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Abstract

Negative customers and signals in open queueing networks were originally introduced by Gelenbe, and Chao and Pinedo. Open networks of queues with positive customers and negative customers have been shown to have product form solutions for the state probabilities in equilibrium. The goal of this paper is to demonstrate that an algebraic topological interpretation can be applied to networks of queues with positive and negative customers to explain the existence of such product form solutions. In particular it is shown that this is true for certain fundamental queueing networks.

Key Words: Algebraic topology, negative customer, product form solution and queueing network.

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1 Introduction

Networks of queues have been the subject of much research in the past. A simple analytical result for the state probabilities of networks of queues in equilibrium is the product form solution. In the product form solution, the joint state probability of the states of a network of queues is simply expressed as a product of functions of the individual queues of the network. This was first demonstrated by Jackson [5] for an open network of state independent queues. The functions of the individual queues are just the marginal state probabilities. This result was generalized by Jackson [6] and extended by Gordon and Newell [4] to closed networks. Baskett, Chandy, Muntz and Palacios [1] then generalized the families of queueing networks with product form solutions.

However, a product form solution for most networks is more often the exception rather than the rule. Quite often, no closed form solution can be found and only an iterative or recursive solution exists. The question that arises then is why do product form solutions exist for some networks and do not exist for others. To answer this, Lazar and Robertazzi [8] and Lazar [7] discussed a method through which it is possible to distinguish between a network with a product form solution and one without. Their method involves the state transition diagram (lattice) of the network. Basically, a product form solution for a network of queues exists when a certain algebraic topological structure called a *building block* which is defined below, can be isolated from the state transition diagram. Using building blocks, the state probabilities relative to a state probability of a reference state can be easily found. The state probability of a specific state is obtained by tracing a "path" from the specific state back to the reference state. The concept of building blocks provides an elegant tool through which it can be determined whether a product form solution exists. It also provides a method to decide what modifications need to be made to a queueing network protocol so that the modified network has a product form solution. Building blocks can also be used to explain why certain queueing models have product form solutions and why others do not.

In recent years Gelenbe [3] introduced a new type of customer called a negative customer. There are two possible ways that a negative customer can be created. The first instance is an external arrival from outside the system. The other case occurs when a normal customer leaves a queue and where there is a certain probability that upon departure it may become a negative customer. When a negative customer arrives at a queue, it does not act like a normal customer. Instead, it instantly removes a normal customer (if there are any present) from the queue and vanishes from the system. Chao and Pinedo [2] refined this idea by defining a type of customer called a signal. When a signal arrives at a queue, there is a fixed probability that one of the customers in the queue will complete its service and leave the queue. Which customer is chosen to complete its service depends on the number of customers in the queue and their respective classes. The probability that the chosen customer finishes its service would also depend on the number of customers in the queue and the class of the customer chosen. From here on, a normal customer will be referred to as a positive customer and the term negative customer will include signals as defined by Chao and Pinedo. One subtle difference between the two models is that Gelenbe's negative customer just reduces the queue length. On the other hand the signal as defined by Chao and Pinedo causes a positive customer to complete service, thus contributing to the throughput of the system. In both cases though, if a negative customer or signal arrives at an empty queue, it just vanishes and nothing happens. Another consequence of the above difference between negative customers and signals is that the traffic equations for Gelenbe's model are non-linear but in Chao's model, they are linear.

Negative customers can be used to model several situations in communication networks. One example would be in buffer management where negative customers can be used to model congestion control schemes implementing Quality of Service (QOS) constraints. In a typical QOS implementation, packets in a queue that have exceeded a QOS delay threshold should be removed to allow more recent packets to enter the queue. Negative customers can be used to represent the mechanism through which packets can be removed. Another example would be a packet network with a priority scheme where some classes of packets have a higher priority than others. The arrival of negative customers and the resultant departure of a customer can be used to model the activity of higher priority packets. Negative customer networks have also been proposed for neural network modeling by Gelenbe [3].

The most important common characteristic of the above two models is that they both have product form solutions. One question that naturally arises is can an algebraic topological interpretation be applied to a network of queues of both negative and positive customers to find building blocks and hence a product form solution? In this paper we find that, in at least certain fundamental cases, it can be.

In section 2, some terms that are needed to explain algebraic topological interpretations are defined. The system description is given in section 3 followed by examples of different queue topologies in sections 4, 5 and 6. Some concluding remarks will be made in section 7.

2 Algebraic Topological Interpretation

The main goal in an algebraic topological interpretation of the product form solution is to determine the existence of building blocks. The reason is that the existence of building blocks implies the existence of a product form solution. In order to do this several terms are first defined. A detailed introduction to these terms and examples illustrating them can be found in Robertazzi [9]. Another useful concept called local balance will also be explained. Note that this paper deals with queues in equilibrium only.

Definition 1: Probability Flux: *In equilibrium, the probability flux of a transition is the average number of times per second that the transition is transited by the system state. It is equal to the rate of the transition multiplied by the state probability of the state that the transition originates from.*

This definition of probability flux allows the phenomenon of local balance to be explained. Local balance in equilibrium occurs when the sum of probability flux on a set of transitions entering a state equals the sum of probability flux on a set of transitions leaving the state. Note that this differs from the concept of global balance which states that the probability flux due to all transitions incident on a state is equal to the probability flux due to all transitions leaving the state. The existence of local balance will be used later to determine the existence of building blocks.

Definition 2: Circulatory Structure: *The pattern of probability flux in a state transition diagram is called the circulatory structure of the state transition diagram.*

Definition 3: Isolated Circulation: *An isolated circulation is the probability flux of a subset of transition edges which is conserved if the subset of transitions is embedded into the overall state transition diagram.*

There is a very useful property that can be directly attributed to an isolated circulation. When an isolated circulation occurs in a state transition diagram, the relative state equilibrium properties of the states adjacent to the isolated circulation can be solved without having to take into consideration the rest of the states in the state transition diagram.

Definition 4: Cyclic Flow: *If the probability flux for the isolated circulation forms a closed loop, it is then called a cyclic flow.*

A cyclic flow along L edges or transitions is simply a cyclic flow of length L. A "closed loop" in the above definition refers to a set of disjoint transitions that begin and end on the same node.

Definition 5: Building Block: *A building block is a set of transition edges and associated nodes that form a closed system such that the flux on the building block edges is conserved at the building block nodes if the building block is embedded in the overall state transition diagram.*

Put another way, building blocks and the isolated circulations that flow through them can be removed from the state transition diagram without affecting the rest of the state transition diagram and circulatory structure except for a renormalization. They can then be solved to find the relative state probabilities.

If a state transition diagram can be decomposed (aggregated) into (from) a collection of building blocks the implication is that a product form solution exists. This is because for each building block the adjacent states can be solved in relation to each other for the equilibrium state probabilities. Any state probability can be calculated by solving building blocks for their relative state probabilities along a path from the state of interest to a reference state (usually an empty queues state). The solutions generate a multiplicative recursion that is in fact the product form solution.

In order to find the building blocks of a system, the matching transition "pairs" at a node should first be found. They can usually be found by finding local balance among the transitions. A transition "pair" is a set of transitions that can be paired together to obtain local balance. These transitions are not necessarily a proper pair but could instead consist of several disjoint transitions. Once the transition pairs can be found, then the building blocks can be "constructed" from these transition pairs. The building block is found by identifying the probability flux flow that forms the edges of the building blocks. The flow can be found by tracing transitions through the adjacent states where the probability flux is conserved.

Consider the example in Figure 1. Here only the transitions of interest entering or leaving the state (i,j) are shown. There would be other transitions that are not shown. The probability flux due to transition a is found to be equal to the sum of the probability fluxes due to transitions b, c, and d. Thus, there is local balance along the boundary labeled A in Figure 1 and a and (b, c, d) form a transition "pair". Transition a can now be decomposed into subtransitions a_b, a_c and a_d so as to form one to one matching transition pairs as shown in Figure 2.

The proportion of probability flux from transition a matching the probability flux from transition d is associated with the transition a_d. This is the first step in finding the building block, i.e. finding a possible starting transition of the flow. The proportion of the probability flux due to transition d is now known and the process of tracing the rest of the building block can begin by tracing the transition a to (i+1,j) and finding the transition or subtransition that gives the same probability flux. The process continues to the next node until a return is made to the starting node. At each step, local balance must be satisfied. To carry out the preceding procedure all the state probabilities must be known in advance. This can be done by using Chao's product form equations or Gelenbe's product form equations. The complete process of finding a building block for a system of queues will be illustrated later.

3 Systems of Queues Description

The systems of queues under investigation have the following characteristics: positive customers arrive at queue i at the rate Λ_i and negative customers arrive at queue i at the rate λ_i. The arrival processes are assumed to be Poisson in nature. Each queue follows a First In First Out (FIFO) discipline and the ith queue has an exponential server with a service rate μ_i. The characteristics of negative customers are chosen to conform to the signals as defined by Chao and Pinedo. There is no differentiation as to whether a customer leaves a system due to a service completion or a negative customer arrival. The service completion time of a customer is independent of the amount of service that it has already received since the service times are assumed to be negative exponential random variables. When a negative customer arrives at a queue, one customer that is already present in the queue is selected randomly to leave. The probability that the chosen customer leaves due to the arrival of a negative customer is chosen to be one. When a customer leaves a queue i for another queue j, it may change its type. The probability of it remaining a positive customer is p_{ij}^{+} and the probability of it changing to a negative customer is p_{ij}^{-}. Thus,

$$\sum_{x=-,+} p_{ij}^{x} = 1 \tag{3.1}$$

Three queueing systems will be examined. The first system consists of two queues in series or in tandem. All external arrivals enter the first queue only and all positive and negative departing customers leaving queue Q1 enter queue Q2. Customers can only depart from the system through queue Q2. There is no feedback from queue Q2 to queue Q1.

The second system has two queues in parallel. Customers leaving each queue can either exit the system with the probability d_i (where i refers to queue i) or go to queue j with the probability p_{ij}^{+}. Only positive customers will be fed back due to the existence of a phenomenon called the "ping-pong" effect which will be discussed in detail in the section 5.

The third and final system is a single queue with multiple class customers entering the system. For simplicity, a positive customer system with only two classes and negative customer arrivals will be examined. The state variables are more complex since the queueing discipline is FIFO and thus there is a need to keep track of the order in which customers arrive into the system.

4 Topology One: Series Queues

The first system of queues to be examined is a series combination of two queues with the output of queue Q1 connected to the input of queue Q2. The queueing system is illustrated in Figure 3. Positive and negative customers arrive to Q1 with mean Poisson rates Λ_1 and λ_1, respectively. Customers leaving Q1 enter Q2 as either positive or negative customers with probability p^+ and p^-, respectively. There are no external arrivals into queue Q2. The type of customer leaving queue Q2 is unimportant since it leaves the system. The state transition diagram is illustrated in Figure 4.

The x coordinates in the state transition diagram are the number of customers in Q1 and the y coordinates are the number of customers in Q2. The bottom-most row states in the state transition diagram have transitions moving horizontally from right to left. That is, from a state i to a state $i-1$ the service rate is $(\mu_1 + \lambda_1)p^-$. These transitions represent a customer leaving queue Q1 for Q2. This could be either through a service completion or an arrival of a negative customer in queue Q1. Since Q2 is empty when the state is in the bottom-most row, the net effect is of a customer leaving the system. On the other hand for the other rows, when a customer leaves Q1 and becomes a negative customer arriving at a non-empty Q2, then the net effect is of two customers leaving the system, hence the transition from state (i,j) to state (i-1,j-1) at a rate $p^-(\lambda_1 + \mu_1)$. The other transition of interest is from a state (i,j) to a state (i-l,j+1) which occurs when a positive customer moves from Q1 to Q2 at a rate of $p^+(\lambda_1 + \mu_1)$.

4.1 Local Balance

Using the product form equations of Chao and Pinedo [2], the probability distribution of the number of customers in queue Q1 is given by:

$$\Pi(n_i) = (1 - \frac{\Lambda_1}{\mu_1 + \lambda_1})(\frac{\Lambda_1}{\mu_1 + \lambda_1})^{n_i} \tag{4.1}$$

The probability distribution of the number of customers in queue Q2 is given by:

$$\Pi(n_j) = (1 - \frac{\Lambda_1 p^+}{\mu_2 + \Lambda_1 p^-})(\frac{\Lambda_1 p^+}{\mu_2 + \Lambda_1 p^-})^{n_j} \tag{4.2}$$

Consider the state (i,0) in the bottom-most row. There are three incoming transitions and one outgoing transition (Figure 5). Local balance conditions are satisfied across boundary A and boundary B (Figure 5).

Now consider the states in the left-most column of the state transition diagram (Figure 6). Since state (0,0) is accounted for by the bottom most row, only states starting from state (0,1) and above need to be considered.

Consider state (0,j). There are three transitions from (0,j+1), (1,j+1) and (1,j-1) that end on this state. It can easily be shown that the probability flux of the three incoming transitions and one single outgoing transition from (0,j) to (1,j) show local balance across boundary A (Figure 6) and thus form transition pairs. Note that Figure 6 does not show the transition Λ_1 from state (0,j) to (1,j). Instead it shows how the transition should be decomposed in order to illustrate the building blocks which will be discussed in the next subsection. In a similar manner local balance can be shown to exist at every node in the state transition diagram.

4.2 Formulation of building blocks

The building blocks for the bottom-most row will be determined first. A possible building block with a cyclic length two labeled *a* in Figure 5 goes from (i,0) to (i+1,0) and then back to (i,0). Another possible building block of length three labeled b starts from (i,0) goes to (i+1,0) then to state (i,1) and then back to (i,0). Finally another suspected building block labeled c and of length four has the path (i,0), (i+1,0), (i,1), (i+1,1) and back to (i,0). Now it remains to verify that these potential building blocks contain truly cyclic flows by checking that the probability flux is conserved along all the edges. Conservation will first be established for the probability flux from (i,0) to (i+1,0). The probability flux due to flow a is:

$$(1-\frac{\Lambda_1}{\mu_1+\lambda_1})(1-\frac{\Lambda_1 p^+}{\mu_2+\Lambda_1 p^-})(\frac{\Lambda_1}{\mu_1+\lambda_1})^{i+1}(\mu_1+\lambda_1)p^- \tag{4.3}$$

Since the probability flux of the returning transition can be show to be equal to this, the suspected building block is indeed a building block. The probability flux due to the cyclic flow b is

$$(1-\frac{\Lambda_1}{\mu_1+\lambda_1})(1-\frac{\Lambda_1 p^+}{\mu_2+\Lambda_1 p^-})(\frac{\Lambda_1}{\mu_1+\lambda_1})^{i}(\frac{\Lambda_1 p^+}{\mu_2+\Lambda_1 p^-})\mu_2 \tag{4.4}$$

and this is conserved along the edges of the cyclic flow indicating that this is a building block. Finally the probability flux due to the cyclic flow c is

$$(1-\frac{\Lambda_1}{\mu_1+\lambda_1})(1-\frac{\Lambda_1 p^+}{\mu_2+\Lambda_1 p^-})(\frac{\Lambda_1}{\mu_1+\lambda_1})^{i+1}(\frac{\Lambda_1 p^+}{\mu_2+\Lambda_1 p^-})(\mu_1+\lambda_1)p^- \tag{4.5}$$

Summing up these three probability fluxes yields a flux of:

$$(1-\frac{\Lambda_1}{\mu_1+\lambda_1})(1-\frac{\Lambda_1 p^+}{\mu_2+\Lambda_1 p^-})(\frac{\Lambda_1}{\mu_1+\lambda_1})^{i}\Lambda_1 \tag{4.6}$$

This is the probability flux from (i,0) to (i+1,0), indicating that all the building blocks associated with the transition rate of Λ_1 from (i,0) to (i+1,0) have been accounted for. The transition from (i,0) to (i-1,0) and (i-1,1) are part of the building blocks for the node (i-1,0). The above results are applicable to any states along the bottom-most row since the relative differences rather than the absolute numbering of the states are significant.

A portion of the two left-most columns of states is shown in Figure 6. There are two suspected building blocks, one of length three and one of length four which are denoted a and b respectively in Figure 6. The other building block is labeled c but it is associated with state (0,j-1). To verify that a is indeed a building block, the cyclic flow from (0,j) to (1,j) and to (0,j+1) must be checked. The edges do form a cyclic flow and thus the suspected building block is indeed one. Using a similar method, b is also proven to be a building block. Summing up all the contributions from (0,j) to (1,j), results in a probability flux that is equivalent to a probability flux from (0,j) to (1,j) at a rate of Λ_1. There is local balance across boundary A. Thus all the relevant transitions have been accounted for.

This result is applicable to all states in the left-most columns. It can be extended to the succeeding columns in a straight forward manner. Thus it is possible to use this result to decompose a state transition diagram for two queues with a tandem structure and positive and negative customers into elementary building blocks. As discussed previously, a consequence of this building block structure is the existence of a product form solution for the equilibrium state probabilities [9].

5 Topology Two: Parallel Queues

The case of the two parallel queues shown in Figure 7 will now be investigated. A queue Qi has external positive customer arrivals at a rate Λ_i and negative customer arrivals at a rate λ_i. Positive customers are now allowed to move between the two queues. Negative customer feedback is disallowed due to the existence of a phenomenon called the "ping-pong" effect. The "ping-pong" effect occurs when a service completion results in the creation of a negative customer. This negative customer is then fed back to the other queue in turn possibly creating another negative customer. This negative customer can be fed back to the other queue in turn again possibly creating yet another negative customer. This may go on until one queue is empty. All this occurs in zero time. The resulting state transition diagram has transitions that make it unclear whether building blocks can be found. Note that these ping-pong transitions "jump" over substantial parts of the state transition diagram. It is an open problem as to whether such a state transition diagram has a building block structure.

Let d_i be the probability that a customer leaves the system from queue Qi. The x ordinates of the state transition diagram represent the number of customers in queue Q1 and the y ordinates represent the number of customers in queue Q2. From Figure 8, it can be seen that the state transition diagram is symmetrical, unlike that of topology one. Thus there is no need to differentiate the bottom-most row from the other rows. Consider the horizontal transitions from (i,j) to (i-1,j). The transition rate is $(\lambda_1 + \mu_1)d_1$ which is simply the total rate of service of the queue consisting of μ_1, the queue service rate and the "secondary"service rate λ_1 multiplied by the probability that the customer leaves the system, d_1. A similar expression can be obtained for queue Q2. The two diagonal transitions represent the movement of positive customers between the two queues.

5.1 Local Balance

Once again from Chao and Pinedo [2], the probability distribution of the number of customers in queue Q1:

$$\Pi(n_i) = (1 - \frac{\Lambda_1 + \Lambda_2 p_{21}^+}{(\lambda_1 + \mu_1)(1 - p_{12}^+ p_{21}^+)})(\frac{\Lambda_1 + \Lambda_2 p_{21}^+}{(\lambda_1 + \mu_1)(1 - p_{12}^+ p_{21}^+)})^{n_i} \tag{5.1}$$

and the probability distribution of the number of customers in queue Q2 is:

$$\Pi(n_j) = (1 - \frac{\Lambda_2 + \Lambda_1 p_{12}^+}{(\lambda_2 + \mu_2)(1 - p_{12}^+ p_{21}^+)})(\frac{\Lambda_2 + \Lambda_1 p_{12}^+}{(\lambda_2 + \mu_2)(1 - p_{12}^+ p_{21}^+)})^{n_j} \tag{5.2}$$

Consider the three nodes (0,0), (1,0) and (0,1) and transitions between them as shown in Figure 9. Local balance can be found for each of the boundaries A, B and

C. For example the probability flux entering and leaving node (0,0) across boundary C can be shown to be:

$$(1-\frac{\Lambda_1+\Lambda_2 p_{21}^+}{(\lambda_1+\mu_1)(1-p_{12}^+p_{21}^+)})(1-\frac{\Lambda_2+\Lambda_1 p_{12}^+}{(\lambda_2+\mu_2)(1-p_{12}^+p_{21}^+)})(\Lambda_1+\Lambda_2). \tag{5.3}$$

Similarly the probability flux leaving and entering (0,1) across boundary A can be found to be after simplification:

$$(1-\frac{\Lambda_1+\Lambda_2 p_{21}^+}{(\lambda_1+\mu_1)(1-p_{12}^+p_{21}^+)})(1-\frac{\Lambda_2+\Lambda_1 p_{12}^+}{(\lambda_2+\mu_2)(1-p_{12}^+p_{21}^+)})(\frac{\Lambda_2+\Lambda_1 p_{12}^+}{1-p_{12}^+p_{21}^+}). \tag{5.4}$$

The flux leaving and entering (1,0) across boundary B can be found to be after some simplification:

$$(1-\frac{\Lambda_1+\Lambda_2 p_{21}^+}{(\lambda_1+\mu_1)(1-p_{12}^+p_{21}^+)})(1-\frac{\Lambda_2+\Lambda_1 p_{12}^+}{(\lambda_2+\mu_2)(1-p_{12}^+p_{21}^+)})(\frac{\Lambda_1+\Lambda_2 p_{21}^+}{1-p_{12}^+p_{21}^+}). \tag{5.5}$$

The existence of the local balances indicate that the search for building blocks should begin at boundaries A, B and C.

5.2 Formulation of building blocks

The formulation of building blocks is done more easily when split into 2 separate cases: 1) symmetrical and 2) asymmetrical flow. The symmetrical case arises when $\Lambda_1 = \Lambda_2$, $\lambda_1 = \lambda_2$ and $p_{12}^+ = p_{21}^+$. Then the probability flux from (0,0) to (1,0) and from (1,0) to (0,0) is equal implying then that the two transitions form a building block of cyclic length two. This is very easily demonstrated by first noting that the probability flux from (1,0) to (0,0) is proportional to:

$$\frac{\Lambda_1+\Lambda_2 p_{21}^+}{(\lambda_1+\mu_1)(1-p_{12}^+p_{21}^+)}(\lambda_1+\mu_1)d_1. \tag{5.6}$$

which after including the probability of state (0,0) and some manipulation becomes:

$$(1-\frac{\Lambda_1+\Lambda_2 p_{21}^+}{(\lambda_1+\mu_1)(1-p_{12}^+p_{21}^+)})(1-\frac{\Lambda_2+\Lambda_1 p_{12}^+}{(\lambda_2+\mu_2)(1-p_{12}^+p_{21}^+)})(\frac{\Lambda_1+\Lambda_2 p_{21}^+-\Lambda_1 p_{12}^+-\Lambda_2 p_{21}^+p_{12}^+}{1-p_{12}^+p_{21}^+}). \tag{5.7}$$

Then if the symmetrical conditions are met, the previous expression just reduces to

$$(1-\frac{\Lambda_1+\Lambda_2 p_{21}^+}{(\lambda_1+\mu_1)(1-p_{12}^+p_{21}^+)})(1-\frac{\Lambda_2+\Lambda_1 p_{12}^+}{(\lambda_2+\mu_2)(1-p_{12}^+p_{21}^+)})\Lambda_1. \tag{5.8}$$

which is just simply the left to right flow from (0,0) to (1,0) thus verifying the existence of the building block. Similarly, other building blocks between (0,0) and (0,1) and between (0,1) and (1,0) can be found. Thus all the building blocks are of cyclic length two. In the asymmetrical case, it is first assumed that:

$$\Lambda_1 p(0,0) > (\mu_1+\lambda_1)d_1 p(1,0) \tag{5.9}$$

$$\Lambda_1 > (\mu_1+\lambda_1)d_1(\frac{\Lambda_1+\Lambda_2 p_{21}^+}{(\lambda_1+\mu_1)(1-p_{12}^+p_{21}^+)}) \tag{5.10}$$

In other words, the "excess" probability flux flows in a counter clockwise direction from (0,0) to (1,0). Then the value of this flow is simply:

$$\Lambda_1 p(0,0) - (\mu_1+\lambda_1)d_1 p(1,0) \tag{5.11}$$

$$\Lambda_1 - (\mu_1+\lambda_1)d_1(\frac{\Lambda_1+\Lambda_2 p_{21}^+}{(\lambda_1+\mu_1)(1-p_{12}^+p_{21}^+)}) = \frac{\Lambda_1(1-d_1) - p_{12}^+p_{21}^+\Lambda_1 - d_1 p_{21}^+\Lambda_2}{(1-p_{12}^+p_{21}^+)} \tag{5.12}$$

Similarly the "excess" probability flux flows in a counter-clockwise direction from (0,1) to (0,0) and is:

$$-\Lambda_2 p(0,0) + (\mu_2+\lambda_2)d_2 p(0,1) \tag{5.13}$$

$$(\mu_2+\lambda_2)d_2(\frac{\Lambda_2+\Lambda_1 p_{12}^+}{(\lambda_2+\mu_2)(1-p_{12}^+p_{21}^+)}) - \Lambda_2 \tag{5.14}$$

This is also equal to the above expression. Thus the "excess" probability flux from (0,0) to (1,0) is also equal to the "excess" probability flux from (0,1) to (0,0).

It then remains to show that there is an equal "excess" probability flux from (1,0) to (0,1) and this is easily done. This result implies that for the symmetrical case, there are again three building blocks of cyclic length two. However there is also a larger building block of length three that covers the "excess" probability flux. This result also applies when the "excess" probability flux is clockwise. The results above have been obtained for the nodes (0,0), (1,0) and (0,1). However, the most significant consideration is the relative difference between the node positions rather than the absolute positions of the nodes themselves. From the state transition diagram, it can be immediately seen that the results can be also applied to any trio of nodes with the same relative differences in coordinates among them. The building blocks for the state transition diagram have been determined since the entire state transition diagram can be replicated starting from these three nodes and associated edges.

6 Topology Three: A multiclass queue

There is only a single queue in this system. There are two classes of positive customers and a single negative customer class that can enter the queue. The queue discipline is First In First Out (FIFO). When a negative customer enters the queue, one of the customers is selected to leave randomly and the class of the customer is not considered when making this choice. The probability that the chosen customer does leave is one. The two classes of positive traffic enter at a rate of Λ_1 and Λ_2 and the negative customers at a rate λ. The queue has a single exponential server that operates at an identical rate μ for both positive classes of customers. There is no feedback and all customers that leave the queue also leave the system. The queue is illustrated in Figure 10 and the state transition diagram is shown in Figure 11.

Due to the FIFO service discipline, there is a need to retain the order in which customers enter the queue. Consequently this leads to a complex state transition diagram. This state transition diagram is in the form of a tree and the maximum number of levels i in this tree would be the maximum number of customers in the queue plus one. The number of states in the ith level would be 2^i so that the number of states increase exponentially. For example, level three is associated with two customers in the queue and there are four states in this level. Some of the transitions are bidirectional. The states are labeled from left to right in the order that the customers have entered the queue. For example, in state 221, the order of customer arrival has been a class two customer followed by another class two customer and finally a class one customer.

Consider the state 12 in Figure 12. It can be reached from five states. First of all, state 12 can be reached from state 1 with the arrival of a class two customer. Hence the transition rate is Λ_2. It can be reached from state 112 with a service completion of the first customer or a negative customer arrival causing any one of the class one customers to leave. Thus the transition rate from state 112 to state 12 is $\mu + \frac{2\lambda}{3}$. From state 121, the only way that the system can reach state 12 is a negative customer arriving and causing the third customer, the class one customer, to leave. Thus the transition rate is just $\frac{\lambda}{3}$. State 12 can also be reached from 122 at a transition rate of $\frac{2\lambda}{3}$ and at a rate of $\mu + \frac{\lambda}{3}$ from state 212. A customer can also depart when the system is in state 12 and there are four possible states that a departing customer can cause the system state to move to.

6.1 Local Balance

Let the number of class one customers in the queue be n_1 and the number of class two customers be n_2. The probability distribution of the number of customers in the queue is given by:

$$\Pi(C) = b\left\{(\frac{\Lambda_1}{\lambda+\mu})^{n_1}(\frac{\Lambda_2}{\lambda+\mu})^{n_2}\right\} \tag{6.1}$$

Here b is given by:

$$b^{-1} = \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} \left\{ \left(\frac{\Lambda_1}{\lambda + \mu}\right)^{n_1} \left(\frac{\Lambda_2}{\lambda + \mu}\right)^{n_2} \right\} \quad (6.2)$$

$$b^{-1} = \sum_{n=0}^{\infty} \frac{(\Lambda_1 + \Lambda_2)^n}{\Pi^n(\lambda + \mu)} \quad (6.3)$$

Using methods that are similar to those used for topology one and topology two it can easily be shown that there is local balance across the boundaries named A, B, C, D, E, F and G by equating the probability flux across the boundaries (Fig. 12). Since the bottom half of the state transition is identical except for a relabeling of the states, any results obtained for the top half also apply to the bottom half of the state transition diagram, due to considerations of symmetry.

6.2 Building Blocks

By simply checking at the local balance boundaries, it is evident that a building block with a cyclic length of two exists on all the outer-most transitions of the state transition diagram. For level two states, Figure 13 shows the building blocks across the boundary D for the level two states.

The probability flux due to the transition rate of Λ_2 from state 1 to state 12 can be split into two components: one component matches the probability flux due to the transition rate of $\frac{\lambda}{2}$ from state 12 to state 1 and the other component matches the probability flux due to the transition rate of $\mu + \frac{\lambda}{2}$ from state 21 back to state 1. Since it already known that there is local balance across the boundary D, then it can be established that there is a building block of length two from state 1 to state 12 and back to state 1 again. This suspected building block is labeled *a* in the diagram (Figure 13). There is also a building block of length four that starts from state 1, goes to state 12 then to state 2 and state 21 and finally back to 1. This building block is labeled *b* in Figure 13. Both are indeed building blocks. Thus, for this level of the tree, state 1 has three building blocks associated with it, two of cyclic length two and one of cyclic length four.

The building blocks for the two level three states state 11 and state 12 are shown in Figure 14. For level three states, the determination of the building blocks is more involved. Considering state 11, building block a is already known to be a building block since it is on the outer branch of the tree. There are three other transitions that are incident on state 11 namely transition of rate $\frac{\lambda}{3}$ from state 112, transition of rate $\frac{\lambda}{3}$ from state 121 and transition of rate $\mu + \frac{\lambda}{3}$ from state 211. It can be shown that the probability flux due to the rightward transition labeled b in Figure 14 is equal to the probability flux from the three transitions (leftward) b, c and d. Thus for state 11, in this level of the tree, there are two building blocks of cyclic length two, one of length four and one of length six. Once again using cyclic flows, these are proven to be building blocks. Similar sets of correspondences can be shown for state 12.

The building blocks for the subsequent levels will not be illustrated here due to increasing complexity of the state transition diagram. However, as the number of states in a level increases, the cyclic length of the building blocks increase. This is in contrast to the series and the parallel queues where it is possible to construct the entire state transition diagram from the same type of building blocks.

7 Conclusion

In this paper, the building blocks for three common topologies of networks of queues with product form solutions for the equilibrium state probabilities have been determined. The conclusion that can be drawn is that it is possible, as it is with the case with queues with only positive customers, that an algebraic topological interpretation can be applied to certain cases of product form networks of queues with both positive and negative customers to explain the existence of product form solutions.

An open question is to determine whether it is possible to always find a building block structure for all networks of negative and positive customer queues that are known to have product form solutions.

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Figure Captions for *Algebraic Topology of Negative Customer Networks* by E. Foo and T.G. Robertazzi.

Figure 1: Local balance at a node.

Figure 2: Decomposed state transition pairs at a node.

Figure 3: Two queues, Q1 and Q2, in series.

Figure 4: State transition diagram for topology one.

Figure 5: Building blocks for the bottom most row for topology one.

Figure 6: Building blocks for the left most column for topology one.

Figure 7: Two queues, Q1 and Q2, in parallel.

Figure 8: State transition diagram for topology two.

Figure 9: Building block for topology two.

Figure 10: Two positive customer classes and negative customer class queue.

Figure 11: State transition diagram for multiclass customer queue.

Figure 12: Local balance boundaries for two class customer queue showing only 3 levels.

Figure 13: Two building blocks for level two states.

Figure 14: Building blocks for level three.

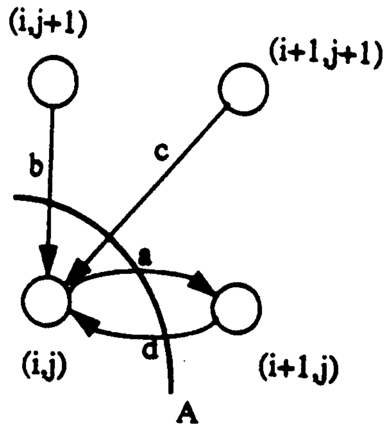

Figure 1: Local balance at a node

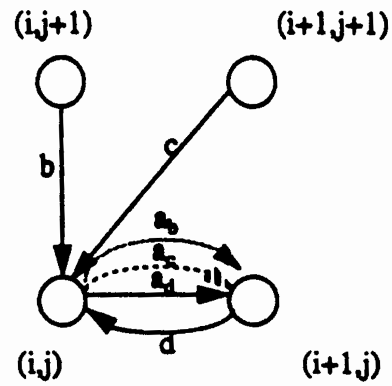

Figure 2: Decomposed state transitions pairs at a node

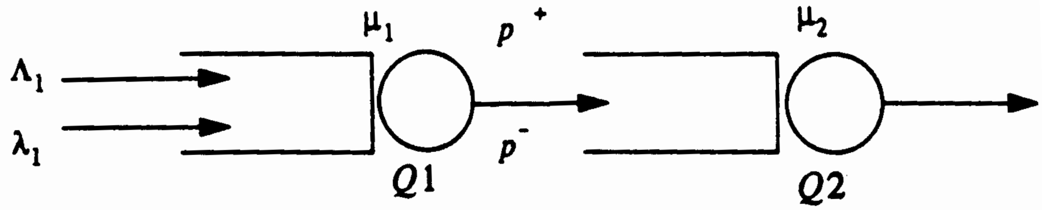

Figure 3: Two queues, Q1 and Q2 in series

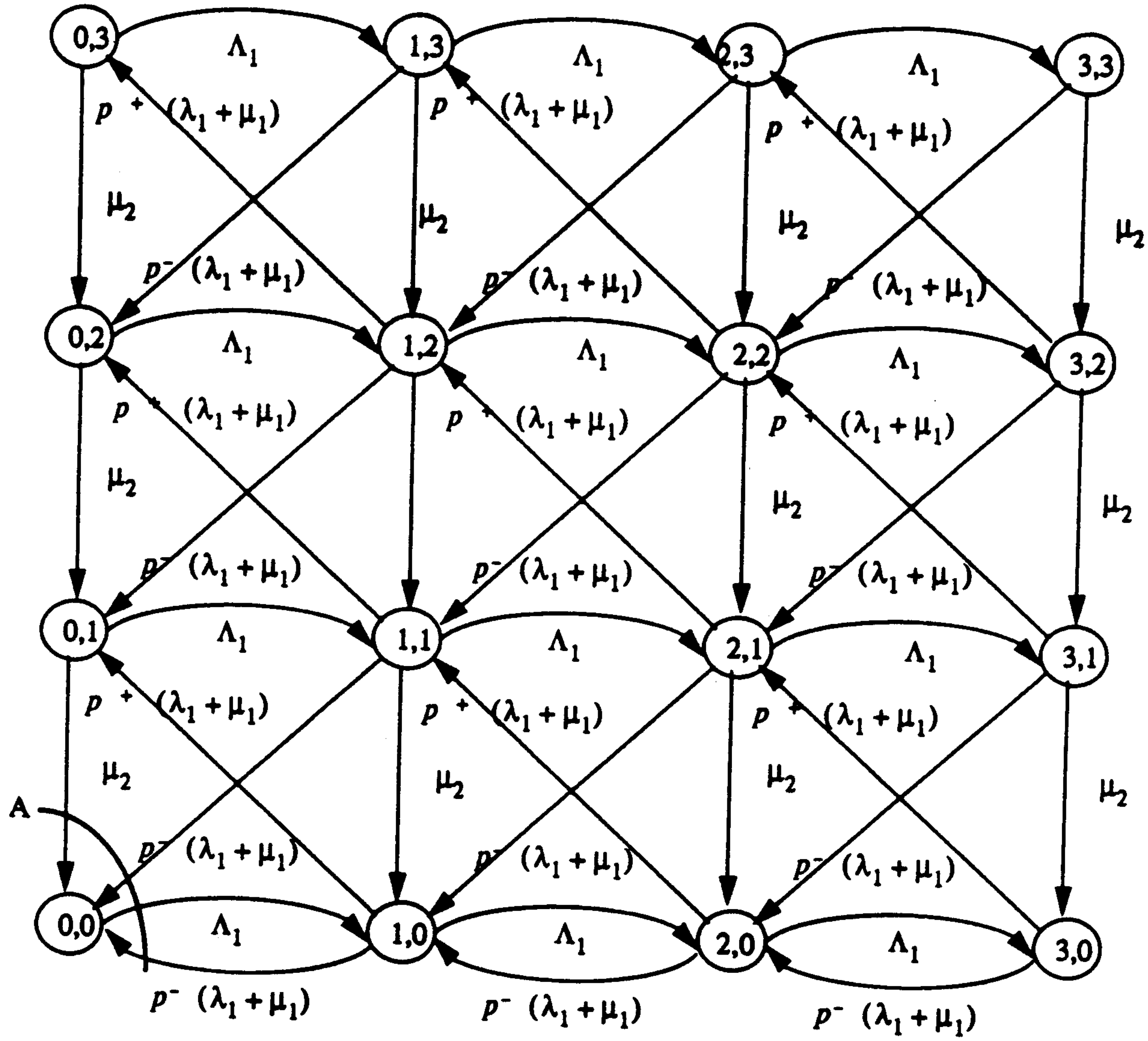

Figure 4: State Transition diagram for topology one

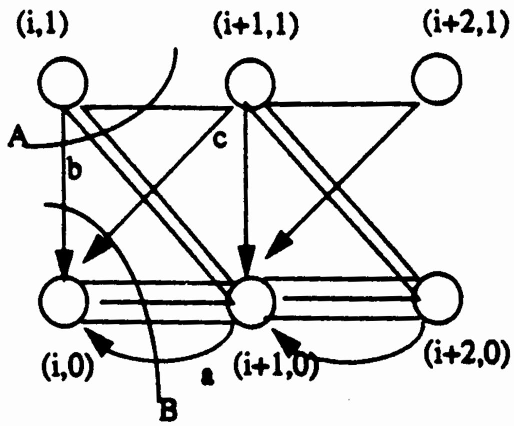

Figure .5: Building blocks for the bottom most row for topology 1

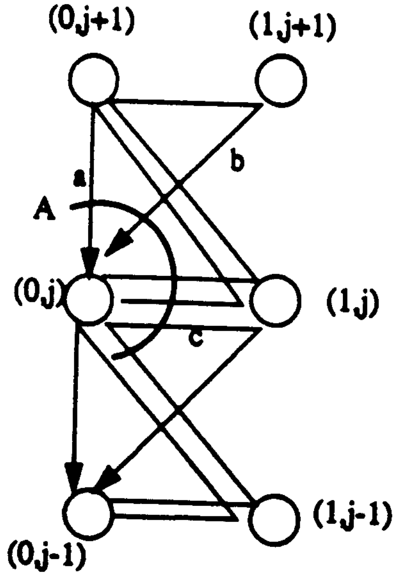

Figure 6: Building blocks for the left most column for topology 1

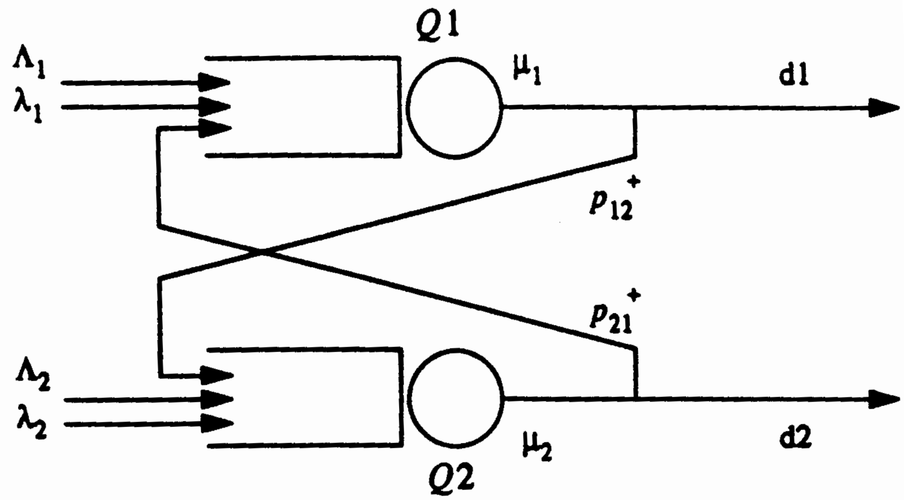

Figure 7: Two queue Q1 and Q2 in parallel

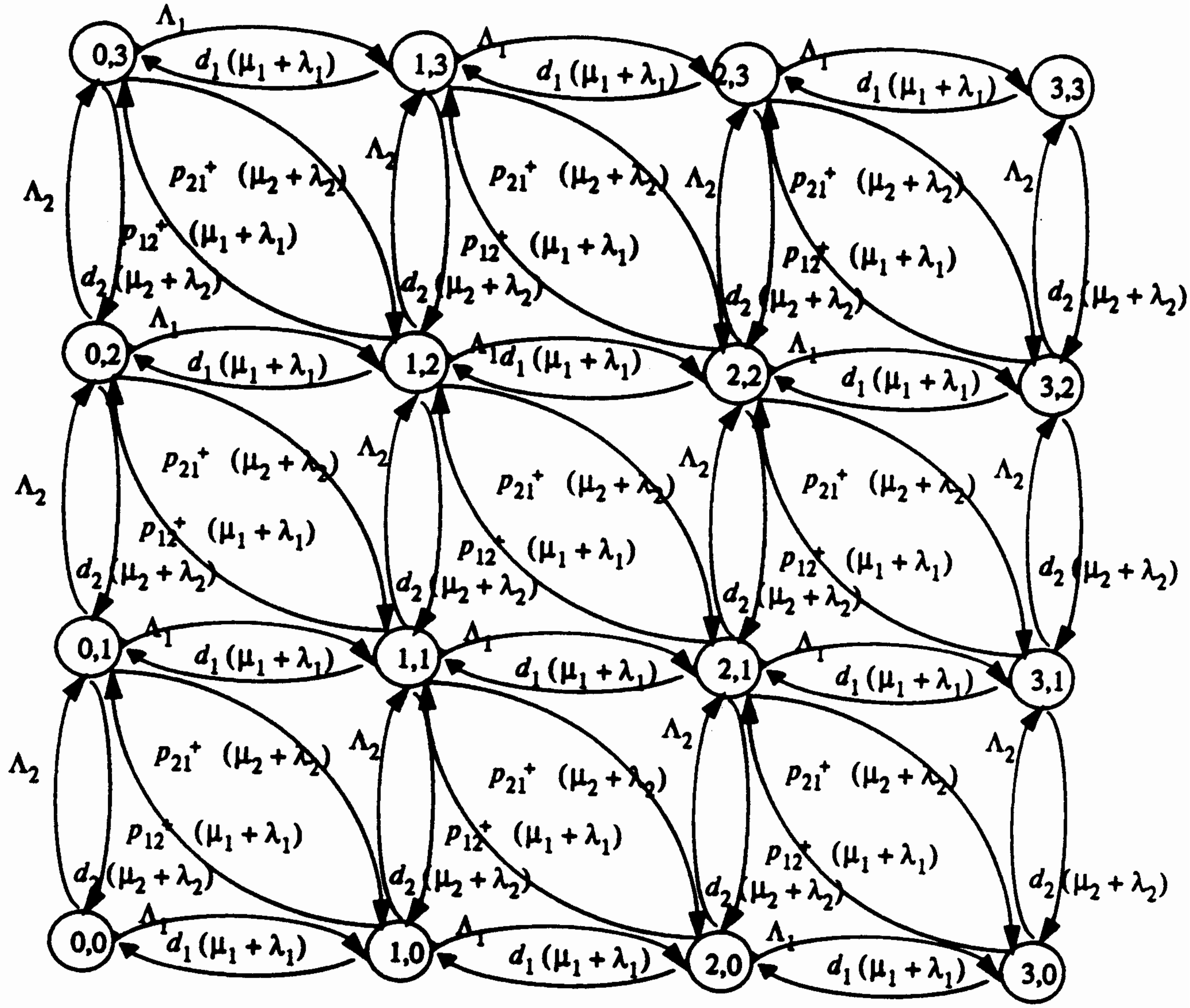

Figure 8: State transition diagram for topology 2

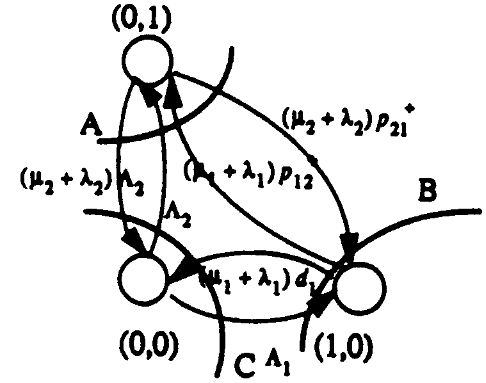

Figure 9: Building block for topology 2

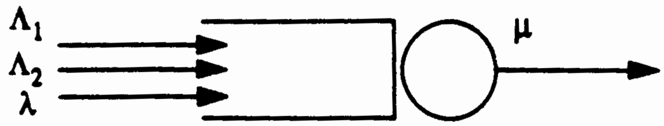

Figure 10: Two positive customer classes and negative customer class queue

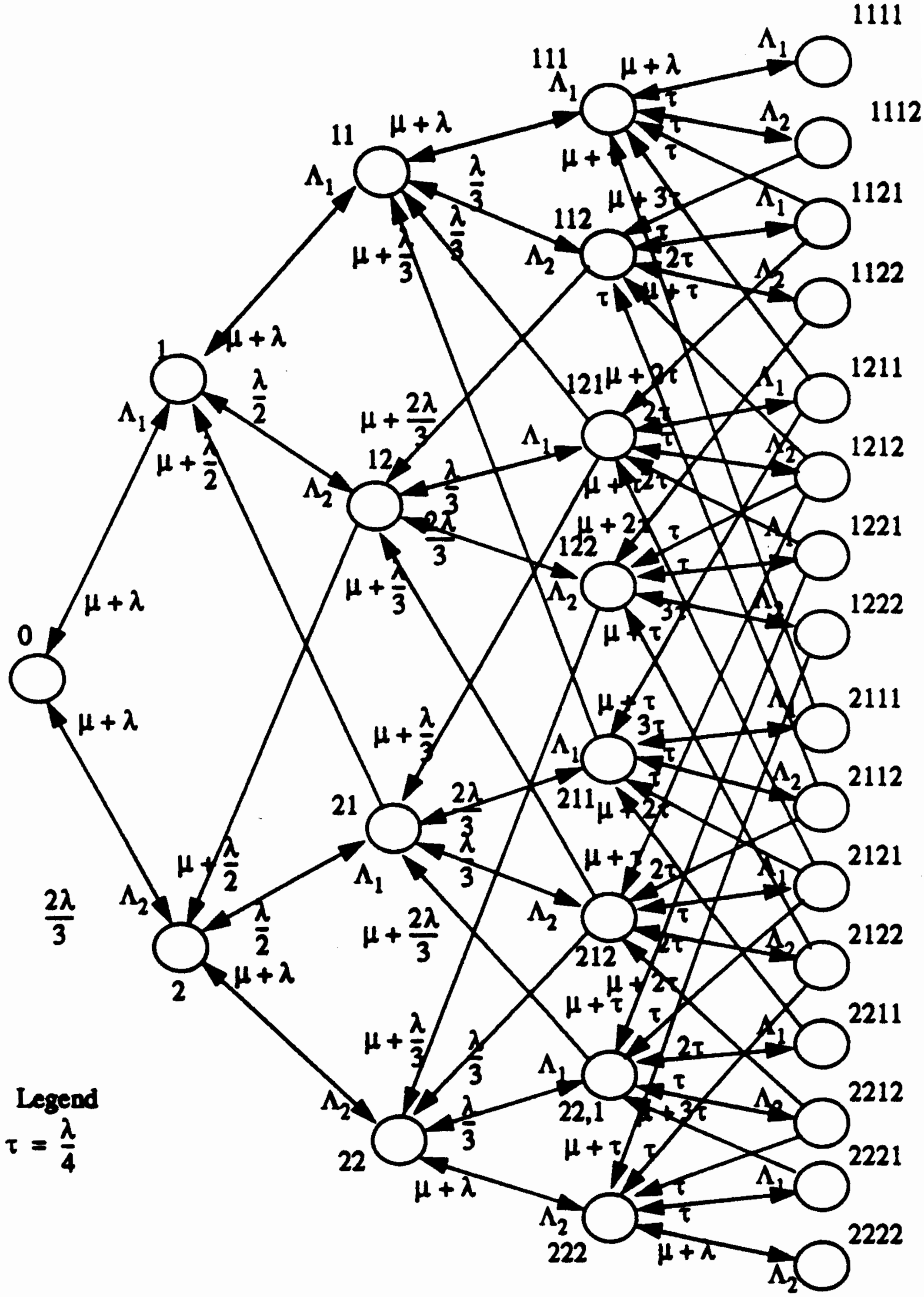

Figure 11: State transition diagram for multiclass customer queue

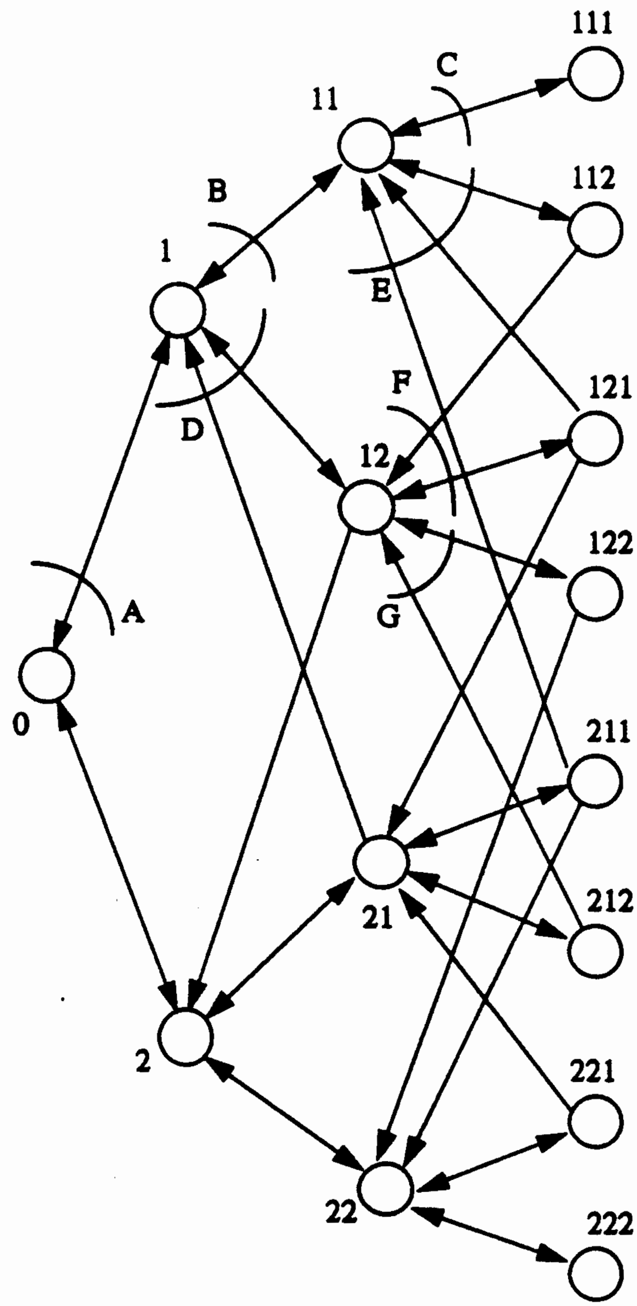

Figure 12: Local balance boundaries for two class customer queue showing only 3 levels

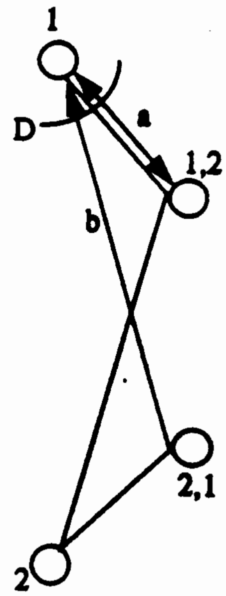

Figure 13: Two building blocks for level two states

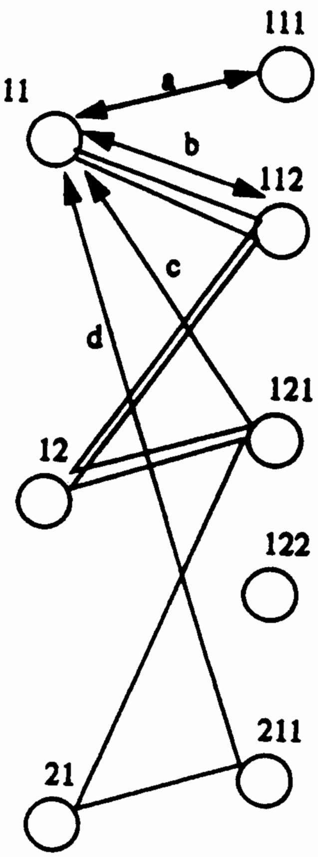

Figure 14: Building blocks for level three