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A CRITERION FOR TRANSITIVITY IN TRANSFINITE CONNECTEDNESS

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Abstract — This work establishes a sufficient set of conditions under which λ-connectedness in a transfinite graph is transitive and thereby is an equivalence relation between branches. This in turn insures that λ-sections do not overlap and that they partition the transfinite graph.

1 Introduction

Transfinite graphs possess many connectedness concepts that are meaningless for conventional graphs. For example, two nodes that are infinitely far apart in a transfinite graph can only be connected by transfinite paths that pass through infinite extremities of conventional subgraphs of that transfinite graph. Moreover, there is a hierarchy of connectedness concepts, namely, "λ-connectedness" for each finite or denumerably infinite ordinal λ. As λ increases, λ-connectedness weakens in the sense that, when $\lambda_1 < \lambda_2$, λ_1-connectedness implies λ_2-connectedness, but not conversely. Thus, two branches may be λ_2-connected but not λ_1-connected.

λ-connectedness is a reflexive and symmetric binary relation between branches, but it need not be transitive [3, Section 3]. As a result, λ-connectedness need not be an equivalence relation between branches, and this has undesirable ramifications. For example, it may prevent unique node voltages in the theory of transfinite electrical networks. A sufficient condition for the transitivity of λ-connectedness was established in [3]. In this work we establish another one. Neither criterion implies the other. Moreover, the one presented herein, although more complicated, is more easily established.

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The prerequisites for an understanding of this work are completely covered in [1] and [3]. The reference [2] can be used in place of [1] and in fact provides a more general discussion. The present paper is written as a sequel to [3]. Of course, ideas that are newly introduced herein are precisely defined below.

We will establish our results for every natural number rank μ, as well as for the ranks $\vec{\omega}$ and ω. Results for still higher ranks are obtained through obvious modifications. In particular, when the rank is a successor ordinal, one need only follow the development for a natural number rank. Similarly, when the rank is an arrow rank $\vec{\theta}$ or a limit ordinal θ, the development to follow is that for $\vec{\omega}$ or respectively ω. (See [3, Section 4] for the definition of an arrow rank $\vec{\theta}$.)

2 The Main Results

In the following ν will denote a countable ordinal, and γ and μ will denote natural numbers. $\gamma+$ will denote any unspecified rank no less than γ and no larger than ν, and $\gamma-$ will be any unspecified rank no less than 0 and less than γ. Thus, $0 \leq (\gamma-) < \gamma \leq (\gamma+) \leq \nu$. $\mathcal{G}^\nu$ will be a ν-graph. $\mathcal{S}^{\gamma-1}$ will be a $\gamma - 1$-section.

Conditions 2.1. *Let μ be a fixed natural number such that $0 < \mu \leq \nu$. For each natural number γ such that $0 \leq \gamma \leq \mu$ and for every two $(\gamma+)$-nodes $n_1^{\gamma+}$ and $n_2^{\gamma+}$ that are incident to the same $(\gamma-1)$-section $\mathcal{S}^{\gamma-1}$, there exists a two-ended (i.e., finite) γ-path P^γ such that either* **(a)** *its traversed tips — other than those incident to $n_1^{\gamma+}$ and $n_2^{\gamma+}$ — are not shorted to any tips not traversed by P^γ or* **(b)** *P^γ satisfies the following three conditions:*

(b1) *All the branches of P^γ are embraced by $\mathcal{S}^{\gamma-1}$.*

(b2) *One terminal node of P^γ is embraced by $n_1^{\gamma+}$, and the other by $n_2^{\gamma+}$.*

(b3) *P^γ does not meet any boundary node of a $((\gamma-1)+)$-section other than $n_1^{\gamma+}$ and $n_2^{\gamma+}$.*

With regard to the two conditions (a) and (b), (a) does not require that P^γ be embraced by $\mathcal{S}^{\gamma-1}$, whereas (b) does do so. The simultaneous satisfaction of both conditions is allowed. (a) can be restated in more suggestive terms by saying that P^γ is an isolated path totally disjoint form the rest of $\mathcal{G}^\nu$ except at its terminal nodes. (b) requires that the only way one can leave $\mathcal{S}^{\gamma-1}$ at a node of P^γ is at a terminal node of P^γ.

When $\gamma = 0$, $\mathcal{S}^{\gamma-1}$ is a (-1)-section, that is, a single branch. Any two nodes incident to that branch will automatically satisfy Conditions 2.1.

Theorem 2.2. *Let $\mathcal{G}^\nu$ be such that Conditions 2.1 are satisfied. Then, for each γ such that $0 \leq \gamma \leq \mu$, γ-connectedness is transitive in $\mathcal{G}^\nu$; that is, it is a transitive binary relation among the branches of $\mathcal{G}^\nu$ — as well as among the nodes of $\mathcal{G}^\nu$.*

Proof. The transitivity of γ-connectedness for the branches will follow from that for the nodes. Choose γ and μ as in Conditions 2.1. Let n_a, n_b, and n_c be three totally disjoint nodes of any ranks, possibly different ranks. Consider the two statements

(i) n_a and n_b are γ-connected, and n_b and n_c are γ-connected.

(ii) n_a and n_c are γ-connected.

We shall prove that (i) implies (ii) inductively, and thereby the transitivity of γ-connectedness. This is true for $\gamma = 0$ because 0-connectedness is always transitive. (Moreover, Conditions 2.1 are always satisfied when $\gamma = 0$, as was noted above.) So, assume that (i) implies (ii), where now $0 \leq \gamma < \mu$. Hence, γ-connectedness is transitive, and γ-sections partition $\mathcal{G}^\nu$. We shall show that (i) implies (ii) for γ replaced by $\gamma + 1$.

Let $P_{ab}^{\gamma+1}$ be a two-ended $(\gamma+1)$-path connecting n_a and n_b and oriented from n_a to n_b, and let $P_{ba}^{\gamma+1}$ be the same path but with the reversed orientation. Also let $P_{bc}^{\gamma+1}$ be a two-ended $(\gamma+1)$-path connecting n_b to n_c and oriented from b to c. We shall construct a two-ended $(\gamma+1)$-path connecting n_a to n_c.

Since $P_{ba}^{\gamma+1}$ and $P_{bc}^{\gamma+1}$ are two-ended $(\gamma+1)$-paths, they each traverse only finitely many γ-tips. All other tips traversed by these paths will have ranks less than γ. If $P_{ba}^{\gamma+1}$ and $P_{bc}^{\gamma+1}$ meet at a node n, they will meet there with four tips (two for each), except when n is n_b, in which case they will meet at n_b with two tips (one for each). If all four tips (or both tips when $n = n_b$) have ranks less than γ, they will have representatives that are γ-connected; in this case, we shall say that $P_{ba}^{\gamma+1}$ and $P_{bc}^{\gamma+1}$ *meet inside a γ-section.* $P_{ba}^{\gamma+1}$ and $P_{bc}^{\gamma+1}$ can meet inside a γ-section infinitely many times because they can traverse infinitely many tips of ranks less than γ. On the other hand, if at least one of those four (or two) tips has a rank γ, we shall say that $P_{ba}^{\gamma+1}$ and $P_{bc}^{\gamma+1}$ *meet with a γ-tip*; $P_{ba}^{\gamma+1}$ and $P_{bc}^{\gamma+1}$ can meet with a γ-tip only finitely many times since they traverse only finitely many γ-tips.

Now let us consider the various ways $P_{ba}^{\gamma+1}$ and $P_{bc}^{\gamma+1}$ may meet.

Case 1: They may meet only at n_b, in which case $P_{ab}^{\gamma+1} \cup P_{bc}^{\gamma+1}$ is a two-ended $(\gamma+1)$-path connecting n_a and n_b. So, assume in the following three cases that $P_{ba}^{\gamma+1}$ and $P_{bc}^{\gamma+1}$ meet at at least one node totally disjoint from n_b.

Case 2: If n_a and n_c are both incident to the same γ-section, then, since $\gamma < \mu$ now, we can invoke Condition 2.1 to conclude that n_a and n_c are $(\gamma+1)$-connected. So, assume in the following two cases that n_a and n_c are not incident to the same γ-section.

Case 3: Let us now trace $P_{ba}^{\gamma+1}$ and $P_{bc}^{\gamma+1}$ starting at n_b. Assume these paths meet inside at least one γ-section (possibly infinitely often). They will do so only within finitely many γ-sections. Let $\mathcal{S}_l^\gamma$ be the last such γ-section that $P_{ba}^{\gamma+1}$ meets and let n_d be the last boundary node of $\mathcal{S}_l^\gamma$ that $P_{ba}^{\gamma+1}$ meets. Let $P_{ad}^{\gamma+1}$ be the $(\gamma+1)$-path obtained by tracing $P_{ab}^{\gamma+1}$ from n_a to n_d. $P_{ad}^{\gamma+1}$ will be the trivial path $\{n_a\}$ or $\{n_d\}$ if n_a embraces n_d or conversely. Also, let n_e be the last boundary node of $\mathcal{S}_l^\gamma$ that $P_{bc}^{\gamma+1}$ meets. If n_d and n_e are totally disjoint, we can invoke Conditions 2.1 to assert the existence of a two-sided $(\gamma+1)$-path $P_{de}^{\gamma+1}$ that connects n_d and n_e and does not meet $P_{ad}^{\gamma+1}$ except terminally at n_d. $P_{de}^{\gamma+1}$ may also be a trivial path. Let $P_{ec}^{\gamma+1}$ be the (possibly trivial) path obtained by tracing along $P_{bc}^{\gamma+1}$ from n_e to n_c. If $P_{ad}^{\gamma+1}$ and $P_{ec}^{\gamma+1}$ do not meet, then $P_{ad}^{\gamma+1} \cup P_{de}^{\gamma+1} \cup P_{ec}^{\gamma+1}$ is a two-ended $(\gamma+1)$-path connecting n_a to n_c. If $P_{ad}^{\gamma+1}$ and $P_{ec}^{\gamma+1}$ do meet, they will do so with a γ-tip, and there will be a first node n_f at which $P_{ad}^{\gamma+1}$ meets $P_{ec}^{\gamma+1}$ — "first" with respect to the orientation of $P_{ad}^{\gamma+1}$ from n_a to n_d. Possibly $n_f = n_a$ or $n_f = n_c$. Now, let $P_{af}^{\gamma+1}$ (or $P_{fc}^{\gamma+1}$) be the path obtained by tracing along $P_{ab}^{\gamma+1}$ from n_a to n_f (respectively, by tracing along $P_{bc}^{\gamma+1}$ from n_f to n_c) Then, $P_{af}^{\gamma+1} \cup P_{fc}^{\gamma+1}$ is the $(\gamma+1)$-path we seek. In the event that $n_f = n_a$ or $n_f = n_c$, one of these two paths will be trivial.

Case 4: Finally, assume that $P_{ba}^{\gamma+1}$ and $P_{bc}^{\gamma+1}$ do not meet inside any γ-section. Then, there is a last node n_f ("last" with respect to the orientation of $P_{ba}^{\gamma+1}$) at which $P_{ba}^{\gamma+1}$ and $P_{bc}^{\gamma+1}$ meet with a γ-tip. Then, as in Case 3, $P_{af}^{\gamma+1} \cup P_{fc}^{\gamma+1}$ is the desired path.

This exhausts all possibilities and completes the proof. ♣

Corollary 2.3. *Let $\mathcal{G}^\nu$ be such that Conditions 2.1 are satisfied. Also, let θ be any ordinal such that $\gamma < \theta \leq \nu$. Then, any θ-section is partitioned by the γ-sections it embraces. (In particular, $\mathcal{G}^\nu$ is partitioned by the γ-sections.)*

Proof. Since γ-connectedness is obviously a reflexive and symmetric binary relation between branches, it follows from Theorem 2.2 that it is an equivalence relation between the branches of $\mathcal{G}^\nu$. Moreover, any two-ended γ-path that connects two branches of a θ-section will remain within that θ-section — whence our conclusion. ♣

In order to extend our results to $\vec{\omega}$-connectedness and ω-connectedness, we need other kinds of "sections." An *$\vec{\omega}$-section* (or *ω-section*) is a reduced graph induced by a maximal set of branches that are pairwise $\vec{\omega}$-connected (respectively, ω-connected).

Corollary 2.4. *If Condition 2.1 holds for every natural number μ, then $\vec{\omega}$-connectedness is transitive in $\mathcal{G}^\nu$ ($\nu \geq \vec{\omega}$), and $\vec{\omega}$-sections partition any θ-section whenever $\theta > \vec{\omega}$.*

Proof. $\vec{\omega}$-connectedness between two nodes means that the two nodes are μ-connected for some natural number μ depending on the choice of the two nodes. The transitivity and partitioning properties of $\vec{\omega}$-connectedness thereby follow from those of μ-connectedness. ♣

As before, $\vec{\omega}+$ (or $\omega+$) will denote any unspecified rank no less than $\vec{\omega}$ (respectively, ω) and no larger than ν.

Conditions 2.5. *Conditions 2.1 hold for every natural number μ and in addition the following is required with $\omega \leq \nu$. For every two $(\omega+)$-nodes $n_1^{\omega+}$ and $n_2^{\omega+}$ that are incident to the same $\vec{\omega}$-section $\mathcal{S}^{\vec{\omega}}$, there exists a two-ended ω-path P^ω such that either* **(a)** *its traversed tips — other than those incident to $n_1^{\omega+}$ and $n_2^{\omega+}$ — are not shorted to any tips not traversed by P^ω or* **(b)** *P^ω satisfies the following three conditions:*

(b1) *All the branches of P^ω are embraced by $\mathcal{S}^{\vec{\omega}}$.*

(b2) *One terminal node of P^ω is embraced by $n_1^{\omega+}$, and the other by $n_2^{\omega+}$.*

(b3) *P^ω does not meet any boundary node of an $(\vec{\omega}+)$-section other than $n_1^{\omega+}$ and $n_2^{\omega+}$.*

These last two conditions (a) and (b) are the same as those of Conditions 2.1 but with the following replacements: γ is replaced by ω, and $\gamma - 1$ by $\vec{\omega}$.

Corollary 2.6. *Assume $\mathcal{G}^\nu$ $(\nu \geq \omega)$ satisfies Conditions 2.5. Then, ω-connectedness is transitive in $\mathcal{G}^\nu$, and ω-sections partition any θ-section whenever $\theta > \omega$.*

The proof of this corollary is the same as that of Theorem 2.2 but with obvious changes in wording and notations, such as the replacement of Conditions 2.1 by Conditions 2.4 and the replacements of symbols indicated just above.

3 A Final Note

The results obtained in this paper are different from those of [3]. In particular, both Conditions 2.1 and 2.5 allow nondisconnectable nonopen tips that are not shorted. Hence, those conditions do not imply Condition 6.1 of [3], which prohibited such tips. Conversely, the prohibition of such tips does not imply Conditions 2.1 or 2.5. Thus, even though Theorem 2.2 has the same conclusion as Theorem 6.2 of [3], neither theorem subsumes the other, and the same is true with regard to their corollaries.

References

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[3] A.H.Zemanian, Connectedness in Transfinite Graphs and the Existence and Uniqueness of Node Voltages, *Discrete Mathematics*, in press.