QUASIUNIFORMIZATION OF BITOPOLOGICAL SPACES

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It has been shown by Thampuran (2) that the concept of complete regularity of a topological space can be generalized so as to apply to bitopological spaces. Thampuran (3) has also shown that a gage can be defined for a quasiuniformity and that its bitopological space is completely regular. It is shown in this paper that a completely regular bitopology $(\mathcal{T},\mathcal{T}')$ is generated by the family G of all quasimetrics, continuous relative to $(\mathcal{T},\mathcal{T}')$, that $(\mathcal{T},\mathcal{T}')$ is the coarsest bitopology which makes every member of G continuous, that the quasiuniformity $\mathcal{U}$ generated by G is the finest one with bitopology $(\mathcal{T},\mathcal{T}')$ and that G is the gage of $\mathcal{U}$. Also, every bitopology $(\mathcal{T},\mathcal{T}')$ has a finest completely regular bitopology (S,S') coarser than $(\mathcal{T},\mathcal{T}')$ and that (S,S') is the bitopology of all the quasimetrics that are continuous, relative to $(\mathcal{T},\mathcal{T}')$.

Unless otherwise specified the terminology used in this paper has the same meaning as in Kelley (1).

Definition 1. Let M be a set and $\mathcal{T},\mathcal{T}'$ two topologies for M. The ordered triple $(M,\mathcal{T},\mathcal{T}')$ is said to be a bitopological space. $\mathcal{T}$ and $\mathcal{T}'$ are said to be the left and right topologies of this bitopological space. We will also call $(\mathcal{T},\mathcal{T}')$ a bitopology for M.

When there is no possibility of confusion we will denote this bitopological space by M.

Definition 2. A function d from the Cartesian product $M \times M$ to the non-negative reals is said to be a quasimetric iff for all x,y,z in M

$$(1) \quad d(x,x) = 0 \text{ and}$$

$$(2) \quad d(x,y) \leq d(x,z) + d(z,y).$$

(M,d) is said to be a quasimetric space.

Let $\mathcal{T}$ be the family of all subsets T of M such that x in T implies the set $S(d,x,r) = \{ y: d(y,x) < r\} \subset T$ for some $r > 0$; then $\mathcal{T}$ is a topology for M. Let $\mathcal{T}'$ be the family of all subsets T of M such that x in T implies the set $S'(d,x,r) = \{y: d(x,y) < r\} \subset T$ for some $r > 0$; then $\mathcal{T}'$ is also a topology for M.

Definition 3. $\mathcal{T}$ and $\mathcal{T}'$ as defined in the preceding paragraph are said to be the left and right topologies of d and $(M,\mathcal{T},\mathcal{T}')$ is called the bitopological space of d; we will call $(\mathcal{T},\mathcal{T}')$ the bitopology of d.

Denote the reals by R and define a quasimetric m for R as follows: $m(x,y) = \max \{y - x, 0\}$ for all x,y in R. Let $\mathcal{R}$ and $\mathcal{R}'$ be the left and right topologies of m.

Definition 4. m is said to be the usual quasimetric for the reals and $(R,\mathcal{R},\mathcal{R}')$ the usual bitopological space for the reals.

Definition 5. Let $(M,\mathcal{T},\mathcal{T}')$, $(N,\mathcal{S},\mathcal{S}')$ be bitopological spaces and f a function from M to N. f is said to be continuous iff f is both $\mathcal{T}$-$\mathcal{S}$ and $\mathcal{T}'$-$\mathcal{S}'$ continuous. f is said to be a homeomorphism iff f is one to one, f is continuous and f^{-1} is continuous. M and N are said to be homeomorphic iff there is a homeomorphism between them.

Theorem 1. Let $(M,\mathcal{T},\mathcal{T}')$ be a bitopological space and d is a quasimetric for M. Take $L = M \times M$, $\mathcal{L} = \mathcal{T}' \times \mathcal{T}$ and $\mathcal{L}' = \mathcal{T} \times \mathcal{T}'$. Then d considered as a function from $(L,\mathcal{L},\mathcal{L}')$ to $(R,\mathcal{R},\mathcal{R}')$ is continuous iff for each x in M and each $r > 0$ the set $S(d,x,r) \in \mathcal{T}$ and $S'(d,x,r) \in \mathcal{T}'$.

Proof. Let x in M and $r > 0$ imply $S(d,x,r) \in \mathcal{T}$ and $S'(d,x,r) \in \mathcal{T}'$. Let $(x,y) \in L$ and $r > 0$. Then $A = S'(d,x,r) \times S(d,y,r)$ is a $\mathcal{L}$-neighborhood of (x,y). Let $(u,v) \in A$. Now $d(x,y) \le d(x,u) + d(u,v) + d(v,y)$. This implies that $d(x,y) - d(u,v) \le d(x,u) + d(v,y) < r + r = 2r$. Hence d is $\mathcal{L}$-$\mathcal{R}$ continuous. We can prove similarly that d is also $\mathcal{L}'$-$\mathcal{R}'$ continuous.

To prove the converse let us assume d is continuous, $x \in M$ and $r > 0$. Then d is $\mathcal{L}'$-$\mathcal{R}'$ continuous at (x,x) and so there is a neighborhood A of (x,x) such that (u,v) in A implies $d(u,v) - d(x,x) < r$ or $d(u,v) < r$. There is now a $\mathcal{T}$-neighborhood B of x and there is a $\mathcal{T}'$-neighborhood C of x such that $B \prec C \subset A$. If $u \in B$ then $d(u,x) < r$ since $x \in C$ and so $B \subset S(d,x,r)$. If $y \in S(d,x,r)$ then there is $t > 0$ such that $S(d,y,t) \subset S(d,x,r)$ and $S(d,y,t)$ contains a $\mathcal{T}$-neighborhood of y. Therefore $S(d,x,r)$ is a $\mathcal{T}$-neighborhood of each of its points and so $S(d,x,r)$ is $\mathcal{T}$-open. It can similarly be proved that $S'(d,x,r)$ is $\mathcal{T}'$-open.

In the proof of the converse of Theorem 1 we used only the condition that d is continuous at each (x,x) for $x \in M$. We therefore have:

Corollary 1. d is continuous iff d is continuous on the diagonal $\{(x,x): x \in M\}$.

Corollary 2. Let $\mathcal{T}$ and $\mathcal{T}'$ be the left and right topologies of d in Theorem 1. Then d from $(L,\mathcal{L},\mathcal{L}')$ to $(R,\mathcal{R},\mathcal{R}')$ is continuous.

Hereafter continuity of a quasimetric d, relative to $(\mathcal{T},\mathcal{T}')$, will be used in the sense of Theorem 1.

Lemma 1. Let $(M,\mathcal{T},\mathcal{T}')$ be a bitopological space and f a function from M to R. For x,y in M write $d(x,y) = \max\{f(y)-f(x),0\}$. Then d is a quasimetric for M and d is continuous iff f is continuous.

Proof. It is clear d is a quasimetric for M. f is $\mathcal{T}$-$\mathcal{R}$ continuous iff $S(d,x,r) = \{y: f(x)-f(y) < r\} \in \mathcal{T}$ for each x in M and each $r > 0$. Hence f is $\mathcal{T}$-$\mathcal{R}$ continuous iff d is $\mathcal{T}$-$\mathcal{R}$ continuous. The part for $\mathcal{T}'$-$\mathcal{R}'$ continuity can be proved in the same way.

Lemma 2. Let f be a function from the bitopological space $(M,\mathcal{T},\mathcal{T}')$ to the usual bitopological space $(R,\mathcal{R},\mathcal{R}')$ for the reals. Define d as in Lemma 1.

Let $f^{-1}\mathcal{R}$ and $f^{-1}\mathcal{R}'$ denote the inverses under f of $\mathcal{R}$ and $\mathcal{R}'$. Then $f^{-1}\mathcal{R}$ and $f^{-1}\mathcal{R}'$ are the left and right topologies of d.

Let $K = [0,1]$ denote the closed unit interval. Then m restricted to K is a quasimetric for K and let K have the bitopological space of this restriction of m to K.

Definition 6. A bitopological space $(M,\mathcal{T},\mathcal{T}')$ or $(\mathcal{T},\mathcal{T}')$ is said to be completely regular iff for $A,B \subset M$,

(1) A is $\mathcal{T}$-closed and y not in A imply there is a continuous function f from M to K such that $fA = 0$ and $f(y) = 1$ and

(2) B is $\mathcal{T}'$-closed and x not in B imply there is a continuous function g from M to K such that $g(x) = 0$ and $gB = 1$.

Definition 7. Let $(\mathcal{T},\mathcal{T}')$, (S,S') be two bitopologies for a set M. We will say $(\mathcal{T},\mathcal{T}')$ is finer than (S,S') or (S,S') is coarser than $(\mathcal{T},\mathcal{T}')$ iff S is a subfamily of $\mathcal{T}$ and S' is a subfamily of $\mathcal{T}'$.

Theorem 2. Let $(M,\mathcal{T},\mathcal{T}')$ be a completely regular bitopological space. Then there is a family F of quasimetrics for M such that $(\mathcal{T},\mathcal{T}')$ is the coarsest bitopology, for M, making each member of F continuous.

Proof. Let A be a $\mathcal{T}$-closed subset of M and let y be not in A. There is then a continuous function f from M to K such that $fA = 0$ and $f(y) = 1$. For all u,v in M take $d(u,v) = \max\{f(v)-f(u),0\}$. Then d is a quasimetric for M, the bitopology of d is coarser than $(\mathcal{T},\mathcal{T}')$ and x in A implies $d(x,y)=1$. For each $\mathcal{T}$-closed subset A of M and each y not in A there is a quasimetric d with these properties; let D be the family of all such quasimetrics. Let $\mathcal{B}$ be the family of all $S(d,x,r)$ for d in $D, x \in M$ and $r > 0$. Obviously $\mathcal{B}$ is a subfamily of $\mathcal{T}$. Let $T \in \mathcal{T}$ and $y \in T$. Then the complement C of T is $\mathcal{T}$-closed and so there is a d in D such that $d(x,y) = 1$ for all $x \in C$. Hence

$x \in S(d,x,\frac{1}{2}) \subset T$ and so $\mathcal{B}$ is a base for $\mathcal{T}$. And each $S'(d,x,r) \in \mathcal{T}'$ for $x \in M$ and $r > 0$.

For each $\mathcal{T}'$-closed set B and each x not in B there is a quasimetric e for M such that $S(e,u,r) \in \mathcal{T}$, $S'(e,u,r) \in \mathcal{T}'$ for each u in M and each $r > 0$ and $d(x,v) = 1$ for each v in B; let E be the family of all such quasimetrics. Then the family of all $S'(e,x,r)$ for $e \in E, x \in M$ and $r > 0$ is a base for $\mathcal{T}'$.

Let F be the union of D and E. It is easy to see that $(\mathcal{T},\mathcal{T}')$ is the coarsest bitopology making each member of F continuous. This completes the proof.

Let $\mathcal{U}$ be a quasiuniformity for M, $\mathcal{U}'$ the family of all inverses of members of $\mathcal{U}$ and d a quasimetric for M. Thampuran (3) has proved that d considered as a function from $(M \times M, \mathcal{U}' \times \mathcal{U})$ to (R,m) is uniformly continuous iff $\{(x,y): d(x,y) < r\}$ is a member of $\mathcal{U}$ for each $r > 0$. Uniform continuity of a quasimetric will hereafter be used in this sense. The following corollary to Theorem 2 then follows easily.

Corollary. Let $\mathcal{U}$ be the quasiuniformity generated by F. Then $(\mathcal{T},\mathcal{T}')$ is the bitopology of $\mathcal{U}$ and each d in F is uniformly continuous.

Theorem 3. Let $(M,\mathcal{T},\mathcal{T}')$ be completely regular and G the family of all continuous quasimetrics for M. Denote by $\mathcal{U}$ the quasiuniformity generated by G. Then $(\mathcal{T},\mathcal{T}')$ is the bitopology of $\mathcal{U}$, i.e.,$(\mathcal{T},\mathcal{T}')$ is the coarsest bitopology making each member of G continuous. Also G is the gage of $\mathcal{U}$.

Proof. The F of Theorem 2 is a subfamily of G and the bitopology of $\mathcal{U}$ is clearly coarser than $(\mathcal{T},\mathcal{T}')$ and so the first part of the theorem follows. The second part can be proved by noticing that each member of G is in the gage of $\mathcal{U}$ and that each member of the gage of $\mathcal{U}$ is continuous and so is in G.

Corollary. Let $\mathcal{V}$ be a quasiuniformity, for M, having the same bitopology as that of $\mathcal{U}$. Then the gage of $\mathcal{V}$ is a subfamily of G and hence $\mathcal{U}$ is the finest quasiuniformity having the bitopology $(\mathcal{T},\mathcal{T}')$.

Definition 8. A family G of quasimetrics for a set M is said to be the gage of a bitopology iff there is a bitopology $(\mathcal{T},\mathcal{T}')$ for M such that G is the family of all continuous quasimetrics, relative to $(\mathcal{T},\mathcal{T}')$. We will also call G the gage of $(\mathcal{T},\mathcal{T}')$.

Definition 9. Let F be a family of quasimetrics for a set M and let $\mathcal{U}$ be the quasiuniformity, for M, generated by F. The bitopology $(\mathcal{T},\mathcal{T}')$ of $\mathcal{U}$ is said to be the bitopology of F or the bitopology generated by F. We will also say that F generates the gage of $(\mathcal{T},\mathcal{T}')$.

Let $(M,\mathcal{T},\mathcal{T}')$ be a bitopological space and G the family of all continuous quasimetrics, relative to $(\mathcal{T},\mathcal{T}')$. Then the bitopology (S,S') of G is clearly coarser than $(\mathcal{T},\mathcal{T}')$. But these bitopologies coincide iff $(M,\mathcal{T},\mathcal{T}')$ is completely regular.

Theorem 4. A bitopological space $(M,\mathcal{T},\mathcal{T}')$ is completely regular iff $(\mathcal{T},\mathcal{T}')$ is the bitopology of its gage.

Let $(M,\mathcal{T},\mathcal{T}')$ be a bitopological space and G the gage of $(\mathcal{T},\mathcal{T}')$. Then the bitopology of G is the finest completely regular bitopology coarser than $(\mathcal{T},\mathcal{T}')$.

Theorem 5. Let $(M,\mathcal{T},\mathcal{T}')$ be a bitopological space. Then there is a finest completely regular bitopology (S,S') coarser than $(\mathcal{T},\mathcal{T}')$ and (S,S') is the bitopology of the gage of $(\mathcal{T},\mathcal{T}')$.

Let F be a family of quasimetrics for a set M and $(\mathcal{T},\mathcal{T}')$ the bitopology of F. Then the family of all sets of the form $S(d,x,r)$ for d in F, x in M and $r > 0$ is a subbase for $\mathcal{T}$ and the family of all sets of the form $S'(d,x,r)$ for d in F, x in M and $r > 0$ is a subbase for $\mathcal{T}'$. A quasimetric e for M is continuous iff $S(e,x,r) \in \mathcal{T}$ and $S'(e,x,r) \in \mathcal{T}'$ for each x in M and each $r > 0$ and so is continuous iff for each x in M and each $r > 0$ there is $t > 0$ and there is a finite number $d_1, \cdots, d_n$ of members of F such that

$S(d_1,x,t) \cap \cdots \cap S(d_n,x,t) \subset S(e,x,r)$ and

$S'(d_1,x,t) \cap \cdots \cap S'(d_n,x,t) \subset S'(e,x,r)$.

Theorem 6. Let F be a family of quasimetrics for a set M and let G be the gage generated by F. Then a quasimetric d belongs to G iff x in M and $r > 0$ imply there is $t > 0$ and there is a finite subfamily $d_1, \cdots, d_n$ of F such that $\cap\{S(d_i,x,t) : i = 1,\cdots,n\} \subset S(d,x,r)$ and

$$\cap\{S'(d_i,x,t) : i = 1,\cdots,n\} \subset S'(d,x,r).$$

Definition 10. Let $(M,\mathcal{T},\mathcal{T}')$ be a bitopological space and N a subset of M. Denote by $\mathcal{S},\mathcal{S}'$ the relativizations of $\mathcal{T},\mathcal{T}'$ to N. Then $(N,\mathcal{S},\mathcal{S}')$ is said to be a subspace of $(M,\mathcal{T},\mathcal{T}')$.

Definition 11. Let I be an index set and $(M_i,\mathcal{T}_i,\mathcal{T}_i')$ for $i \in I$ be a family of bitopological spaces. Let $M,\mathcal{T},\mathcal{T}'$ be respectively the Cartesian product of M_i, the product of the left topologies $\mathcal{T}_i$ and the product of the right topologies $\mathcal{T}_i'$ for i in I. Then the bitopological space $(M,\mathcal{T},\mathcal{T}')$ is said to be the product of the bitopological spaces $(M_i,\mathcal{T}_i,\mathcal{T}_i')$ for i in I.

Definition 12. A bitopological space $(M,\mathcal{T},\mathcal{T}')$ is said to be quasi-Hausdorff iff x,y are distinct points of M imply they have either disjoint $\mathcal{T},\mathcal{T}'$-neighborhoods respectively or they have disjoint $\mathcal{T}',\mathcal{T}$-neighborhoods respectively.

Let $(M,\mathcal{T},\mathcal{T}')$ be a completely regular space and G the gage of $(\mathcal{T},\mathcal{T}')$. Then $(\mathcal{T},\mathcal{T}')$ is the coarsest bitopology for M such that the identity function from $(M,\mathcal{T},\mathcal{T}')$ to (M,d) is continuous for each d in G. Let $P = \times\{M_d : d \in G\}$ where $M_d = M$ for each d in G. Let $\mathcal{O},\mathcal{O}'$ be respectively the products of the left and right topologies of M_d, for d in G. Assign to P the bitopologies $(\mathcal{O},\mathcal{O}')$. For u in P let u_d denote the projection of u into the d-th coordinate space and let f be the function from M to P defined by $f(x)_d = x$ for all x in M and all d in G. Now f is continuous iff f followed by projection into each coordinate space is continuous. Hence $(\mathcal{T},\mathcal{T}')$ is the coarsest bitopology for M such that f is continuous. But f is one to one and therefore f is a homeomorphism.

Now suppose the bitopological space $(M,\mathcal{T},\mathcal{T}')$ is quasi-Hausdorff. Use the notation of the preceding paragraph. Thampuran (3) has proved that the quasimetric space (M,d) is isometric under a function h_d to a quasi-Hausdorff quasimetric space (M_{d^*},d^*). Hence $(\mathcal{T},\mathcal{T}')$ is the coarsest bitopology for M such that each h_d is continuous. Denote by N the Cartesian product of M_{d^*} for d in G and by h the function from M to N given by $h(x)_d = h_d(x)$. For the left and right topologies of N take the products respectively of the left and right topologies of d^* for d in G. Then $(\mathcal{T},\mathcal{T}')$ is the coarsest bitopology for which h is continuous. Let x,y be two distinct points of M. If $h(x) = h(y)$ then $h_d(x) = h_d(y)$ for each d in G and hence $d(x,y) = 0 = d(y,x)$ for each d in G; but this implies $(M,\mathcal{T},\mathcal{T}')$ is not quasi-Hausdorff. Therefore h is one to one and then we easily see h is a homeomorphism. Thus we have:

Theorem 7. Let $(M,\mathcal{T},\mathcal{T}')$ be completely regular and G the gage of $(\mathcal{T},\mathcal{T}')$. Then M is homeomorphic to a subspace (the diagonal, in fact) of the product of all the quasimetric spaces (M,d) for d in G. If $(M,\mathcal{T},\mathcal{T}')$ is quasi-Hausdorff then M is homeomorphic to a subspace of the product of all the quasi-Hausdorff quasimetric spaces (M_{d^*},d^*) for d in G.

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