for each nonnegative integer k, $D^k R_\varepsilon(x)$ converges to zew as $\varepsilon \to 0+$ uniformly on every compact subset of $0 < x < \infty$.

Proof: Since $\sqrt{z}\, J_\mu(z)$ is bounded on $0 < z < \infty$, $R_\varepsilon(x)$ converges uniformly to zew on $0 < x < \infty$. By repeatedly differentiating under the integral sign, we obtain, for k=o, 1, 2, ...,

$$(-x^{-1}D)^k x^{-\mu-\frac{1}{2}} R_\varepsilon(x) = \int_0^\varepsilon \phi(y)\, y^{\mu+\frac{1}{2}+2k} \frac{J_{\mu+k}(xy)}{(xy)^{\mu+k}}\, dy$$

This is valid because $J_n(z) / z^n$ also is bounded on $0 < z < \infty$. It follows that the left-hand side converges uniformly to zero on $0 < x < \infty$ as $\varepsilon \to 0+$. Moreover, we can write

$$(x^{-1}D)^k x^{-\mu-\frac{1}{2}} R_\varepsilon = x^{-\mu-\frac{1}{2}} \left[a_{k0} \frac{R_\varepsilon}{x^{2k}} + a_{k1} \frac{DR_\varepsilon}{x^{2k-1}} + \dots + a_{kk} \frac{D^k R_\varepsilon}{x^k}\right],$$

where the a's denote constants. Our conclusion follows from this by induction.

Lemma 5-3: Under the hypothesis of theorem 2,

$$\langle f(x), \sqrt{xy}\, J_\mu(xy) \rangle \qquad (5\text{-}4)$$

is a smooth function on $0 < y < \infty$ that satisfies the inequality (5-3).

Proof: The expression (5-4) is a smooth function on $0 < y < \infty$ since the support of f is a compact subset of I and since $\sqrt{xy}\, J_\mu(xy)$ is smooth for $0 < x < \infty$ and $0 < y < \infty$ [13; Sec. 2.7]. Also, note that for every nonnegative integer

$$(x^{-1}D_x)^\nu \left[x^{-\mu-\frac{1}{2}} \sqrt{xy}\, J_\mu(xy)\right] = (-1)^\nu y^{\mu+\frac{1}{2}+2\nu} \frac{J_{\mu+\nu}(xy)}{(xy)^{\mu+\nu}}$$

and that $J_{\mu+\nu}(z)/z^{\mu+\nu}$ is bounded on $0 < z < \infty$ by, say, B_ν. Moreover, if $\lambda(x)$ is a testing function in D_I that is equal to 1 on a neighborhood of the support of f, then

$$\left|x^m (x^{-1}D_x)^k \left[\lambda(x)\, x^{-\mu-\frac{1}{2}} \sqrt{xy}\, J_\mu(xy)\right]\right| = \left|\sum_{\nu=0}^{k} \binom{k}{\nu} \left[x^m (x^{-1}D_x)^{k-\nu} \lambda(x)\right] \left[(x^{-1}D_x)^\nu x^{-\mu-\frac{1}{2}} \sqrt{xy}\, J_\mu(xy)\right]\right|$$
$$\leq \sum_{\nu=0}^{k} \binom{k}{\nu} B_\nu C_{m\nu}\, y^{\mu+\frac{1}{2}+2\nu}$$

where $C_{m\nu}$ is a bound on $x^m (x^{-1}D_x)^{k-\nu}\lambda(x)$ for $0 < x < \infty$. Since (5-4) is the same as $<f(x), \lambda(x) \sqrt{xy}\, J_\mu(xy)>$, our conclusion now follows from lemma 3-1.

Proof of theorem 2: The only thing that remains to be shown is (5-2). For every $\phi \in \mathcal{H}_\mu$,

$$<<f(x), \sqrt{xy}\, J_\mu(xy)>, \phi(y)> = \lim_{\varepsilon \to 0+} \int_\varepsilon^\infty <f(x), \lambda(x) \sqrt{xy}\, J_\mu(xy)> \phi(y)\, dy$$

We can interchange the integration on y with the "inner product" on x (see, for example, [13; corollary 5.3-26, p.121]) to obtain

$$\lim_{\varepsilon \to 0+} <f(x), \lambda(x) \int_\varepsilon^\infty \phi(y) \sqrt{xy}\, J_\mu(xy)\, dy>.$$

In view of lemma 5-2, this is equal to

$$<f(x), \int_0^\infty \phi(y) \sqrt{xy}\, J_\mu(xy)dy> = <\mathfrak{h}_\mu f, \phi>.$$

Q. E. D.

6. Some Operation-transform Formulas.

Theorem 3: Let $\mu \geq -\frac{1}{2}$. If $f \in \mathcal{H}'_\mu$, then

$$\mathfrak{h}_{\mu+1}(-xf) = N_\mu \mathfrak{h}_\mu f, \tag{6-1}$$

$$\mathfrak{h}_{\mu+1}(N_\mu f) = -y\,\mathfrak{h}_\mu f \tag{6-2}$$

$$\mathfrak{h}_\mu(-x^2 f) = M_\mu N_\mu \mathfrak{h}_\mu f, \tag{6-3}$$

$$\mathfrak{h}_\mu(M_\mu N_\mu f) = -y^2\,\mathfrak{h}_\mu f. \tag{6-4}$$

If $f \in \mathcal{H}'_{\mu+1}$, then

$$\mathfrak{h}_\mu(xf) = M_\mu \mathfrak{h}_{\mu+1} f \tag{6-5}$$

$$\mathfrak{h}_\mu(M_\mu f) = y\,\mathfrak{h}_{\mu+1} f. \tag{6-6}$$

Proof: To prove (6-2), let $\Theta \in \mathcal{H}_{\mu+1}$ and $\phi = \mathfrak{h}_{\mu+1}\Theta$. By noting that $\mathcal{H}_{\mu+1} \subset \mathcal{H}_{\mu-1}$ (lemma 2-1) and $\mathcal{H}'_{\mu-1} \subset \mathcal{H}'_{\mu+1}$ (note (iv) Sec. 3), and by using notes (i) and (iii) of Sec. 4 Eq. (2-11), and lemmas 2-4 and 2-6, we may write

$$<\mathfrak{h}_{\mu+1} N_\mu f, \phi> = <N_\mu f, \Theta> = <f, -M_\mu \Theta> = <\mathfrak{h}_\mu f, -y\phi>$$
$$= <-y\,\mathfrak{h}_\mu f, \phi>,$$

which is the desired result.

Eq. (6-1) can be obtained from (6-2) as follows. Let $F = \mathfrak{h}_\mu f$

and note that $\mathfrak{h}_\mu \mathfrak{h}_\mu$ is the identity operator. Then,

$$N_\mu \mathfrak{h}_\mu F = \mathfrak{h}_{\mu+1} \mathfrak{h}_{\mu+1} N_\mu f = \mathfrak{h}_{\mu+1} (-y \mathfrak{h}_\mu f) = \mathfrak{h}_{\mu+1} (-y F).$$

This is the same as (6-1) with f and x replaced by F and y respectively.

Similarly, for $\mathcal{Q} \in \mathcal{H}_\mu$ and $\phi = \mathfrak{h}_\mu \mathcal{Q}$, (6-6) is obtained by using notes (ii) and (iv) of Sec. 4 and Eq. (2-7) as follows.

$$\langle \mathfrak{h}_\mu M_\mu f, \phi \rangle = \langle M_\mu f, \mathcal{Q} \rangle = \langle f, -N_\mu \mathcal{Q} \rangle$$
$$= \langle \mathfrak{h}_{\mu+1} f, y\phi \rangle = \langle y\, \mathfrak{h}_{\mu+1} f, \phi \rangle.$$

Then, by setting $F = \mathfrak{h}_{\mu+1} f$, we can derive (6-5) by

$$M_\mu \mathfrak{h}_{\mu+1} F = \mathfrak{h}_\mu \mathfrak{h}_\mu M_\mu f = \mathfrak{h}_\mu y\, \mathfrak{h}_{\mu+1} f = \mathfrak{h}_\mu y F.$$

Eqs. (6-3) and (6-4) follow directly from these results.

7. An Application to Ordinary Differential Equations.

Let P(x) be a polynomial with no roots on $-\infty < x \leq 0$ and let $\mu \geq -\frac{1}{2}$. We can use our distributional Hankel transformation to solve a differential equation of the form.

$$P\,(M_\mu N_\mu)\; u = g \qquad (0 < x < \infty), \tag{7-1}$$

where g is a given distribution in $\mathcal{H}_\mu'$ and u is an unknown distribution which is also required to be in $\mathcal{H}_\mu'$. Applying $\mathfrak{h}_\mu$ and invoking (6-4), we get

$$P\,(-y^2)\;\; U\,(y) = G\,(y),$$

where U and G are the Hankel transforms of u and g.

Since $1/P(-y^2)$ is a multiplier in $\mathcal{H}_\mu$ (see note (i) of Sec. 4 and lemma 2-3), our solution is given by

$$u(x) = \mathfrak{h}_\mu^{-1} \frac{G(y)}{P(-y^2)}.$$

That is, u is the distribution in $\mathcal{H}_\mu'$ that assigns to each $\phi \in \mathcal{H}_\mu$ the member

$$<u, \phi> = \left\langle \frac{G(y)}{P(-y^2)} , \phi(y) \right\rangle$$

$$= \left\langle g(x) , \int_0^\infty \frac{\phi(y)}{P(-y^2)} \sqrt{xy}\, J_\mu(xy)\, dy \right\rangle$$

By lemma 5-1 the solution u is unique in .

In a similar way we can solve simultaneous equations of the form

$$\underset{\sim}{P} (M_\mu N_\mu) \underset{\sim}{u} = \underset{\sim}{g},$$

where $\underset{\sim}{P}$ is an n x n matrix whose elements are polynomials and $\underset{\sim}{u}$ and $\underset{\sim}{g}$ are respectively unknown and given n x 1 vectors whose elements are distributions in $\mathcal{H}_\mu'$. In this case the determinant of $\underset{\sim}{P}$ (x) must have no roots on $-\infty < x \leq 0$.

As an extension of this technique, we can allow the polynomial P (x) to have some roots on $-\infty < x < 0$, say, at $x = \zeta_1^2, \ldots, -\zeta_k^2$ ($\zeta_\nu > 0$) with multiplicities $m_1 \ldots, m_k$ respectively. That is,

$$P(x) = Q(x)\,(x+\xi_1^2)^{m_1} \dots (x+\xi_k^2)^{m_k},$$

where $Q(x)$ has no roots on $-\infty < x < 0$. In this case a solution u to (7-1) can be found if $g \in \mathcal{H}_\mu'$ is such that

$$\mathfrak{h}_\mu g = G(y) = F(y)\,(-y^2+\xi_1^2)^{m_1} \dots (-y^2+\xi_k^2)^{m_k},$$

where $F(y) \in \mathcal{H}_\mu'$. Indeed,

$$u = \mathfrak{h}_\mu^{-1}\,\frac{F(y)}{Q(-y^2)}$$

However, this solution is no longer unique in $\mathcal{H}_\mu'$ since we can add to it any complementary solution in $\mathcal{H}_\mu'$. Such a complementary solution has the form [13; Sec. 3-2]

$$\mathfrak{h}_\mu^{-1} \sum_{\rho=1}^{k} \sum_{\nu=0}^{m_\rho-1} c_{\rho\nu} D_y^\nu\, \delta(y-\xi_\rho) = \sum_{\rho=1}^{k} \sum_{\nu=0}^{m_\rho-1} c_{\rho\nu} D_y^\nu \left[\sqrt{xy}\, J_\mu(xy)\right]\Big|_{y=\xi_\rho}$$

Here, the $c_{\rho\nu}$ are arbitrary constants and $\delta(y-\xi_\rho)$ denotes the delta functional concentrated at the point $y=\xi_\rho$.

Differential equations of the type

$$P(B_\mu)\, v = f, \tag{7-2}$$

where v and f are known and given distributions on $0 < x < \infty$ and

$$B_\mu = D^2 + \frac{1}{x}D - \frac{\mu^2}{x^2},$$

can also be solved by means of $\mathfrak{h}_\mu$ if we first multiply v and f by $\sqrt{x}$. This is because

$$P(B_\mu) = \frac{1}{\sqrt{x}}\, P(M_\mu N_\mu)\, \sqrt{x}$$

so that, with $u=\sqrt{x}\,v$ and $g=\sqrt{x}\,f$, (7-2) is the same as (7-1).

Finally, we note that we can relax the condition that P(x) be different from zero at x=0 if we restrict g somewhat further. For example, the differential equation $M_\mu N_\mu u=g$ can be solved uniquely in $\mathcal{H}_\mu^2$ $(\mu \geq -\frac{1}{2})$ if we require that $\mathfrak{h}_\mu g \in \mathcal{H}_{\mu-2}^2$. Indeed, applying $\mathfrak{h}_\mu$ and noting (ii) of Sec. 4, we get

$$U(y) = -G(y)/y^2 \in \mathcal{H}_\mu^2 \quad \text{and} \quad u=\mathfrak{h}_\mu^{-1} U \in \mathcal{H}_\mu^2.$$

8. Another Application: A Boundary-value Problem with a Distributional Boundary Condition. The problem we wish to solve is the following: Find a function v(r,z) on the domain, $0 < r < \infty$, $0 < z < \infty$, that satisfies Laplace's equation (in cylindrical coordinates with no θ variation)

$$\frac{\partial^2}{\partial r^2} v + \frac{1}{r}\frac{\partial}{\partial r} v + \frac{\partial^2}{\partial z^2} v = 0 \tag{8-1}$$

and the following boundary conditions:

(a) As $z \to 0+$, v (r,z) converges in some generalized sense to the distribution f(r) whose support is a compact subset of $0 < r < \infty$.

(b) As $z \to \infty$, v(r,z) converges to zero uniformly on $0 < r < \infty$.

(c) As $r \to \infty$, v(r,z) converges to zero for every $z > 0$.

(d) As $r \to 0+$, v(r,z) remains finite.

We will formally derive the solution by using the zero-order Hankel transformation and then will prove that the solution satisfies (8-1) and the boundary condition. First, however, we set

$u(r,z)=\sqrt{r}\,v(r,z)$ and $g(r)=\sqrt{r}\,f(r)$.

Eq. (8-1) then becomes

$$M_o N_o u + \frac{\partial^2}{\partial z^2} u = r^{-\frac{1}{2}} \frac{\partial}{\partial r} r \frac{\partial}{\partial r} r^{-\frac{1}{2}} u + \frac{\partial^2}{\partial z^2} u = 0. \qquad (8\text{-}2)$$

By applying h_o with respect to r, using (6-4), and formally interchanging h_o with $\partial^2/\partial z^2$ we get

$$-\rho^2 U(\rho, z) + \frac{\partial^2}{\partial z^2} U(\rho, z) = 0,$$

where $U(p,z) = h_o[u(r,z)]$. In view of the boundary condition at z=0 and invoking theorem 2.

$$A(p) = h_o\, g(r) = \langle g(x), \sqrt{xp}\, J_o(xp)\rangle$$

By theorem 2, (8-3) is an absolutely integrable smooth function on $0 < p < \infty$ when $z > 0$. Thus, we may apply the ordinary inverse Hankel transformation to get the solution,

$$u(r,z) = \int_0^\infty \langle g(x), \sqrt{xp}\, J_o(xp)\rangle\, e^{-pz} \sqrt{rp}\, J_o(rp)\, dp \qquad (8\text{-}4)$$
$$(0 < r < \infty,\ 0 < z < \infty).$$

That (8-4) satisfies the differential equation (8-2) can be shown by differentiating (8-4) under the integral sign, a procedure that is justified by theorem 2 and the facts that $yJ_o(y)$ and $yJ_1(y)$ are bounded on $0 < y < \infty$.

Now, we shall show that boundary condition (a) is satisfied. For any $\mathcal{U} \in \mathcal{H}_o$ and for $\Phi = h_o \mathcal{U}$, the definition of the distributional

Hankel transformation yields

$$\langle u(r,z), \varphi(r)\rangle = \int_0^{\infty} \langle g(x), \sqrt{xp}\, J_0\,(xp)\rangle\, e^{-pz}\, \phi(p)\, dp.$$

The last integral converges uniformly for $0 \leq z < \infty$ because of theorem 2 and the fact that $\phi(p)$ is smooth, bounded on every interval of the form $0 < p < R$ $(R < \infty)$, and of rapid descent as $p \to \infty$. Since the integrand is a continuous function of z, we may take a limit under the integral sign to get

$$\lim_{z \to 0+} \langle u(r,z), \varphi(r)\rangle = \int_0^{\infty} \langle g(x), \sqrt{xp}\, J_0\,(xp)\rangle\, \phi(p)\, dp = \langle g(r), \varphi(r)\rangle.$$

Thus, in the sense of convergence in $\mathcal{H}_0'$, $u(r,z)$ converges to $g(r)$ as $z \to 0+$. Therefore, $v(r,z)$ converges in a distributional way to $f(r)$.

Next, we have from (8-4) that

$$|u(r,z)| \leq \sqrt{r} \int_0^{\infty} |\langle g(x), \sqrt{xp}\, J_0\,(xp)\rangle\, e^{-pz}\, \sqrt{p}\, J_0\,(rp)|\, dp. \qquad (8\text{-}5)$$

Because $J_0(rp)$ is bounded on $0 < rp < \infty$, we may again take a limit under the integral sign to get, for $z \to \infty$,

$$|v(r,z)| = r^{-\frac{1}{2}}\, |u(r,z)| \to 0$$

uniformly on $0 < r < \infty$. This verifies boundary condition (b).

Eq. (8-4) also shows that $v(r,z)$ remains finite as $r \to 0+$ because the integral in $u\,(r,z)/\sqrt{r}$ is bounded on $0 < r < \infty$ and

$Z \leq z < \infty$ for every $Z > 0$.

Finally, boundary condition (c) can be derived from an analogue to the Riemann-Lebesgue lemma [7; p. 457]; namely, if the locally integrable function $h(p)$ is absolutely integrable on $0 < p < \infty$, then as $r \to \infty$

$$\int_0^\infty h(p)\,\sqrt{p}\,J_0\,(rp)\,dp = o\,(1/\sqrt{r}).$$

PART II

THE HANKEL TRANSFORMATION OF CERTAIN DISTRIBUTIONS OF RAPID GROWTH

9. The Testing Function Space $\mathcal{B}_{\mu,b}$

Let b be a real positive number. $\mathcal{B}_{\mu,b}$ is the space of all smooth complex-valued functions $\varphi(x)$ on $0<x<\infty$ such that $\varphi(x) = 0$ for $b < x < \infty$ and $(x^{-1}D)^k x^{-\mu-\frac{1}{2}}\varphi$ is bounded on $0 < x < \infty$. $\mathcal{B}_{\mu,b}$ is a linear space to which we assign the topology generated by the countable set of seminorms,

$$\gamma_k^{\mu}(\varphi) = \sup_{0<x<\infty} |(x^{-1}D)^k x^{-\mu-\frac{1}{2}}\varphi| \qquad (k=0, 1, 2,\ldots).$$

An equivalent topology is generated by the following countable set of norms.

$$\tau_k^{\mu}(\varphi) = \max_{0\le\nu\le k} \gamma_\nu^{\mu}(\varphi) \qquad (k=0, 1, 2, \ldots).$$

It follows that $\mathcal{B}_{\mu,b}$ is a Hausdorff locally convex linear topological space that satisfies the first countability axiom [14; Chap. 1].

Lemma 9-1: The norms τ_p^{μ} and τ_q^{μ} are in concordance for every choice of p and q. (That is, whenever $\{\varphi_\nu\}_{\nu=1}^{\infty}$ is a Cauchy sequence under each norm and converges to zero under one of the norms, then it converges to zero under the other norm as well.)

Proof: For definiteness, assume $p<q$. Since $\tau_p^{\mu}(\varphi_\nu)\le\tau_q^{\mu}(\varphi_\nu)$,

it follows that convergence to zero under the τ_q^μ norm implies convergence to zero under the τ_p^μ norm.

Conversely, let $\{\varphi_\nu\}_{\nu=1}^\infty$ converge to zero under the τ_p^μ norm. Therefore, $(x^{-1}D)^k\, x^{-\mu-\frac{1}{2}}\varphi_\nu \to 0$ uniformly for each k=0, 1, ...,p. It follows from an inductive argument that $D^k\varphi_\nu \to 0$ for each k=0, 1, ..., p uniformly on every compact subset Ω of $(0,\infty)$. The fact that $\{\varphi_\nu\}$ is a Cauchy sequence under τ_q^μ similarly implies that $\{D^k\varphi_\nu\}_{\nu=1}^\infty$ converges uniformly on every Ω for k=p+1, p+2, ..., q. If the limits of these latter sequences were not the zero function, then $D^k\varphi_\nu$ could not converge to zero on $(0,\infty)$ for every k=0, ..., p. Hence, $(x^{-1}D)^k\, x^{-\mu-\frac{1}{2}}\varphi_\nu$ converges pointwise to zero on $(0,\infty)$ for k=p+1, ..., q. Since $\{\varphi_\nu\}_{\nu=1}^\infty$ is a Cauchy sequence under τ_k^μ (k=p+1, ..., q), it now follows that $\tau_k^\mu(\varphi_\nu) \to 0$ for k=p+1, ..., q. Q. E. D.

Altogether then, $\mathcal{B}_{\mu,b}$ is a countably normed space [14; p.6].

<u>Lemma 9-2:</u> $\mathcal{B}_{\mu,b}$ <u>is a sequentially complete space</u>.

This lemma follows directly from the proof of lemma 2-2 and the fact that all members of $\mathcal{B}_{\mu,b}$ are identically zero for $b < x < \infty$.

We finally note that, for b<c, $\mathcal{B}_{\mu,b} \subset \mathcal{B}_{\mu,c}$ and the topology induced on $\mathcal{B}_{\mu,b}$ by $\mathcal{B}_{\mu,c}$ is identical to the topology of $\mathcal{B}_{\mu,b}$.

Lemma 9-3: If q is an even positive integer, then $\mathcal{B}_{\mu+q,b} \subset \mathcal{B}_{\mu,b}$, and the topology of $\mathcal{B}_{\mu+q,b}$ is stronger than that induced on it by $\mathcal{B}_{\mu,b}$.

Proof: A little computation shows that for $\phi \in \mathcal{B}_{\mu+2,b}$

$$\gamma_k^{\mu}(\phi) \le 2k\,\gamma_{k-1}^{\mu+2}(\phi) + b^2\,\gamma_k^{\mu+2}(\phi),$$

which implies our assertion.

Lemma 9-4: The operation $\phi \to x\phi$ is an isomorphism from $\mathcal{B}_{\mu,b}$ onto $\mathcal{B}_{\mu+1,b}$.

Proof: This follows from the fact that $\gamma_k^{\mu+1}(x\phi) = \gamma_k^{\mu}(\phi)$.

Lemma 9-5: The operation $\phi \to N_\mu \phi$ is a continuous linear mapping of $\mathcal{B}_{\mu,b}$ into $\mathcal{B}_{\mu+1,b}$.

Proof: $\gamma_k^{\mu+1}(N_\mu \phi) = \gamma_{k+1}^{\mu}(\phi)$

Lemma 9-6: The operation $\phi \to M_\mu \phi$ is a continuous linear mapping of $\mathcal{B}_{\mu+1,b}$ into $\mathcal{B}_{\mu,b}$.

Proof: Some computation shows that for $\phi \in \mathcal{B}_{\mu+1,b}$

$$\gamma_k^{\mu}(M_\mu \phi) \le 2(\mu+k+1)\,\gamma_k^{\mu+1}(\phi) + b^2\,\gamma_{k+1}^{\mu+1}(\phi),$$

from which the lemma follows.

Lemma 9-7: Let $\theta(x)$ be a smooth function on $0 < x < \infty$ such that $(x^{-1}D)^k \theta$ is bounded on $0 < x < X$ for some X. Then, $\varphi \to \theta\varphi$ is a continuous linear mapping of $\mathcal{B}_{\mu,b}$ into $\mathcal{B}_{\mu,b}$.

Proof:

$$\gamma_k^{\mu}(\theta\varphi) \leq \sum_{\nu=0}^{k} \binom{k}{\nu} \gamma_{\nu}^{\mu}(\varphi) \sup_{0<x<b} |(x^{-1}D)^{k-\nu}\theta|$$

This implies the lemma.

10. The Testing Function Space $\mathcal{B}_{\mu}$.

For any fixed μ, $\mathcal{B}_{\mu}$ denotes the strict inductive limit of the $\mathcal{B}_{\mu,b}$ where now b traverses a monotonically increasing sequence of positive numbers and tends to ∞. Thus, $\varphi \in \mathcal{B}_{\mu}$ if and only if $\varphi \in \mathcal{B}_{\mu,b}$ for some b. A basis of neighborhoods of 0 in $\mathcal{B}_{\mu}$ is the collection of all convex sets W in $\mathcal{B}_{\mu}$ such that, for any b, $W \cap \mathcal{B}_{\mu,b}$ is a neighborhood of 0 in $\mathcal{B}_{\mu,b}$.

An equivalent topology can be set up in $\mathcal{B}_{\mu}$ by using the following neighborhood basis of 0 in $\mathcal{B}_{\mu}$. Let $\varepsilon = \{\varepsilon_0, \varepsilon_1, \varepsilon_2, \dots\}$ be a sequence of decreasing positive numbers tending to 0, and let $m = \{m_0, m_1, m_2, \dots\}$ be a sequence of increasing nonnegative numbers tending to ∞. $V_{\mu}(m,\varepsilon)$ is the set of all $\varphi \in \mathcal{B}_{\mu}$ such that, for each $\nu = 0, 1, 2, \dots,$ we have for $x \geq \nu$

$$|(x^{-1}D)^k x^{-\mu-\frac{1}{2}}\varphi| < \varepsilon_{\nu} \qquad (k = 0, 1, \dots, m_{\nu}).$$

The collection of all such $V_{\mu}(m,\varepsilon)$ is the basis of neighborhoods of 0 in $\mathcal{B}_{\mu}$, which generates the equivalent topology in $\mathcal{B}_{\mu}$. Since we shall make no use of this result, we omit its proof.

The following conclusions are immediately obtained from the theory of strict inductive limits of linear topological spaces. The sequence $\{\phi_\nu\}$ converges to ϕ in $\mathcal{B}_\mu$ if and only if ϕ and all ϕ_ν are in $\mathcal{B}_{\mu,b}$ for some fixed b, and $\phi_\nu \to \phi$ in $\mathcal{B}_{\mu,b}$ [14; p.21, theorem 34]. Since each $\mathcal{B}_{\mu,b}$ is sequentially complete, $\mathcal{B}_\mu$ is too. A linear operator $\mathfrak{F}$ from $\mathcal{B}_\mu$ into a locally convex topological linear space X is continuous if and only if $\mathfrak{F}\phi_\nu \to 0$ in X whenever the sequence $\{\phi_\nu\}$ converges to zero in $\mathcal{B}_\mu$ [14; p. 18, theorem 29 and p. 20, theorem 33].

<u>Lemma 10-1</u>: <u>$\mathcal{B}_\mu$ is a subspace of $\mathcal{H}_\mu$. The topology of $\mathcal{B}_\mu$ is stronger than the topology induced on it by $\mathcal{H}_\mu$. With the latter topology $\mathcal{B}_\mu$ is everywhere dense in $\mathcal{H}_\mu$.</u>

<u>Proof</u>: First, it is obvious that $\mathcal{B}_\mu$ is a subspace of $\mathcal{H}_\mu$. Second, consider a set A of all $\phi \in \mathcal{B}_\mu$ such that $\gamma^\mu_{m,k}(\phi) < \varepsilon$ for fixed m and k. This is a convex set in $\mathcal{B}_\mu$. Its intersection with each $\mathcal{B}_{\mu,b}$ contains a neighborhood of 0 in $\mathcal{B}_{\mu,b}$ because $\gamma^\mu_{m,k}(\phi) \le b^m \gamma^\mu_k(\phi)$. It follows that the intersection of $\mathcal{B}_\mu$ with every neighborhood of 0 in $\mathcal{H}_\mu$ contains a neighborhood of 0 in $\mathcal{B}_\mu$.

Finally, we wish to show that $\mathcal{B}_\mu$ is everywhere dense in $\mathcal{H}_\mu$. Let $\lambda(x)$ be a smooth function that equals 1 for $-\infty < x < 1$ and equals 0 for $2 < x < \infty$. Then $\lambda_\nu \phi = \lambda(x-\nu)\phi \in \mathcal{B}_\mu$ (ν = 1, 2, 3, ...) whenever $\phi \in \mathcal{H}_\mu$. Moreover, some computation shows that, for every m and k, $\gamma^\mu_{m,k}(\phi - \lambda_\nu \phi) \to 0$ as $\nu \to \infty$. This implies that every neighborhood in $\mathcal{H}_\mu$ of a given $\varphi \in \mathcal{H}_\mu$ contains some member of $\mathcal{B}_\mu$. Q. E. D.

11. The Distribution Space $\mathcal{B}_\mu'$.

Let f be a linear functional on $\mathcal{B}_\mu$ and let $\langle f, \varphi \rangle$ be the number that f assigns to $\varphi \in \mathcal{B}_\mu$. It follows from a statement of the preceding section that f is continuous on $\mathcal{B}_\mu$ if and only if $\langle f, \varphi_\nu \rangle \to 0$ whenever the sequence $\{\varphi_\nu\}$ converges to 0 in $\mathcal{B}_\mu$. $\mathcal{B}_\mu'$ is the dual of $\mathcal{B}_\mu$ (i.e., the space of all continuous linear functionals on $\mathcal{B}_\mu$). We shall use only the weak topology of $\mathcal{B}_\mu'$, that is, the topology assigned to it by the seminorms

$$\zeta_\varphi(f) = |\langle f, \varphi \rangle|,$$

where the index φ traverses $\mathcal{B}_\mu$. Since each $\mathcal{B}_{\mu,b}$ is a sequentially complete countably normed space, $\mathcal{B}_\mu'$ is also sequentially complete [14; p. 22, theorem 37].

Note that D_I is a subspace of $\mathcal{B}_\mu$ for every μ and that the topology of D_I is stronger than the topology induced on D_I by $\mathcal{B}_\mu$. As a consequence of these facts, the restrictions of the members of $\mathcal{B}_\mu'$ to D_I comprise a subspace of D_I', and convergence in $\mathcal{H}_\mu'$ implies the weak convergence of these restrictions in D_I'.

If $\mathfrak{N}$ is an operator on $\mathcal{B}_{\mu,b}$ into $\mathcal{B}_{\nu,b}$, then the adjoint operator $\mathfrak{N}^*$ is a mapping of $\mathcal{B}_\nu'$ into $\mathcal{B}_\mu'$ defined by

$$\langle \mathfrak{N}^* f, \varphi \rangle = \langle f, \mathfrak{N}\varphi \rangle.$$

Thus, multiplication-by-x is a self-adjoint operator, whereas N_μ and M_μ are adjoints of one another.

The next five lemmas follow directly from lemmas 9-3 to 9-7.

Lemma 11-1: If q is an even positive integer, then $\mathcal{B}'_\mu \subset \mathcal{B}'_{\mu+q}$, and the topology of $\mathcal{B}'_\mu$ is stronger than the topology induced on it by $\mathcal{B}'_{\mu+q}$.

Lemma 11-2: The operation $f \to xf$ is an isomorphism from $\mathcal{B}'_{\mu+1}$ into $\mathcal{B}'_\mu$.

Lemma 11-3: The operation $f \to N_\mu f$ is a continuous linear mapping of $\mathcal{B}'_\mu$ into $\mathcal{B}'_{\mu+1}$.

Lemma 11-4: The operation $f \to M_\mu f$ is a continuous linear mapping of $\mathcal{B}'_{\mu+1}$ into $\mathcal{B}'_\mu$.

Lemma 11-5: Let θ be defined as in lemma 9-7. Then $f \to \theta f$ is a continuous linear mapping of $\mathcal{B}'_\mu$ into $\mathcal{B}'_\mu$. (This mapping is defined by $\langle \theta f, \varphi \rangle = \langle f, \theta\varphi \rangle$ where $\varphi \in \mathcal{B}_\mu$.)

Lemma 11-6: $\mathcal{H}'_\mu$ is a subspace of $\mathcal{B}'_\mu$. The topology of $\mathcal{H}'_\mu$ is stronger than the topology induced on $\mathcal{H}'_\mu$ by $\mathcal{B}'_\mu$. With the latter topology, $\mathcal{H}'_\mu$ is everywhere dense in $\mathcal{B}'_\mu$.

Proof: The first two statements follow directly from the first two statements of lemma 10-1.

Next let λ be a smooth function which is equal to 1 for $-\infty < x < 1$ and is equal to 0 for $2 < x < \infty$. Set $\lambda_\nu(x) = \lambda(x-\nu)$, and let $f \in \mathcal{B}'_\mu$. It follows that $\lambda_\nu f \in \mathcal{H}'_\mu$. Moreover, if $\theta \in \mathcal{B}_\mu$, The next five lemmas follow directly from lemmas 9-3 to 9-7.

Lemma 11-1: If q is an even positive integer, then $\mathcal{B}'_\mu \subset \mathcal{B}'_{\mu+q}$, and the topology of $\mathcal{B}'_\mu$ is stronger than the topology induced on it by $\mathcal{B}'_{\mu+q}$.

Lemma 11-2: The operation $f \to xf$ is an isomorphism from $\mathcal{B}'_{\mu+1}$ into $\mathcal{B}'_\mu$.

Lemma 11-3: The operation $f \to N_\mu f$ is a continuous linear mapping of $\mathcal{B}'_\mu$ into $\mathcal{B}'_{\mu+1}$.

Lemma 11-4: The operation $f \to M_\mu f$ is a continuous linear mapping of $\mathcal{B}'_{\mu+1}$ into $\mathcal{B}'_\mu$.

Lemma 11-5: Let θ be defined as in lemma 9-7. Then $f \to \theta f$ is a continuous linear mapping of $\mathcal{B}'_\mu$ into $\mathcal{B}'_\mu$. (This mapping is defined by $\langle \theta f, \varphi \rangle = \langle f, \theta\varphi \rangle$ where $\varphi \in \mathcal{B}_\mu$.)

Lemma 11-6: $\mathcal{H}'_\mu$ is a subspace of $\mathcal{B}'_\mu$. The topology of $\mathcal{H}'_\mu$ is stronger than the topology induced on $\mathcal{H}'_\mu$ by $\mathcal{B}'_\mu$. With the latter topology, $\mathcal{H}'_\mu$ is everywhere dense in $\mathcal{B}'_\mu$.

Proof: The first two statements follow directly from the first two statements of lemma 10-1.

Next let λ be a smooth function which is equal to 1 for $-\infty < x < 1$ and is equal to 0 for $2 < x < \infty$. Set $\lambda_\nu(x) = \lambda(x-\nu)$, and let $f \in \mathcal{B}'_\mu$. It follows that $\lambda_\nu f \in \mathcal{H}'_\mu$. Moreover, if $\theta \in \mathcal{B}_\mu$, then $\lambda_\nu \theta \to \theta$ in $\mathcal{B}_\mu$, and $\langle \lambda_\nu f, \theta\rangle = \langle f, \lambda_\nu \theta\rangle \to \langle f, \theta\rangle$. Thus, every neighborhood of $f \in \mathcal{B}'_\mu$ in the $\mathcal{B}'_\mu$ topology contains a member of $\mathcal{H}'_\mu$; that is, $\mathcal{H}'_\mu$ is everywhere dense in $\mathcal{B}'_\mu$. Q. E. D.

12. The Testing Function Space $\mathcal{Y}_{\mu,b}$.

Let y and w be real variables and let $\eta = y + iw$. As before, μ and b are fixed real numbers with $b > 0$. We define a topological linear space $\mathcal{Y}_{\mu,b}$ as follows. ϕ is a member of $\mathcal{Y}_{\mu,b}$ if and only if $\eta^{-\mu-\frac{1}{2}}\phi(\eta)$ is an even entire function of η and for each nonnegative integer k the quantity

$$\alpha^{\mu}_{b,k}(\phi) = \sup_{\eta} \left| e^{-b|w|}\eta^{2k-\mu-\frac{1}{2}}\phi(\eta)\right|$$

exists (i.e., is finite). The topology of $\mathcal{Y}_{\mu,b}$ is the one generated by using the $\alpha^{\mu}_{b,k}$ $(k=0, 1, 2, \ldots)$ as seminorms. An equivalent topology is generated by the following set of norms.

$$\beta^{\mu}_{b,k}(\phi) = \max_{0 \le \nu \le k} \alpha^{\mu}_{b,\nu}(\phi) \qquad (k=0, 1, 2, \ldots)$$

Thus, $\mathcal{Y}_{\mu,b}$ is a Hausdorff locally convex topological linear space that satisfies the first countability axiom.

Lemma 12-1: The norms $\beta^{\mu}_{b,p}$ and $\beta^{\mu}_{b,q}$ are in concordance for every choice of p and q.

Proof: Again for definiteness, let $p < q$. Also, let $\{\phi_\nu\}$ be a Cauchy sequence under both norms and let $\beta^{\mu}_{b,p}(\phi_\nu) \to 0$. Hence, $\alpha^{\mu}_{b,0}(\phi_\nu) \to 0$. This means that the pointwise limit of the $\eta^{-\mu-\frac{1}{2}}\phi_\nu(\eta)$, is the identically zero function. Moreover, for each $k=0, 1, \ldots, q$, $e^{-b|w|}\eta^{2k-\mu-\frac{1}{2}}\phi_\nu(\eta)$ converges uniformly on the entire η -plane. It follows that $\beta^{\mu}_{b,q}(\phi_\nu) \to 0$ Q. E. D.

It follows now that $\mathcal{Y}_{\mu,b}$ is a countably normed space [14; p. 6].

<u>Lemma 12-2</u>: $\mathcal{Y}_{\mu,b}$ <u>is a sequentially complete space</u>.

<u>Proof</u>: Let $\{\phi_\nu\}$ converge in $\mathcal{Y}_{\mu,b}$. By the seminorm $\alpha^{\mu}_{b,k}$, the $\phi_\nu(\eta)\eta^{-\mu-\frac{1}{2}}$ converge uniformly on every bounded domain Ω of the η -plane. Therefore, there exists a limit function ϕ such that $\eta^{-\mu-\frac{1}{2}}\phi_\nu \to \eta^{-\mu-\frac{1}{2}}\phi$ uniformly on Ω. Since each $\eta^{-\mu-\frac{1}{2}}\phi_\nu(\eta)$ is even and entire, $\eta^{-\mu-\frac{1}{2}}\phi$ is also even and entire. Also, since $|e^{-b|w|}\eta^{2k-\mu-\frac{1}{2}}\phi_\nu|$ is bounded on the η -plane uniformly for all ν, the limit function ϕ satisfies a similar bound. Thus, $\phi \in \mathcal{Y}_{\mu,b}$.

<u>Lemma 12-3</u>: <u>If</u> $b<c$, <u>then</u> $\mathcal{Y}_{\mu,b} \subset \mathcal{Y}_{\mu,c}$, <u>and the topology generated on</u> $\mathcal{Y}_{\mu,b}$ <u>by</u> $\mathcal{Y}_{\mu,c}$ <u>is identical to the topology of</u> $\mathcal{Y}_{\mu,b}$.

<u>Proof</u>: Let $\phi \in \mathcal{Y}_{\mu,b}$. By the Phragmen-Lindeloff theorem [15; p. 177 we have for $w \geq 0$

$$|e^{ib\eta}\eta^{2k-\mu-\frac{1}{2}}\phi(\eta)| = e^{-bw}|\eta^{2k-\mu-\frac{1}{2}}\phi(\eta)| \leq \sup_y |y^{2k-\mu-\frac{1}{2}}\phi(y)|$$

and for $w \leq 0$

$$|e^{-ib\eta}\eta^{2k-\mu-\frac{1}{2}}\phi(\eta)| = e^{bw}|\eta^{2k-\mu-\frac{1}{2}}\phi(\eta)| \leq \sup_y |y^{2k-\mu-\frac{1}{2}}\phi(y)|.$$

Thus,

$$\alpha^{\mu}_{b,k}(\phi) = \sup_\eta |e^{-b|w|}\eta^{2k-\mu-\frac{1}{2}}\phi(\eta)| \leq \sup_y |y^{2k-\mu-\frac{1}{2}}\phi(y)| \leq \alpha^{\mu}_{c,k}(\phi).$$

Conversely,

$$\alpha^{\mu}_{c,k}(\phi) \leq \sup_{\eta} |e^{-c|\omega|} \eta^{2k-\mu-\frac{1}{2}} \phi(\eta)| \leq \sup_{\eta} |e^{-b|\omega|} \eta^{2k-\mu-\frac{1}{2}} \phi(\eta)| = \alpha^{\mu}_{b,k}(\phi).$$

Hence, $\alpha^{\mu}_{c,k}(\phi) = \alpha^{\mu}_{b,k}(\phi)$. Q. E. D.

Another topology can be constructed in $\mathcal{Y}_{\mu,b}$ by using the seminorms

$$\zeta^{\mu}_{a,P}(\phi) = \sup_{\eta \in C_a} |P(\eta) \eta^{-\mu-\frac{1}{2}} \phi(\eta)|$$

where P is an even polynomial and C_a is a strip of width 2a symmetrically placed around the real axis.

Lemma 12-4: <u>The topologies generated by the seminorms</u> $\alpha^{\mu}_{b,k}$ <u>and</u> $\zeta^{\mu}_{a,p}$ <u>are equivalent</u>.

Proof: It is obvious that the topology generated by the $\alpha^{\mu}_{b,k}$ is stronger than the topology generated by the $\zeta^{\mu}_{a,p}$. Conversely, the Phragmen-Lindehoff theorem shows, as in the proof of lemma 12-3, that

$$\sup_{\eta} |e^{-b|\omega|} \eta^{2k-\mu-\frac{1}{2}} \phi(\eta)| \leq \sup_{y} |y^{2k-\mu-\frac{1}{2}} \phi(y)|$$

The right-hand side is in turn less than

$$\sup_{\eta \in C_a} |\eta^{2k-\mu-\frac{1}{2}} \phi(\eta)|.$$

Consequently, the topology due to the $\zeta^{\mu}_{a,p}$ is stronger than the topology due to the $\alpha^{\mu}_{b,k}$. Q. E. D.

<u>Theorem 4</u>: <u>For</u> $\mu \geq -\frac{1}{2}$, $\mathfrak{h}_\mu$ <u>is an isomorphism from</u> $\mathcal{B}_{\mu,b}$ <u>onto</u> $\mathcal{Y}_{\mu,b}$.

<u>Proof</u>: First, assume that $\varphi \in \mathcal{B}_{\mu,b}$. Then

$$\Phi(\eta) = \int_0^b \varphi(x)\sqrt{\eta x}\, J_\mu(\eta x)\, dx$$

where $\phi(x) \in L_2(0,b)$. By Griffith's theorem, $\eta^{-\mu-\frac{1}{2}}\Phi(\eta)$ is an even entire function of η. Furthermore, an argument similar to the one given in the proof of lemma 2-8 shows that

$$\eta^{2m-\mu-\frac{1}{2}}\Phi(\eta) = \int_0^b x^{2m+2\mu+1}\left[(x^{-1}D)^{2m} x^{-\mu-\frac{1}{2}}\varphi(x)\right](x\eta)^{-\mu} J_{\mu+2m}(x\eta)\, dx \qquad (12\text{-}1)$$

$$(m = 0, 1, 2, \ldots)$$

From the asymptotic formula

$$J_\nu(z) \sim \sqrt{\frac{2}{\pi z}}\cos\left(z - \frac{\nu\pi}{2} - \frac{\pi}{4}\right) \qquad (|z| \to \infty;\ -\pi < \arg z \leq \pi)$$

and from the fact that $z^{-\mu}J_{\mu+2m}(z)$ is an entire function and hence bounded on every bounded domain of the z-plane, it follows that for all x and η

$$|e^{-b|w|}(x\eta)^{-\mu} J_{\mu+2m}(x\eta)| < C_{m\mu}$$

where $C_{m\mu}$ is a constant independent of x and η. With this result (12-1) can be converted into

$$\alpha^\mu_{b,m}(\Phi) \leq C_{m\mu} b^{2(m+\mu+1)} \gamma^\mu_{2m}(\varphi).$$

Thus, the linear transformation $\mathfrak{h}_\mu$ is a continuous mapping of $\mathcal{B}_{\mu,b}$ into $\mathcal{Y}_{\mu,b}$.

Conversely, let $\Phi(\eta) \in \mathcal{Y}_{\mu,b}$. It follows from Griffith's theorem that $\Phi(\eta)$ is the μth-order Hankel transform of a function $\varphi \in L_2(0,\infty)$ which is zero almost everywhere for $x>b$. Since the Hankel transformation is self-reciprocal, we have

$$\varphi(x) = \int_0^\infty \Phi(y) \sqrt{xy}\, J_\mu(xy)\, dy$$

By formally differentiating under the integral sign, we obtain

$$(-1)^k (x^{-1}D_x)^k x^{-\mu-\frac{1}{2}}\varphi(x) = \int_0^\infty y^{2k+\mu+\frac{1}{2}} \Phi(y)\, (xy)^{-\mu-k} J_{\mu+k}(xy)\, dy \tag{12-2}$$

Since $\Phi(y)$ is of rapid descent as $y \to \infty$ and $z^{-\mu-k} J_{\mu+k}(z)$ is bounded for $0 < z < \infty$, the last integral converges uniformly for $0 < x < \infty$. It follows that the left-hand side of (12-2) is continuous and bounded on $0 < x < \infty$ for every k. An inductive argument now shows that φ is smooth on $0 < x < \infty$. Thus, $\varphi \in \mathcal{B}_{\mu,b}$.

From (12-2) we also get

$$\gamma_k^\mu(\varphi) \le \left[\sup_{0<y<\infty} |(1+y^2)\, y^{2k+\mu+\frac{1}{2}} \phi(y)|\right] \left[\sup_{0<z<\infty} |z^{-\mu-k} J_{\mu+k}(z)|\right] \int_0^\infty \frac{dy}{1+y^2}$$

$$= K \sup_{0<y<\infty} |(1+y^2)\, y^{2k+2\mu+1}\, y^{-\mu-\frac{1}{2}} \phi(y)|$$

where K is a constant. If n is an integer no less than $k+\mu+\frac{1}{2} \geq 0$, then $y^{2k+2\mu+1} < (1+y^2)^n$ for $y > 0$. Hence,

$$\gamma_k^\mu(\varphi) \le K \sup_{0<y<\infty} |(1+y^2)^{n+1} y^{-\mu-\frac{1}{2}} \phi(y)|$$

$$\le K \sup_{\eta} |e^{-b|\omega|} (1+\eta^2)^{n+1} \eta^{-\mu-\frac{1}{2}} \phi(\eta)|$$

$$\le K \sum_{\nu=0}^{n+1} \binom{n+1}{\nu} \alpha_{b,\nu}^\mu(\phi)$$

Thus, $\mathfrak{h}_\mu$ is a continuous mapping of $\mathcal{Y}_{\mu,b}$, into $\mathcal{B}_{\mu,b}$.

The linearity and uniqueness of $\mathfrak{h}_\mu$ complete the proof.

In the next two lemmas we allow μ to be once again any real number.

Lemma 12-5: If q is an even positive integer, then $\mathcal{Y}_{\mu+q,b} \subset \mathcal{Y}_{\mu,b}$ and the topology of $\mathcal{Y}_{\mu+q,b}$ is stronger than the topology induced on it by $\mathcal{Y}_{\mu,b}$.

Proof: For $\phi \in \mathcal{Y}_{\mu+2,b}$ and k=0, 1, 2, ..., $\alpha_{b,k}^{\mu}(\phi) = \alpha_{b,k+1}^{\mu+2}(\phi)$.
Q. E. D.

Lemma 12-6: The operation $\phi \to \eta\phi$ is an isomorphism from $\mathcal{Y}_{\mu,b}$ onto $\mathcal{Y}_{\mu+1,b}$.

Proof: $\alpha_{b,k}^{\mu+1}(\eta\phi) = \alpha_{b,k}^{\mu}(\phi)$. Q. E. D.

Lemma 12-7: If $\mu \geq -\frac{1}{2}$, the operation $\phi \to N_{\mu}\phi$ is an isomorphism from $\mathcal{Y}_{\mu,b}$ onto $\mathcal{Y}_{\mu+1,b}$.

Proof: Invoking lemma 9-4, theorem 4, and Eq. (2-6) (for which we use analytic continuation from the positive y axis onto the η plane), we see that $N_{\mu} = -h_{\mu+1}\, x\, h_{\mu}$ and that h_{μ} is an isomorphism from $\mathcal{Y}_{\mu,b}$ onto $\mathcal{B}_{\mu,b}$, multiplication-by-x is an isomorphism from $\mathcal{B}_{\mu,b}$ onto $\mathcal{B}_{\mu+1,b}$, and $-h_{\mu+1}$ is an isomorphism from $\mathcal{B}_{\mu+1,b}$ onto $\mathcal{Y}_{\mu+1,b}$. Q. E. D.

Lemma 12-8: If $\mu \geq -\frac{1}{2}$, $\phi \to M\phi$ is a continuous linear mapping of $\mathcal{Y}_{\mu+1,b}$ into $\mathcal{Y}_{\mu,b}$.

Proof: Invoking lemmas 9-3 and 9-4, theorem 4, and Eq. (2-10) (for which we use analytic continuation from the positive y axis onto the η plane), we see that $M_\mu = \mathfrak{h}_\mu \times \mathfrak{h}_{\mu+1}$ and that $\mathfrak{h}_{\mu+1}$ is an isomorphism from $\mathcal{Y}_{\mu+1,b}$ onto $\mathcal{B}_{\mu+1,b}$, multiplication-by-x is a continuous linear mapping of $\mathcal{B}_{\mu+1,b}$ into $\mathcal{B}_{\mu,b}$, and $\mathfrak{h}_\mu$ is an isomorphism from $\mathcal{B}_{\mu,b}$ onto $\mathcal{Y}_{\mu,b}$. Q. E. D.

Lemma 12-9: Let $\Upsilon(\eta)$ be an even entire function such that

$$|\Upsilon(\eta)| < C e^{a|\omega|}(1+|\eta|^{2m})$$

where C and a are real positive constants and m is a nonnegative integer. Then, $\phi \to \Upsilon\phi$ is a continuous linear mapping of $\mathcal{Y}_{\mu,b}$ into $\mathcal{Y}_{\mu,a+b}$.

Proof:

$$\alpha^{\mu}_{a+b,k}(\Upsilon\phi) \le \sup\left|\frac{e^{-a|\omega|}\Upsilon(\eta)}{1+|\eta|^{2m}}\right| \sup\left|e^{-b|\omega|}(1+|\eta|^{2m})\eta^{2k-\mu-\frac{1}{2}}\phi(\eta)\right|$$
$$\le C\left[\alpha^{\mu}_{b,k}(\phi) + \alpha^{\mu}_{b,k+m}(\phi)\right]$$

Q. E. D.

13. The Testing Function Space $\mathcal{Y}_\mu$.

$\mathcal{Y}_\mu$ denotes the strict inductive limit of the $\mathcal{Y}_{\mu,b}$ where b traverses a monotonically increasing sequence of positive numbers tending to ∞. A basis of neighborhoods of 0 in $\mathcal{Y}_\mu$ is the collection of all convex set N in $\mathcal{Y}_\mu$ such that, for every b,

$N \cap \mathcal{Y}_{\mu,b}$ is a neighborhood of 0 in $\mathcal{Y}_{\mu,b}$. Thus, a sequence $\{\phi_\nu\}$ converges to ϕ in $\mathcal{Y}_\mu$ if and only if ϕ and all ϕ_ν are in $\mathcal{Y}_{\mu,b}$ for some b and $\phi_\nu \to \phi$ in $\mathcal{Y}_{\mu,b}$ [14; p. 21, theorem 34]. $\mathcal{Y}_\mu$ is also sequentially complete [14; p. 21, theorem 36]. A linear operator $\mathcal{V}$ from $\mathcal{Y}_\mu$ into a locally topological linear space X is continuous if and only if $\mathcal{V}\phi_\nu \to 0$ in X whenever the sequence $\{\phi_\nu\}$ converges to zero in $\mathcal{Y}_\mu$ [14; p. 18, theorem 29 and p. 20, theorem 33]. Since both $\mathcal{B}_\mu$ and $\mathcal{Y}_\mu$ are locally convex topological linear spaces, the following theorem is a consequence of theorem 4 and our comments concerning continuous linear operators on $\mathcal{B}_\mu$ and $\mathcal{Y}_\mu$.

<u>Theorem 5: $\mathfrak{h}_\mu$ is an isomorphism from $\mathcal{B}_\mu$ onto $\mathcal{Y}_\mu$</u>.

<u>Lemma 13-1: $\mathcal{Y}_\mu$ is a subspace of $\mathcal{H}_\mu$. The topology of $\mathcal{Y}_\mu$ is stronger than the topology induced on it by $\mathcal{H}_\mu$. With the latter topology $\mathcal{Y}_\mu$ is everywhere dense in $\mathcal{H}_\mu$.</u>

<u>Proof</u>: Our proof uses the facts that $\mathfrak{h}_\mu$ is an isomorphism from $\mathcal{B}_\mu$ onto $\mathcal{Y}_\mu$, as well as from $\mathcal{H}_\mu$ onto itself. Then, by lemma 10-1,

$$\varphi \in \mathcal{Y}_\mu \Rightarrow \varphi = \mathfrak{h}_\mu^{-1}\phi \in \mathcal{B}_\mu \subset \mathcal{H}_\mu \Rightarrow \phi = \mathfrak{h}_\mu \varphi \in \mathcal{H}_\mu.$$

Hence, $\mathcal{Y}_\mu \subset \mathcal{H}_\mu$.

Next, let Ω be a neighborhood of 0 in $\mathcal{H}_\mu$. $\Omega \cap \mathcal{Y}_\mu$ is a neighborhood of 0 in the induced topology of $\mathcal{Y}_\mu$. On the other hand, $\mathfrak{h}_\mu^{-1}(\Omega)$ is also a neighborhood of 0 in $\mathcal{H}_\mu$, and $\mathfrak{h}_\mu^{-1}(\Omega \cap \mathcal{Y}_\mu) = \mathfrak{h}_\mu(\Omega) \cap \mathcal{B}_\mu$.

By lemma 10-1, there exists a neighborhood Λ of 0 in $\mathcal{B}_\mu$ such that $\Lambda \subset h_\mu^{-1}(\Omega) \cap \mathcal{B}_\mu$. Hence, $h_\mu(\Lambda)$ is a neighborhood of 0 in $\mathcal{Y}_\mu$ and $h_\mu(\Lambda) \subset h_\mu h_\mu^{-1}(\Omega \cap \mathcal{Y}_\mu) = \Omega \cap \mathcal{Y}_\mu$. This verifies the second statement of the lemma.

The last statement follows from the last statement of lemma 10-1.

<u>14</u>. <u>The</u> <u>Generalized</u> <u>Function</u> <u>Space</u> $\mathcal{Y}_\mu'$.

$\mathcal{Y}_\mu'$ is the dual of $\mathcal{Y}_\mu$; that is, $\mathcal{Y}_\mu'$ is the space of all continuous linear functionals on $\mathcal{Y}_\mu$. We again use the notation $\langle F, \phi \rangle$ to denote the number that $F \in \mathcal{Y}_\mu'$ assigns to $\phi \in \mathcal{Y}_\mu$. We shall employ only the weak topology of $\mathcal{Y}_\mu'$. From a standard theorem [14; p. 22, theorem 37] we have that $\mathcal{Y}_\mu'$ is sequentially complete.

The following lemmas are consequences of lemmas 12-5 to 12-9. The operators in lemmas 14-2 to 14-5 are defined in the customary way as adjoints of operators on $\mathcal{Y}_\nu$ for suitably chosen ν.

<u>Lemma 14-1</u>: <u>If</u> q <u>is</u> <u>an</u> <u>even</u> <u>positive</u> <u>integer,</u> <u>then</u> $\mathcal{Y}_\mu' \subset \mathcal{Y}_{\mu+q}'$, <u>and</u> <u>the</u> <u>topology</u> <u>of</u> $\mathcal{Y}_\mu'$ <u>is</u> <u>stronger</u> <u>than</u> <u>the</u> <u>topology</u> <u>induced</u> <u>on</u> <u>it</u> <u>by</u> $\mathcal{Y}_{\mu+q}'$.

<u>Lemma 14-2:</u> <u>The</u> <u>operation</u> $F \rightarrow \eta F$ <u>is</u> <u>an</u> <u>isomorphism</u> <u>from</u>

$\mathcal{Y}_{\mu+1}'$ onto $\mathcal{Y}_\mu'$.

Lemma 14-3: If $\mu \geq -\frac{1}{2}$, the operation $F \to N_\mu F$ is a continuous linear mapping of $\mathcal{Y}_\mu'$ into $\mathcal{Y}_{\mu+1}'$.

Lemma 14-4: If $\mu \geq -\frac{1}{2}$, the operation $F \to M_\mu F$ is an isomorphism from $\mathcal{Y}_{\mu+1}'$ onto $\mathcal{Y}_\mu'$.

Lemma 14-5: Let Υ be defined as in lemma 12-9. Then, $F \to \Upsilon F$ is a continuous linear mapping of $\mathcal{Y}_\mu'$ into $\mathcal{Y}_\mu'$.

15. The Hankel Transformation of $\mathcal{B}_\mu'$.

Henceforth, we assume that $\mu \geq -\frac{1}{2}$. For $f \in \mathcal{B}_\mu'$, $\varphi \in \mathcal{B}_\mu$, and $\phi = h_\mu \varphi \in \mathcal{Y}_\mu$, we define the Hankel transform $F = h_\mu f$ by

$$\langle F, \phi \rangle = \langle f, \varphi \rangle. \tag{15-1}$$

Clearly, $F \in \mathcal{Y}_\mu'$, and h_μ is a continuous linear mapping of $\mathcal{B}_\mu'$ into $\mathcal{Y}_\mu'$. Equation (15-1) also allows us to define the inverse Hankel transform $f = h_\mu^{-1} F \in \mathcal{B}_\mu'$ when it is $F \in \mathcal{Y}_\mu'$ that is given. It readily follows that h_μ is a one-to-one mapping of $\mathcal{B}_\mu'$ onto $\mathcal{Y}_\mu'$ and that h_μ^{-1} is also continuous and linear on $\mathcal{Y}_\mu'$. In other words, we have

Theorem 6: h_μ is an isomorphism from $\mathcal{B}_\mu'$ onto $\mathcal{Y}_\mu'$.

Lemma 15-1: $\mathcal{N}_\mu^s$ is a subspace of $\mathcal{Y}_\mu^s$. The topology of $\mathcal{N}_\mu^s$ is stronger than the topology induced on it by $\mathcal{Y}_\mu^s$. With the induced topology $\mathcal{N}_\mu^s$ is everywhere dense in $\mathcal{Y}_\mu^s$.

Proof: This follows directly from lemmas 11-6 and 13-1, theorem 6, and the fact that h_μ is a topological automorphism on $\mathcal{N}_\mu^s$.

Theorem 7: Let $\mu \geq -\frac{1}{2}$. If $f \in \mathcal{B}_\mu^s$ and $F = h_\mu f \in \mathcal{Y}_\mu^s$, then

$$h_{\mu+1}(xf) = -N_\mu h_\mu f \, ; \quad h_{\mu+1}^{-1}(\eta F) = -N_\mu h_\mu^{-1} F \tag{15-2}$$

$$h_{\mu+1}(N_\mu f) = -\eta h_\mu f \, ; \quad h_{\mu+1}^{-1}(N_\mu F) = -x h_\mu^{-1} F \tag{15-3}$$

$$h_\mu(x^2 f) = -M_\mu N_\mu h_\mu f \, ; \quad h_\mu^{-1}(\eta^2 F) = -M_\mu N_\mu h_\mu^{-1} F \tag{15-4}$$

$$h_\mu(M_\mu N_\mu f) = -\eta^2 h_\mu f \, ; \quad h_\mu^{-1} M_\mu N_\mu F = -x^2 h_\mu^{-1} F \tag{15-5}$$

If $f \in \mathcal{B}_{\mu+1}^s$ and $F = h_{\mu+1} f \in \mathcal{Y}_\mu^s$ then

$$h_\mu(xf) = M_\mu h_{\mu+1} f \, ; \quad h_\mu^{-1} \eta F = M_\mu h_{\mu+1}^{-1} F \tag{15-6}$$

$$h_\mu(M_\mu f) = \eta h_{\mu+1} f \, ; \quad h_\mu^{-1}(M_\mu F) = x h_{\mu+1}^{-1} F \tag{15-7}$$

Proof: Our proof makes use of lemmas 2-7, 9-3 to 9-6, 11-1 to 11-4, 12-5 to 12-8 and 14-1 to 14-4, and theorems 5 and 6. With lemma 2-7 we also employ analytic continuation from the positive y axis onto the η plane.

If $a \in \mathcal{B}_{\mu+1}$, $\phi = h_{\mu+1} a$ and $f \in \mathcal{B}_\mu^s$, then

$$\langle h_{\mu+1} xf, \phi \rangle = \langle xf, a \rangle = \langle f, xa \rangle = \langle h_\mu f, M_\mu h_{\mu+1} a \rangle = \langle -N_\mu h_\mu f, \phi \rangle .$$

This establishes the first part of (15-2). The second part of (15-3) is gotten by applying $h_{\mu+1}^{-1}$ to the first part of (15-2).

Also,

$$\langle h_{\mu+1} N_\mu f, \phi\rangle = \langle N_\mu f, \varphi\rangle = \langle f, -M_\mu \varphi\rangle = \langle h_\mu f, -\eta h_{\mu+1} \varphi\rangle = \langle -\eta h_\mu f, \phi\rangle,$$

This gives the first part of (15-3), from which the second part of (15-2) is obtained by applying $h_{\mu+1}^{-1}$.

Next, let $\varphi \in B_\mu$, $\phi = h_\mu \varphi$, and $f \in B'_{\mu+1}$. Then,

$$\langle h_\mu x f, \phi\rangle = \langle x f, \varphi\rangle = \langle f, x\varphi\rangle = \langle h_{\mu+1} f, -N_\mu \phi\rangle = \langle M_\mu h_{\mu+1} f, \phi\rangle,$$

which yields the first part of (15-6). The application of h_μ^{-1} yields the second part of (15-7).

Furthermore,

$$\langle h_\mu M_\mu f, \phi\rangle = \langle M_\mu f, \varphi\rangle = \langle f, -N_\mu \varphi\rangle = \langle h_{\mu+1} f, \eta\phi\rangle = \langle \eta h_{\mu+1} f, \phi\rangle,$$

which verifies the first part of (15-7). The second part of (15-6) follows from an application of h_μ^{-1} .

Finally, (15-4) and (15-5) are obtained by appropriately combining (15-2), (15-3), (15-6), and (15-7). Q. E. D.

We conclude this section with the determination of some specific Hankel transforms. First note that, if γ is a nonzero complex parameter, then $\sqrt{x\gamma}\, J_\mu(x\gamma)$ $(\mu \geq -\frac{1}{2})$ is in B'_μ . Hence, for $\varphi \in B_\mu$ and $\phi h_\mu \varphi$,

$$\langle \sqrt{x\gamma}\, J_\mu(x\gamma), \varphi(x)\rangle = \phi(\gamma) = \langle \delta(\eta-\gamma), \phi(\eta)\rangle.$$

(Here, the right-hand equality serves as the definitions for $\delta(\eta-\gamma) \in \mathcal{Y}_\mu'$.) Thus, we have

$$h_\mu[\sqrt{x\gamma}\, J_\mu(x\gamma)] = \delta(\eta-\gamma) \qquad (15\text{-}8)$$

Note that, if $\operatorname{Im} \gamma \neq 0$, then $\delta(\eta-\gamma)$ has no sense as a functional on $\mathcal{H}_\mu$. Thus, h_μ in (15-8) must be interpreted as the extension of the Hankel transformation to $\mathcal{B}_\mu'$.

Furthermore,

$$\left\langle \frac{\partial^\nu}{\partial \gamma^\nu} \sqrt{x\gamma}\, J_\mu(x\gamma), \phi(x) \right\rangle = \int_0^\infty \frac{\partial^\nu}{\partial \gamma^\nu} \left[\sqrt{x\gamma}\, J_\mu(x\gamma)\right] \phi(x)\, dx.$$

It is permissible to interchange the integration and differentiation here [15; p.99], and this yields $\partial^\nu \phi / \partial \gamma^\nu$. We define $\delta^{(\nu)}(\eta-\gamma) \in \mathcal{H}_\mu'$ through

$$\langle \delta^{(\nu)}(\eta-\gamma), \phi(\eta) \rangle = (-1)^\nu \frac{\partial^\nu \phi(\gamma)}{\partial \gamma^\nu}.$$

We have hereby established that

$$h_\mu \frac{\partial^\nu}{\partial \gamma^\nu} \sqrt{x\gamma}\, J_\mu(x\gamma) = (-1)^\nu \delta^{(\nu)}(\eta-\gamma) \quad (\gamma \neq 0, \mu \geq -\tfrac{1}{2}) \tag{15-9}$$

A number of other Hankel transforms can be obtained by using the formulas of theorem 7.

16. An Application : The General Solution of a Homogeneous Bessel-type Differential Equation.

We wish to determine the general solution in $\mathcal{B}_\mu'$ of the following homogeneous differential equation,

$$P(M_\mu N_\mu) f = \left[a_m (M_\mu N_\mu)^m + a_{m-1} (M_\mu N_\mu)^{m-1} + \dots + a_0\right] f(x) = 0 \tag{16-1}$$

where m is a positive integer, the a's are constants with $a_m \neq 0$, $0 < x < \infty$, and $\mu \geq -\frac{1}{2}$. It is known that the distributional solutions are none other than the classical solutions [14; p. 74].

Moreover, if the roots of the polynomial $P(-\eta^2)$ are $\eta = \gamma_n \neq 0$ with the multiplicities k_n, $(n = \pm 1, \ldots, \pm q, \gamma_n = -\gamma_{-n}, k_n = k_{-n})$, then an easy computation shows that a solution to (16-1) is

$$f(x) = \sum_{n=1}^{q} \sum_{\nu=0}^{k_n - 1} b_{n\nu} \frac{\partial^\nu}{\partial \eta^\nu} \left[\sqrt{x\eta}\, J_\mu(x\eta)\right] \Big|_{\eta=\gamma_n} \qquad (16\text{-}2)$$

where the $b_{n\nu}$ denote arbitrary constants. We shall employ our generalized Hankel transformation to verify that (16-1) has no other solutions in $\mathcal{B}'_\mu$.

By applying h_μ to (16-1) and invoking (15-5), we obtain

$$P(-\eta^2)F(\eta) = (\eta^2 - \gamma_1^2)^{k_1}(\eta^2 - \gamma_2^2)^{k_2} \cdots (\eta^2 - \gamma_q^2)^{k_q} F(\eta) = 0 \qquad (16\text{-}3)$$

<u>Lemma 16-1</u>: <u>In order for $X(\eta) \in \mathcal{Y}_{\mu,b}$ to have the form $X(\eta) = P(-\eta^2)\phi(\eta)$, where $\phi(\eta)$ is also in $\mathcal{Y}_{\mu,b}$, it is necessary and sufficient that</u>

$$X^{(\nu)}(\pm\gamma_n) = 0 \quad (\nu = 0, 1, \ldots, k_n - 1;\ n = 1, 2, \ldots, q). \qquad (16\text{-}4)$$

The proof of this lemma is almost identical to that of lemma 1 in [13; Sec. 7.10].

<u>Lemma 16-2</u>: <u>Let</u> $r = \max k_n$. <u>Let</u> $\Lambda_n(\eta)$ $(n = \pm 1, \pm 2, \ldots, \pm q)$ <u>be fixed entire functions</u> (<u>not necessarily even</u>) <u>such that</u>:

1. $|e^{-b|\omega|}\eta^k \Lambda_n(\eta)| < C_k$ $(k = 0, 1, 2, \ldots)$, <u>when the</u> C_k <u>are constants</u>.
2. $\Lambda_n(-\eta) = \Lambda_{-n}(\eta)$
3. <u>At each point</u> $\eta = \gamma_\ell$ $(\ell \neq n)$, Λ_n <u>has a zero with a multiplicity of at least</u> r. <u>At</u> $\eta = \gamma_n$, $\Lambda_n = 1$ <u>and</u> $\Lambda_n^{(\nu)} = 0$ <u>for</u> $\nu = 1, 2, \ldots, r$.

<u>Finally, for any</u> $\psi \in \mathcal{Y}_{\mu,b}$ <u>let</u>

$$\hat{\Psi}^{(\nu)}(\eta) = D^\nu \left[\eta^{-\mu-\frac{1}{2}} \psi(\eta)\right]$$

<u>Then,</u> Ψ <u>can be uniquely represented in the form</u>

$$\Psi(\eta) = X(\eta) + \eta^{\mu+\frac{1}{2}} \sum_{n=1}^{q} \sum_{\nu=0}^{k_n-1} \frac{1}{\nu!} \Psi^{(\nu)}(\gamma_n) [\Lambda_n(\eta)(\eta-\gamma_n)^{\nu} + (-1)^{\nu} \Lambda_{-n}(\eta)(\eta-\gamma_{-n})^{\nu}] \quad (16\text{-}5)$$

<u>where</u> $X(\eta)$ <u>is in</u> $\mathcal{Y}_{\mu,b}$ <u>and satisfies</u> (16-4).

<u>Proof</u>: That there exist Λ_n satisfying the above conditions can be shown as follows. Let Θ be an entire function satisfying condition 1.

Then,

$$\Xi(\eta) = \Theta(\eta) \prod_{\substack{\ell=\pm 1 \\ \ell \neq n}}^{\pm n} (\eta - \gamma_\ell)^r$$

possesses the desired zeros. Finally, we can determine a polynomial H of degree r such that $H(\gamma_n)\Xi(\gamma_n) = 1$ and $D^{\nu} H \Xi \big|_{\eta=\gamma_n} = 0$ $(\nu = 1, 2, \ldots, r)$. Then, $\Lambda_n = H\,\Xi$ is the function we seek.

Next,

$$[\Lambda_n(\eta)(\eta-\gamma_n)^{\nu} + (-1)^{\nu} \Lambda_{-n}(\eta)(\eta-\gamma_{-n})^{\nu}] \quad (16\text{-}6)$$

is an even entire function, and its product with $\eta^{\mu+\frac{1}{2}}$ is in $\mathcal{Y}_{\mu,b}$. Consequently, $X(\eta)$, which is uniquely determined by (16-5), is also in $\mathcal{Y}_{\mu,b}$. That $X(\eta)$ satisfies (16-4) can be shown by differentiating the product of (16-5) with $\eta^{-\mu-\frac{1}{2}}$.

<u>Lemma 16-3</u>: <u>A necessary and sufficient condition for an</u> $F \in \mathcal{Y}'_{\mu}$ <u>to satisfy</u> (16-3) <u>is that</u>

$$F(\eta) = \sum_{n=1}^{q} \sum_{\nu=0}^{k_n-1} (-1)^{\nu} b_{n\nu}\, \delta^{(\nu)}(\eta - \gamma_n) \quad (16\text{-}7)$$

where the $b_{n\nu}$ are arbitrary constants.

Proof: That (16-7) implies (16-3) is clear. Conversely, for any $\phi \in \mathcal{Y}_{\mu,b}$ we have from (16-3) and lemmas 12-9 and 14-5 that $\langle P(-\eta^2)F, \phi\rangle = \langle F, P(-\eta^2)\phi\rangle = 0$. Moreover, for any $\Psi \in \mathcal{Y}_{\mu,b}$, lemma 16-2 allows us to write

$$\langle F, \Psi\rangle = \langle F, \chi\rangle + \sum_{n=1}^{q} \sum_{\nu=0}^{k_n-1} c_{n\nu}\, \hat{\Psi}^{(\nu)}(\gamma_n)$$

where

$$c_{n\nu} = \langle F(\eta), \frac{\eta^{\mu+\frac{1}{2}}}{\nu!}\left[\Lambda_n(\eta)(\eta-\gamma_n)^\nu + (-1)^\nu \Lambda_{-n}(\eta)(\eta-\gamma_{-n})^\nu\right].$$

By lemma 16-1, χ has the form $P(-\eta^2)\phi$, and therefore $\langle F, \chi\rangle = 0$. Writing $\hat{\Psi}^{(\nu)}$ in terms of $\Psi^{(\nu)}$, we see that $F(\eta)$ has the form of (16-7). Note that (16-3) is satisfied by (16-7) for any choice of the $b_{n\nu}$. Q. E. D.

In view of (16-3) and lemma 16-3, the general solution of the homogeneous equation (16-1) is obtained by taking the inverse Hankel transform of (16-7). In accordance with (15-9), we get (16-2).

17. Another Application: An Operational Calculus.

h_μ generates an operational calculus for the solution u in $\mathcal{B}_\mu'$ of differential equations of the form

$$(M_\mu N_\mu)^k u = g \qquad (\mu \geq -\tfrac{1}{2}) \tag{17-1}$$

where $g \in \mathcal{B}_\mu'$ is such that $h_\mu g \in \mathcal{Y}_{\mu-2k}'$. There are many such g's (see lemma 20). By applying h_μ to (17-1) and invoking lemma 14-2, we get the following as the solution in $\mathcal{B}_\mu^{\prime}$ of (17-1).

$$u(x) = \mathfrak{h}_\mu^{-1} \frac{G(\eta)}{(-\eta^2)^k}$$

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