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NONSTANDARD TRANSFINITE GRAPHS AND NETWORKS OF HIGHER RANKS

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Abstract — In Chapter 8 of the book, "Graphs and Networks: Transfinite and Nonstandard," (published by Birkhauser-Boston in 2004), nonstandard versions of transfinite graphs and of electrical networks having such graphs were defined and examined but only for the first two ranks, 0 and 1, of transfiniteness. In the present work, these results are extended to higher ranks of transfiniteness. Such is done in detail for the natural-number ranks and also for the first transfinite ordinal rank ω. Results for still higher ranks of transfiniteness can be established in much the same way. Once the transfinite graphs of higher ranks are established, theorems concerning the existence of hyperreal operating points and the satisfaction of Kirchhoff's laws in nonstandard networks of higher ranks can be proven just as they are for nonstandard networks of the first rank.

Key Words: Nonstandard graphs, nonstandard electrical networks, hyperreal operating points, Kirchhoff's laws

1 Introduction

In a prior publication [2, Chapter 8] we defined and examined nonstandard versions of graphs that are conventionally infinite as well as those that are transfinite but only of the first rank of transfiniteness. We also examined nonstandard, resistive, electrical networks having such graphs and established an existence theorem for their operating points (i.e., their hyperreal current-voltage regimes) as well as Kirchhoff's laws for those nonstandard networks. In this work, we shall extend these results to graphs and networks having higher ranks of transfiniteness. We do so in detail for the natural-number ranks and also for the first transfinite-ordinal rank ω. Results for still higher ranks of transfiniteness can be established in virtually the same way; the development for the successor-ordinal ranks (resp. limit-ordinal ranks) are virtually the same as that for the natural-number ranks (resp. the rank ω).

All this is accomplished through a recursive analysis proceeding along increasing ordinal ranks. The first two steps of that recursion concern the ranks 0 and 1. These have been explicated in [2, Chapter 8] and will not be repeated here. Our notation and terminology is the same as that used in [2].

2 Nonstandard μ-Graphs

Let μ be a natural number no less than 2. Our development of a nonstandard μ-graph starts with a given sequence $\langle G_n^\mu : n \in \boldsymbol{N} \rangle$, where

$$G_n^\mu = \{X_n^0, B_n, X_n^1 \ldots, X_n^\mu\}$$

is a standard transfinite graph of rank μ. Here, we are defining the branches (i.e., the members of B_n) as pairs of 0-nodes (i.e., members of X_n^0). This differs from the definition of branches given in [2, page 6] based upon elementary tips but only in a nonessential way. We can indeed use all the ideas and results given in [2, Chapter 2].[1] Thus,

$$G_n^{\mu-1} = \{X_n^0, B_n, X_n^1, \ldots, X_n^{\mu-1}\}$$

is the $(\mu-1)$-graph of G_n^μ.

The *extremities* of $G_n^{\mu-1}$ are taken to be its $(\mu-1)$-tips and also the exceptional elements of the μ-nodes of G_n^μ. (The exceptional element, if it exists, of a μ-node x^μ is the unique node of rank less than μ contained in x^μ; see [2, page 11].) Let $\mathcal{T}_n^{\mu-1}$ be the set of $(\mu-1)$-tips of $G_n^{\mu-1}$. Let a typical μ-node of G_n^μ be denoted by $x_{n,k}^\mu$. We are indexing those μ-nodes by k, and we let K be the index set for those μ-nodes. In accordance with the partitioning defined by the $x_{n,k}^\mu$, $\mathcal{T}_n^{\mu-1}$ is partitioned into subsets $\mathcal{T}_{n,k}^{\mu-1}$, where $\mathcal{T}_{n,k}^{\mu-1}$ is the set of all the $(\mu-1)$-tips in $x_{n,k}^\mu$. Thus,

$$\mathcal{T}_n^{\mu-1} = \cup_{k\in K} \mathcal{T}_{n,k}^{\mu-1},$$

[1] In this regard, see how the definition $G^1 = \{X^0, B, X^1\}$ of a 1-graph given in [2, page 163] differs from the definition $\mathcal{G}^1 = \{\mathcal{B}, \mathcal{X}^0, \mathcal{X}^1\}$ of a 1-graph given in [2, page 8] in that nonessential way.

where K serves also as the index set for that partitioning.

If a μ-node $x_{n,k}^{\mu}$ of G_n^{μ} has an exceptional element $x_{n,k}^{\alpha}$ $(\alpha < \mu)$, let $\mathcal{Z}_{n,k}$ denote the singleton set $\mathcal{Z}_{n,k} = \{x_{n,k}^{\alpha}\}$. Otherwise, let $\mathcal{Z}_{n,k} = \emptyset$. In either case, by the definition of any standard μ-node $x_{n,k}^{\mu}$, $\mathcal{Z}_{n,k} \cap \mathcal{Z}_{n,l} = \emptyset$ whenever $k \neq l$, and we have

$$x_{n,k}^{\mu} \;=\; \mathcal{T}_{n,k}^{\mu-1} \cup \mathcal{Z}_{n,k}.$$

If e_n and f_n are two extremities in the same μ-node $x_{n,k}^{\mu}$ of G_n^{μ}, we say that e_n and f_n are *shorted together*, and we write $e_n \asymp f_n$ to denote this fact.

Out next objective is to make an ultrapower construction of the nonstandard μ-nodes and thereby obtain the nonstandard μ-graph ${}^*G^{\mu}$. We already have at hand the nonstandard 0-graph ${}^*G^0 = \{{}^*X^0, {}^*B\}$ [2, page 155] and the nonstandard 1-graph ${}^*G^1 = \{{}^*X^0, {}^*B, {}^*X^1\}$ [2, page 164]. So recursion will yield the nonstandard μ-graphs

$$ {}^*G^{\mu} \;=\; \{{}^*X^0, {}^*B, {}^*X^1, \ldots, {}^*X^{\mu}\}, \tag{1}$$

where ${}^*X^{\mu}$ is the set of nonstandard μ-nodes.

To this end, let $\mathcal{F}$ be any chosen and fixed nonprincipal ultrafilter. Let $\langle e_n \rangle$ be a sequence where each e_n is an extremity of $G_n^{\mu-1}$. Two such sequences $\langle e_n \rangle$ and $\langle f_n \rangle$ are said to be *equivalent* if $e_n = f_n$ for almost all n (modulo $\mathcal{F}$); i.e., $\{n\colon e_n = f_n\} \in \mathcal{F}$. This partitions the set of all extremities into equivalence classes; indeed, reflexivity and symmetry are obvious and for transitivity we have, with $\langle g_n \rangle$ being another sequence of extremities,

$$\{n\colon e_n = f_n\} \cap \{n\colon f_n = g_n\} \subseteq \{n\colon f_n = g_n\}$$

so that $\langle e_n \rangle$ is also equivalent to $\langle g_n \rangle$. Each such equivalence class will be called a *nonstandard extremity* and denoted by $\mathbf{e} = [e_n]$, where $\langle e_n \rangle$ is any representative of that equivalence class.

Given any sequence $\langle e_n \rangle$ of extremities, let $N_{t^{\mu-1}}$ be the set of all n for which e_n is a $(\mu-1)$-tip of $G_n^{\mu-1}$, and let N_x be the set of all n for which e_n is an exceptional element of a μ-node of G_n^{μ}. Consequently, $N_{t^{\mu-1}} \cup N_x = \mathbb{N}$ and $N_{t^{\mu-1}} \cap N_x = \emptyset$. So, exactly one of $N_{t^{\mu-1}}$ and N_x is a member of $\mathcal{F}$. If it is $N_{t^{\mu-1}}$ (resp. N_x), we define $\langle e_n \rangle$ as being a representative of a *nonstandard* $(\mu-1)$*-tip* $\mathbf{t}^{\mu-1} = [e_n]$ (resp. a representative of a *nonstandard exceptional element*) $\mathbf{x} = [e_n]$).

In the latter case of an exceptional element, the e_n are nodes of $G_n^{\mu-1}$ for almost all n, but they need not be of the same rank; their ranks can vary through values no larger than $\mu - 1$. There are no more than finitely many such ranks. Let K be the finite set of such ranks, and let F_k denote the set of all n for which the rank has the value k. The sets F_k are finitely many, pairwise disjoint, and their union is a member of $\mathcal{F}$. Therefore, exactly one of those sets $F(\rho)$ is a member of $\mathcal{F}$ [2, page 19, fact (4)]. Consequently, we can identify the rank of $\mathbf{x}$ as the rank ρ of that unique set $F(\rho)$, and so we may denote $\mathbf{x}$ as $\mathbf{x}^\rho$.

Next step: Let $\mathbf{e} = [e_n]$ and $\mathbf{f} = [f_n]$ be two nonstandard extremities. Let $N_{ef} = \{n\colon e_n \asymp f_n\}$ and $N_{ef}^c = \{n : e_n \not\asymp f_n\}$. So, exactly one of N_{ef} and N_{ef}^c is a member of $\mathcal{F}$. If it is N_{ef} (resp. N_{ef}^c), we say that $\mathbf{e}$ is *shorted to* $\mathbf{f}$, and we write $\mathbf{e} \asymp \mathbf{f}$ (resp. we say that $\mathbf{e}$ is *not shorted to* $\mathbf{f}$, and we write $\mathbf{e} \not\asymp \mathbf{f}$). Also, we take it that $\mathbf{e}$ is *shorted to itself.* This shorting is an equivalence relation for the set of all nonstandard extremities. Indeed, reflexivity and symmetry are again obvious, and transitivity follows from

$$\{n\colon e_n \asymp f_n\} \cap \{n\colon f_n \asymp g_n\} \subseteq \{n\colon e_n \asymp g_n\}.$$

The resulting equivalence classes are defined to be the *nonstandard* μ*-nodes*, and we use the boldface notation $\mathbf{x}^\mu$ to denote a typical one.

Various properties of standard nodes transfer directly to nonstandard nodes. For instance, if the nonstandard node μ-node $\mathbf{x}^\mu$ has a nonstandard exceptional element $\mathbf{x}^\rho = [e_n]$ $(\rho < \mu)$, we have that for almost all n, $e_n \asymp f_n$, where $\mathbf{f} = [f_n]$ is a nonstandard $(\mu - 1)$-tip in $\mathbf{x}^\mu$; that is, every nonstandard exceptional element is shorted to at least one nonstandard $(\mu - 1)$-tip. This also implies that every nonstandard μ-node has at least one nonstandard $(\mu - 1)$-tip.

For similar reasons, the exceptional element of a nonstandard μ-node cannot be the exceptional element of any other nonstandard μ-node, and no nonstandard μ-node can have two or more nonstandard exceptional elements.

Let ${}^*X^\mu$ denote the set of all nonstandard μ-nodes as determined by the given sequence $\langle G_n^\mu \rangle$ of standard μ-graphs. By recursion we can now define the nonstandard μ-graph ${}^*G^\mu$

as the $(\mu + 2)$-tuple given by (1) above.

3 Nonstandard Graphs of Rank $\vec{\omega}$

We now take it that our recursive construction of the μ-graphs can be continued indefinitely through all the natural-numbers ranks. That is, given any sequence $\langle G_n^{\vec{\omega}} \rangle$ of standard $\vec{\omega}$-graphs,[2] we can construct as in the preceding section each set ${}^*X^\mu$ of nonstandard μ-nodes from the sequence $\langle G_n^\mu \rangle$, where G_n^μ is the μ-graph of the $\vec{\omega}$-graph $G_n^{\vec{\omega}}$, this being so for every $\mu \in \boldsymbol{N}$. Furthermore, for the sake of simplicity, we shall assume that none of the $G_n^{\vec{\omega}}$ contains $\vec{\omega}$-nodes. As a result, the nonstandard graph ${}^*G^{\vec{\omega}}$ we shall specify in a moment will not possess any nonstandard $\vec{\omega}$-node.[3]

Thus, we can now define the *nonstandard $\vec{\omega}$-graph* ${}^*G^{\vec{\omega}}$ as the sequence

$$ {}^*G^{\vec{\omega}} \;=\; \{{}^*X^0, {}^*B, \ldots, {}^*X^\mu, \ldots\}, \tag{2} $$

where the entries ${}^*X^\mu$ extend throughout all the natural-numbers $\mu \in \boldsymbol{N}$.

4 Nonstandard Graphs of Rank ω

We now start with a given sequence $\langle G_n^\omega \rangle$ of standard ω-graphs:

$$ G_n^\omega \;=\; \{X_n^0, B_n, X_n^1 \ldots, X_n^\mu, \ldots, X_n^\omega\}, $$

where none of the G_n^ω has any $\vec{\omega}$-node—in accordance with our assumption in Section 3. We have

$$ G_n^{\vec{\omega}} \;=\; \{X_n^0, B_n, X_n^1 \ldots, X_n^\mu, \ldots\} $$

as the $\vec{\omega}$-graph of G_n^ω. The *extremities* of $G_n^{\vec{\omega}}$ are its $\vec{\omega}$-tips and the exceptional elements of the ω-nodes of G_n^ω. Those exceptional elements, if they exist, are nodes of $G_n^{\vec{\omega}}$ with natural-number ranks.

Given any sequence $\langle e_n \rangle$ of extremities, one from each $G_n^{\vec{\omega}}$, let $N_{t\vec{\omega}}$ be the set of all n for which e_n is an $\vec{\omega}$-tip, and let N_x be the set of all n for which e_n is an exceptional

[2] See [2, Section 2.5] for the definition of such a $\vec{\omega}$-graph.

[3] It may be possible to construct an equivalence class $[x_n^{\vec{\omega}}]$ (modulo $\mathcal{F}$) of standard $\vec{\omega}$-nodes to obtain nonstandard $\vec{\omega}$-nodes $\mathbf{x}^{\vec{\omega}}$, but there is a problem concerning the ranks of the embraced nodes of the $x_n^{\vec{\omega}}$. We have not resolved this matter.

element. Thus, $N_{t\vec{\omega}} \cap N_x = \emptyset$ and $N_{t\vec{\omega}} \cup N_x = \boldsymbol{N}$. Consequently, exactly one of $N_{t\vec{\omega}}$ and N_x is a member of $\mathcal{F}$. If it is $N_{t\vec{\omega}}$ (resp. N_x), $\langle e_n \rangle$ is a representative of a *nonstandard $\vec{\omega}$-tip* $\mathbf{t}^{\vec{\omega}} = [e_n]$ (resp. a representative of a *nonstandard exceptional element* $\mathbf{x} = [e_n]$). In either case, we also refer to $\mathbf{e} = [e_n]$ as a *nonstandard extremity.*

Now, let $\mathbf{e} = [e_n]$ and $\mathbf{f} = [f_n]$ be two nonstandard extremities. Let $N_{ef} = \{n\colon e_n \asymp f_n\}$ and $N_{ef}^c = \{n\colon e_n \not\asymp f_n\}$. So, exactly one of N_{ef} and N_{ef}^c is a member of $\mathcal{F}$. If it is N_{ef} (resp. N_{ef}^c), we say that the nonstandard extremities $\mathbf{e} = [e_n]$ and $\mathbf{f} = [f_n]$ are *shorted together*, and we write $\mathbf{e} \asymp \mathbf{f}$ (resp. $\mathbf{e}$ and $\mathbf{f}$ are *not shorted together*, and we write $\mathbf{e} \not\asymp \mathbf{f}$). Also we take it that $\mathbf{e}$ is shorted to itself, i.e., $\mathbf{e} \asymp \mathbf{e}$. This shorting is an equivalence relation on the set of nonstandard extremities, whose transitivity is shown by

$$\{n\colon e_n \asymp f_n\} \cap \{n\colon f_n \asymp g_n\} \subseteq \{n\colon e_n \asymp g_n\}$$

as usual. The resulting equivalence classes are the *nonstandard ω-nodes*; typically, they will be denoted by $\mathbf{x}^\omega$.

Note that each nonstandard ω-node may or may not have a nonstandard exceptional element. In the event that it does have one, say, $\mathbf{x} = [e_n]$, where the e_n are standard nodes $x_n^{\mu_n}$ of natural-number ranks μ_n for almost all n, there are two cases to consider. In the first case, the ranks μ_n assume only finitely many values for almost all n. As was argued in Section 2, there will be exactly one rank ρ for which $\{n\colon \mu_n = \rho\} \in \mathcal{F}$. This allows us to identify the rank of $\mathbf{x}$ as being ρ, and we now write $\mathbf{x}^\rho$ for that nonstandard exceptional element $\mathbf{x}$.

The other case arises when, for every $N \in \mathcal{F}$, the set $\{\mu_n : n \in N\}$ assumes infinitely many values. In this case, we can identify the rank $\rho = [\mu_n]$ of $\mathbf{x}$ as being a hypernatural number that is not a standard number, and we may denote $\mathbf{x}$ by $\mathbf{x}^\rho$ again.

Here, too, every nonstandard ω-node $\mathbf{x}^\omega$ possesses at least one nonstandard $\vec{\omega}$-tip. Also, if $\mathbf{x}^\omega$ possesses an exceptional element $\mathbf{x}^\rho$, that node $\mathbf{x}^\rho$ will be shorted to at least one nonstandard $\vec{\omega}$-tip, and moreover $\mathbf{x}^\rho$ will not be the nonstandard exceptional element of any other nonstandard ω-node. Furthermore, $\mathbf{x}^\omega$ cannot have two or more different nonstandard exceptional elements. These facts, too, follow directly from the properties of standard ω-nodes.

Finally, let ${}^*X^\omega$ denote the set of nonstandard ω-nodes induced by the originally assumed sequence $\langle G_n^\omega \rangle$ of standard ω-graphs. We may now define the nonstandard ω-graph ${}^*G^\omega$ induced by $\langle G_n^\omega \rangle$ to be the set[4]

$$ {}^*G^\omega \;=\; \{ {}^*X^0, {}^*B, {}^*X^1, \ldots, {}^*X^\mu, \ldots, {}^*X^\omega \} \tag{3} $$

5 Nonstandard Resistive Electrical Networks

Let ${}^*G^\nu$ be a nonstandard graph of rank ν, where $1 \le \nu \le \omega$, induced by a sequence $\langle G_n^\nu \rangle$ of standard ν-graphs. ${}^*G^\nu$ can be converted into a nonstandard, resistive, electrical network ${}^*\mathbf{N}^\nu$ by assigning positive resistances to every branch of every G_n^ν and voltage sources to some of those branches, which we take to be in the Thevenin form (i.e., each branch voltage source is connected in series with the branch resistance). This induces hyperreal-valued branch resistances $\mathbf{r_b}$ and hyperreal branch voltage sources $\mathbf{e_b}$ in the nonstandard branches $\mathbf{b}$ of ${}^*G^\nu$, where again in any branch the hyperreal voltage source is connected in series with the hyperreal resistance if that source exists there. This inducement of ${}^*\mathbf{N}^\nu$ is exactly the same as that for the nonstandard network ${}^*\mathbf{N}^1$ of rank 1 explained in [2, pages 165-166].

The question arises as to whether ${}^*\mathbf{N}^\nu$ has a hyperreal operating point, that is, a set of hyperreal branch currents $\mathbf{i_b}$ and a set of hyperreal branch voltages $\mathbf{v_b}$ satisfying in each nonstandard branch $\mathbf{b}$ Ohm's law:

$$ \mathbf{v_b} \;=\; \mathbf{r_b i_b} \;-\; \mathbf{e_b} $$

and Tellegen's equation in a certain way. The answer is "yes," and it is established in virtually the same way as it was established for nonstandard 1-networks ${}^*\mathbf{N}^1$ in [2, Section 8.9]. In fact, Theorem 8.9-2 holds word-for-word with ${}^*\mathbf{N}^\nu$ as it does for ${}^*\mathbf{N}^1$. Now, however, the solution space ${}^*\mathcal{L}$ is based upon loops of ranks up to ν instead of up to 1. Similarly, nonstandard versions of Kirchhoff's current law and Kirchhoff's voltage law[5] again hold as stated in Theorems 8.9-3 and 8.9-4 respectively, with ${}^*\mathbf{N}^1$ replaced by ${}^*\mathbf{N}^\nu$. Since all of this

[4] Since we have assumed that no G_n^ω possesses any $\vec{\omega}$-nodes, a set ${}^*X^{\vec{\omega}}$ of nonstandard $\vec{\omega}$-nodes does not appear in the set (3).

[5] Please see the website www.ee.sunysb.edu/~zeman (in particular, the Errata for [2]) for a rather obvious correction to pages 167 and 168 of that book.

is virtually identical to the development given in [2, Section 8.9], we will say no more about it, other that to note that these nonstandard results for linear networks can be extended to nonlinear resistive networks by exploiting Duffin's theorems noted in the last paragraph of that Section 8.9 of [2].

6 Nonstandard Graphs and Networks of Still Higher Ranks

Let us briefly remark that all our results can be extended to still higher ranks. The results for successor-ordinal ranks can be obtained by mimicking our development for natural-number ranks. For limit-ordinal ranks, the development mimics that given above for rank ω.

References

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