



STATE UNIVERSITY OF NEW YORK  
AT STONY BROOK

COLLEGE OF  
ENGINEERING

Report No. 28

ON THE FLUX OF HEAT THROUGH  
LAMINAR, THIN FILMS

by

Edward E. O'Brien

February 1965

*Spec*  
*TAI*  
*N532*  
*NO. 28*  
*C.2*

On the Flux of Heat Through Laminar, Thin Films<sup>1</sup>

by

Edward E. O'Brien  
Associate Professor  
Department of Thermal Sciences  
State University of New York  
Stony Brook, L. I. N. Y.

<sup>1</sup> The research reported in this paper was supported by the National Science Foundation under N. S. F. Grant 31-82.

## Abstract

Dynamically passive transfer of heat across a thin film supporting a progressive, periodic surface wave is approached from a Lagrangian viewpoint. For most fluids the fluctuations in temperature of any fluid point is shown to be negligible over the time scale of the passage of a wave and thus a very close approximation to the heat flux can be easily obtained for waves described by material variables when the boundaries are material surfaces and temperature is a constant on the boundaries. Specific flux and profile results are deduced for Gerstner waves and shallow gravity waves.

## On the Flux of Heat Through Laminar, Wavy Films

### 1. Introduction

A situation of some interest and considerable complexity arises when an interface takes the form of a train of progressive waves. A great deal of progress has been made in the last decade toward understanding the physics of the generation of such waves from the quiescent state [1], [2]. Experimentally it has long been observed, for example in condensation studies, that interfacial waves are the norm rather than the exception and a recent article [3] has reviewed and cataloged the kinds of waves that have been obtained and the conditions under which they occur.

It frequently becomes of interest, in situations in which wave trains are generated, to ask to what extent has the existence of the waves affected the flux of heat through the interface. For example McAdams [4] apparently recommends an assumption of 20% increase in flux due to waves for film type condensation on vertical tubes. We consider a specific boundary value problem in two space dimensions in which the temperature at the upper boundary is maintained a constant and the temperature at the other boundary of the film is also held fixed at a lower temperature so that cellular motion due to bouyancy instabilities does not arise. No transfer of matter at the surface is considered although it appears that for many condensing phenomena

the most significant influence of condensation may be to maintain a fixed temperature. Certainly this is true for the situation in which the condensing film is in a quasi-steady state. That is for situations in which the wave train constitutes a perturbation about a steady (thermal and hydrodynamic) plane horizontal condensation layer. For the type of problem just described one can identify several mechanisms each of which leads to an augmentation of the magnitude of the flux. The surface area is evidently more extensive than that of the undisturbed surface and likewise the isotherms are, in the mean, closer together. Furthermore the particle motions which correspond to the surface traveling waves are periodic trajectories with a possible drift in the direction of wave propagation [5] and thus they can perhaps perform the function of carrying a high temperature from near the hotter surface and essentially dumping it in the colder regions at the bottom of the trajectory, and then reversing the procedure for the remainder of the cycle. In the quasi-steady situation with a homogeneous film and wave system they are condemned to this role with an invariable temperature cycle for each particle. The role of diffusion by continuous movements, so crucial in turbulent transport, can be seen to be of no interest in this situation. If the temperature of each particle were to be conserved in the motion (essentially a low peclet number requirement), it is

the continued increase in time of the mean displacement of a particle which constitutes turbulent transport and when the particle is in periodic motion in the direction of the mean flux one obtains no mean contribution to the flux from this particle motion.

In an incompressible fluid the energy equation takes the form

$$\frac{\partial T}{\partial t} + \frac{\partial(u_i T)}{\partial x_i} = D \nabla^2 T \quad (1.1),$$

where viscous dissipation has been ignored and the effect of temperature variation on density is not included.  $D$  is the diffusivity of heat.

Consider a film of liquid with constant temperature  $T_1$  maintained on its upper surface and a constant temperature  $T_2$  maintained at its lower surface.  $T_2$  is less than  $T_1$ . Let a progressive wave train consisting of periodic waves of wave length  $\lambda$  occupy the space from  $-\infty$  to  $+\infty$  in the horizontal direction on either surface or both surfaces. Then we ask for the flux of heat through such a moving geometry. Great difficulties exist in the exact solution of (1.1) for finite amplitude waves which are the ones that are likely to be of most interest. In fact it appears evident that two length ratios should play a major role in determining  $Q$ , the ratio of wave amplitude to wave length and the ratio of wave length to total film depth, and unless both are  $O(1)$  one expects no significant flux increase over that through a quiescent film.

Our objective is to obtain the value of  $Q$  for a class of representative finite amplitude waves and for that purpose, particularly in view of the boundary conditions being applied at a material surface, a Lagrangian formulation is natural. Also there exists a closed-form solution in material variables to the hydrodynamic progressive wave problem in the absence of viscosity which will therefore be the natural velocity field to assume for the purpose of studying heat transfer across finite waves. In section (4) we show that for one case we have studied the temperature profile for this wave system, the Gerstner waves [6] is indistinguishable from an extended shallow gravity wave solution under the same thermal, and equivalent geometric, boundary conditions.

## 2. Energy Equation for the Gerstner Wave

The Gerstner wave is a solution of the inviscid incompressible hydrodynamic equations presented in terms of Lagrangian variables  $a$  and  $b$ . The trajectory of any particle whose material coordinates are  $a$  and  $b$  is given by

$$\begin{aligned} x(a,b) &= a + k^{-1} e^{kb} \sin k(a+ct) \\ y(a,b) &= b - k^{-1} e^{kb} \cos k(a+ct) \end{aligned} \quad (2.1) ;$$

where  $c$  is the wave speed,  $c = g^{\frac{1}{2}}k^{-\frac{1}{2}}$ , and  $k$  the wave number. Thus a physical interpretation of  $(a, b)$  is that it is the center of the circle in which the particle moves.

The Jacobian of the transformation,  $J = \frac{\partial(x, y)}{\partial(a, b)}$ , is  $1 - e^{2kb}$ , from which it is taken that  $b$  can have all values less than zero.

The shapes of such waves are given by Lamb and are reproduced in Figure 1. Some significant features are: (1) any surface  $b = \text{const}$  is a trochoid and is a possible free surface and, naturally, a material surface; (2) the limiting surface  $b = 0$  forms a cusp; (3) such a wave exhibits considerable vorticity and is thus unlikely to be generated by conservative mechanisms; and (4)  $b = \text{constant}$  is essentially a horizontal surface for  $|b|$  greater than  $2k^{-1}$ .

For our purposes (3) is probably not a serious drawback since the major mechanisms of heat transfer are the gross straining of the isotherms and the periodic motion of individual particles both of which occur in the Gerstner wave in a manner quantitatively similar to all progressive surface waves. There is no drift of particles in the Gerstner wave but this is also not expected to be a significant matter for heat transfer across an homogeneous wave system.

The Gerstner wave has considerable analytical advantages over other kinds of wave solutions. Namely the Jacobian is time independent and, within the



approximation to be developed in section(3), a closed form solution to the temperature field can be obtained for all amplitudes, wavelengths and depths that Gerstner waves can exhibit.

In terms of the variables  $a$ ,  $b$  and  $t$ , as defined in (2.1), (1.1) becomes

$$\begin{aligned}
 T_t(a,b,t) = DJ^{-2} & \left\{ \left[ (1+e^{2kb}) - 2e^{kb} \cos k(a+ct) \right] T_{aa} \right. \\
 & + \left[ (1+e^{2kb}) + 2e^{kb} \cos k(a+ct) \right] T_{bb} \\
 & - 4e^{kb} \sin k(a+ct) T_{ab} - \frac{4ke^{3kb}}{(1-e^{2kb})} \sin k(a+ct) T_a \\
 & \left. + \left[ \frac{4ke^{2kb}}{(1-e^{2kb})} (1+e^{2kb}) + \frac{4ke^{3kb}}{(1-e^{2kb})} \cos k(a+ct) \right] T_b \right\}
 \end{aligned}
 \tag{2.2},$$

where subscripts refer to partial derivatives. For

example  $T_{ab} = \frac{\partial^2 T}{\partial a \partial b}$ .

In nondimensional form with  $\alpha = ka$ ,

$\beta = kb$ ,  $\tau = kct$  we obtain

$$\begin{aligned}
 T_z(\alpha,\beta,\tau) = \frac{Dk}{c} J^{-2} & \left\{ \left[ 1+e^{2\beta} - 2e^{\beta} \cos(\alpha+\tau) \right] T_{\alpha\alpha} + \left[ 1+e^{2\beta} + 2e^{\beta} \cos(\alpha+\tau) \right] T_{\beta\beta} \right. \\
 & - 4e^{\beta} \sin(\alpha+\tau) T_{\alpha\beta} - 4e^{3\beta} (1-e^{2\beta})^{-1} \sin(\alpha+\tau) T_{\alpha} \\
 & \left. + \left[ \frac{4e^{2\beta}}{1-e^{2\beta}} + \frac{4e^{3\beta}}{1-e^{2\beta}} \cos(\alpha+\tau) \right] T_{\beta} \right\}
 \end{aligned}
 \tag{2.3}.$$

The boundary conditions are

$$T(\beta_1) = T_1 \quad \text{a constant} \quad (2.4),$$

$$T(\beta_2) = T_2 \quad \text{a constant} \quad (2.5),$$

$$T \text{ is a periodic function of } (\alpha + \tau) \quad (2.6),$$

where  $\beta = \beta_1$  is the upper surface

and  $\beta = \beta_2$  is the lower surface.

Equation (2.3) is a linear partial differential equation with periodic coefficients, a situation for which there seems to be little theory. The Eulerian form of the energy equation is of course of just the same type but here there are several advantages. As previously mentioned the boundary conditions for the quasi-steady situation are, in the Lagrangian frame, of the simplest type, but more significantly a natural and accurate first approximation suggests itself from this viewpoint.

The parameter  $\frac{Dk}{c}$  is a ratio between the velocity at which molecular diffusion can be considered to travel and the speed of the wave and one expects it, for ordinary liquids, to be extremely small except for very large wave numbers. In fact if surface tension is to be included this parameter would be small for all wave lengths. The smallness of the ratio reflects the fact that in the time scale of one period

the diffusive loss or gain of temperature by a particle must be minute. For the diffusion of heat in water at room temperature if the wave length of the wave be 20 cms. then  $\frac{Dk}{c}$  can be computed to be approximately  $3 \times 10^{-6}$ . For such a case we expect that the final quasi-steady solution will be one in which the particle temperature is effectively constant during the motion. That is

$$T_{\tau}(\alpha, \beta, \tau) \approx 0$$

or  $T \approx T(\alpha, \beta)$  .

But by the boundary condition, equation (2.6), the mean temperature must be a function only of  $\beta$  and the fluctuation temperature must be as weak a function of  $\alpha$  as it is of  $\tau$  . Hence we can conclude that the mean particle temperature depends only on the  $\beta$  coordinate and that the fluctuating temperature of a given particle is almost rigorously zero during a cycle of the motion.

Now any function  $T(\beta)$  satisfies  $T_{\tau} \equiv 0$  and evidently an other criterion must be invoked to determine its form. For example we expect the long time effect of diffusion is at least to make  $T$  a continuous function of its argument and of course the boundary conditions (2.4) and (2.5) are to be satisfied.

To obtain the quasi-steady state form of  $T(\beta)$  it is postulated that any dependence of  $T$  on  $(\alpha + \tau)$  is infinitesimal and that the contributions of the periodic

coefficients in (2.3) with the fluctuating part of the particle temperature field is negligible. A more formal argument in favor of the above postulate can be presented by expanding the temperature field in the form

$$T(\alpha, \beta, z) = \sum_{n=0}^{\infty} f_n(\beta) \cos n(\alpha+z) + \sum_{n=0}^{\infty} g_n(\beta) \sin n(\alpha+z).$$

Substituting this series form into the governing equation (2.3) and collecting coefficients of the same harmonic function the general terms are of the form

$$g_n(\beta) = -\frac{Dk}{c} \left\{ \mathcal{L}_1^{(n)} f_n + \mathcal{L}_2^{(n)} f_{n-1} + \mathcal{L}_3^{(n)} f_{n+1} \right\}, \quad n \geq 1 \quad (2.7),$$

$$f_n(\beta) = \frac{Dk}{c} \left\{ \mathcal{L}_1^{(n)} g_n + \mathcal{L}_2^{(n)} g_{n-1} + \mathcal{L}_3^{(n)} g_{n+1} \right\}, \quad n \geq 1 \quad (2.8),$$

where

$$\mathcal{L}_1^{(n)} = (1-e^{2\beta})^{-3} \left\{ (1-e^{4\beta}) \left( \frac{d^2}{d\beta^2} - n^2 \right) + 4e^{2\beta} \frac{d}{d\beta} \right\},$$

$$\mathcal{L}_2^{(n)} = e^{\beta} (1-e^{2\beta})^{-2} \left\{ -\left( \frac{d^2}{d\beta^2} + (n-1)^2 \right) + 2(n-1) \left( \frac{d}{d\beta} + e^{2\beta} (1-e^{2\beta})^{-1} \right) - 2e^{2\beta} (1-e^{2\beta})^{-1} \frac{d}{d\beta} \right\},$$

and

$$\mathcal{L}_3^{(n)} = e^{\beta} (1-e^{2\beta})^{-2} \left\{ -\left( \frac{d^2}{d\beta^2} + (n+1)^2 \right) + 2(n+1) \left( \frac{d}{d\beta} + e^{2\beta} (1-e^{2\beta})^{-1} \right) - 2e^{2\beta} (1-e^{2\beta})^{-1} \frac{d}{d\beta} \right\}.$$

Also  $g_0 = 0$  (2.9),

and  $f_0$  satisfies  $L_1^{(0)} f_0 = -L_3^{(0)} f_1$  (2.10).

In particular from (2.7), (2.8) and (2.9)

$g_1$  and  $f_1$  satisfy the following equations

$$g_1 = -\frac{Dk}{c} \left\{ L_1^{(1)} f_1 + L_2^{(1)} f_0 + L_3^{(1)} f_2 \right\} \quad (2.11),$$

$$f_1 = \frac{Dk}{c} \left\{ L_1^{(1)} g_1 + L_3^{(1)} g_2 \right\} \quad (2.12).$$

Boundary conditions take the form

$$\begin{aligned} f_0(\beta_1) &= T_1, \\ f_0(\beta_2) &= T_2, \end{aligned} \quad (2.13).$$

$$f_n(\beta_1) = g_n(\beta_1) = f_n(\beta_2) = g_n(\beta_2) = 0; \quad n \neq 0.$$

The operators do not contain  $\frac{Dk}{c}$  and are generally smoothing. Hence in view of the boundary condition it

is pertinent to assume all  $f_n$  and  $g_n$  are  $O(1)$ .

From (2.11) and (2.12) we see that  $f_1$  and  $g_1$  are  $O\left(\frac{Dk}{c}\right)$ .

Thus to the first approximation, which neglects terms of order  $\frac{Dk}{c}$ ,  $f_0$  satisfies

$$\begin{aligned} L_1^{(0)} f_0 &= 0, \\ f_0(\beta_1) &= T_1, \\ f_0(\beta_2) &= T_2 \end{aligned} \quad (2.14).$$

Integration is straightforward and yields as a solution

$$\frac{T_1 - T(\beta)}{T_1 - T_2} = \frac{\ln\left(\frac{\cosh \beta_1}{\cosh \beta}\right)}{\ln\left(\frac{\cosh \beta_1}{\cosh \beta_2}\right)} \quad (2.15),$$

$$\text{and } T_\beta = -\frac{(T_1 - T_2)}{\ln\left(\frac{\cosh \beta_1}{\cosh \beta_2}\right)} \frac{(1 - e^{2\beta})}{(1 + e^{2\beta})} \quad (2.16).$$

Since  $f_2 = O\left(\frac{Dk}{c}\right)g_1 = O\left(\left[\frac{Dk}{c}\right]^2\right)$  and  $g_2 = O\left(\frac{Dk}{c}\right)f_1 = O\left(\left[\frac{Dk}{c}\right]^2\right)$ , application of equations (2.7) and (2.8) shows  $f_n$  and  $g_n$  are  $O\left(\left[\frac{Dk}{c}\right]^n\right)$ . Since  $\frac{Dk}{c}$  has a value  $O(10^{-6})$  for water it is evident that in this case there is no need to proceed beyond the zeroth approximation as given by equation (2.15). In general for arbitrary values of  $\frac{Dk}{c}$  the system defined by (2.11), (2.12) and (2.13) is evidently vastly complicated. For the remainder of the paper we restrict our attention to those substances\* and circumstances in which the parameter  $\frac{Dk}{c}$  is several orders of magnitude less than one.

### 3. Evaluation of Heat Flux

Since  $\beta = \beta_1$  is the free surface the instantaneous heat flux  $Q$  through a wave length of surface of unit width is given by the integral

$$Q = D \int_0^{\frac{2\pi}{k}} |\nabla T| \eta_i \frac{\partial x_i}{\partial a} da,$$

where  $\eta_i$  is the unit vector normal to the surface,  $x_i$  is the Eulerian position vector of the surface, and all quantities are to be evaluated at  $\beta = \beta_1$ .

---

\* Even for the more usual liquid metals  $D$  is no greater than  $10^3 \cdot D_{\text{water}}$ .

Thus 
$$Q = D \int_0^{\frac{2\pi}{k}} |\nabla T| \{x_s x_a + y_s y_a\} da$$

or 
$$Q = D \int_0^{\frac{2\pi}{k}} |\nabla T| \{x_a^2 + y_a^2\} a_s da.$$

But 
$$a_s = \{x_a^2 + y_a^2\}^{-1/2}$$

and hence 
$$Q = D \int_0^{\frac{2\pi}{k}} |\nabla T| \{x_a^2 + y_a^2\}^{1/2} da.$$

Also 
$$|\nabla T| = \{T_x^2 + T_y^2\}^{1/2}$$

which transforms to (a,b) in the following way

$$T_x = (1 - e^{2\beta})^{-1} (-y_a T_b + y_b T_a),$$

$$T_y = (1 - e^{2\beta})^{-1} (x_a T_b - x_b T_a).$$

So that to the accuracy of the zeroth order solution we find

$$|\nabla T| = (1 - e^{2\beta})^{-1} \{x_a^2 + y_a^2\}^{1/2} T_b.$$

Thus

$$Q = D (1 - e^{2\beta})^{-1} T_b \int_0^{\frac{2\pi}{k}} [x_a^2 + y_a^2] da.$$

$\beta = \beta_1$

From (2.1) this becomes

$$Q = 2\pi D \frac{(1+e^{2\beta_1})}{(1-e^{2\beta_1})} T_{\beta} \Big|_{\beta=\beta_1}$$

or from (2.16)

$$Q = -2\pi D (T_1 - T_2) \left\{ \ln \frac{\cosh \beta_1}{\cosh \beta_2} \right\}^{-1} \quad (3.1)$$

It is useful to compare such a heat flux with the flux one would observe through an equivalent slab, that is a quiescent film obtained by allowing the wavy film to come to rest. If  $b = b_1$  is the wave surface then the height of the equivalent quiescent surface is given [7] by  $y = y_1$ , where  $b_1 - y_1 = (2k)^{-1} e^{2kb_1}$ . If  $\Delta y$  is the dimensionless equivalent slab depth corresponding to the surfaces  $\beta = \beta_1$ , and  $\beta = \beta_2$  we have

$$\Delta y = k(y_1 - y_2) = \beta_1 - \beta_2 - \frac{1}{2} (e^{2\beta_1} - e^{2\beta_2})$$

The heat flux per unit wave length per unit width through such an equivalent slab is given by

$$Q_{\text{slab}} = 2\pi D (T_1 - T_2) (\Delta y)^{-1} \quad (3.2)$$

and thus finally

$$\frac{Q}{Q_{\text{slab}}} = \frac{[(\beta_2 - \beta_1) + \frac{1}{2}(e^{2\beta_1} - e^{2\beta_2})]}{\ln \left( \frac{\cosh \beta_1}{\cosh \beta_2} \right)} \quad (3.3)$$



### Some Numerical Results for Large Amplitude Waves

Consider the maximum amplitude wave which occurs when  $\beta_1$  approaches zero,

$$\text{Then } \frac{Q}{Q_{\text{slab}}} = \frac{\beta_2 + \frac{1}{2}(1 - e^{2\beta_2})}{-\ln \cosh \beta_2}$$

If we further consider films of sufficient depth that  $\beta_2 \leq -2$ , then  $\cosh \beta_2 \approx \frac{e^{-\beta_2}}{2}$  and  $1 - e^{2\beta_2} \approx 1$ ,

$$\text{thus } \frac{Q}{Q_{\text{slab}}} \approx \frac{(\beta_2 + \frac{1}{2})}{\beta_2 + \ln 2} \approx \frac{(\beta_2 + \frac{1}{2})}{(\beta_2 + .7)} \quad (3.4)$$

For  $\beta_2 = -1$  or  $|b| = \frac{\lambda}{2\pi}$ , where  $\lambda$  is the wave length, equation (3.3) predicts a heat flux ratio of 1.31,

$$\text{for } \beta_2 = -2 \quad \frac{Q}{Q_{\text{slab}}} = 1.15,$$

$$\text{and for } \beta_2 = -3 \quad \frac{Q}{Q_{\text{slab}}} = 1.08,$$

when  $\beta_2 = 2\pi$  the depth  $|b|$  equals the wave length and

$$\frac{Q}{Q_{\text{slab}}} = 1.03.$$

### Maximum Heat Flux Ratio:

A maximum heat flux should occur at  $\beta_1 = 0$  and  $\beta_2 \rightarrow 0$

$$\text{Then } \frac{Q}{Q_{\text{slab}}} = \frac{\beta_2 + \frac{1}{2}(1 - [1 + 2\beta_2 + \frac{1}{2}(2\beta_2)^2 + \dots])}{-\ln \cosh \beta_2},$$

$$\text{or } \frac{Q}{Q_{\text{slab}}} = \frac{\beta_2^2}{\ln \cosh \beta_2}$$

Now  $\cosh \beta_2 = 1 + \frac{1}{2} \beta_2^2$  for  $\beta_2$  small and therefore

$$\ln \cosh \beta_2 = \frac{1}{2} \beta_2^2 \quad \text{as } \beta_2 \rightarrow 0 .$$

Thus, finally

$$\frac{Q}{Q_{\text{slab}}} \rightarrow 2 \quad , \quad \beta_1 = 0 \quad , \quad \beta_2 \rightarrow 0 .$$

Hence, as one would expect, the maximum heat flux increase is for a thin ribbon of wavy film and the flux in this case is just doubled.

Therefore, for a Gerstner wave,  $1 \leq \frac{Q}{Q_{\text{slab}}} \leq 2$ .

In Figure 1 the Gerstner wave, with lines of constant  $\beta$  inscribed and identified, is presented with the object of relating the above results to an Eulerian frame. Detailed calculation of Eulerian quantities such as temperature at a point is of course possible but hardly more informative than Figure 1 when combined with the Lagrangian temperature solution (2.15). For the special case  $\beta_1 = 0$ ,  $\beta_2 = -3$  the solution (2.15) is shown in Figure 2 and the instantaneous Eulerian profiles are plotted at several cross sections of Figure 1.

#### 4. Shallow Gravity Waves

A classical hydrodynamic solution for the shallow gravity wave of small slope exists and from it approximate particle trajectories are calculable. These trajectories turn out to be ellipses and it is possible to write an approximate wave solution in terms of material variables

which correspond to the geometric center of the ellipses in close analogy to the Gerstner wave formulation of the previous sections. We find that the solution takes the form [8]

$$\begin{aligned} x(a,b) &= a + A \frac{\cosh kh(b+h)}{\sinh kh} \sin k(a+ct), \\ y(a,b) &= b - A \frac{\sinh kh(b+h)}{\sinh kh} \cos k(a+ct), \end{aligned} \quad (4.1),$$

where  $A$  is an arbitrary constant related to the wave amplitude,  $h$  is the mean depth of the film and  $c$  is again the wave speed.

The solution for the flux in the quasi-steady state in which uniform temperatures are applied at  $b = 0$  and  $b = -h$  follows precisely as for the Gerstner wave. The Jacobian in this instant is not independent of time and a considerable algebraic complexity is the result.

However applying the postulate of section (2) that isotherms should coincide with lines given by  $b = \text{constant}$  it is found that the zeroth order solution is described by

$$\begin{aligned} & \left[ 1 - \frac{k^4 A^4}{4 \sinh^4 kh} - \frac{k^4 A^4}{8 \sinh^4 kh} \cosh 4k(b+h) \right] T_{bb} \\ & + \left[ \frac{2k^3 A^2 \sinh^2 k(b+h)}{\sinh^2 kh} \right] T_b = 0. \end{aligned}$$

Considerable simplification occurs in the event that

$$\frac{4 \sinh^4 kh}{k^4 A^4} \gg 1,$$

which is certainly the normal case. For example if

$$h = \frac{\pi}{k} = \frac{\lambda}{2}$$

$$\text{and } A = \frac{1}{k} = \frac{\lambda}{2\pi} \text{ then } \frac{4 \sinh^4 kh}{A^4 k^4} \approx 200.$$

Under this condition and with the boundary conditions

$$T(0) = T_1, T(-h) = T_2 \quad \text{the solution can be written}$$

$$\frac{T_1 - T(\beta)}{T_1 - T_2} = \frac{\beta - \ln \left| \frac{1 + P \tanh(\beta - \beta_2)}{1 - P \tanh(\beta - \beta_2)} \right| + \ln \left| \frac{1 - P \tanh \beta_2}{1 + P \tanh \beta_2} \right|}{\ln \left| \frac{1 - P \tanh \beta_2}{1 + P \tanh \beta_2} \right|} \quad (4.2),$$

where

$$P = \left( \frac{\frac{2 \sinh^2 kh}{k^2 A^2} - 1}{\frac{2 \sinh^2 kh}{k^2 A^2} + 1} \right)^{1/2}, \quad \beta = kb$$

$$\text{and } \beta_2 = -kh.$$

To show the similarity between the Gerstner wave with surfaces  $\beta_1 = 0$ ,  $\beta_2 = -3$  and that of the shallow gravity wave with  $A = \frac{1}{k}$  and  $\beta_2 = -3$ , the temperature as a function of  $\beta$  is tabulated in Table 1 for both cases and it is evident that there is no significant difference between them. The choice  $A = \frac{1}{k} = \frac{\lambda}{2\pi}$  is determined by

noting that for  $\beta = 0$  in the Gerstner wave the amplitude of the wave is just  $\frac{1}{k}$ , as can be seen from (2.1).

Although the shallow gravity wave particle trajectory as given by (4.1) is not strictly valid for large amplitudes the above result reinforces the notion that Gerstner waves of equivalent amplitude to wave length and depth to wave length of any non-turbulent progressive wave system will yield useful flux estimates.

##### 5. General Lagrangian Formulation

The Gerstner wave of sections (1), (2) and (3) and the shallow gravity wave of section (4) can both be included under a more general Lagrangian description.

Suppose the periodic particle motion can be described by

$$\begin{aligned} x(a,b) &= a + f(b) \sin k(a+ct) \\ y(a,b) &= b + g(b) \cos k(a+ct) \end{aligned} \quad (5.1)$$

Then  $(a,b)$  is again the center of the motion of a certain particle which can therefore be identified by its unique description  $(a,b)$ .

Proceeding as before we obtain the following general statement for the energy equation in material variables.

$$\begin{aligned}
T_{\pm}(a,b,t) = D J^{-2} & \left\{ \left[ 1 + \frac{1}{2}(g'^2 + f'^2) + 2g' \cos \eta + \frac{1}{2}(g'^2 - f'^2) \cos 2\eta \right] T_{aa} \right. \\
& + \left[ 1 + \frac{1}{2}k^2(q^2 + f^2) + 2kf \cos \eta + \frac{1}{2}k^2(f^2 - q^2) \cos 2\eta \right] T_{bb} \\
& - 2 \left[ (f' - kg) \sin \eta + \frac{k}{2}(ff' - gg') \sin 2\eta \right] T_{ab} \\
& - \left[ (f'' + kg') \sin \eta - \frac{1}{2}(kf f'' - kff'' - kg'g'' + kgg'') \sin 2\eta \right] T_a \\
& + \left[ k^2(ff' + gg') + (k^2g + kf') \cos \eta \right] T_b \\
& + \frac{J_b}{J} \left[ (f - kg) \sin \eta + \frac{1}{2}k(ff' - gg') \sin 2\eta \right] T_a \\
& - \frac{J_a}{J} \left[ 1 + \frac{1}{2}(g'^2 + f'^2) + 2g' \cos \eta + \frac{1}{2}(g'^2 - f'^2) \cos 2\eta \right] T_a \\
& + \frac{J_a}{J} \left[ (f' - kg) \sin \eta + \frac{k}{2}(ff' - gg') \sin 2\eta \right] T_b \\
& - \frac{J_b}{J} \left[ 1 + \frac{1}{2}k^2(q^2 + f^2) + 2kf \cos \eta + \frac{1}{2}k^2(f^2 - q^2) \cos 2\eta \right] T_b
\end{aligned}
\tag{5.2}$$

In the above equation  $' = \frac{d}{d\beta}$ ,

$$J = \frac{\partial(x,y)}{\partial(a,b)}, \quad J_a = \frac{\partial J}{\partial a}, \quad J_b = \frac{\partial J}{\partial b} \quad \text{and} \quad \eta = k(a+ct).$$

The dependence of  $J$  and its derivatives on  $t$  complicates the resulting equations describing the zeroth solution  $T(\beta)$  but it is a straight forward, if tedious, matter to obtain by the previous technique

$$\begin{aligned}
& \left\{ \left[ 1 + \frac{1}{2}(FG' + F'G) \right] \left[ 1 + \frac{1}{2}(F^2 + G^2) \right] + (F + G')F + \frac{1}{8}(F^2 - G^2)(FG' - F'G) \right\} T_{\beta\beta} \\
& + \left\{ [FF' + GG'] \left[ 1 + \frac{1}{2}(FG' + F'G) \right] + \frac{1}{2}(G + F')(F + G') - [F'G' + \frac{1}{2}FG'' + \frac{1}{2}F''G] \left[ 1 + \frac{1}{2}(F^2 + G^2) \right] \right. \\
& \quad \left. + \frac{1}{8}(FG'' - F''G)(F^2 - G^2) + \frac{1}{4}(F'G - FG')(FF' - GG') - \frac{1}{2}(F + G')(F' - G) \right\} T_{\beta} \\
& \equiv 0 \quad (5.3)
\end{aligned}$$

where  $F = kf$ ,  $G = kg$ ,  $\beta = kb$  and  $' = \frac{d}{d\beta}$ .

The boundary conditions for the quasi-steady problem are as before

$$T(\beta_1) = T_1$$

$$T(\beta_2) = T_2 \quad (5.4).$$

## 6. Unsteady Heat Transfer

The postulate adopted in section (2) which depends on the smallness of the ratio between the time scale for a particle period and the time scale associated with molecular diffusion over the diameter of the particle orbit is equally useful in estimating unsteady transfer

of heat across the same kinds of waves. Over the time of one period the contribution of molecular diffusion to the temperature change of a particle is negligible and we may, as before, ignore the contribution of the fluctuating component of particle temperature over such a time scale. Thus as far as molecular diffusion is concerned we may again, provided the boundary conditions involve  $T = \text{constant}$  on constant  $b$  surfaces, consider lines of constant  $b$  to be isotherms and ignore the infinitesimal fluctuation in particle temperature associated with the passage of each wave.

Thus for the Gerstner wave we have the zeroth order solution, which now is permitted to suffer a drift in temperature over time scales appropriate to molecular diffusion, described by the following system of equations

$$T_t(\beta, t) = D k^2 J^{-3} \left\{ (1 - e^{4\beta}) T_{\beta\beta} + 4e^{2\beta} T_{\beta} \right\} \quad (6.1),$$

$$T(\beta, 0) = T_0(\beta) \quad (6.2),$$

$$T(\beta_1, t) = T_1 \quad t \geq 0 \quad (6.3)$$

and  $T(\beta_2, t) = T_2 \quad t \geq 0 \quad (6.4).$

For the case  $T_0(\beta) = T_1$ ,  $\beta_1 = 0$ ,  $\beta_2 = -2$ , a numerical solution is shown in Figure 3.



The behavior is qualitatively as expected. The time scale for obtaining a quasi-steady state is just  $\frac{h^2}{D}$  where  $h$  is the mean depth of the wave.

## 7. Summary

The major effect in the augmentation of heat flux through a surface which supports a progressive wave is shown to be the stretching of isothermal surfaces and the consequent narrowing of the distance between any pair of isotherms. Molecular diffusion in liquids will normally be too slow to cause significant dumping by conduction in a single cycle so that the basic phenomenon in such a case is one in which the particles can be considered to retain their equilibrium temperature throughout the entire cycle. This condition, if the boundary conditions can be suitably stated in Lagrangian form, leads to a simple and accurate solution in terms of material variables from which heat flux information can be easily extracted.

For the Gerstner waves which are taken as kinematically typical of laminar, homogeneous, progressive waves the flux through the wave is specifically calculated and is shown to be not less than the flux through an equivalent slab and no greater than double such a flux. Extension of the basic postulate to waves described by fairly general particle trajectories and to certain transient heat flux problems is shown to be quite straight forward.

Acknowledgements

Calculations for the shallow gravity wave of section (4) were carried out by Mr. Walter Demske. The author is grateful for his assistance and for the numerous discussions of the problem carried on with Dr. Michael Bentwich. This work was supported by N. S. F. Grant No. 31-82.

References

1. Phillips, O. M., J. Fl. Mechs. 2, 417, 1957.
2. Miles, J., J. Fl. Mechs. 3, 185, 1957.
3. Ostrach, S. and Koestel, A., Presented at the 6th National Heat Transfer Conference, Boston, A.I.Ch.E. preprint No. 45, 1963.
4. McAdams, W. H., "Heat Transmission", McGraw-Hill, p. 337, 1954.
5. Longuet-Higgins, Phil. Trans. Roy. Soc. A 245, 535, 1953.
6. Lamb, H., "Hydrodynamics", Dover, 423, 1945.
7. Milne-Thompson, "Theoretical Hydrodynamics", 424, 1960.
8. Lamb, H., "Hydrodynamics", Dover, 367, 1945.



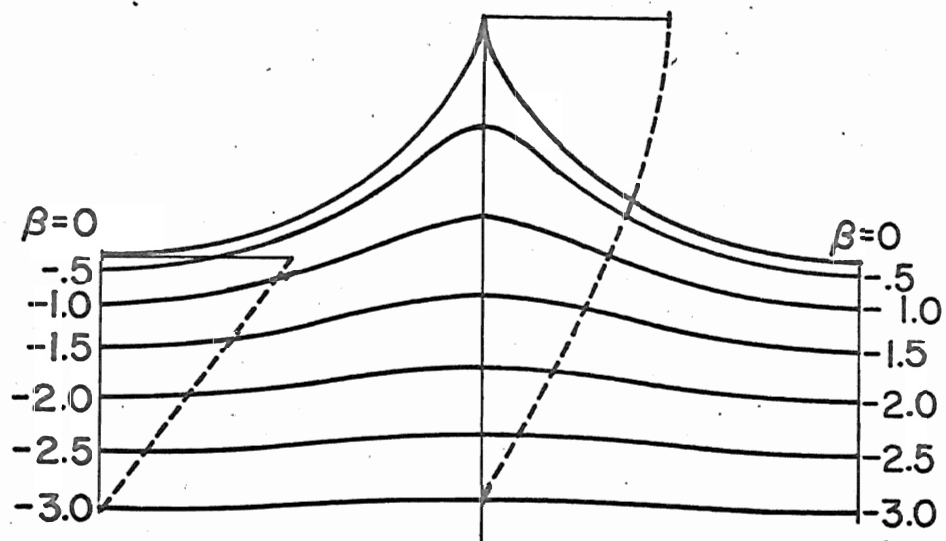
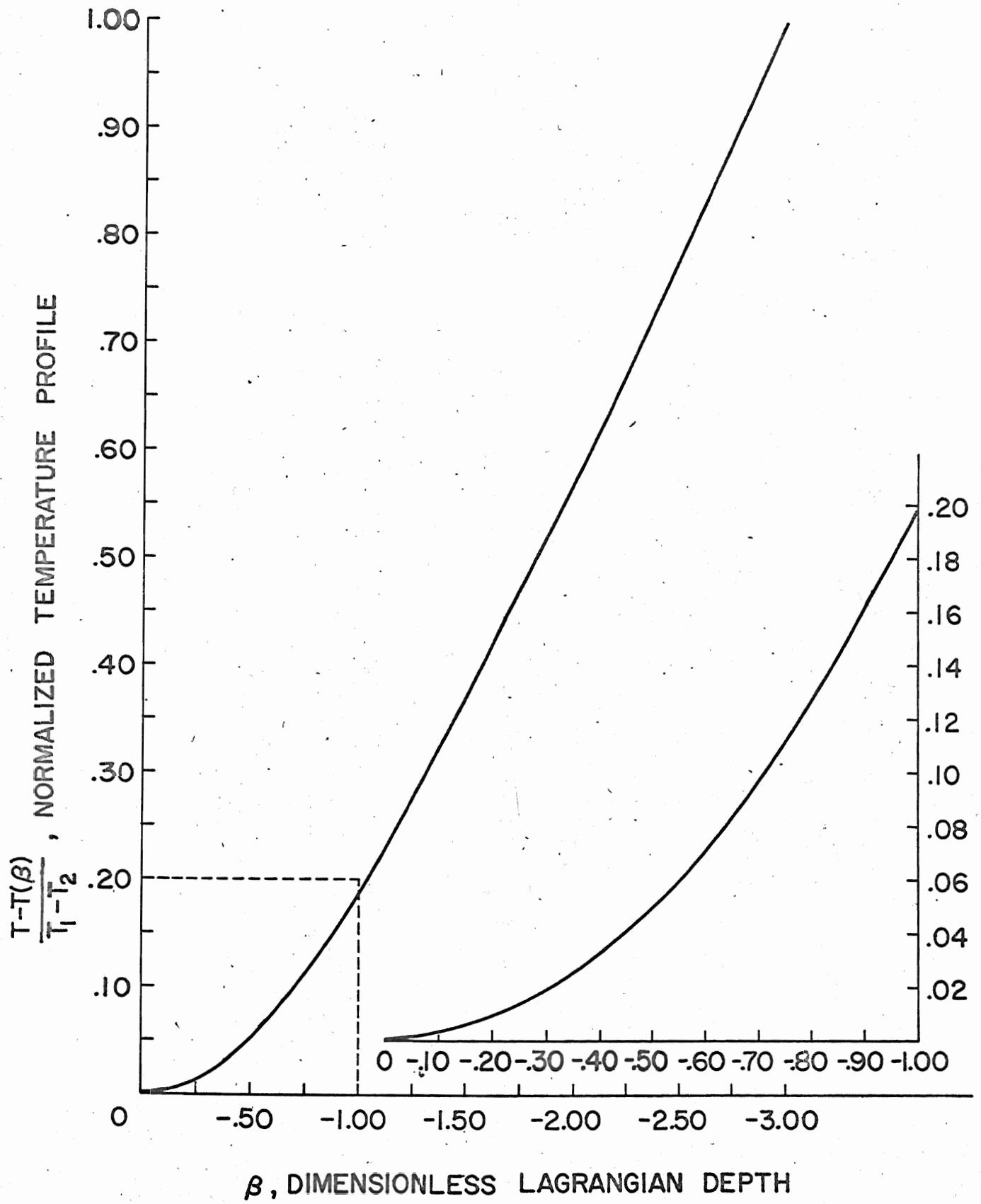


Figure 1 Gerstner Waves [6]

- - - - Two Quasi-steady  
Normalized Temperature  
Profiles



$\beta$ , DIMENSIONLESS LAGRANGIAN DEPTH

Figure 2 Temperature Profile for Gerstner Wave ( $\beta_1 = 0, \beta_2 = -3$ )

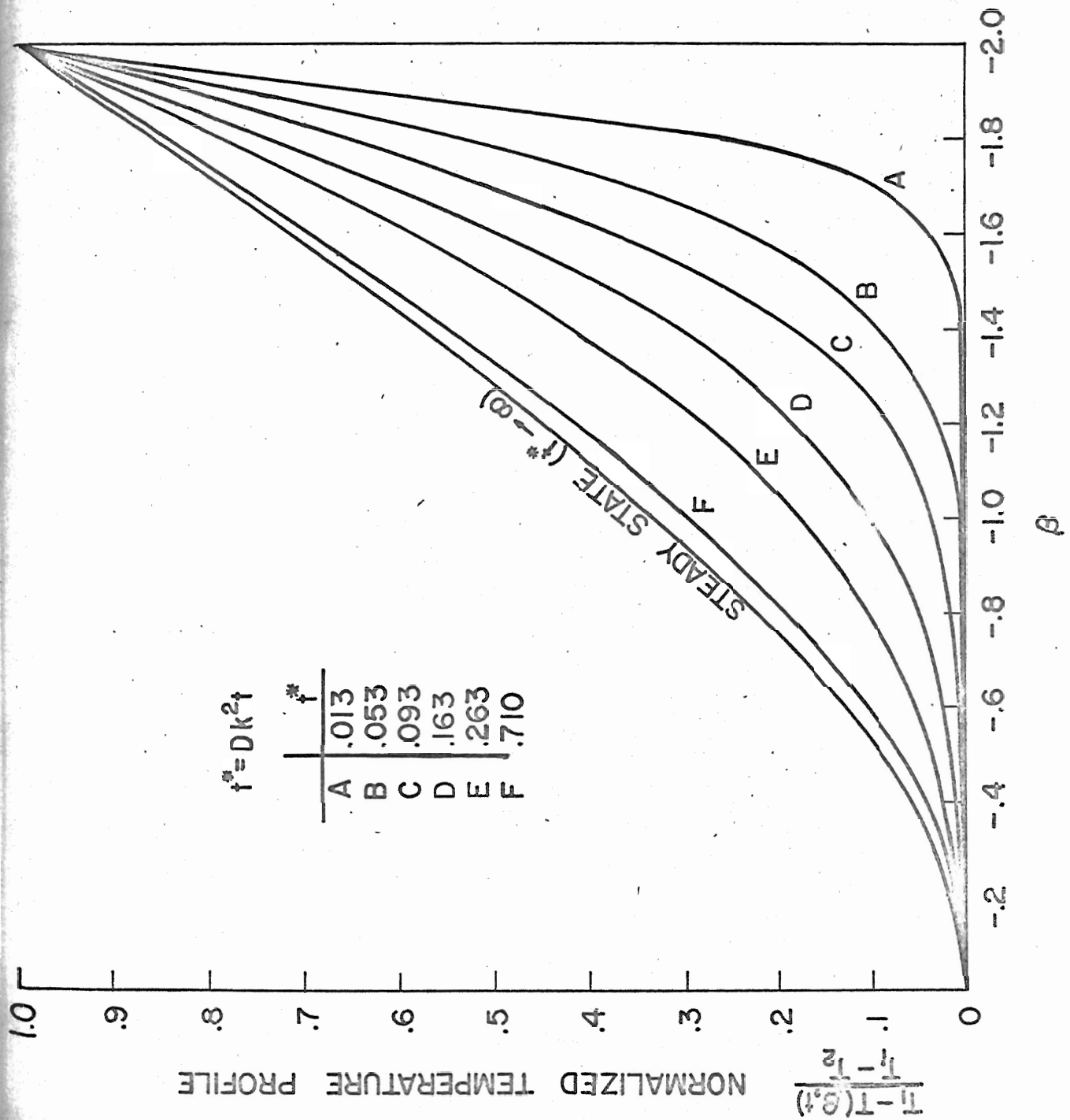


Figure 2 Transient Temperature Response ( $\beta_1 = 0, \beta_2 = -2, T(\beta, 0) = T_1$ ).

Handwritten signature: *James E. A. Brown*

$\beta$	$\frac{T_1 - T(\beta)}{T_1 - T_2}$	
	GRAVITY WAVE $A = \frac{1}{K} \cdot B_2 = -3$	GERSTNER WAVE $\beta_1 = 0 \cdot \beta_2 = -3$
-.10	.0017	.0021
-.20	.0087	.0086
-.30	.0191	.0192
-.40	.0337	.0337
-.50	.0521	.0520
-.60	.074	.074
-.70	.098	.098
-.80	.126	.126
-.90	.156	.156
-1.00	.187	.183
-1.25	.276	.275
-1.50	.372	.370
-1.75	.472	.471
-2.00	.575	.574
-2.25	.680	.679
-2.50	.786	.785
-2.75	.894	.892
-3.00	1.000	1.000

Table I Normalized Temperature for Equivalent Gerstner and Gravity Waves