CONTINUOUS POINT SOURCE DIFFUSION IN A SLIGHTLY UNSTABLE BOUNDARY LAYER

By

David A. Mandell
and
Edward E. O'Brien
College of Engineering
State University of New York at Stony Brook

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ABSTRACT

An entirely Lagrangian similarity analysis is used to determine the mean trajectory and mean ground level concentration resulting from continuous sources located at the origin in a slightly unstable boundary layer. The basic equations obtained reduce to the free convection equations for $L \rightarrow -0$ and to the neutral stratification equations for $L \rightarrow -\infty$, where L is the Monin-Obukhov length scale characterizing the instability. The predictions are that the mean trajectories are somewhat more sensitive to a slightly unstable condition than has been previously suggested and also that the mean ground level concentration drops off more rapidly with distance from the source. The second of these predictions has some experimental support.

INTRODUCTION

In this paper, the results obtained by Batchelor (1964) for the case of neutral stratification in the constant stress region of the boundary layer are extended to the case of a slightly unstable boundary layer. The basic equations are obtained by assuming Lagrangian similarity. The corrections to neutral stratification depend upon a characteristic length L, which was first used by Monin and Obukhov (1954). For $L \to -\infty$, the present results reduce to the neutral stratification results of Batchelor and for $L \to 0$ (from the negative side), the basic equations reduce to the free convection equations of Yaglom (1965).

Previously, Gifford (1962) obtained results for diffusion from point and line sources in a diabatic surface layer by using a Lagrangian description, but introducing an Eulerian function (the log-plus-linear law of Monin and Obukhov). Cermak (1963) also uses a Lagrangian and Eulerian description to obtain diffusion results. Cermak includes the effects of a nonzero source height, while Gifford takes the source to be at $Z = 0$. The source is also taken to be at $Z = 0$ in the present work. Both of these analyses require knowledge of an empirical function. These empirical functions do not enter into the first order corrections to the neutral stratification case obtained herein.

The basic equations used by Gifford, Cermak and Yaglom do not reduce to the correct equations for both $L \to -\infty$ and $L \to 0$.

Basic Equations

It will be assumed that Lagrangian similarity exists. From dimensional reasoning, the following two equations are obtained for $\overline{X}$ and $\overline{Z}$ respectively:

$$\frac{d^2\overline{X}}{dt^2} = \frac{u_*}{t}\, g\left(\frac{u_* t}{L}\right) \qquad (1)$$

$$\frac{d^2\overline{Z}}{dt^2} = \frac{1}{\sqrt{t}}\left(\frac{q\, g}{c_p\, \rho_o\, T_o}\right)^{\frac{1}{2}} f\left(\frac{u_* t}{L}\right) \qquad (2)$$

where g and f are two unknown universal functions, u_* $\left(= \frac{\sqrt{\tau}}{\sqrt{\rho_o}}\right)$ is the friction velocity, L $\left(= \frac{-u_*^3}{\frac{k\, q\, g}{T_o\, \rho_o\, c_o}} \right)$ is a characteristic length, t is the time since the particle left the origin, τ is the constant shear stress; ρ_o, T_o and c_p are reference density, temperature and specific heat respectively, k is von Karman's constant, g is the acceleration of gravity and q is the vertical heat flux, positive upward. $X(t)$ and $Z(t)$ are the directions downwind and vertically upwards respectively.

In the neutral stratification case $L \rightarrow -\infty$. This results from $q = 0$ and u_* remaining finite. In this case equations (1) and (2) become

$$\frac{d^2 \bar{X}}{dt^2} = \frac{u_*}{t} g(0)$$

$$\frac{d^2 \bar{Z}}{dt^2} = 0$$

If $g(0) = a$, the above two equations are identical to the equations given by Batchelor (a is a dimensionless absolute constant defined by Batchelor and Batchelor's t_1 is zero for a particle released at $Z = 0$).

In the limit of free convection $u_* \rightarrow 0$ and q remains finite. Therefore, from the definition of L, $\frac{u_*}{L} \rightarrow -\infty$ as $u_* \rightarrow 0$. Equations (1) and (2) become

$$\frac{d^2 \bar{X}}{dt^2} = 0 \qquad \text{or} \qquad \bar{X} = c_1 t + c_2 \tag{3}$$

$$\frac{d^2 \bar{Z}}{dt^2} = \left(\frac{q\, g}{c_p \rho_o T_o} \right)^{\frac{1}{2}} \frac{1}{\sqrt{t}} f(-\infty) \tag{4}$$

If the constants of integration c_1 and c_2 equal U_o and 0 respectively (U_o is the constant wind velocity of Yaglom); and, $f(-\infty) = \frac{3}{4} c$ (c is a universal constant used by Yaglom), then equations (3) and (4) are identical to the free convection results obtained by Yaglom. The free convection limit will not be considered further here.

Slightly Unstable Solution

By using the definition of L, equation (2) can be rewritten as

$$\frac{d^2 \bar{Z}}{dt^2} = \left(- \frac{u_*^3}{kL} \right)^{\frac{1}{2}} \frac{1}{\sqrt{t}} f \left(\frac{u_* t}{L} \right) \tag{5}$$

The functions $f\left(\frac{u_* t}{L}\right)$ and $g\left(\frac{u_* t}{L}\right)$ can be expanded in powers of $\frac{u_* t}{L}$ about the zero value provided f and g are analytic functions. Since Yaglom has shown that $\frac{u_* t}{L}$ is a function of $\frac{\bar{Z}}{L}$, only small values of $\frac{\bar{Z}}{L}$ will be considered. In the unstable case under consideration $-\infty < L < 0$. Equations (1) and (5) become

$$\frac{d^2 \bar{X}}{dt^2} = \frac{u_*}{t} g(0) + \frac{u_*^2 g'(0)}{L} + \text{higher order terms}$$

$$\frac{d^2 \bar{Z}}{dt^2} = \left(\frac{-u_*^3}{k L} \right)^{\frac{1}{2}} \frac{1}{\sqrt{t}} f(0) + \left(\frac{-u_*^3}{k L} \right)^{\frac{1}{2}} \frac{u_*}{L} \sqrt{t}\, f'(0) + \text{higher order terms}$$

Only terms of order $\frac{\sqrt{\bar{Z}}}{\sqrt{L}}$ will be retained. Therefore,

$$\frac{d^2 \bar{X}}{dt^2} = \frac{a\, u_*}{t} \tag{6}$$

$$\frac{d^2 \bar{Z}}{dt^2} = \frac{b_1\, u_*^{3/2}}{\sqrt{L}} \frac{1}{\sqrt{t}} \tag{7}$$

where $g(0) = a$ and $\frac{f\,(0)}{\sqrt{-k}} = b_1$.

$\bar{Z}$ (t) should reduce to Batchelor's result for $L \rightarrow -\infty$. This determines the constants of integration when equation (7) is integrated with respect to time. Equation (7) becomes

$$\bar{Z}\,(t) = \frac{4}{3} \frac{b_1 u_*^{3/2}}{\sqrt{L}}\, t^{3/2} + b\, u_*\, t \tag{8}$$

where b is the proportionality constant used by Batchelor. (Batchelor estimates b to be between .1 and .2).

Equation (8) is a cubic equation in the $\sqrt{t}$ which can be solved for $t\,(\bar{Z})$ by using the standard techniques for cubic equations. Two of the roots result in negative t for positive $\bar{Z}$ and small $\frac{\bar{Z}}{L}$ and thus are not physically possible. The third root, for small $\frac{\bar{Z}}{L}$, is

$$t = \frac{\bar{Z}}{b\, u_*} \left[1 - \frac{4}{3} \left(\frac{b_1}{b} \right) \frac{\sqrt{\bar{Z}}}{\sqrt{b\, L}} + 0 \left(\frac{1}{L} \right) \right] \tag{9}$$

$\bar{X}$ (t) can be found from equation (6). The result is

$$\bar{X}\,(t) = a\, u_*\, t \log t - a\, u_*\, t \tag{10}$$

The constants of integration are taken equal to zero so that the result is identical to Batchelor's result for large L. Substitution of equation (9) into equation (10) gives $\overline{X}(\overline{Z})$, after some algebra:

$$\overline{X}(\overline{Z}) = \frac{a}{b}\overline{Z}\left(\log\frac{\overline{Z}}{bu_*} - 1\right) - \frac{4}{3}\left(\frac{a}{b}\right)\left(\frac{b_1}{b}\right)\overline{Z}\frac{\sqrt{\overline{Z}}}{\sqrt{b}\,L}\log\frac{\overline{Z}}{b\,u_*} \qquad (11)$$

For $L \rightarrow -\infty$, this is identical to Batchelor's result since $\frac{1}{b\,u_*} = \frac{c}{Z_o}$, where Z_o is a length characteristic of the surface roughness, when the source is at $Z = 0$. (c is a constant defined by Batchelor.)

Mean Concentration

The mean concentration from a continuous point source is

$$c_p(X, y, Z) = Q\int_o^{\infty}\frac{1}{\overline{Z}^3}\,\psi\left(\frac{X-\overline{X}}{\overline{Z}}, \frac{y}{\overline{Z}}, \frac{Z}{\overline{Z}}, \frac{\overline{Z}}{L}\right)dt$$

where Q is the rate of release of particles from the source and ψ is the probability that a particle will be at position X, y, Z at time t. The function ψ can be expanded in a series about $\frac{\overline{Z}}{L}$ to give

$$\psi\left(\frac{X-\overline{X}}{\overline{Z}}, \frac{y}{\overline{Z}}, \frac{Z}{\overline{Z}}, \frac{\overline{Z}}{L}\right) = \psi\left(\frac{X-\overline{X}}{\overline{Z}}, \frac{y}{\overline{Z}}, \frac{Z}{\overline{Z}}, 0\right) + \frac{\overline{Z}}{L}\psi'\left(\frac{X-\overline{X}}{\overline{Z}}, \frac{y}{\overline{Z}}, \frac{Z}{\overline{Z}}; 0\right)$$

+ higher order terms

To order $\frac{\sqrt{\overline{Z}}}{\sqrt{L}}$, only the first term on the right side remains. Therefore,

$$c_p(X, y, Z) = Q\int_o^{\infty}\frac{1}{\overline{Z}^3}\,\psi\left(\frac{X-\overline{X}}{\overline{Z}}, \frac{y}{\overline{Z}}, \frac{Z}{\overline{Z}}\right)dt$$

The mean ground level concentration in the slightly unstable case can be found in the same manner as Batchelor's solution for the neutral case, the only difference being the use of equation (11) instead of Batchelor's $\overline{X}\,(\overline{Z})$. The concentration at ground level from a point source is

$$c_p(X, 0, 0) \sim \frac{Q}{a\, u_*} \left[\frac{1}{\overline{Z}^2 \log \frac{\overline{Z}}{b u_*}} \right]_{\overline{X} = X} \left\{ 1 + \frac{\frac{4}{3}\left(\frac{b_1}{b}\right) \frac{\sqrt{\overline{Z}}}{\sqrt{bL}}}{\log \frac{\overline{Z}}{b u_*}} \right\}_{\overline{X} = X} \int_{-\infty}^{\infty} \psi(\;, 0, 0)\, d$$

or

$$c_p(X, 0, 0) \sim \frac{Q}{u_* X^2} \left\{ \log \frac{\overline{Z}}{b\, u_*} \left[1 - \frac{4}{3}\left(\frac{b_1}{b}\right) \frac{\sqrt{\overline{Z}}}{\sqrt{bL}} \left(2 - \frac{1}{\log \frac{\overline{Z}}{b\, u_*}} \right) \right] \right\}_{\overline{X} = X} \quad (12)$$

For the continuous line source

$$c_l(X, 0) \sim \frac{Q}{u_* X} \left[1 - \frac{4}{3}\left(\frac{b_1}{b}\right) \frac{\sqrt{\overline{Z}}}{\sqrt{bL}} \left(1 - \frac{1}{\log \frac{\overline{Z}}{b\, u_*}} \right) \right]_{\overline{X} = X} \quad (13)$$

Results

In terms of Cermak's notation, equation (11) for the trajectory becomes, in dimensionless form

$$b\,k\,\xi = \zeta(\log\zeta - 1) - \frac{4}{3}\,\frac{b_1}{b\sqrt{b}}\,\sqrt{\alpha}\,\zeta^{3/2}\,\log\zeta \qquad (13)$$

where $k = \frac{1}{a}$ is von Karman's constant, $\xi = \frac{\overline{X}}{Z_o}$, $\zeta = \frac{\overline{Z}}{Z_o}$ and $\alpha = \frac{Z_o}{L}$. The corresponding equation obtained by Cermak is (with the source height zero)

$$b\,k\,\xi = \zeta(\log\zeta - 1) + \alpha\left(-\zeta + \frac{7}{16}\zeta^2 + \frac{1}{8}\zeta^2\log\zeta\right)$$

By using equation (13) and the data given by Cermak the universal constant $|b_1|$ is estimated to be .01.

Cermak gives his mean ground level concentration results in the form $c_p \propto X^{m_{cp}}$ for the continuous point source and a similar expression for the line source. The exponents, m_{cp} and $m_{c\ell}$, vary with X.

Table I shows typical results for neutral stratification, Cermak's analysis (with the source height zero) and the present analysis for $\alpha = -10^{-4}$:

		$b k \xi$			$-m_{cp}$			$-m_{c\ell}$		
ζ	$\frac{\bar{Z}}{\lvert L \rvert}$ ($= \lvert\alpha\rvert \zeta$)	Neutral ($\alpha = 0$)	Cermak	Present Analysis	Neutral	Cermak	Present Analysis	Neutral	Cermak	Present Analysis
100	.01	360	360	340	1.74	1.74	1.77	.953	.953	.964
500	.05	2600	2570	2310	1.81	1.82	1.92	.974	.976	1.01
1000	.1	5900	5770	4970	1.84	1.85	1.98	.978	.990	1.02

Table I Comparison of Results for Neutral and Slightly Unstable Boundary Layers

It is clear from the table that the mean trajectory is higher for unstable stratification than for neutral stratification and the present analysis predicts higher values than does Cermak. As would be expected, the mean ground level concentration decreases more rapidly with distance from the source for the unstable case and it also decreases more rapidly for the present analysis than for Cermak's. His calculated $-m_{cp}$ is lower than the experimental values he gives. Since the present analysis results in larger values of $-m_{cp}$ than Cermak, it appears as if the larger corrections to neutral stratification indicated by the present analysis have some experimental support.

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Appendix - References

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