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Matrix Two-Dimensional Spectral Factorization

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Abstract

The existence of an asymmetric half-plane spectral factorization for a non-negative two-dimensional matrix-valued spectral density is established under general conditions ; namely, that the density and the logarithm of the determinant of the density are absolutely integrable on the torus. The proof consists of first obtaining a symmetric half-plane factorization by applying a one-dimensional spectral factorization algorithm, and then modifying the factors to get the desired form. An efficient algorithm for calculating the two-dimensional spectral factorization is thereby obtained; Wilson's one-dimensional factorization algorithm appears to be the most suitable for this purpose. In the case where the initial array is of finite extent, it is shown that the factors have constant (minimal) order in the 'causal' direction. It follows that if a finite-support array has a quarter-plane spectral factorization, the spectral factors must also be of finite support.

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1 Introduction

The problem of spectral factorization is fundamental in numerous areas of circuit, systems, and signal theory, and has been extensively studied by many researchers [1, 2, 3]. In two-dimensional signal processing, it arises in the design of recursive filters from an amplitude response specification, and in the derivation of autoregressive models for stationary stochastic processes [4, 5, 6, 7, 8, 9, 10]. Each of these problems has its multivariable or multichannel counterpart which is also of importance; in this context, the problem is to find asymmetric half-plane spectral factors for a non-negative square matrix-valued function on the torus.

The purpose of this paper is to establish the existence of such spectral factors under general conditions, and to give an efficient algorithm which calculates these factors. The conditions considered (in addition to the obvious positivity condition) are that the original matrix-valued function, and the logarithm of its determinant, are absolutely integrable on the torus.

It is well-known, of course, that in the multidimensional case the factors can not in general be of finite order [11]; from the scalar case , the best which can be hoped for is the following [4]. If the original array has nonzero terms from $-N$ to N in the vertical direction, and from $-M$ to M in the horizontal direction, and the asymmetric half-plane decomposition corresponds to horizontal scanning, then each approximate factor should have order N in the vertical direction, and infinite order in the horizontal direction. It is shown below that this continues to hold in the matrix case.

It follows from the proof that any one-dimensional spectral factorization algorithm can be used to give a two-dimensional algorithm; however, algorithms based on Newton's method [12, 13] have some properties which make them particularly suitable in the present context; they are also quite efficient in the absence of roots on the unit circle.

Finally, we note that although we can always assume in practice that the density to be factored is rational, the proof is given under general hypotheses for two reasons: first, since in the multidimensional case the factors are in general nonrational, there is no simplification in assuming a rational density; and second, a general proof shows that the assumption of rationality is not essential, and so the properties of the factors are unlikely to be critically dependent of the order. The latter point is particularly important in that most rational models are actually simplified approximations to much higher-order models.

The paper is organized as follows: in section 2, the notation is introduced, and in section 3 the algorithm is described and some comments on its computational complexity and convergence rate are given. Examples are given in section 4. The last section contains conclusions and mentions some open problems; proofs are contained in the appendix.

2 Definitions and Notation

The following notation will be used throughout.

The unit torus, will be denoted by T^2:

$$T^2 = \{(z_1, z_2) \mid |z_1| = 1 \text{ and } |z_2| = 1\}$$

The set of absolutely integrable p by p matrix-valued functions on T^2 will be denoted by $L_1(T^2)$, and the subset of square-integrable such functions will be denoted by $L_2(T^2)$. The set of absolutely integrable scalar-valued functions on T^2 will be denoted by $L_1^s(T^2)$. Elements of $L_1(T^2)$ and $L_2(T^2)$ will be denoted by uppercase letters, and their two-dimensional inverse Z-transforms by the corresponding lowercase letters. The unit matrix will be denoted by 1. As usual, an upper $*$ denotes conjugate transpose. The inverse Z-transform as used here is defined by

$$f(m,n) = 1/4\pi^2 \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} F(e^{-j\theta_1}, e^{-j\theta_2}) d\theta_1 d\theta_2$$

The asymmetric half-plane which defines causal filters will be denoted by A; we will take this to be set

$$A = \{(k,l) \mid l > 0\} \cup \{(k,l) \mid l = 0 \text{ and } k >= 0\}$$

we will refer to the second coordinate (the l direction) as the vertical, or causal, direction, and the first coordinate (the k direction) as the horizontal, or noncausal, direction. An element F of $L_1(T^2)$ is causal if, and only if, $f(k,l) = 0$ when (k,l) is not in A.

A causal element G of $L_1(T^2)$ is said to be outer (or minimum-phase) if $\log|\det\ G(e^{j\theta_1}, e^{j\theta_2})| \in L_1^s(T^2)$ and

$$\log|\det\ g(0,0)| = 1/4\pi^2 \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \log|\det\ G(e^{j\theta_1}, e^{j\theta_2})|\ d\theta_1 d\theta_2.$$

In the case where $G(z_1, z_2)$ is a polynomial in $z_1$and z_2, this is equivalent [9, p. 590] to $G(z_1, z_2)$ having no zeros on the open unit bidisk defined by

$$U^2 = \{(z_1, z_2) \mid |z_1| \leq 1 \text{ and } |z_2| \leq 1\}$$

Given a positive Hermitian element H of $L_1(T^2)$, a spectral factorization of H will mean a factorization of H as

$$H = GG^*$$

where G is causal, stable (i.e., in $L_2(T^2)$) and outer. This factorization, when it exists, is unique up to multiplication by a constant, unitary matrix; it will be made unique by assuming that $g(0,0)$ is lower triangular, with a non-negative, real diagonal.

Note that in the above definitions, stability of a transfer function has been defined in terms of square-integrability, rather than absolute summability of the inverse Z-transform; although this is not strictly correct, it has been done to simplify the presentation, since, in the general case the spectral factors may not have an absolutely summable inverse Z-transform. (This will happen, for example, if H is discontinuous on T^2, since absolute summability of the inverse Z-transform implies continuity on T^2)

With this background, the algorithm can now be described.

3 Spectral Factorization Algorithm

The main result of this paper is the following.

Theorem 1 *Suppose that K is a non-negative p by p matrix-valued element of $L_1(T^2)$, and that* $\log(\det\ K)\ \in L_1^s(T^2)$.

Then K has a spectral factorization:

$$K = GG^*$$

Further, if the inverse Z-transform, $k(m,n)$, of K is zero outside the set $\{(m,n) \mid -N \leq n \leq N\}$, then the inverse Z-transform, $g(m,n)$, of G is zero outside the set $\{(m,n) \mid 0 \leq n \leq N\}$.

The detailed proof of this theorem is contained in the appendix; here the following remarks may be made.

First, the proof actually provides a two-dimensional matrix spectral factorization algorithm; the spectral factor G can be found as follows.

For each $\theta_1 \in [-\pi, \pi]$, let $K_{\theta_1}(\theta) = K(\theta_1, \theta)$. Then (using any of the known one-dimensional spectral factorization algorithms) find the spectral factorization of each $K_{\theta_1}(\theta)$:

$$K_{\theta_1} = G_{\theta_1} G^*_{\theta_1}$$

Now let

$$P(\theta_1) = 1/2\pi \int_{-\pi}^{\pi} G_{\theta_1}(\theta) d\theta$$

and let $F(\theta_1)$ be the one-dimensional spectral factor of $P(\theta_1)P^*(\theta_1)$. Then the required two-dimensional spectral factor is given by

$$G(\theta_1, \theta_2) = G_{\theta_1}(\theta_2) P^{-1}(\theta_1) F(\theta_1)$$

The two-dimensional factorization can therefore be calculated by means of a family of one-dimensional factorizations.

Second, the step in which the $G_{\theta_1}(\theta)$ are calculated essentially gives the noncausal symmetric half-plane factorization; the only adjustment needed is to take $g_{\theta_1}(0,0)$ to be positive Hermitian instead of upper triangular. Thus the algorithm finds both the noncausal symmetric half-plane factors and the causal asymmetric half-plane factors. (It may be worth remarking that the three-term factorization into causal symmetric factors and a one-dimensional factor does not always exist in the general non-algebraic case.)

Third, the theorem has a consequence for which is of some interest, as follows.

Corollary 1 *If K (in the theorem above) has a* quarter-plane *factorization G, and $k(m,n)$ is zero outside the set $\{(m,n) \mid -N \leq n \leq N$ and $-M \leq m \leq M\}$, then $g(m,n)$ is zero outside the set $\{(m,n) \mid 0 \leq n \leq N$ and $0 \leq m \leq M\}$.*

In other words, if a polynomial *non-negative matrix has quarter-plane spectral factors, then the spectral factors are also polynomial.*

Proof:

From the theorem, the half-plane spectral factor $\tilde{G}$ must have the property that $\tilde{g}(m,n)$ is zero outside the the set $\{(m,n) \mid 0 \leq n \leq N\}$. If we interchange the role of the two coordinates, we get another half-plane spectral factor $\hat{G}$ with the property that $\hat{g}(m,n)$ is zero outside the the set $\{(m,n) \mid 0 \leq m \leq M\}$. Since the quarter-plane spectral factor G is a half-plane spectral factor in both cases, and the spectral factors are unique, (assuming that the $g(0,0)$ have all been normalized in the same way) it follows that G must coincide with both $\tilde{G}$ and $\hat{G}$, and so the support of g must be contained in the intersection of the supports of $\tilde{g}$ and $\hat{g}$, as required.

4 Examples

4.1 Implementation Details

As was mentioned previously, the implementation reduces to a family of one-dimensional factorizations. In theory, this is an infinite family parameterized by the unit circle; in practice, of course, it is discretized. Any one-dimensional factorization algorithm may be used; however, since there is a family of problems to be solved here, it is advantageous to use an algorithm which is easily adapted to continuation methods. This is especially true since the continuation version of the spectral factorization problem is an additive decomposition problem which is easily solved by using a Fast Fourier

Transform. From this point of view, the most natural one-dimensional spectral factorization algorithm to use is Wilson's [12, 13]; this was used for all of the examples below. The only real disadvantage of this is that it has problems with zeros on the circle.

When the additive decomposition is implemented by means of the FFT on a q by r grid, and the number of operations required to calculate this FFT is J_{qr}, then the number of operations required per iteration of Wilson's spectral factorization algorithm is $O(p^2 J_{qr} + p^3 qr)$, since both matrix multiplication and inversion are $O(p^3)$ operations. Thus, for Cooley-Tukey-type FFT algorithms, the number of operations per iteration is $O(p^2 qr \log qr + p^3 qr)$. Typically, the first factorization requires ten or fewer iterations, and subsequent factorizations require about five, in the absence of zeros on the circle.

Since the spectral factor is in general of infinite support in the non-causal direction, it is necessary to take q to be the full horizontal width of the two-dimensional signal under consideration; however, if the original matrix density K is of finite order, then the factor G has a vertical order no larger than that of K, and so in many cases it will be possible to take r to be considerably smaller than the vertical dimension of the signal.

For reasons of space, only low-order examples will be presented here. In all of these examples, calculations were done in single precision, the DFT was taken on a 32 by 32 grid, and, in the absence of zeros on the unit circle, the one-dimensional spectral factorization algorithm was run until the the update term was a floating-point zero; divergence in the case of zeros on the circle was prevented by stopping at 12 iterations.

4.2 Example 1

In order to test the algorithm, the following (z-domain) matrix, which is known to be asymmetric half-plane stable, was chosen.

$$G(z_1, z_2) = \begin{pmatrix} 1 + 0.5z_1 + 0.5z_2 - 0.125z_1z_2 & z_2 + .25\bar{z}_1z_2 \\ 0.5 & 1 \end{pmatrix}$$

Then GG^* was calculated and evaluated on the unit torus, to give

$$GG^*(e^{j\theta_1}, e^{j\theta_2}) = K(e^{j\theta_1}, e^{j\theta_2}) = \begin{pmatrix} K_{11}(\theta_1, \theta_2) & K_{12}(\theta_1, \theta_2) \\ K_{21}(\theta_1, \theta_2) & K_{22}(\theta_1, \theta_2) \end{pmatrix}$$

where

$$\begin{aligned} K_{11}(\theta_1, \theta_2) &= 2.578125 + 1.375\cos\theta_1 + 0.875\cos\theta_2 - 0.25\cos(\theta_1 + \theta_2) \\ &\quad +0.5\cos(\theta_1 - \theta_2) \\ K_{12}(\theta_1, \theta_2) &= 0.5 + 0.25e^{j\theta_1} + 1.25e^{j\theta_2} - 0.0625e^{j\theta_1}e^{j\theta_2} + 0.25e^{-j\theta_1}e^{j\theta_2} \\ K_{21}(\theta_1, \theta_2) &= 0.5 + 0.25e^{-j\theta_1} + 1.25e^{-j\theta_2} - 0.0625e^{-j\theta_1}e^{-j\theta_2} \\ &\quad +0.25e^{j\theta_1}e^{-j\theta_2} \end{aligned}$$

and

$$K_{22}(\theta_1, \theta_2) = 1.25$$

The algorithm was then applied to K and yielded G with errors of the order of 10^{-7}.

The number of iterations in the one-dimensional Wilson factorization algorithm was typically five or six to final convergence for each value of θ_1.

4.3 Example 2

To test the algorithm in the situation where finite-order factors did not exist, the constant term in the K of example 1 was changed to

$$\begin{pmatrix} 3 & 0.5 \\ 0.5 & 1.5 \end{pmatrix}$$

and the factors of the resulting K were calculated. As expected, the factors were no longer of finite order; they were zero, however, (to an accuracy of the order of 10^{-6}) outside the set indicated by theorem 1. Again, the one- dimensional algorithm was allowed to run to final convergence; in this case, the typical number of iterations was four or five.

4.4 Example 3

In order to test a case with a zero on the torus, the same procedure as in example 1 was followed, using

$$K(e^{j\theta_1}, e^{j\theta_2}) = \begin{pmatrix} K_{11}(\theta_1, \theta_2) & K_{12}(\theta_1, \theta_2) \\ K_{21}(\theta_1, \theta_2) & K_{22}(\theta_1, \theta_2) \end{pmatrix}$$

where

$$\begin{aligned} K_{11}(\theta_1, \theta_2) &= 2.8125 + 1.75\cos\theta_1 + 0.75\cos\theta_2 - 0.5\cos(\theta_1 + \theta_2) \\ &\quad -0.5\cos(\theta_1 - \theta_2) \\ K_{12}(\theta_1, \theta_2) &= 0.5 + 0.25e^{j\theta_1} + 1.25e^{j\theta_2} - 0.125e^{j\theta_1}e^{j\theta_2} + 0.5e^{-j\theta_1}e^{j\theta_2} \\ K_{21}(\theta_1, \theta_2) &= 0.5 + 0.25e^{-j\theta_1} + 1.25e^{-j\theta_2} - 0.125e^{-j\theta_1}e^{-j\theta_2} \\ &\quad +0.5e^{j\theta_1}e^{-j\theta_2} \end{aligned}$$

and

$$K_{22}(\theta_1, \theta_2) = 1.25$$

which has a zero at $z_1 = z_2 = -1$.

In this case, when $\theta_1$reached the value at which the zero occurred, the algorithm failed to converge; for this reason also, it was preferable to re-initialize the algorithm at each step of θ_1, rather than using the the factors from the previous θ_1. In addition, the number of iterations was limited to 12; when these steps were taken, the algorithm

reconstructed G to an accuracy of the order of 10^{-3}; this is about what might be expected, since $10^{-3} \approx 32^2$..

To test the effects of zeros close to, but not quite on, the torus, example 3 was repeated with a random perturbation of the coefficients of the order of 10^{-7}. In this case the algorithm converged after 20 iterations; the values of G obtained, however, still had errors of the order of 10^{-3}. The algorithm therefore produces approximately the same errors for zeros on the torus as for zeros very close to the torus; the convergence behavior of the algorithm, however, appears to be a very sensitive indicator of whether the zeros are merely close to the torus, or actually numerically indistinguishable from being on the torus.

5 Conclusions and Discussion

The existence of asymmetric half-plane matrix spectral factors has been established under general hypotheses, and an efficient algorithm for calculating these factors has been given. It has also been shown that if a polynomial matrix density has a quarter-plane spectral factorization, then the spectral factors are also polynomial. Among the remaining problems the following seem to be the most important: first, to find a method for obtaining approximate factors which are of finite order in the horizontal direction; and second, to improve the performance when there are zeros or singularities (especially indeterminacies) on the torus. In connection with the latter problem, it should be emphasized that the Wilson algorithm used here is by no means the only efficient way of implementing the one-dimensional factorizations. Finally, it is of some interest to extend the algorithm to higher dimensions, since moving multisensor and multispectral moving images are of some importance (for example, in self-navigating vehicles and HDTV), and are among the multidimensional applications where real-time processing, and therefore spectral factorization, are genuinely necessary.

A Appendix: Proof of Theorem 1

The idea of the proof has already been outlined in the discussion following the statement of the theorem; the main task remaining is to verify the conditions necessary for the existence of the one-dimensional factorizations, and to verify that the two-dimensional factor is indeed spectral.

The unit circle will be denoted by T, and $L_1(T)$, etc., will have meanings corresponding to those used for T^2.

If, for each $\theta_1 \in [-\pi, \pi]$, the function K_{θ_1} is defined on T by

$$K_{\theta_1}(\theta) = K(\theta_1, \theta)$$

then, since $K \in L_1(T^2)$ and $\log(\det K) \in L_1^s(T^2)$, it follows immediately that $K_{\theta_1} \in L_1(T)$ and $\log(\det K_{\theta_1}) \in L_1^s(T)$ for almost all θ_1, and so, for almost all θ_1, K_{θ_1} has a unique spectral factorization

$$K_{\theta_1} = G_{\theta_1} G_{\theta_1}^*$$

where G_{θ_1} has the properties:

$$G_{\theta_1} \in L_2(T) \tag{1}$$

$$\log(\det G_{\theta_1}) \in L_1^s(T) \tag{2}$$

$$\log|\det g_{\theta_1}(0)| = 1/2\pi \textstyle\int_{-\pi}^{\pi} \log|\det G_{\theta_1}(e^{j\theta})|\, d\theta \tag{3}$$

$$g_{\theta_1}(n) = 0 \text{ for } n < 0 \tag{4}$$

and $g_{\theta_1}(0)$ is lower triangular with positive real diagonal.

Also, if $k(m,n) = 0$ for $|n| > N$, it follows immediately that $k_{\theta_1}(n) = 0$ for $|n| > N$, and so $g_{\theta_1}(n) = 0$ for $n > N$, for almost all θ_1.

Note that, apart from the normalization of $g_{\theta_1}(0)$, $G_{\theta_1}(\theta_2)$ is the noncausal symmetric half-plane factorization of $K(\theta_1, \theta_2)$.

To find the asymmetric half-plane factorization, we define

$$P(\theta_1) = 1/2\pi \int_{-\pi}^{\pi} G_{\theta_1}(\theta)\, d\theta = g_{\theta_1}(0)$$

In order to establish that PP^* has a one-dimensional spectral factorization, we need to show that

$$PP^* \in L_1(T) \tag{5}$$

$$\log(\det PP^*) \in L_1^s(T) \tag{6}$$

The second of these properties follows immediately from

$$\begin{aligned}
\int_{-\pi}^{\pi} \log(\det P(\theta_1)P^*(\theta_1))d\theta_1 &= \int_{-\pi}^{\pi} \log(\det g_{\theta_1}(0)g^*_{\theta_1}(0))d\theta_1 \\
&= \int_{-\pi}^{\pi} 1/2\pi \int_{-\pi}^{\pi} \log|\det G_{\theta_1}(e^{j\theta})|^2 d\theta d\theta_1 \\
&= 1/2\pi \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \log\det K(\theta_1,\theta_2)\, d\theta_1 d\theta_2 \\
&< \infty
\end{aligned}$$

where the second equation follows from equation 3 above.

For the first property, it is sufficient to show that $\int_{-\pi}^{\pi} \|P(\theta_1)\|^2 d\theta_1 < \infty$, where $\|Q\|$ denotes the matrix norm defined by $\|Q\|^2 = \mathrm{Tr}QQ^*$. We have

$$\begin{aligned}
\|P(\theta_1)\| &= \|1/2\pi \int_{-\pi}^{\pi} G_{\theta_1}(\theta)d\theta\| \\
&\leq 1/2\pi \int_{-\pi}^{\pi} \|G_{\theta_1}(\theta)\|d\theta \\
&\leq \left(1/2\pi \int_{-\pi}^{\pi} \|G_{\theta_1}(\theta)\|^2 d\theta\right)^{1/2}
\end{aligned}$$

from the Schwartz inequality.

It then follows that

$$\begin{aligned}
\int_{-\pi}^{\pi} \|P(\theta_1)\|^2 d\theta_1 &\leq 1/2\pi \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \mathrm{Tr}K(\theta_1,\theta)d\theta d\theta_1 \\
&< \infty
\end{aligned}$$

Thus $PP^* \in L_1^s(T)$, and so PP^* has a spectral factorization

$$P(\theta_1)P^*(\theta_1) = F(\theta_1)F^*(\theta_1)$$

where F satisfies the conditions 1 – 4 above, and is again made unique by the assumption that $f(0)$ is lower triangular with positive real diagonal. Noting that property 6 implies that $P(\theta_1)^{-1}$ exists for almost all θ_1, we define

$$H(\theta_1, \theta_2) = G_{\theta_1}(\theta_2)P(\theta_1)^{-1}F(\theta_1)$$

It now follows immediately that $HH^* = K$, and $H \in L_2(T^2)$. To show that H is the asymmetric half-plane spectral factor, it remains only to show that H is outer, and that $h(m,n)$ has its support on the asymmetric half-plane.

To verify the support of $h(m,n)$, we have

$$\begin{aligned} h(m,n) &= 1/4\pi^2 \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} H(\theta_1,\theta_2) e^{-jm\theta_1} e^{-jn\theta_2} d\theta_1 d\theta_2 \\ &= 1/4\pi^2 \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} G_{\theta_1}(\theta_2) P(\theta_1)^{-1} F(\theta_1) e^{-jm\theta_1} e^{-jn\theta_2} d\theta_1 d\theta_2 \\ &= 1/2\pi \int_{-\pi}^{\pi} \left(1/2\pi \int_{-\pi}^{\pi} G_{\theta_1}(\theta_2) e^{-jn\theta_2} d\theta_2\right) P(\theta_1)^{-1} F(\theta_1) e^{-jm\theta_1} d\theta_1 \\ &= 1/2\pi \int_{-\pi}^{\pi} g_{\theta_1}(n) P(\theta_1)^{-1} F(\theta_1) e^{-jm\theta_1} d\theta_1 \end{aligned}$$

and the last integral is zero for $n < 0$, since the integrand is identically zero. Also, if $k(m,n) = 0$ for $|n| > N$, we know that $g_{\theta_1}(n) = 0$ for $n > N$, for almost all θ_1, and so we get $h(m,n) = 0$ for $n > N$ also.

For $n = 0$ we have

$$\begin{aligned} h(m,0) &= 1/2\pi \int_{-\pi}^{\pi} \left(1/2\pi \int_{-\pi}^{\pi} G_{\theta_1}(\theta_2) d\theta_2\right) P(\theta_1)^{-1} F(\theta_1) e^{-jm\theta_1} d\theta_1 \qquad (7) \\ &= 1/2\pi \int_{-\pi}^{\pi} F(\theta_1) e^{-jm\theta_1} d\theta_1 \qquad (8) \\ &= 0 \text{ for } m < 0 \qquad (9) \end{aligned}$$

since F is a spectral factor.

We therefore have shown that $h(m,n)$ has asymmetric half-plane support, and that if $k(m,n) = 0$ for $|n| > N$, then $h(m,n) = 0$ for $n > N$

To show that H is outer, we have, from equation 8 and the fact that F and G_{θ_1} are outer,

$$\begin{aligned}
\log\det h(0,0)h^*(0,0) &= \log\left|\det 1/2\pi \textstyle\int_{-\pi}^{\pi} F(\theta_1)d\theta_1\right|^2 \\
&= 1/2\pi \textstyle\int_{-\pi}^{\pi} \log\det F(\theta_1)F^*(\theta_1)d\theta_1 \\
&= 1/2\pi \textstyle\int_{-\pi}^{\pi} \log\det g_{\theta_1}(0)g^*_{\theta_1}(0)d\theta_1 \\
&= 1/4\pi^2 \textstyle\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \log\det G_{\theta_1}(\theta)G^*_{\theta_1}(\theta)d\theta d\theta_1 \\
&= 1/4\pi^2 \textstyle\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \log\det H(\theta_1,\theta_2)H^*(\theta_1,\theta_2)d\theta_1 d\theta_2
\end{aligned}$$

where the last equation follows from the fact that $HH^* = K = GG^*$.

It follows immediately that H is outer, and so is the required spectral factor.

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