Report #276

ON IEVLEV'S CLOSURE OF THE FINITE-DIMENSIONAL PROBABILITY DENSITY EQUATIONS FOR TURBULENT REACTIVE FLOWS.

E. E. O'Brien

1. Introduction

The only closed formulation for the evolution of a turbulent flow field from an initially prescribed statistical state is at the level of probability density functionals[1,2] and no useful solutions are known except in the somewhat uninteresting low Reynolds number limit of the final period of decay.[3] The dynamics of turbulence have subsequently been approached through two approximate techniques. The more active of these has been the moment closure approximation, in which a closed set of integro-differential equations for low order moments of the velocity field replace the finite set of moment equations derived directly from the equations of conservation of mass, momentum and energy. A useful review of approximations in this category can be found in a monograph by Leslie.[4] There have also been several proposals for closing the infinite hierarchy of equations for the finite dimensional probability density functions of the velocity field.[5,6] These are of special interest in the study of reactive flows because the reactive term is automatically closed in a probability density formulation whereas it has proved to be very difficult to approximate in a moment formulation.

In the following sections we pursue Ievlev's proposal[7,8] for closing the finite dimensional probability density equations for the velocity field in turbulence. The closure is made at the two point level although the method can be applied for any finite number of dimensions and presumably will become more accurate as the number of dimensions is increased. We have assumed homogeneous and isotropic turbulent and scalar fields, again for simplicity. The method is, in principle, applicable to shear flows although, as we shall see below, geometric complexity due to the pressure term is significant even with the above assumptions.

Approximations to the probability density formulation at the single point level already exist.[9,10] By necessity, they entail strong assumptions concerning the role of molecular diffusion and, in shear flows, the role of pressure fluctuations. Both of these phenomena at a point involve neighboring fluid elements and therefore must be accounted for artificially in a single point formulation. It is a purpose of the present approach to obtain a more rational description of molecular diffusion and the role of pressure by examining the consequences of Ievlev's closure at the two point level and thereby avoiding ad hoc assumptions above the time and length scales of these phenomena.

It should be pointed out that the closure is non-unique in the sense that the version of a variational method to be used in a numerical solution was not specified by Ievlev. In this report we choose the simplest possible technique to obtain a solution satisfying the

equations of constraint, with the option of introducing the variational method if, as seems likely, the properties of this solution fail to be adequate in some important aspect.

Let $a^{(\alpha)}$ represent the random variables $(\underset{\sim}{u}, \phi_{(m)})$, at the point $\underset{\sim}{x}^{(\alpha)}$, $\alpha = 1, \ldots, n$, where $\underset{\sim}{u}$ is the velocity vector, $\phi_{(m)}$, $m = 1, \ldots, N$ represents the set of N scalar quantities which appear in the conservation equations.

$$\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} = -\frac{1}{\rho}\frac{\partial p}{\partial x_i} + \nu \frac{\partial^2 u_i}{\partial x_j \partial x_j} \qquad i = 1, 2, 3 \qquad (1)$$

$$\frac{\partial \phi_{(m)}}{\partial t} + u_j \frac{\partial \phi_{(m)}}{\partial x_j} = D_{(m)} \frac{\partial^2 \phi_{(m)}}{\partial x_j \partial x_j} + R_{(m)}(\phi_1, \ldots, \phi_N) \qquad m = 1, \ldots, N \qquad (2)$$

$R_{(m)}$ is the term representing the rate of production of $\phi_{(m)}$ and $D_{(m)}$ is the diffusivity of $\phi_{(m)}$.

Let $f_n(a^{(1)}, \underset{\sim}{x}^{(1)}, \ldots, a^{(\alpha)}, x^{(\alpha)}, \ldots, a^{(n)}, \underset{\sim}{x}^{(n)}, t)$ denote the probability density of the various values of the random variables a at all n spatial locations and time t. Monin, Ievlev, and others, have shown that the quantity f_n obeys the equations

$$\frac{\partial f_n}{\partial t} + \sum_{\gamma=1}^{n} \frac{\partial}{\partial x_k^{(\gamma)}}\left[f_n u_k^{(\gamma)}\right] + \sum_{m=1}^{N} \frac{\partial}{\partial \phi_{(m)}}\left[f_n R_{(m)}\right]$$

$$= -\sum_{\gamma=1}^{n} \frac{\partial}{\partial u_k^{(\gamma)}}\left[f_n B_k^{(\gamma)}\right] - \sum_{m=1}^{N} \frac{\partial}{\partial \phi_{(m)}}\left[f_n C_{(m)}^{(\gamma)}\right] \qquad (1)$$

$$\int u_k^{(\gamma)} \frac{\partial f_n}{\partial x_k^{(\gamma)}} \, d\underset{\sim}{u}^{(\gamma)} = 0 \qquad \gamma = 1, \ldots, n. \qquad (2)$$

where

$$B_k^{(\gamma)} = \lim_{\underset{\sim}{x}^{n+1} \to \underset{\sim}{x}^{(\gamma)}} \left\{ \nu \frac{\partial^2 E(u_k^{(n+1)} | A_n)}{\partial x_q^{(n+1)} \partial x_q^{(n+1)}} - \frac{1}{\rho} \frac{\partial}{\partial_k^{(n+1)}} E(p^{(n+1)} | A_n) \right\}$$

and

$$C^{(\gamma)}_{(m)} = \lim_{\underset{\sim}{x}^{n+1} \to \underset{\sim}{x}^{(\gamma)}} \left\{ D_{(m)} \frac{\partial^2 E(\phi^{(n+1)}_{(m)} | A_n)}{\partial x_q^{(n+1)} \partial x_q^{(n+1)}} \right\}.$$

The superscript (n+1) means the quantity is evaluated at the $n+1^{th}$ point. A_n is the set of variables $\left\{\underset{\sim}{x}^{(\alpha)}, \phi^{(\alpha)}_{(m)}\right\}$ for all n points and E(A|B) represents the conditional expected value of variables A given the values of variables B. Through incompressibility $p^{(n+1)}$ can be written in terms of $\underset{\sim}{u}^{(n+1)}$.

Equation (2) is the p.d.f. form of the incompressibility condition $\nabla \cdot \underset{\sim}{u} = 0$. Equation (1) has the appearance of a Liouville equation but this is deceptive since the coefficients $B_k^{(\gamma)}$ and $C^{(\gamma)}_{(m)}$ are not functions of just the generalized coordinates of the finite dimensional space under consideration. In fact, $B_k^{(\gamma)}$, $C^{(\gamma)}_{(m)}$ clearly depend on f_{n+1} and therefore represent terms which must be approximated to obtain a closed formulation for f_n. There is no Hamiltonian for the strongly dissipative system represented by (1), although there exists a Liouville-like equation, first derived by Hopf,[1] for the probability density functional of turbulence in the absence of advected scalar fields. The extension to the probability density functional for the concentrations of reactive species has been developed by Dopazo.[11]

The Ievlev closure hypothesis replaces, in Equation (1), the exact expressions for $B_k^{(\gamma)}$ and $C^{(\gamma)}_{(m)}$ by approximations $B_k'^{(\gamma)}$ and $C'^{(\gamma)}_{(m)}$ which preserve the following specific properties of equation (1) and $f_n(A_n)$ its resulting solution.

1. If, at the initial time t = 0, f_n satisfies the continuity equation (2), it continues to satisfy it for all subsequent time.
2. The function f_n satisfies the normalization condition at any time t if it is normed at t = 0.
3. f_n is a real non-negative quantity.
4. Equation (1), when integrated over all values of $a^{(\alpha)}$ at any of the n points, becomes the exact equation for f^{n-1}.
5. The exact Friedman-Keller equations for the m point moments, $m \leq n$, are obtained from Equation (1).
6. The solution f_n retains the proper behavior when any of the n points coalesce with each other or become infinitely distant from the others.

This last result, generally called the coincidence property, is only approximately satisfied in Lundgren's closure, which is one of the few p.d.f. approximations that have been seriously examined to date.[12]

Ievlev's prescription for obtaining $B_k'^{(\gamma)}$ and $C'^{(\gamma)}_{(m)}$ can be found in Ref. (2) for the general problem described above. For the purposes of examining the details of his method and for extending it to obtain solutions we now restrict the study to incompressible turbulence without advected scalar fields. It is to be noted however that the only terms dependent on f^{n+1} which arise from the scalar fields, that is $C^{(\gamma)}_{(m)}$, are in fact molecular diffusion contributions. The viscous diffusion of momentum, the first term in $B_k^{(\gamma)}$, is a prototype of this kind of term so it is reasonable to expect to lose little insight by confining our attention to the turbulence dynamics. We shall see, as one might expect, that the pressure term is a much more cumbersome quantity to manipulate than molecular diffusion and probably will be the important item constraining the practical use of Ievlev's method for multipoint distributions.

2. The Two Point p.d.f. Equation for Incompressible Turbulence

For incompressible turbulence it can be shown that (1) and (2) reduce to the following equations for the two point joing velocity p.d.f.

$$\frac{\partial f_2}{\partial t}(\underset{\sim}{u}^{(1)},\underset{\sim}{u}^{(2)};\underset{\sim}{x}^{(1)},\underset{\sim}{x}^{(2)},t) + \sum_{\gamma=1}^{2} \frac{\partial}{\partial x_k^{(\gamma)}}\left[f_2 u_k^{(\gamma)}\right] = -\sum_{\gamma=1}^{2} \frac{\partial}{\partial u_k^{(\gamma)}}\left[f_n B_k^{(\gamma)}\right] \qquad (3)$$

$$\int u_k^{(\gamma)} \frac{\partial f_2}{\partial x_k^{(\gamma)}} d\underset{\sim}{u}^{(\gamma)} = 0 \qquad \gamma = 1,\ 2 \qquad (4)$$

where

$$B_k^{(\gamma)} = \lim_{\underset{\sim}{x}^{(3)}\to\underset{\sim}{x}^{(\gamma)}} \left\{ \nu \frac{\partial^2 E(u_k^{(3)}|\underset{\sim}{u}^{(1)},\underset{\sim}{u}^{(2)},\underset{\sim}{x}^{(1)},\underset{\sim}{x}^{(2)})}{\partial x_q^{(3)}\partial x_q^{(3)}} - \frac{1}{\rho}\frac{\partial}{\partial x_K^{(3)}} E(p^{(3)}|\underset{\sim}{u}^{(1)},\underset{\sim}{u}^{(2)},\underset{\sim}{x}^{(1)},\underset{\sim}{x}^{(2)}) \right\}$$

The first step in the Ievlev proposed closure is to express $B_k'^{(\gamma)}$ the approximate replacement for $B_k^{(\gamma)}$ by a series of functions of the form

$$B_k'^{(\gamma)} = \sum_{\alpha=1}^{2} \psi_{k\alpha}^{(\gamma)} + \sum_{\alpha\neq(\gamma)} u_{m\alpha} \frac{\partial \lambda_{k\alpha}^{(\gamma)}}{\partial x_{m\alpha}} \qquad (5)$$

where $\psi_{k\alpha}^{(\gamma)}$ and $\lambda_{k\alpha}^{(\gamma)}$ are functions of the coordinates of all n points, time and the velocities of all points in question except α. They are therefore functions of the random velocities at n-1 points plus the coordinates of all n spatial points and time (if the flow is statistically unsteady).

A crucial set of constraints on the form of $\psi_{k\alpha}^{(\gamma)}$ and $\lambda_{k\alpha}^{(\gamma)}$ is obtained by noting the exact result

$$\int_{\underset{\sim}{u}^{(\alpha)}} B_k^{(\gamma)} f_n d\underset{\sim}{u}^{(\alpha)} = f_{n-1}\left(A_{(\alpha)}\right) \lim_{\underset{\sim}{x}^{(n+1)} \to \underset{\sim}{x}^{(\gamma)}} \times \left\{ \nu \frac{\partial^2 E(u_k^{(n+1)} | A_{(\alpha)})}{\partial x_q^{(n+1)} \partial x_q^{(n+1)}} - \frac{1}{\rho} \frac{\partial E(p^{(n+1)} | A_\alpha)}{\partial x_k^{(n+1)}} \right\}$$

where α represents one of the n locations $\underset{\sim}{x}^{(\alpha)}$, $1 \le \alpha \le n$.

$A_{(\alpha)}$ has the meaning of the set of velocities, positions and time associated with all points except the α point, the point at which the velocity values have been integrated.

If we assume homogeneity and, for the moment, suppress time dependence this integral condition applied to the point $\underset{\sim}{x}^{(1)}$ for a two point distribution can be written as follows:

$$\int_{\underset{\sim}{u}^{(1)}} B_k^{(\gamma)} f_2 d\underset{\sim}{u}^{(1)} = f_1\left(\underset{\sim}{u}^{(2)}\right) \lim_{\underset{\sim}{x}^{(3)} \to \underset{\sim}{x}^{(\gamma)}} \times \left\{ \nu \frac{\partial^2 E(u_k^{(3)} | \underset{\sim}{u}^{(2)}, \underset{\sim}{x}^{(2)})}{\partial x_q^{(3)} \partial x_q^{(3)}} - \frac{1}{\rho} \frac{\partial E(p^{(3)} | \underset{\sim}{u}^{(2)}, \underset{\sim}{x}^{(2)})}{\partial x_k^{(3)}} \right\} \qquad (6)$$

A similar exact result applies to

$$\int_{\underset{\sim}{u}^{(2)}} B_k^{(\gamma)} f_2 d\underset{\sim}{u}^{(2)}.$$

Ievlev demands that $B_k^{\prime(\gamma)}$ satisfies the same equation (6) as does $B_k^{(\gamma)}$. If we make use of (5) we obtain

$$\int_{\underset{\sim}{u}^{(1)}} \left\{ \sum_{\alpha=1}^{2} \psi_{k\alpha}^{(\gamma)} + \sum_{\alpha \ne \gamma} u_{m\alpha} \frac{\partial \lambda_{k\alpha}^{(\gamma)}}{\partial x_{m\alpha}} \right\} f_2 d\underset{\sim}{u}^{(1)} = f_1\left(\underset{\sim}{u}^{(2)}\right) \lim_{\underset{\sim}{x}^{(3)} \to \underset{\sim}{x}^{(\gamma)}} \left\{ \nu \frac{\partial^2 E(u_k^{(3)} | \underset{\sim}{u}^{(2)}, \underset{\sim}{x}^{(2)})}{\partial x_q^{(3)} \partial x_q^{(3)}} - \frac{1}{\rho} \frac{\partial}{\partial x_k^{(3)}} E(p^{(3)} | \underset{\sim}{u}^{(2)}, \underset{\sim}{x}^{(2)}) \right\} \qquad (7)$$

and similarly for

$$\int_{\underset{\sim}{u}^{(2)}} B_k^{\prime(\gamma)} f_2 d\underset{\sim}{u}^{(2)}.$$

It is clear that the right hand side of equation (7) is determined by the two point distribution function f_2. In fact the right hand side can be rewritten in the form

$$\lim_{\underset{\sim}{x}^{(3)} \to \underset{\sim}{x}^{(\gamma)}} \left\{ \nu \nabla^2_{\underset{\sim}{x}^{(3)}} \int u_k^{(3)} f_2(\underset{\sim}{u}^{(3)}, \underset{\sim}{u}^{(2)}, \underset{\sim}{x}^{(3)}, \underset{\sim}{x}^{(2)}) d\underset{\sim}{u}^{(3)} - \frac{1}{\rho} \frac{\partial}{\partial x_k^{(3)}} \int p^{(3)} f_2(\underset{\sim}{u}^{(3)}, \underset{\sim}{u}^{(2)}, \underset{\sim}{x}^{(3)}, \underset{\sim}{x}^{(2)}) d\underset{\sim}{u}^{(3)} \right\}$$

where the explicit dependence of $p^{(3)}$ on the $\underset{\sim}{u}$ field will have to be introduced.

By considering each of the points in turn, i.e. $\gamma = 1,2$, equation (7) can be written more explicitly as follows:

$$\int_{\underset{\sim}{u}^{(1)}} \left\{ \psi_{k1}^{(1)} + \psi_{k2}^{(1)} + u_{m2} \frac{\partial \lambda_{k2}^{(1)}}{\partial x_{m2}} \right\} f_2 d\underset{\sim}{u}^{(1)} = \lim_{\underset{\sim}{x}^{(3)} \to \underset{\sim}{x}^{(1)}} \left\{ \nu \nabla^2_{x^{(3)}} \int u_k^{(3)} f_2(\underset{\sim}{u}^{(3)}, \underset{\sim}{u}^{(2)}) d\underset{\sim}{u}^{(3)} - \frac{1}{\rho} \frac{\partial}{\partial x_k^{(3)}} \int p^{(3)} f_2(\underset{\sim}{u}^{(3)}, \underset{\sim}{u}^{(2)}) d\underset{\sim}{u}^{(3)} \right\} \qquad (8a)$$

$$\int_{\underset{\sim}{u}^{(1)}} \left\{ \psi_{k1}^{(2)} + \psi_{k2}^{(2)} + u_{m1} \frac{\partial \lambda_{k1}^{(2)}}{\partial x_{m1}} \right\} f_2 d\underset{\sim}{u}^{(1)} = \lim_{\underset{\sim}{x}^{(3)} \to \underset{\sim}{x}^{(2)}} \left\{ \nu \nabla^2_{x^{(3)}} \int u_k^{(3)} f_2(\underset{\sim}{u}^{(3)}, \underset{\sim}{u}^{(2)}) d\underset{\sim}{u}^{(3)} - \frac{1}{\rho} \frac{\partial}{\partial x_k^{(3)}} \int p^{(3)} f_2(\underset{\sim}{u}^{(3)}, \underset{\sim}{u}^{(2)}) d\underset{\sim}{u}^{(3)} \right\} \qquad (8b)$$

A similar pair of equations apply to

$$\int_{\underset{\sim}{u}^{(2)}} B_k^{\prime(\gamma)} f_2 d\underset{\sim}{u}^{(2)}.$$

Now the limit process applied to (8a) can be carried out immediately to yield

$$\int_{\underset{\sim}{u}^{(1)}} \left\{\psi_{k1}^{(1)} + \psi_{k2}^{(1)} + u_{m2}\frac{\partial\lambda_{k2}^{(1)}}{\partial x_{m2}}\right\} f_2 d\underset{\sim}{u}^{(1)} = \nu\nabla^2_{\underset{\sim}{x}^{(1)}} \int_{\underset{\sim}{u}^{(1)}} u_k^{(1)} f_2(\underset{\sim}{u}^{(1)},\underset{\sim}{u}^{(2)}) d\underset{\sim}{u}^{(1)} - \frac{1}{\rho}\frac{\partial}{\partial x_k^{(1)}}\int_{\underset{\sim}{u}^{(1)}} p^{(1)} f_2(\underset{\sim}{u}^{(1)},\underset{\sim}{u}^{(2)}) d\underset{\sim}{u}^{(1)} \qquad (9a)$$

No comparable simplification for (8b) is possible.

$$\int_{\underset{\sim}{u}^{(1)}} \left\{\psi_{k1}^{(2)} + \psi_{k2}^{(2)} + u_{m1}\frac{\partial\lambda_{k1}^{(2)}}{\partial x_{m1}}\right\} f_2 d\underset{\sim}{u}^{(1)} = \lim_{\underset{\sim}{x}^{(3)}\to\underset{\sim}{x}^{(2)}} \left\{\nu\nabla^2_{\underset{\sim}{x}^{(3)}} \int u_k^{(3)} f_2(\underset{\sim}{u}^{(3)},\underset{\sim}{u}^{(2)}) d\underset{\sim}{u}^{(3)} - \frac{1}{\rho}\frac{\partial}{\partial x_k^{(3)}}\int p^{(3)} f_2(\underset{\sim}{u}^{(3)},\underset{\sim}{u}^{(2)}) d\underset{\sim}{u}^{(3)}\right\} \qquad (9b)$$

Similarly

$$\int_{\underset{\sim}{u}^{(2)}} \left\{\psi_{k1}^{(1)} + \psi_{k2}^{(1)} + u_{m2}\frac{\partial\lambda_{k2}^{(1)}}{\partial x_{m2}}\right\} f_2 d\underset{\sim}{u}^{(2)} = \lim_{\underset{\sim}{x}^{(3)}\to\underset{\sim}{x}^{(1)}} \left\{\nu\nabla^2_{\underset{\sim}{x}^{(3)}} \int u_k^{(3)} f_2(\underset{\sim}{u}^{(3)},\underset{\sim}{u}^{(1)}) d\underset{\sim}{u}^{(3)} - \frac{1}{\rho}\frac{\partial}{\partial x_k^{(3)}}\int p^{(3)} f_2(\underset{\sim}{u}^{(3)},\underset{\sim}{u}^{(1)}) d\underset{\sim}{u}^{(3)}\right\} \qquad (9c)$$

$$\int_{\underset{\sim}{u}^{(2)}} \left\{\psi_{k1}^{(2)} + \psi_{k2}^{(2)} + u_{m1}\frac{\partial\lambda_{k1}^{(2)}}{\partial x_{m1}}\right\} f_2 d\underset{\sim}{u}^{(2)} = \nu\nabla^2_{\underset{\sim}{x}^{(2)}} \int u_k^{(2)} f_2(\underset{\sim}{u}^{(2)},\underset{\sim}{u}^{(1)}) d\underset{\sim}{u}^{(2)} - \frac{1}{\rho}\frac{\partial}{\partial x_k^{(2)}}\int p^{(2)} f_2(\underset{\sim}{u}^{(2)},\underset{\sim}{u}^{(1)}) d\underset{\sim}{u}^{(2)} \qquad (9d)$$

Equation (9) represents a set of 4 vector equations of constraint for 6 undetermined three-dimensional vectors $\psi_{k1}^{(1)}$, $\psi_{k1}^{(2)}$, $\psi_{k2}^{(1)}$, $\psi_{k2}^{(2)}$,

$\lambda_{k1}^{(2)}$ and $\lambda_{k2}^{(1)}$.

The continuity equation (2) can be shown to become (Ievlev)

$$\int_{\underset{\sim}{u}^{(\gamma)}} u_m^{(\gamma)} \frac{\partial}{\partial x_m^{(\gamma)}}\left\{f_2\left(\sum_{\alpha=1}^{2}\psi_{k\alpha}^{(p)} + \sum_{\alpha\neq p} u_{\ell\alpha}\frac{\partial\lambda_{k\alpha}^{(p)}}{\partial x_{\ell\alpha}}\right)\right\} d\underset{\sim}{u}^{(\gamma)} = 0 \quad p \neq (\gamma) \qquad (10)$$

Choosing $\gamma = 1$ then 2 we get

$$\int_{\underset{\sim}{u}^{(1)}} u_m^{(1)} \frac{\partial}{\partial x_m^{(1)}}\left[f_2\left\{\psi_{k1}^{(2)} + \psi_{k2}^{(2)} + u_{\ell 1}\frac{\partial\lambda_{k1}^{(2)}}{\partial x_{\ell 1}}\right\}\right] d\underset{\sim}{u}^{(1)} \equiv 0 \qquad (11a)$$

and

$$\int_{\underset{\sim}{u}^{(2)}} u_m^{(2)} \frac{\partial}{\partial x_m^{(2)}}\left[f_2\left\{\psi_{k1}^{(1)} + \psi_{k2}^{(1)} + u_{\ell 2}\frac{\partial\lambda_{k2}^{(1)}}{\partial x_{\ell 2}}\right\}\right] d\underset{\sim}{u}^{(2)} = 0 \qquad (11b)$$

Equations (11a) and (11b) complete the set of 6 vector equations of constraint for the 6 vectors to be determined.

3. A Simple Solution

In a homogeneous turbulence

$$B_k'^{(1)}(\underset{\sim}{u}^{(2)},\underset{\sim}{x}^{(2)},\underset{\sim}{x}^{(1)},t) = B_k'^{(1)}(\underset{\sim}{u}^{(2)},r,(\underset{\sim}{u}^{(2)}\cdot \underset{\sim}{r}),t)$$

and

$$B_k'^{(1)}(\underset{\sim}{u}^{(2)},r,(\underset{\sim}{u}^{(2)}\cdot \underset{\sim}{r}),t) = B_k'^{(2)}(\underset{\sim}{u}^{(1)},r,(\underset{\sim}{u}^{(1)}\cdot \underset{\sim}{r}),t)$$

Hence it is sufficient to solve equations (9a), (9c) and (11a) for the variables $\psi_{k1}^{(1)}$, $\psi_{k2}^{(1)}$ and $\lambda_{k2}^{(1)}$. The remaining three functions can then be found by symmetry arguments.

One solution of (9a) and (9c) can be constructed as follows. Equation (9a) can be written

$$\int_{\underset{\sim}{u}^{(1)}} \left\{\psi_{k1}^{(1)} + \psi_{k2}^{(1)} + u_{m2}\frac{\partial\lambda_{k2}^{(1)}}{\partial x_{m2}}\right\} f_2 d\underset{\sim}{u}^{(1)} = \int_{\underset{\sim}{u}^{(1)}} \left\{\nu\nabla^2_{\underset{\sim}{x}^{(1)}}(u_k^{(1)} f_2(\underset{\sim}{u}^{(1)},\underset{\sim}{u}^{(2)})) - \frac{1}{\rho}\frac{\partial}{\partial x_k^{(1)}} p^{(1)} f_2\right\} d\underset{\sim}{u}^{(1)} .$$

On equating integrands one obtains formally

$$\left\{\psi_{k1}^{(1)} + \psi_{k2}^{(1)} + u_{m2}\frac{\partial\lambda_{k2}^{(1)}}{\partial x_{m2}}\right\} f_2 = \left\{\nu\nabla^2_{\underset{\sim}{x}^{(1)}}(u_k^{(1)} f_2) - \frac{1}{\rho}\frac{\partial}{\partial x_k^{(1)}}(p^{(1)} f_2)\right\}. \tag{12a}$$

Similarly from (9c) we have, renaming $\underset{\sim}{x}^{(3)}$ as $\underset{\sim}{x}^{(2)}$ on the right hand side,

$$\left\{\psi_{k1}^{(1)} + \psi_{k2}^{(1)} + u_{m2}\frac{\partial\lambda_{k2}^{(1)}}{\partial x_{m2}}\right\} f_2 = \lim_{\underset{\sim}{x}^{(2)}\to \underset{\sim}{x}^{(1)}} \left\{\nu\nabla^2_{\underset{\sim}{x}^{(2)}}\left\{u_k^{(2)} f_2(\underset{\sim}{u}^{(2)},\underset{\sim}{u}^{(1)})\right\} - \frac{1}{\rho}\frac{\partial}{\partial x_k^{(2)}}\left\{p^{(2)} f_2(\underset{\sim}{u}^{(2)},\underset{\sim}{u}^{(1)})\right\}\right\}. \tag{12b}$$

On observing that the integral of the right hand side of (12a) over all values of the velocity at $\underset{\sim}{x}^{(2)}$ yields the null vector because of homogeneity and, likewise, an integral of the right hand side of (12b) over all values of the velocity at $\underset{\sim}{x}^{(1)}$ is also a null vector, one can state that the following expression is a solution of equations (9a) and (9c).

$$\left\{\psi_{k1}^{(1)} + \psi_{k2}^{(1)} + u_{m2}\frac{\partial\lambda_{k2}^{(1)}}{\partial x_{m2}}\right\} f_2 = \left\{\nu\nabla^2_{\underset{\sim}{x}^{(1)}}[u_k^{(1)} f_2] - \frac{1}{\rho}\frac{\partial}{\partial x_k^{(1)}}[p^{(1)} f_2]\right\}$$
$$+ \lim_{\underset{\sim}{x}^{(2)}\to \underset{\sim}{x}^{(1)}} \left\{\nu\nabla^2_{\underset{\sim}{x}^{(2)}}\left\{u_k^{(2)} f_2(\underset{\sim}{u}^{(2)},\underset{\sim}{u}^{(1)})\right\} - \frac{1}{\rho}\frac{\partial}{\partial x_k^{(2)}}\left\{p^{(2)} f_2(\underset{\sim}{u}^{(2)},\underset{\sim}{u}^{(1)})\right\}\right\}. \tag{13}$$

Such a solution is not unique. For example any function $\chi_k^{(1)}(\underset{\sim}{u}^{(1)},\underset{\sim}{u}^{(2)})$ which vanishes when integrated over $\underset{\sim}{u}^{(1)}$ or $\underset{\sim}{u}^{(2)}$ can be added to the right hand side of (13) and still have it satisfy (9a) and (9c).

Finally it is necessary to show that equation (13), considered as a solution for

$$\left\{\psi_{k1}^{(1)} + \psi_{k2}^{(1)} + u_{m2}\frac{\partial\lambda_{k2}^{(1)}}{\partial x_{m2}}\right\} f_2,$$

satisfies the incompressibility condition, equation (11b). Note first that the last two terms on the right hand side of (13) are functions of only the spatial coordinate $\underset{\sim}{x}^{(1)}$ after the limit is taken. Thus the operator $\partial/\partial x_m^{(2)}$ applied to these two terms yields zero.

The remaining terms on the right hand side of (13) become, when substituted into (11b)

$$\nu \frac{\partial^2}{\partial x_m^{(1)}\partial x_m^{(1)}}\left\{u_k^{(1)}\int_{u^{(2)}} u_m^{(2)} \frac{\partial}{\partial x_m^{(2)}} f_2\, du^{(2)}\right\}$$

$$-\frac{1}{\rho}\frac{\partial}{\partial x_k^{(1)}}\left\{p^{(1)}\int_{u^{(2)}} u_m^{(2)}\frac{\partial}{\partial x_m^{(2)}} f_2\, du^{(2)}\right\}$$

Both integrals are identically zero since f_2 satisfies equation (2).

Hence equation (13) represents a solution for $B_k'^{(1)} f_2$ which satisfies all of Ievlev's criteria. Similarly

$$B_k'^{(2)} f_2 = \left\{\nu\nabla^2_{x^{(2)}}\left(u_k^{(2)} f_2\right) - \frac{1}{\rho}\frac{\partial}{\partial x_k^{(2)}}\left(p^2 f_2\right)\right\}$$

$$+ \lim_{x^{(1)}\to x^{(2)}}\left\{\nu\nabla^2_{x^{(1)}}\left(u_k^{(1)} f_2(u^{(2)},u^{(1)})\right) - \frac{1}{\rho}\frac{\partial}{\partial x_k^{(1)}}\left(p^{(1)} f_2(u^{(2)},u^{(1)})\right)\right\} \quad (14)$$

If one incorporates the arbitrary function $\chi_k^{(\gamma)}\ (u^{(1)},u^{(2)})$ in (13) and (14) as discussed following (13), the incompressibility condition on it is

$$\int_{u^{(\alpha)}} u_m^{(\gamma)} \frac{\partial}{\partial x_m^{(\gamma)}} \chi_k^{(p)}\, du^{(\gamma)} = 0 \qquad \gamma = 1, 2;\quad p = 2, 1;\quad p \neq \gamma$$

To this condition must be added the constraints already discussed.

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$$\int_{u^{(\gamma)}} \chi_k^{(p)} \cdot du^{(\gamma)} = 0 \qquad p = 1, 2;\quad \gamma = 1, 2$$

The solutions represented by (13) and (14) in effect assumes $\chi_k^{(p)} \equiv 0$, $p = 1, 2$.

4. Explicit Form of the Simplest Solution

Expressions (13) and (14) are formal statements of a possible expression for the Ievlev approximates $B_k'^{(1)}$ and $B_k'^{(2)}$. In this section we shall derive their form for isotropic turbulence. It is convenient to return to the operator-form equation (9). For example (9a) can be written

$$\int_{u^{(1)}} B_k'^{(1)} f_2\, du^{(1)} = \int_{u^{(1)}} \nu\nabla^2_{x^{(1)}}\left[u_k^{(1)} f_2(u^{(1)},u^{(2)})\right] du^{(1)}$$

$$- \frac{1}{\rho}\int_{u^{(1)}} \frac{\partial}{\partial x_k^{(1)}}\, p^{(1)} f_2(u^{(1)},u^{(2)})\, du^{(1)} \qquad (9a)$$

The last term can be expressed in an alternate form, namely

$$-\frac{1}{\rho}\frac{\partial}{\partial x_k^{(1)}}\left\{E(p^{(1)}|u^{(2)}) f_1(u^{(2)})\right\},$$

where $f_1(u^{(2)})$ is the single point velocity p.d.f. at $x^{(2)}$. But since, from the Navier-Stokes equation,

$$\frac{1}{\rho}\frac{\partial E(p^{(1)}|A_1)}{\partial x_k^{(1)}\partial x_k^{(1)}} = -\frac{\partial^2 E(u_m^{(1)}u_\ell^{(1)}|A_1)}{\partial x_m^{(1)}\partial x_\ell^{(1)}},$$

for an infinite media equation (9a) can be put in the form

$$\int_{u^{(1)}} B_k'^{(1)} f_2\, du^{(1)} = \int_{u^{(1)}} \nu\nabla^2_{x^{(1)}}\left[u_k^{(1)} f_2(u^{(1)},u^{(2)})\right] du^{(1)}$$

$$+ \frac{1}{4\pi}\frac{\partial}{\partial x_k^{(1)}}\int_{x^{(3)}} \frac{1}{|x^{(3)}-x^{(1)}|}\, \frac{\partial^2 E(u_m^{(3)}u_\ell^{(3)}|A_1)}{\partial x_m^{(3)}\partial x_\ell^{(3)}}\, f_1(u^{(2)})\, dx^{(3)}$$

or

$$\int_{u^{(1)}} B_k'^{(1)} f_2\, du^{(1)} = \int_{u^{(1)}} \nu u_k^{(1)}\nabla^2_{x^{(1)}} f_2(u^{(1)},u^{(2)})\, du^{(1)}$$

$$+ \frac{1}{4\pi}\int_{u^{(3)}} \frac{\partial}{\partial x_k^{(1)}}\int_{x^{(3)}} \left(\frac{1}{|x^{(3)}-x^{(1)}|}\right) u_m^{(3)}u_\ell^{(3)}\, \frac{\partial^2}{\partial x_m^{(3)}\partial x_\ell^{(3)}}\, f_2(u^{(3)},u^{(2)})\, dx^{(3)}\, du^{(3)}$$

Appealing to isotropy, we define

$$\mathbf{x}^{(3)} - \mathbf{x}^{(2)} = \mathbf{r} \quad \text{and} \quad \mathbf{x}^{(2)} - \mathbf{x}^{(1)} = \mathbf{R}.$$

and note

$$\frac{\partial}{\partial x_k^{(2)}} = -\frac{\partial}{\partial r_k} = -\frac{\partial}{\partial x_k^{(3)}}, \quad \mathbf{x}^{(3)} - \mathbf{x}^{(1)} = \mathbf{R} + \mathbf{r}$$

Therefore

$$\int_{\mathbf{u}^{(1)}} B_k'^{(1)} f_2 d\mathbf{u}^{(1)} = \int_{\mathbf{u}^{(1)}} \nu u_k^{(1)} \frac{\partial^2}{\partial r_m \partial r_m} f_2(\mathbf{u}^{(1)}, \mathbf{u}^{(2)}) d\mathbf{u}^{(1)}$$

$$+ \frac{1}{4\pi} \int_{\mathbf{u}^{(3)}} \int_{\mathbf{r}} \frac{r_k + R_k}{|\mathbf{r}+\mathbf{R}|^3} u_m^{(3)} u_\ell^{(3)} \frac{\partial^2}{\partial r_m \partial r_\ell} f_2(\mathbf{u}^{(3)}, \mathbf{u}^{(2)}) d\mathbf{r} d\mathbf{u}^{(3)}$$

The right hand side can be written formally as an integral over $\mathbf{u}^{(1)}$ in the following way

$$\int_{\mathbf{u}^{(1)}} B_k'^{(1)} f_2 d\mathbf{u}^{(1)} = \int_{\mathbf{u}^{(1)}} \{\nu u_k^{(1)} \frac{\partial^2}{\partial r_m \partial r_m} f_2(\mathbf{u}^{(1)}, \mathbf{u}^{(2)})$$

$$+ \frac{1}{4\pi} f_1(\mathbf{u}^{(1)}) \int_{\mathbf{u}^{(3)}} \int_{\mathbf{r}} \frac{r_k + R_k}{|\mathbf{r}+\mathbf{r}|^3} u_m^{(3)} u_\ell^{(3)} \frac{\partial^2}{\partial r_m \partial r_\ell} f_2(\mathbf{u}^{(3)}, \mathbf{u}^{(2)}) d\mathbf{r} d\mathbf{u}^{(3)}\} d\mathbf{u}^{(1)}$$

or, simply

$$B_k'^{(1)} f_2 = \nu u_k^{(1)} \frac{\partial^2}{\partial r_m \partial r_m} f_2(\mathbf{u}^{(1)}, \mathbf{u}^{(2)})$$

$$+ \frac{1}{4\pi} f_1(\mathbf{u}^{(1)}) \int_{\mathbf{u}^{(3)}} \int_{\mathbf{r}} \frac{r_k + R_k}{|\mathbf{r}+\mathbf{R}|^3} u_m^{(3)} u_\ell^{(3)} \frac{\partial^2}{\partial r_m \partial r_\ell} f_2(\mathbf{u}^{(3)}, \mathbf{u}^{(2)}) d\mathbf{r} d\mathbf{u}^{(3)} \tag{15}$$

Similarly from (9c) we can deduce

$$B_k'^{(1)} f_2 = \nu u_k^{(2)} \lim_{r\to 0} \frac{\partial^2}{\partial r_m \partial r_m} f_2(\mathbf{u}^{(2)}, \mathbf{u}^{(1)})$$

$$+ \frac{f_1(\mathbf{u}^{(2)})}{4\pi} \int_{\mathbf{u}^{(3)}} \int_{\mathbf{r}} \frac{r_k}{r^3} u_m^{(3)} u_\ell^{(3)} \frac{\partial^2}{\partial r_m \partial r_\ell} f_2(\mathbf{u}^{(3)}, \mathbf{u}^{(1)}) d\mathbf{r} d\mathbf{u}^{(3)} \tag{16}$$

where, in this case $\mathbf{r} = \mathbf{x}^{(3)} - \mathbf{x}^{(1)}$. As before the full simple solution for $B_k'^{(1)} f_2$ can be obtained by adding the right hand sides of (15) and (16) since the integral constraints (9a) and (9c) remain valid under such an addition. Finally we have

$$B_k'^{(1)} f_2 = \nu\Big(u_k^{(1)} \frac{\partial^2}{\partial r_m \partial r_m} f_2(\mathbf{u}^{(1)}, \mathbf{u}^{(2)}) + u_k^{(2)} \lim_{r\to 0} \frac{\partial^2}{\partial r_m \partial r_m} f_2(\mathbf{u}^{(2)}, \mathbf{u}^{(1)})\Big)$$

$$+ \frac{1}{4\pi} \int_{\mathbf{u}^{(3)}} \int_{\mathbf{r}} \frac{f_1(\mathbf{u}^{(1)})(r_k + R_k) u_m^{(3)} u_\ell^{(3)}}{|\mathbf{r}+\mathbf{R}|^3} \frac{\partial^2}{\partial r_m \partial r_\ell} f_2(\mathbf{u}^{(3)}, \mathbf{u}^{(2)}) d\mathbf{r} d\mathbf{u}^{(3)}$$

$$+ \frac{1}{4\pi} \int_{\mathbf{u}^{(3)}} \int_{\mathbf{r}} \frac{f_1(\mathbf{u}^{(2)}) r_k}{r^3} u_m^{(3)} u_\ell^{(3)} \frac{\partial^2}{\partial r_m \partial r_\ell} f_2(\mathbf{u}^{(3)}, \mathbf{u}^{(1)}) d\mathbf{r} d\mathbf{u}^{(3)} \tag{17}$$

Similarly from (14), by symmetry or by an entirely analogous procedure to that just completed, we can obtain an expression for the only other term in (3) which must be approximated, $B_k'^{(2)} f_2$.

$$B_k'^{(2)} f_2 = \nu\Big(u_k^{(2)} \frac{\partial^2}{\partial r_m \partial r_m} f_2(\mathbf{u}^{(2)}, \mathbf{u}^{(1)}) + u_k^{(1)} \lim_{r\to 0} \frac{\partial^2}{\partial r_m \partial r_m} f_2(\mathbf{u}^{(1)}, \mathbf{u}^{(2)})\Big)$$

$$+ \frac{1}{4\pi} \int_{\mathbf{u}^{(3)}} \int_{\mathbf{r}} \frac{f_1(\mathbf{r}^{(2)})(r_k - R_k)}{|\mathbf{r}-\mathbf{R}|^3} u_m^{(3)} u_\ell^{(3)} \frac{\partial^2}{\partial r_m \partial r_\ell} f_2(\mathbf{u}^{(3)}, \mathbf{u}^{(1)}) d\mathbf{r} d\mathbf{u}^{(3)}$$

$$+ \frac{1}{4\pi} \int_{\mathbf{u}^{(3)}} \int_{\mathbf{r}} \frac{f_1(\mathbf{u}^{(1)}) r_k}{r^3} u_m^{(3)} u_\ell^{(3)} \frac{\partial^2}{\partial r_m \partial r_\ell} f_2(\mathbf{u}^{(3)}, \mathbf{u}^{(2)}) d\mathbf{r} d\mathbf{u}^{(3)} \tag{18}$$

where $\underset{\sim}{R} = \underset{\sim}{x}^{(2)} - \underset{\sim}{x}^{(1)}$, as before, and $\underset{\sim}{r}$ is the vector distance between the two spatial points whose velocities are the independent variables for f_2.

It is now possible to solve equation (3) and such a solution will preserve the properties listed in the introduction. It remains to be shown if adequate dynamical characteristics of turbulence will also be produced. It is our intention to carry out a numerical solution of the p.d.f. equations, starting from an initially prescribed state, to evaluate the usefulness of the Ievlev closure and this proposed simple solution in particular.

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