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THE ECCENTRICITIES, BLOCKS, AND CENTERS OF TRANSFINITE GRAPHS WHOSE NODES OF HIGHEST RANK ARE PRISTINE

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Abstract — Eccentricities, blocks, and centers are examined for those transfinite connected graphs whose nodes of highest rank are "pristine" in the sense that those nodes do not embrace nodes of lower ranks. When there are only finitely many such nodes of highest rank, the infinitely many nodes of all ranks have eccentricities of the form $\omega^{\nu} \cdot p$, where ω is the first transfinite ordinal, ν is the rank of the nodes of highest rank, and p lies in a finite set of natural numbers. Furthermore, two known theorems concerning finite conventional graphs are extended transfinitely as follows: The center is contained in a single block of highest rank. Also, when every loop of the transfinite graph is confined within a section of highest rank, the center either is a single node of highest rank, or is the set of internal nodes of a section of rank one less than the highest rank, or is the union of the latter two kinds of centers.

Key Words: Graphical eccentricities, graphical blocks, graphical centers, transfinite graphs, distances in graphs.

1 Introduction

In a prior work [5] it was established that an ordinal-valued metric $d: \mathcal{M} \rightsquigarrow \aleph_1$ can be assigned to a set $\mathcal{M}$ of nodes in a transfinite connected graph $\mathcal{G}^{\nu}$ of rank ν. $\mathcal{M}$ contains all the nonsingleton nodes of $\mathcal{G}^{\nu}$ (that is, those nodes containing two or more extremities of subgraphs) and may contain some—but not necessarily all—the singleton nodes, as well. To each pair of nodes in $\mathcal{M}$, d assigns a countable ordinal representing a finite or transfinite "distance" between those nodes. That work also examined the generalizations of eccentricities and centers for transfinite graphs, as well as other related concepts.

The present work is a continuation of that study [5]. It establishes a finite range of possible transfinite ordinals for the eccentricities of the nodes of $\mathcal{G}^\nu$. It also extends transfinitely two standard results for finite graphs, namely, the center of a finite graph lies within a block, and, when the graph is a finite tree, the center is either a single node or two adjacent nodes. The transfinite extensions of these results allow loops within $\mathcal{G}^\nu$. However, an additional tree-like structure is required for the last result—but only at the highest rank. To obtain all this, we employ some restrictions on $\mathcal{G}^\nu$; these are stated in the next Section. The open problems arising when these restrictions are relaxed appear to be formidable.

Please note that we use the natural sum of ordinals [1, page 355] whenever ordinals are added, as was the case in [5], too.

2 Some Assumed Conditions

The following assumptions will always be imposed upon the transfinite ν-graph $\mathcal{G}^\nu$ throughout this paper.

Assumptions 2.1.

(a) The rank ν is restricted to $1 \leq \nu \leq \omega$, $\nu \neq \vec{\omega}$.

(b) $\mathcal{G}^\nu$ is ν-connected.

(c) All ν-nodes are pristine.

(d) If two tips are nondisconnectable, they are either shorted together or at least one of them is open.

(e) There are only finitely many boundary ν-nodes.

Let us now explain each of these assumptions.

Assumption 2(a): The ranks of transfinite graphs and of their subgraphs and nodes are of two kinds: ordinals (which may be either natural numbers or countable ordinals) and arrow ranks. Each limit-ordinal rank λ is preceded by an arrow rank $\vec{\lambda}$, which in turn is larger than every ordinal rank less than λ. See [5, Sec. 5] for a definition of an arrow rank.

Since we are restricting ν to $0 \leq \nu \leq \omega$, the only arrow rank we need consider is $\vec{\omega}$, where ω is the first transfinite ordinal. We disallow $\vec{\omega}$-graphs, but $\vec{\omega}$-paths occur by definition in ω-graphs [4, pages 42-43].

The extension of our results to ranks higher than ω simply repeats the arguments needed for the ranks no larger than ν, at least to the extent that ν-graphs can be constructed [4, Sec. 2.5].

Furthermore, we shall restrict our attention to the "maximal" nodes in $\mathcal{G}^\nu$. Let us explain: A transfinite α-node x^α of rank α $(0 \leq \alpha \leq \nu)$ is a set containing one or more "$(\alpha - 1)$-tips" plus possibly a β-node y^β of rank $\beta < \alpha$. We say that y^β is "embraced" by x^α. The β-node may contain a γ-node of rank $\gamma < \beta$, and so forth through finitely many ranks. An $(\alpha - 1)$-tip is an equivalence class of one-way infinite paths (called one-ended paths) that represents an infinite extremity of an $(\alpha - 1)$-graph. When $\alpha = 0$, an $(\alpha - 1)$-tip is one tip of a branch. (See[4, pages 9, 30, 37, and 42] for a thorough discussion of this structure.) On the other hand, an α-node x^α might be embraced by a node of higher rank. When such is not the case, we say that x^α is maximal. Our discussion will concentrate on the maximal nodes, and we will not keep repeating the adjective "maximal." If a considered node is not maximal, we will say so.

Assumption 2(b): Two nodes (resp. two branches) are said to be ν-connected if there is a path of rank ν or less that terminates at those nodes (resp. terminates at 0-nodes of those branches). If this is the case for all pairs of nodes in $\mathcal{G}^\nu$, $\mathcal{G}^\nu$ is said to be ν-connected. (See[4, pages 10, 33, and 43] for the definitions of such transfinite paths.)

Assumption 2.1(c): The nodes of highest rank in $\mathcal{G}^\nu$ are the ν-nodes. In this work, those nodes are not allowed to embrace nodes of lower rank. We refer to this condition by saying that all the ν-nodes are pristine. Since by definition every $\vec{\omega}$-node embraces nodes of lower rank [4, page 37], $\vec{\omega}$-nodes are disallowed by this assumption.

Assumption 2.1(d): ν-connectedness between nodes (or branches) is a binary relationship that is reflexive and symmetric, but unfortunately not always transitive. For instance, node x may be ν-connected to node y, which in turn is ν-connected to node z, but then x need not be ν-connected to z. See [4, Examples 3.1-5 and 3.1-6] for illustrations of this phenomenon. This leads to some formidable difficulties. The latter however, can be avoided if we impose Assumption 2(d), which asserts that two tips having representative paths that meet infinitely often as the tips are approached (i.e., are "nondisconnectable") are embraced by the same node (i.e., are "shorted together")—except possibly when at least one of those tips is "open" (i.e., when that tip is the sole member of a singleton node). Under Assumption 2(d), transitivity of ν-connectedness will hold for all nonsingleton nodes [4, Theorem 3.5-2] and possibly for some singleton nodes[1] as well. $\mathcal{M}$ will denote the set of all such nodes. There are different ways of choosing $\mathcal{M}$ depending upon which singleton nodes are added to the set of nonsingleton nodes. *Henceforth, it is understood that $\mathcal{M}$ has been selected and fixed with all nonsingleton nodes being in $\mathcal{M}$ and that the nodes we refer to are in $\mathcal{M}$.*

The following lemma has been established in [4, Lemma 3.3].

Lemma 2.2. *Given any two nodes (in $\mathcal{M}$), there exists at least one path terminating at those two nodes.*

Assumption 2.1(e): To explain this, we first need to consider the idea of a "$(\nu - 1)$-section" $\mathcal{S}^{\nu-1}$. This is a subgraph [4, page 32] of $\mathcal{G}^\nu$ induced by a maximal set of branches that are pairwise $(\nu - 1)$-connected [4, page 49].[2] By virtue of Assumption 2.1(d), the $(\nu - 1)$-sections partition $\mathcal{G}^\nu$, that is, each branch is in one and only one $(\nu - 1)$-section [4, Corollary 3.5-6]. $\mathcal{S}^{\nu-1}$ contains nodes of all ranks up to an including ν. Because all ν-nodes are now assumed to be pristine, the nodes of $\mathcal{S}^{\nu-1}$ of ranks less than ν are "internal nodes" of $\mathcal{S}^{\nu-1}$, and the ν-nodes of $\mathcal{S}^{\nu-1}$ are "bordering nodes" of $\mathcal{S}^{\nu-1}$ [4, page 81]. We can view the bordering ν-nodes as lying at the infinite extremities of $\mathcal{S}^{\nu-1}$ because they can be reached only along one-ended $(\nu - 1)$-paths of $\mathcal{S}^{\nu-1}$. All the other nodes (i.e. the internal nodes) of $\mathcal{S}^{\nu-1}$ are of ranks less than ν and are not embraced by bordering nodes; hence, they cannot be found at the infinite extremities of $\mathcal{S}^{\nu-1}$. The set $i(\mathcal{S}^{\nu-1})$ of all internal nodes of $\mathcal{S}^{\nu-1}$ will be called the "interior" of $\mathcal{S}^{\nu-1}$.

[1] The singleton nodes are usually inconsequential because they do not contribute to the connectivity of $\mathcal{G}^\nu$, but there are circumstances where some of them might be of importance.

[2] This is the first definition of a $(\nu - 1)$-section. The second equivalent definition requires a correction; it is given in the "Errata for Books" in the web page: www.ee.sunysb.edu/~zeman.

A "boundary node" of $\mathcal{S}^{\nu-1}$ is a bordering node (and hence of rank ν) that contains $(\nu-1)$-tips of $\mathcal{S}^{\nu-1}$ and also $(\nu-1)$-tips of one or more other $(\nu-1)$-sections of $\mathcal{G}^{\nu}$. Thus, a boundary node lies at the infinite extremities of two or more $(\nu-1)$-sections and thereby connects them. A ν-path can pass from the interior of one $(\nu-1)$-section into the interior of another $(\nu-1)$-section only by passing through a boundary node. Assumption 2.1(e) asserts that there are only finitely many boundary nodes in $\mathcal{G}^{\nu}$. However, $\mathcal{G}^{\nu}$ may contain infinitely many non-boundary bordering nodes.

The idea of a "component" is similar to but different from a $(\nu-1)$-section. A component of a subgraph $\mathcal{H}$ of $\mathcal{G}^{\nu}$ is a subgraph of $\mathcal{H}$ induced by a maximal set of branches in $\mathcal{H}$ that are ν-connected [4, page 49]. Because of Assumption 2.1(b), $\mathcal{G}^{\nu}$ has just one component, namely, itself. However, a proper subgraph $\mathcal{H}$ of $\mathcal{G}^{\nu}$ may have many components. For example, if $\mathcal{H}$ consists of two $(\nu-1)$-sections that do not share any boundary nodes, then each of them is a component of $\mathcal{H}$.

3 Eccentricities

A metric d can be defined on all pairs of nodes in $\mathcal{M}$ as follows:

$$d(x,y) \;=\; \min\{|P_{x,y}|: x,y \in \mathcal{M}\} \tag{1}$$

where $|P_{x,y}|$ is the ordinal-valued length for a path $P_{x,y}$ terminating at the nodes x and y and the minimum is taken over all such paths. That minimum exists because the ordinals comprise a well-ordered set. In fact, $d(x,y)$ is a countable ordinal. See [5, Sec. 2] for the definition of the length $|P_{x,y}|$ and [5, Sec. 4] for a proof that d is a metric. Because the minimum is achieved, we can sharpen Lemma 2.2 as follows.

Lemma 3.1. *Given any two nodes x and y, there exists a path $Q_{x,y}$ for which $|Q_{x,y}| = d(x,y)$.*

There may be more than one such path $Q(x,y)$. Each one of them is called an x-to-y *geodesic.*

Lemma 3.2. *If P is a two-ended ν-path, then $|P| = \omega^{\nu}\cdot k$, where k is the number of $(\nu-1)$-tips traversed by P.*

Proof. This follows directly from the definition of the length $|P|$ and the fact that all ν-nodes are pristine. □

Lemma 3.3.

(a) *Let x^ν be a bordering node of a $(\nu-1)$-section $\mathcal{S}^{\nu-1}$, and let z be an internal node of $\mathcal{S}^{\nu-1}$. Then, there exists a one-ended $(\nu-1)$-path $P^{\nu-1}_{z,x}$ within $\mathcal{S}^{\nu-1}$ that terminates at z and reaches x^ν with its one and only $(\nu-1)$-tip. Moreover, the length $|P^{\nu-1}_{z,x}|$ of $P^{\nu-1}_{z,x}$ is ω^ν. $P^{\nu-1}_{z,x}$ is a z-to-x geodesic. Finally, all one-ended $(\nu-1)$-paths within $\mathcal{S}^{\nu-1}$ terminating at an internal node of $\mathcal{S}^{\nu-1}$ and reaching a bordering node of $\mathcal{S}^{\nu-1}$ have the length ω^ν.*

(b) *Let x^ν and y^ν be two bordering nodes of $\mathcal{S}^{\nu-1}$. Then, there exists an endless $(\nu-1)$-path $P^{\nu-1}_{x,y}$ within $\mathcal{S}^{\nu-1}$ that reaches x^ν and y^ν through its two $(\nu-1)$-tips. Moreover, the length $|P^{\nu-1}_{x,y}|$ of $P^{\nu-1}_{x,y}$ is $\omega^\nu\cdot 2$. $P^{\nu-1}_{x,y}$ is an x^ν-to-y^ν geodesic. Finally, all endless $(\nu-1)$-paths in $\mathcal{S}^{\nu-1}$ reaching two bordering nodes of $\mathcal{S}^{\nu-1}$ have the length $\omega^\nu\cdot 2$.*

Proof. For part (a), choose any representative one-ended path $Q^{\nu-1}$ of a $(\nu-1)$-tip of $\mathcal{S}^{\nu-1}$ contained in x^ν. Let R^ρ $(\rho<\nu)$ be a two-ended path that terminates at z and at a node of $Q^{\nu-1}$. Then, $Q^{\nu-1}\cup R^\rho$ contains the asserted path $P^{\nu-1}_{z,x}$ according to [4, Corollary 3.5-4]. $|P^{\nu-1}_{z,x}|=\omega^\nu$ because of Lemma 3.2 and the fact that $P^{\nu-1}_{z,x}$ has exactly one $(\nu-1)$-tip [5, Sec. 2]. Furthermore, any path that terminates at z and x^ν must pass through at least one $(\nu-1)$-tip because x^ν is pristine; therefore, that path must have a length no less than ω^ν. Hence, $P_{z,x}$ is a geodesic. For the final assertion, just note that any such one-ended path traverses exactly one $(\nu-1)$-tip.

Part (b) is proven similarly, but now we use two representative one-ended paths, one for each of x^ν and y^ν. □

Lemma 3.4. *Let x^ν be a bordering node of a $(\nu-1)$-section $\mathcal{S}^{\nu-1}$, let z be an internal node of $\mathcal{S}^{\nu-1}$, and let y be any node of $\mathcal{G}^\nu$. Then, $|d(x^\nu,y)-d(z,y)|\le\omega^\nu$.*

Proof. By Lemma 3.3(a), $d(z,x^\nu)=\omega^\nu$. Since, d is a metric,

$$d(x^\nu,y)\;\le\;d(x^\nu,z)\,+\,d(z,y)\;=\;\omega^\nu\,+\,d(z,y).$$

Also,

$$d(z,y) \;\leq\; d(z,x^\nu) + d(x^\nu,y) \;=\; \omega^\nu + d(x^\nu,y).$$

These inequalities yield the conclusion. □

The *eccentricity* $e(x)$ of a node $x \in \mathcal{M}$ is defined by

$$e(x) \;=\; \sup\{d(x,y)\colon y \in \mathcal{M}\}. \tag{2}$$

When the supremum is achieved at some node $\hat{y} \in \mathcal{M}$, $e(x)$ is a countable ordinal. If the supremum is never achieved, $e(x)$ is an arrow rank. Three examples of the occurrence of arrow-rank eccentricities are given in [5, Examples 6.1, 6.4, and 6.5]. Another is shown in the next example. It may be noted that each of these examples violate either Conditions 2.1(c) or 2.1(e). We will show that, when all of Conditions 2.1 are satisfied, arrow-rank eccentricities do not occur.

Example 3.5. Fig. 1 shows a 1-graph[3] with a single non-pristine 1-node y^1, which embraces a (non-maximal, unlabeled) 0-node. There are two 0-sections; the 0-nodes x_k^0 ($k = 1,2,3,\ldots$) are the internal nodes of one of them, and the 0-nodes $z_{k,p}^0$ ($k = 1,2,3,\ldots; p = 0,1,\ldots,k$) are the internal nodes of the other. The eccentricities are as follows: $e(x_k^0) = \omega \vec{\cdot} 2$, $e(y^1) = \omega$, $e(z_{k,p}) = \omega + k + p$. This 1-graph violates Condition 2.1(c) because y^1 is not pristine. □

We will make use of the following lemma, a more general version of which has been established in [5, Theorem 7.1]. It is a consequence of Assumption 2.1(c).

Lemma 3.6. *All the internal nodes of any $(\nu-1)$-section have the same eccentricity.*

Theorem 3.7. *The eccentricities of all the nodes are contained within the following finite set of ordinals:*

$$\{\omega^\nu \cdot p\colon 1 \leq p \leq 2m+2\} \tag{3}$$

Here, p and m are natural numbers, and m is the number of boundary ν-nodes.

Proof. The eccentricity of any node is at least as large as the distance between any internal node of a $(\nu-1)$-section and any bordering ν-node of that $(\nu-1)$-section. Therefore,

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[3] All our examples examine only 1-graphs. However, any 1-graph can be converted into a ν-graph ($\nu > 1$) by replacing each branch by an endless $(\nu-1)$-path, and this will yield an example with a higher rank.

Lemma 3.3(a) implies that the eccentricity of any node of $\mathcal{G}^\nu$ is at least ω^ν, whence the lower bound in (3). The proof of the upper bound requires more effort.

First of all, we can settle two simple cases by inspection. If $\mathcal{G}^\nu$ consists of a single $(\nu-1)$-section with exactly one bordering ν-node (in $\mathcal{M}$, of course), then all the nodes of $\mathcal{G}^\nu$ have the eccentricity ω^ν. If that one and only $(\nu-1)$-section for $\mathcal{G}^\nu$ has two or more (possibly infinitely many) bordering nodes, the internal nodes have eccentricity ω^ν, and the bordering nodes have eccentricity $\omega^\nu \cdot 2$. In both cases, the conclusion of the theorem is fulfilled with $m = 0$.

We now turn to the general case where $\mathcal{G}^\nu$ has at least one boundary ν-node and therefore at least two $(\nu-1)$-sections. $\mathcal{G}^\nu$ will have a two-ended ν-path of the following form:

$$P^\nu_{0,k} \;=\; \{x_0, P^{\nu-1}_0, x^\nu_1, P^{\nu-1}_1, \ldots, x^\nu_{k-1}, P^{\nu-1}_{k-1}, x_k\} \tag{4}$$

(See [4, pages 34 and 44] for a definition of a ν-path.) Because all ν-nodes are pristine, the x^ν_i $(i = 1, \ldots, k-1)$ are nonsingleton bordering ν-nodes (possibly boundary ν-nodes), and the $P^{\nu-1}_i$ $(i = 2, \ldots, k-2)$ are endless $(\nu-1)$-paths. The same is true of x_0, x_k, $P^{\nu-1}_1$, and $P^{\nu-1}_{k-1}$ if x_0 and x_k are ν-nodes, too. If x_0 (resp. x_k) is of lower rank, then it is an internal node, and $P^{\nu-1}_1$ (resp. $P^{\nu-1}_{k-1}$) is a one-ended $(\nu-1)$-path.

Let us first assume that x_0 and x_k are internal nodes. Let $\mathcal{S}^{\nu-1}_0$ be the $(\nu-1)$-section containing x_0. Let $x^\nu_{i_1}$ be the last ν-node in (4) that is incident to $\mathcal{S}^{\nu-1}_0$. $x^\nu_{i_1}$ will be a boundary ν-node because it is also incident to another $(\nu-1)$-section, say, $\mathcal{S}^{\nu-1}_1$. If need be, we can replace the subpath of (4) between x_0 and x_{i_1} by a ν-path $\{x_0, Q^{\nu-1}_0, x^\nu_{i_1}\}$, where $Q^{\nu-1}_0$ is a one-ended $(\nu-1)$-path and resides in $\mathcal{S}^{\nu-1}_0$, to get a shorter overall ν-path terminating at x_0 and x_k.

Now, let $\mathcal{S}^{\nu-1}_1$ be the next $(\nu-1)$-section after $\mathcal{S}^{\nu-1}_0$ through which our (possibly) reduced path proceeds. Also, let $x^\nu_{i_2}$ be the last ν-node in that path that is incident to $\mathcal{S}^{\nu-1}_1$. $x^\nu_{i_2}$ will be a boundary ν-node incident to $\mathcal{S}^{\nu-1}_1$ and another $(\nu-1)$-section $\mathcal{S}^{\nu-1}_2$. If need be, we can replace the subpath between x_{i_1} and x_{i_2} by a ν-path $\{x^\nu_{i_1}, Q^{\nu-1}_1, x^\nu_{i_2}\}$, where $Q^{\nu-1}_1$ is an endless $(\nu-1)$-path residing in $\mathcal{S}^{\nu-1}_1$. This will yield a still shorter overall ν-path terminating at x_0 and x_k.

Continuing this way, we will find a boundary ν-node $x^\nu_{i_j}$ that is incident to the $(\nu-1)$-

section $\mathcal{S}_j^{\nu-1}$ containing x_k. Finally, we let $Q_j^{\nu-1}$ be a one-ended $(\nu-1)$-path in $\mathcal{S}_j^{\nu-1}$ terminating at $x_{i_j}^{\nu}$ and x_k. Altogether, we will have the following two-ended ν-path, which is not longer than $P_{0,k}^{\nu}$ (actually shorter if the aforementioned replacements were needed).

$$Q_{0,k}^{\nu} = \{x_0, Q_0^{\nu-1}, x_{i_1}^{\nu}, Q_1^{\nu-1}, x_{i_2}^{\nu}, \ldots, x_{i_j}^{\nu}, Q_j^{\nu-1}, x_k\} \tag{5}$$

Because all the ν-nodes herein are boundary nodes and pristine, the length $|Q_{0,k}|$ is obtained simply by counting the $(\nu-1)$-tips traversed by $Q_{0,k}^{\nu}$ and multiplying by ω^{ν} (see Lemma 3.3). We get $|Q_{0,k}^{\nu}| = \omega^{\nu} \cdot (2j)$, where $j \leq m$. Finally, we note that any geodesic path between x_0 and x_k has a length no larger than than $|Q_{0,k}^{\nu}| = \omega^{\nu} \cdot (2j)$.

Next, consider the case where x_0 is a bordering ν-node of $\mathcal{S}_0^{\nu-1}$ and x_k remains an internal node of $\mathcal{S}_j^{\nu-1}$. $P_0^{\nu-1}$ in (4) will be an endless $(\nu-1)$-path residing in some $(\nu-1)$-section $\mathcal{S}_0^{\nu-1}$. We let $x_{i_1}^{\nu}$ be the last ν-node in (4) incident to $\mathcal{S}_0^{\nu-1}$. Otherwise our procedure is as before, and we can now conclude that $|Q_{0,k}^{\nu}| = \omega^{\nu} \cdot (2j+1)$ because the passage from x_0 into $\mathcal{S}_0^{\nu-1}$ traverses one $(\nu-1)$-tip.

The same conclusion, namely, $|Q_{0,k}^{\nu}| = \omega^{\nu} \cdot (2j+1)$ holds if x_k is a bordering ν-node and x_0 is an internal node. Finally, if both x_0 and x_k are bordering ν-nodes, we get $|Q_{0,k}^{\nu}| = \omega^{\nu} \cdot (2j+2)$. For all cases, we can assert that the geodesic between x_0 and x_k has a length no larger than $|Q_{0,k}^{\nu}| = \omega^{\nu} \cdot (2j+2)$.

Now, the eccentricity $e(x_0)$ for x_0 is the supremum of the lengths of all geodesics starting at x_0 and terminating at all other nodes x_k. Since $j \leq m$ where m is the number of boundary ν-nodes in $\mathcal{G}^{\nu}$, we can conclude that $e(x_0) \leq \omega^{\nu} \cdot (2m+2)$, whatever be the node x_0. □

The last proof has also shown that all geodesics in $\mathcal{G}^{\nu}$ will have the form of (5). Moreover, the lengths of all geodesics will reside in the finite set of values (3). Consequently, for every node x_0 of $\mathcal{G}^{\nu}$ there will be at least one geodesic of maximum length starting at x_0 and terminating at some other node z of $\mathcal{G}^{\nu}$. Such a geodesic is called an *eccentric path* for x_0, and z is called an *eccentric node* for x_0. In general, there are many eccentric paths and eccentric nodes for a given x_0.

Corollary 3.8. *Let x^{ν} be any bordering ν-node of a $(\nu-1)$-section $\mathcal{S}^{\nu-1}$ with the eccentricity $e(x^{\nu}) = \omega^{\nu} \cdot k$, and let z be an internal node of $\mathcal{S}^{\nu-1}$ with the eccentricity $e(z) = \omega^{\nu} \cdot p$. Then, $|k-p| \leq 1$.*

Proof. Let $P_{z,x}$ be the two-ended ν-path obtained by appending x^ν to a one-ended $(\nu-1)$-path in $\mathcal{S}^{\nu-1}$ that reaches x^ν and terminates at the internal node z. By Lemma 3.3(a), $P^\nu_{z,x}$ is a z-to-x^ν geodesic, and $|P^\nu_{z,x}| = d(z,x^\nu) = \omega^\nu$. Now, let w be any node. By the triangle inequality for the metric d,

$$d(z,w) \;\le\; d(z,x^\nu) + d(x^\nu,w) \;=\; \omega^\nu + d(x^\nu,w).$$

Next, let w be an eccentric node for z. We get $d(z,w) = e(z)$ and $e(z) \le \omega^\nu + d(x^\nu,w)$. Moreover, $d(x^\nu,w) \le e(x^\nu)$. Therefore,

$$e(z) \;\le\; \omega^\nu + e(x^\nu). \tag{6}$$

By a similar argument with w now being an eccentric node for x^ν, we get

$$e(x^\nu) \;\le\; \omega^\nu + e(z). \tag{7}$$

So, with (6) we have $\omega^\nu \cdot p \le \omega^\nu + \omega^\nu \cdot k = \omega^\nu \cdot (k+1)$, or $p \le k+1$. On the other hand, with (7) we have in the same way $k \le p+1$. Whence our conclusion. □

That $k - p$ can equal 0 is verified by the next example.

Example 3.9. Consider the 1-graph of Fig. 2 consisting of a one-ended 0-path of 0-nodes w^0_k and an endless 0-path of 0-nodes y^0_k connected in series to two 1-nodes x^1 and z^1 as shown. The eccentricities are as follows: $e(w^0_k) = \omega\cdot 3$ for $k = 1,2,3,\ldots$, $e(x^1) = \omega\cdot 2$, $e(y^0_k) = \omega\cdot 2$ for $k = \ldots,-1,0,1,\ldots$, and $e(z^1) = \omega\cdot 3$. Thus, $e(x^1) - e(y^0_k) = 0$, as asserted. □

An immediate consequence of Corollary 3.8 is the following.

Corollary 3.10. *The eccentricities of all the nodes form a consecutive set of values in (3).*

We will need two more results. They hold except for the trivial case where $\mathcal{G}^\nu$ has only one $(\nu-1)$-section and only one ν-node.

Lemma 3.11. *Except for the trivial case just noted, a non-boundary bordering ν-node x^ν of a $(\nu-1)$-section $\mathcal{S}^{\nu-1}$ has an eccentricity that is exactly ω^ν larger than the eccentricity of the internal nodes of $\mathcal{S}^{\nu-1}$.*

Proof. This follows from Lemma 3.6 and the fact that any eccentric path starting at x^ν and entering $\mathcal{S}^{\nu-1}$ must pass through exactly one $(\nu-1)$-tip. □

An *end* $(\nu-1)$*-section* is a $(\nu-1)$-section having exactly one boundary ν-node.

Lemma 3.12. *Except for the trivial case, the eccentricity of the internal nodes of an end* $(\nu-1)$*-section is exactly* ω^ν *larger than the eccentricity of its boundary* ν*-node.*

Proof. Any eccentric path of any node of $i(\mathcal{S}^{\nu-1})$ must pass through that boundary ν-node. So, the argument of the preceding proof works again. □

4 The Center Lies in a ν-Block

This is a known result for finite graphs [2, Theorem 2.2], [3, Theorem 2.9], which we now extend transfinitely. The *center* of $\mathcal{G}^\nu$ is the set of nodes having the minimum eccentricity. To define a "ν-block," we first define the *removal* of a pristine nonsingleton ν-node x^ν to be the following procedure: x^ν is replaced by two or more singleton ν-nodes, each containing exactly one of the $(\nu-1)$-tips of x^ν and with every $(\nu-1)$-tip of x^ν being so assigned. We denote the resulting ν-graph by $\mathcal{G}^\nu - x^\nu$. Then, a subgraph $\mathcal{H}$ of $\mathcal{G}^\nu$ will be called a ν*-block* of $\mathcal{G}^\nu$ if $\mathcal{H}$ is a maximal ν-connected subgraph such that, for every x^ν, all the branches of $\mathcal{H}$ lie in the same component of $\mathcal{G}^\nu - x^\nu$. A more explicit way of defining a ν-block is as follows: For any ν-node x^ν, $\mathcal{G}^\nu - x^\nu$ consists of one or more components. Choose one of those components. Repeat this for every ν-node, choosing one component for each ν-node. Then, take the intersection[4] of all those chosen components. That intersection may be empty, but, if it is not empty, it will be a ν-block of $\mathcal{G}^\nu$. Upon taking all possible intersections of components, one component from each $\mathcal{G}^\nu - x^\nu$, and then choosing the nonempty intersections, we will obtain all the ν-blocks of $\mathcal{G}^\nu$.

Furthermore, we define a *cut-node* as a nonsingleton ν-node x^ν such that $\mathcal{G}^\nu - x^\nu$ has two or more components. It follows that the cut ν-nodes *separate* the ν-blocks in the sense that any path that terminates at two branches in different ν-blocks must pass through at least one cut ν-node. (Otherwise, the two branches would be in the same component of $\mathcal{G}^\nu - x^\nu$ for every x^ν and therefore in the same ν-block.) In summary, we have the following:

[4]This is the subgraph induced by those branches, each of which lie in all the chosen components.

Lemma 4.1. *The ν-blocks of $\mathcal{G}^\nu$ partition*[5] $\mathcal{G}^\nu$, *and the cut ν-nodes separate the ν-blocks.*

Proof. For each x^ν, each branch will be in at least one of the components of $\mathcal{G}^\nu - x^\nu$, and therefore in at least one of the ν-blocks. On the other hand, no branch can be in two different ν-blocks because then there would be a cut ν-node that separates a branch from itself—an absurdity. □

Lemma 4.2. *Each $(\nu-1)$-section $\mathcal{S}^{\nu-1}$ is contained in a ν-block.*

Proof. Since every ν-node is pristine, any two branches of $\mathcal{S}^{\nu-1}$ are connected through a two-ended path of rank no greater than $\nu-1$, and that path will not meet any ν-node. Thus, $\mathcal{S}^{\nu-1}$ will lie entirely within a single component of $\mathcal{G}^\nu - x^\nu$, whatever be the choice of x^ν. By the definition of a ν-block, we have the conclusion. □

By definition, all the bordering nodes of a $(\nu-1)$-section $\mathcal{S}^{\nu-1}$ will be ν-nodes. Moreover, every $(\nu-1)$-section $\mathcal{S}^{\nu-1}$ will have at least one bordering node, and all the bordering nodes of $\mathcal{S}^{\nu-1}$ will be nodes of $\mathcal{S}^{\nu-1}$. Thus, by Lemma 4.2, every ν-block $\mathcal{H}$ will contain the bordering ν-nodes of its $(\nu-1)$-sections, and therefore the rank of $\mathcal{H}$ is ν. So, henceforth we denote $\mathcal{H}$ by $\mathcal{H}^\nu$. In general, a ν-node can belong to more than one $(\nu-1)$-section and also to more than one ν-block.

Example 4.3. Fig. 3 shows a 1-graph in which the P_k^0 $(k = 1,2,3,4,5)$ are endless 0-paths and v^1, w^1, x^1, y^1, and z^1 are 1-nodes. There are two 1-blocks: One of them consists of the P_1^0 along with v^1 and w^1, and the other consists of the P_k^0 $(k = 2,3,4,5)$ along with w^1, x^1, y^1, and z^1. The only cut 1-node is w^1. Also, there are five 0-sections, each consisting of one endless 0-path along with its two bordering 1-nodes. □

Example 4.4. The condition that the ν-nodes are pristine is needed for Lemma 4.2 to hold. For example, Fig. 4 shows a 1-graph with a nonpristine 1-node x^1, three 1-blocks,[6] and two 0-sections. Branch b_1 induces one 1-block, branch b_2 indices another 1-block, and all the branches of the one-ended 0-path P^0 induce the third 1-block. However, b_1 and b_2 together induce a single 0-section $\mathcal{S}^0$, and the branches of P^0 induce another 0-section. $\mathcal{S}^0$

[5]See [4, page 33] for a definition of a partition of $\mathcal{G}^\nu$.

[6]Here, we are extending the definition of a 1-block by requiring that the elementary tips of x^1 also be placed in singleton nodes.

lies in the union of two 1-blocks. □

We are finally ready to verify the title of this section concerning the *center* of $\mathcal{G}^\nu$, which by definition is the set of nodes having the minimum eccentricity. According to Theorem 3.7, such nodes exist. Having set up appropriate definitions and preliminary results for the transfinite case, we can now use a proof that is much the same as that for finite graphs [2, Theorem 2.2], [3, Theorem 2.9].

Theorem 4.5. *The center of $\mathcal{G}^\nu$ lies in a ν-block.*

Proof. Suppose the center of $\mathcal{G}^\nu$ lies in two or more ν-blocks. Let $\mathcal{H}_1^\nu$ and $\mathcal{H}_2^\nu$ be two of them. By Lemma 4.1, there is a cut ν-node x^ν separating them. Let u be an eccentric node for x^ν, and let $P_{x,u}$ be an x^ν-to-u geodesic. Thus, $|P_{x,u}| = e(x^\nu)$. $P_{x,u}$ cannot contain any node different from x^ν in at least one of $\mathcal{H}_1^\nu$ and $\mathcal{H}_2^\nu$, say, $\mathcal{H}_1^\nu$. Let w be a center node in $\mathcal{H}_1^\nu$ other than x^ν, and let $P_{w,x}$ be a w-to-x^ν geodesic. Then, $P_{w.x} \cup P_{x,u}$ is a path whose length satisfies $|P_{w,x} \cup P_{x,u}| = |P_{w,x}| + |P_{x,u}| \geq 1 + e(x)$. This shows that the eccentricity of w is greater than the minimum eccentricity, that is, w is not a center node—a contradiction that proves the theorem. □.

5 The Centers of Cycle-Free ν-Graphs

We now specialize our study to a certain kind of ν-graph that encompasses the class of transfinite trees as a special case. (A *transfinite ν-tree* is a ν-connected ν-graph having no loops.) The kind of ν-graph we now deal with is one having no ν-loop that passes through more than one $(\nu-1)$-section. All other loops are allowed. Let us be more specific.

Because all the ν-nodes are pristine, every loop of rank less than ν must lie within a single $(\nu-1)$-section $\mathcal{S}^{\nu-1}$, that is, all its nodes are internal nodes of $\mathcal{S}^{\nu-1}$. Such loops are allowed. Moreover, a ν-loop might also lie in a single $(\nu-1)$-section $\mathcal{S}^{\nu-1}$ in the sense that all its nodes of ranks less than ν are internal nodes of $\mathcal{S}^{\nu-1}$ and all its ν-nodes are bordering nodes of $\mathcal{S}^{\nu-1}$; thus, all its branches lie in $\mathcal{S}^{\nu-1}$. Such a loop will pass through a closed sequence $\{x_1^\nu, x_2^\nu, \ldots, x_{k-1}^\nu, x_1^\nu\}$ of bordering ν-nodes of $\mathcal{S}^{\nu-1}$ alternating with endless $(\nu-1)$-paths within $\mathcal{S}^{\nu-1}$. The possibility of such endless $(\nu-1)$-paths within $\mathcal{S}^{\nu-1}$ is implied by Lemma 3.3(b). Such ν-loops within $\mathcal{S}^{\nu-1}$ are also allowed. On the other hand, it is possible in general for a ν-loop to pass through two or more $(\nu-1)$-sections. For the sake of a succinct terminology, we shall call the latter kind of ν-loop a *cycle*. We will henceforth assume that $\mathcal{G}^\nu$ is so structured that it does not have any cycle and will say that $\mathcal{G}^\nu$ is *cycle-free*.

In conformity with our definition of an *end* $(\nu-1)$-*section* as a $(\nu-1)$-section having exactly one boundary ν-node, we now define a *non-end* $(\nu-1)$-*section* as a $(\nu-1)$-section having two or more boundary ν-nodes. Note that, when $\mathcal{G}^\nu$ is cycle-free, two $(\nu-1)$-sections cannot share more than one boundary ν-node, for otherwise $\mathcal{G}^\nu$ would contain a cycle. However, still more is implied by the cycle-free condition.

Lemma 5.1. *Assume that $\mathcal{G}^\nu$ is cycle-free. Then, $\mathcal{G}^\nu$ has only finitely many non-end $(\nu-1)$-sections.*

Proof. If there are no non-end $(\nu-1)$-sections, the conclusion is trivially satisfied. So, assume otherwise, and choose any non-end $(\nu-1)$-section. Label it and all its boundary ν-nodes by "1." Label by "2" all the non-end $(\nu-1)$-sections that share boundary ν-nodes with that 1-labeled $(\nu-1)$-section (if such exist), and label their unlabeled boundary ν-nodes by "2," as well. Each 2-labeled section shares exactly one boundary ν-node with the 1-labeled $(\nu-1)$-section and does not share any 2-labeled ν-node with any other 2-labeled $(\nu-1)$-section, for otherwise $\mathcal{G}^\nu$ would contain a cycle. It follows that the number of labeled ν-nodes is no less than the number of labeled $(\nu-1)$-sections. Next, label by "3" all the non-end $(\nu-1)$-sections that share boundary ν-nodes with the 2-labeled sections (if such exist), and label their unlabeled boundary ν-nodes by "3," as well. Again, each 3-labeled $(\nu-1)$-section shares exactly one boundary ν-node with exactly one 2-labeled $(\nu-1)$-section and does not share any 3-labeled ν-node with any other 3-labeled $(\nu-1)$-section, for otherwise $\mathcal{G}^\nu$ would contain a cycle. Here, too, it follows that the number of labeled ν-nodes is no less than the number of labeled $(\nu-1)$-sections. Continue this way until all the non-end $(\nu-1)$-sections have been labeled along with their boundary ν-nodes. At the end, the number of labeled ν-nodes will be no less than the number of $(\nu-1)$-sections. Since there are only finitely many boundary ν-nodes (Condition 2.1(e)) and since these are the ν-nodes that have been labeled, our conclusion follows. □

Our next objective is to replace our cycle-free ν-graph $\mathcal{G}^\nu$ $(\nu \geq 1)$ by a conventional finite tree (i.e., a 0-connected 0-graph having no loop and only finitely many branches), a correspondence that will be exploited in the proof of our final theorem. The nodal eccentricities for $\mathcal{T}^0$ will be related to the nodal eccentricities for $\mathcal{G}^\nu$ in a simple way (see Lemma 5.2 below).

In the following, the index k will number the boundary ν-nodes of $\mathcal{G}^\nu$ (one number for each) as well as sets of non-boundary bordering ν-nodes. The index m will number the interiors of non-end $(\nu-1)$-sections (one number for each interior) as well as sets of the interiors of end $(\nu-1)$-sections.

Consider, first of all, a non-end $(\nu-1)$-section $\mathcal{S}^{\nu-1}$. Each of its boundary ν-nodes x_k^ν is replaced by a 0-node x_k^0 having the same index number k. Also, all the nonboundary bordering ν-nodes of $\mathcal{S}^{\nu-1}$ (if such exist) are replaced by a single 0-node $x_{k'}^0$. Finally, the interior $i(\mathcal{S}^{\nu-1})$ is replaced by a single 0-node y_m^0. A branch is inserted between y_m^0 and each of the x_k^0 and between y_m^0 and $x_{k'}^0$ as well. Thus, $\mathcal{S}^{\nu-1}$ is replaced by a star 0-graph. We view x_k^0 (resp. $x_{k'}^0$, resp. y_m^0) as representing x_k^ν (resp. $x_{k'}^\nu$, resp. any internal node of $\mathcal{S}^{\nu-1}$). For any $y^\gamma \in i(\mathcal{S}^{\nu-1})$, we have $d(y^\gamma, x_k^\nu) = \omega^\nu$ (Lemma 3.3(a)) and $d(y_m^0, x_k^0) = 1$.

Next, consider all the end $(\nu-1)$-sections that are incident to a single boundary node $x_{k''}^\nu$. We represent all of their interiors by a single 0-node $y_{m'}^0$, and we insert a branch between $y_{m'}^0$ and $x_{k''}^0$. If at least one of those end $(\nu-1)$-sections has a non-boundary bordering ν-node $x_{k'''}^\nu$, we represent all of them by another single 0-node $x_{k'''}^0$, and we insert another branch between $y_{m'}^0$ and $x_{k'''}^0$. So, all of these end $(\nu-1)$-sections incident to the chosen $x_{k''}^0$ are represented either by a single branch incident at $x_{k''}^0$ and $y_{m'}^0$ or by two branches in series incident at $x_{k''}^0$, $y_{m'}^0$, and $x_{k'''}^0$. Here, too, we have replaced the said set of end $(\nu-1)$-sections by either a one-branch or two-branch (elementary) star 0-graph. Again, we view $x_{k'''}^0$ (resp. $y_{m'}^0$) as representing any $x_{k'''}^\nu$ (resp. any internal node of the said end $(\nu-1)$-sections). For any internal node y^γ in any one of those end $(\nu-1)$-sections, we have $d(y^\gamma, x_{k''}^\nu) = d(y^\gamma, x_{k'''}^\nu) = \omega^\nu$ and $d(y_{m'}^0, x_{k''}^0) = d(y_{m'}^0, x_{k'''}^0) = 1$.

We now connect all these star 0-graphs together at their end 0-nodes in the same way that the $(\nu-1)$-sections are connected together at their boundary ν-nodes. The result is a finite 0-tree $\mathcal{T}^0$. Indeed, since $\mathcal{G}^\nu$ is cycle-free, $\mathcal{T}^0$ has no loops. Also, by Assumption 2.1(e) and Lemma 5.1, $\mathcal{T}^0$ has only finitely many branches.

Lemma 5.2. *A node z of any rank in $\mathcal{G}^\nu$ has an eccentricity $e(z) = \omega^\nu \cdot p$ if and only if its representative 0-node z^0 in $\mathcal{T}^0$ has the eccentricity $e(z^0) = p$.*

(Here again, p is a natural number.)

Proof. An eccentric path P^ν of any node z of any rank in $\mathcal{G}^\nu$ passes alternately through $(\nu - 1)$-sections and bordering ν-nodes and terminates at z and an eccentric node for z. Because all ν-nodes are pristine, the length $|P^\nu|$ is obtained by counting the $(\nu - 1)$-tips traversed by P^ν and multiplying by ω^ν (Lemma 3.2). Furthermore, corresponding to P^ν there is a unique path Q^0 in $\mathcal{T}^0$ whose nodes x_k^0 and y_m^0 alternate in Q^0 and represent the bordering nodes x_k^ν and interiors of $(\nu - 1)$-sections $\mathcal{S}_m^{\nu-1}$ traversed by P^ν. Each branch of Q^0 corresponds to one traversal of a $(\nu - 1)$-tip in P^ν, and conversely. Thus, we have $|P^\nu| = \omega^\nu \cdot p$ and $|Q^0| = p$, where p is the number of branches in Q^0. Also, since P^ν is an eccentric path in $\mathcal{G}^\nu$, Q^0 is an eccentric path in Q^0. Whence our conclusion. □

Here is our principal result concerning the centers of cycle-free ν-graphs.

Theorem 5.3. *The center of any cycle-free ν-graph $\mathcal{G}^\nu$ has one of the following forms:*

(a) *A single ν-node x^ν.*

(b) *The interior $i(\mathcal{S}^{\nu-1})$ of a single $(\nu - 1)$-section $\mathcal{S}^{\nu-1}$.*

(c) *The set $i(\mathcal{S}^{\nu-1}) \cup \{x^\nu\}$, where $i(\mathcal{S}^{\nu-1})$ is as in (b) and x^ν is one of the bordering ν-nodes of $\mathcal{S}^{\nu-1}$.*

Proof. In the trivial case where $\mathcal{G}^\nu$ has just one $(\nu - 1)$-section and just one ν-node, all the nodes have the same eccentricity and form (c) holds. So, consider the case where $\mathcal{G}^\nu$ has at least two $(\nu - 1)$-sections or at least two ν-nodes. Because of Lemmas 3.11 and 3.12, neither a non-boundary bordering ν-node nor the interior of an end $(\nu - 1)$-section can be in the center because their eccentricities will be larger than the eccentricities of their nearest boundary ν-nodes. Thus, any center node of $\mathcal{G}^\nu$ is either a boundary ν-node or an internal node of a non-end $(\nu - 1)$-section. Also, the correspondence between boundary ν-nodes x_k^ν and their representative 0-nodes x_k^0 in $\mathcal{T}^0$ is a bijection, and so, too, is the correspondence between the interiors $i(\mathcal{S}_m^{\nu-1})$ of non-end $(\nu - 1)$-sections and their representative 0-nodes

y_m^0 in $\mathcal{T}^0$. The eccentricities of these entities in $\mathcal{G}^\nu$ are related to the eccentricities of their representatives in $\mathcal{T}^0$ as stated in Lemma 5.2.

We now invoke an established theorem for finite 0-trees [2, Theorem 2.1]; namely, the center of such a tree is either a single 0-node of a pair of adjacent 0-nodes. When the center of $\mathcal{T}^0$ is a single 0-node x_k^0, form (a) holds. When the center of $\mathcal{T}^0$ is a single 0-node y_m^0, form (b) holds. Finally, when the center of $\mathcal{T}^0$ is a pair of adjacent 0-nodes, one of them will be a 0-node x_k^0 and the other will be a 0-node y_m^0, and thus form (c) holds. □

References

[1] A. Abian, *The Theory of Sets and Transfinite Arithmetic*, W.B. Saunders Co., Philadelphia, PA, 1965.

[2] F. Buckley and F. Harary, *Distances in Graphs*, Addison-Wesley Publishing Co., Redwood City, CA 1990.

[3] G. Chartrand and L. Lesniak, *Graphs and Digraphs*, Third Edition, Chapman & Hall, New York, 1996.

[4] A.H. Zemanian, *Transfiniteness—for Graphs, Electrical Networks, and Random Walks*, Birkhauser, Boston, 1996.

[5] A.H. Zemanian, *Ordinal Distances in Transfinite Graphs*, CEAS Technical Report 790, University at Stony Brook, New York, April, 2001.

Figure Captions

(There are none.)

Fig. 1.

Fig. 2.

Fig. 3.

Fig. 4.

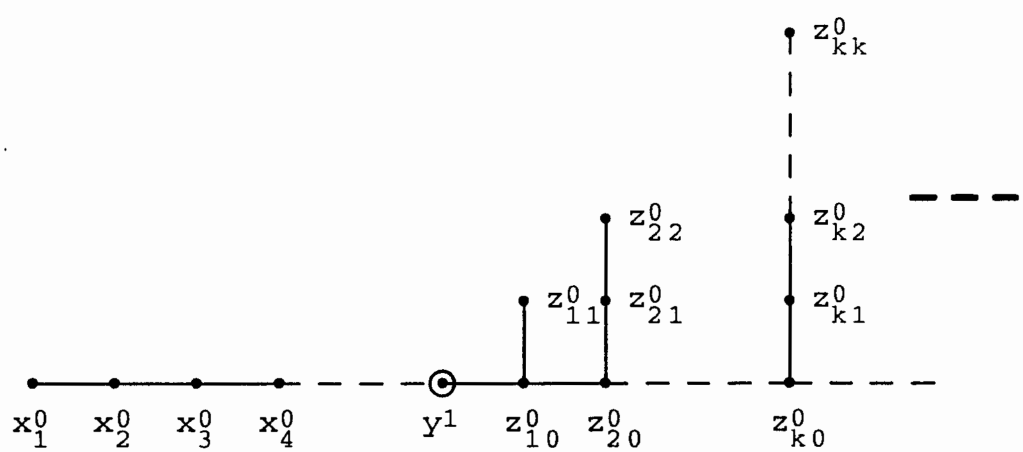

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Fig. 1

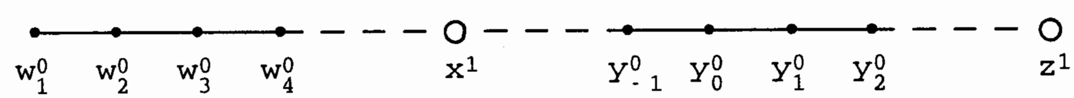

Fig. 2

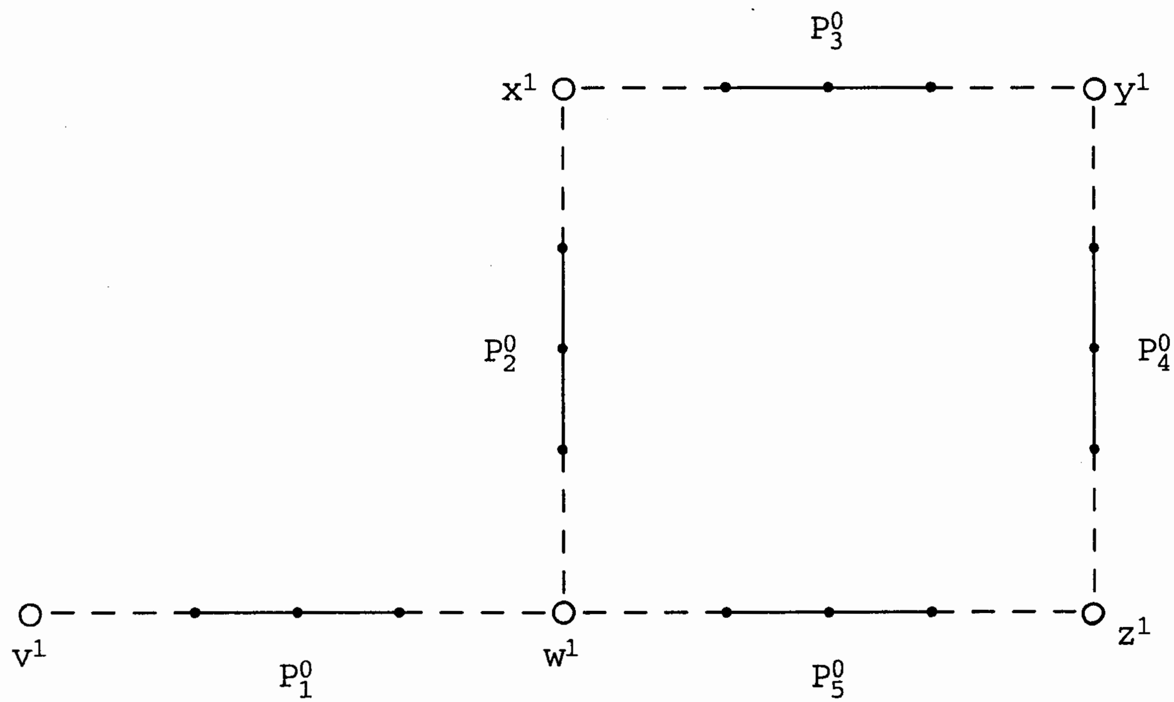

Fig. 3

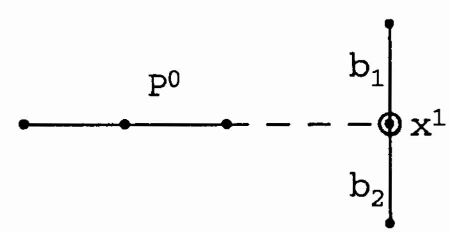

Fig. 4