UNIVERSITY AT STONY BROOK

CEAS Technical Report 787
12/15/00

Lower Bounds on the variance of deterministic signal parameter estimators using Bayesian inference

Yufei Huang

LOWER BOUNDS ON THE VARIANCE OF DETERMINISTIC SIGNAL PARAMETER ESTIMATORS USING BAYESIAN INFERENCE

Yufei Huang

Department of Electrical and Computer Engineering

State University of New York at Stony Brook

Stony Brook, NY, 11794-2350

yfhuang@ece.sunysb.edu

ABSTRACT

1. INTRODUCTION

Cramer-Rao lower bound (CRLB) [1, 2] is a widely used bounds on the variances of unbiased estimators. It was originally developed under the frequentist set up. Under this set up, unknowns are considered deterministic and inferences are drawn from the likelihood distribution. However, when unknowns are random, inferences concerning unknowns are drawn according to Bayesian theory from the posterior distribution which is proportion to the product of the likelihood and the prior distribution. The counterpart of the CRLB in the Bayesian paradigm has also been studied and reported [2, 3] as the lower bound on the posterior variance of the estimator or posterior CRLB (PCRLB).

Although theoretically the Bayesian methods can only be applied to problems whose unknowns are random, in the practical engineering applications, it is often seen that the Bayesian methods are used for the estimation of the deterministic parameters. By this it implies that the deterministic unknowns are taken as if they are random and the prior distributions reflect the constrains of the problem and one's prior knowledge about unknowns. In this case, the only randomness in the problem is still only introduced by observations, hence like other estimators for the deterministic parameters, the performances of the Bayesian estimators should also be evaluated by the variance or the mean square error (MSE) concerning the likelihood function. Indeed they are the most referred criterion for the comparison between the Bayesian estimators and the maximum likelihood estimators (MLE) in practice [4, 5]. Therefore, in order to access the performance of the Bayesian estimators for the deterministic parameters, especially provide the comparison with MLE, it is of both practical and theoretical importance to formulate the lower bound on the variance or MSE of the Bayesian estimators. However, by far, no such bounds have been reported in the literature.

It is the objective for this paper to provide such lower bounds of the Bayesian estimator for the deterministic parameters. We first briefly introduce the basic concept of Bayesian inference as well as CRLB and PCRLB. Then, we scrutinize the derivation of the lower bound of both unbiased and biased estimators for the scalar unknown. Next, the results are further generalized for vector parameters. In addition, we also provide the bound for the case when nuisance parameters exist. Finally, an example is presented to illustrate the obtained bounds.

2. THE PARAMETRIC INFERENCE AND LOWER BOUNDS OF ESTIMATORS

Suppose that noisy observation $\mathbf{y}$ is generated according to some model parameterized by a vector of unknowns $\boldsymbol{\theta}$. The objective of the parametric inference is to provide the accurate estimate of the unknowns $\boldsymbol{\theta}$. In the non-Bayesian approaches, the unknowns $\boldsymbol{\theta}$ are considered as deterministic, and inferences to $\boldsymbol{\theta}$ is drawn from the likelihood distribution $p(\mathbf{y}|\boldsymbol{\theta})$. One such instance is the maximum likelihood estimator (MLE). The performances of any unbiased estimators are usually evaluated by compared the variance (for the scalar parameters) or the covariance matrix (for the vector parameters) of the estimator with certain lower bound. CRLB is the most often refered lower bound. The covariance matrix of the estimate $\hat{\boldsymbol{\theta}}$ is defined as

$$C(\hat{\boldsymbol{\theta}}) = E_{\mathbf{y}}[(\hat{\boldsymbol{\theta}} - \boldsymbol{\theta})(\hat{\boldsymbol{\theta}} - \boldsymbol{\theta})^{\mathsf{T}}] \tag{1}$$

where $E_{\mathbf{y}}[\cdot]$ denote the expectation with respect to the likelihood function $p(\mathbf{y}|\boldsymbol{\theta})$. Then, CRLB indicates that if $p(\mathbf{y}|\boldsymbol{\theta})$ satisfies Wolfowitz's regularity condition [1], the covariance matrix $\mathbf{C}(\hat{\boldsymbol{\theta}})$ satisfies

$$\mathbf{C}_{\hat{\boldsymbol{\theta}}} \geq \mathbf{I}^{-1}(\boldsymbol{\theta}) \tag{2}$$

where $\mathbf{I}(\boldsymbol{\theta})$ denotes the Fisher information matrix which is given by

$$[\mathbf{I}(\boldsymbol{\theta})]_{ij} = -E_{\mathbf{y}}[\frac{\partial^2 \ln p(\mathbf{y}|\boldsymbol{\theta})}{\partial\theta_i\partial\theta_j}] \tag{3}$$

The estimators which achieve CRLB are usually said to be *efficient*.

In the Bayesian approaches, the unknown parameters $\boldsymbol{\theta}$ are considered as random and inferences are drawn from the posterior distribution $p(\boldsymbol{\theta}|\mathbf{y})$ which is proportion to the product of the likelihood distribution $p(\mathbf{y}|\boldsymbol{\theta})$ and the prior distribution $p(\boldsymbol{\theta})$. The minimum mean square error (MMSE) estimator and maximum *a posterior* are the most popular Bayesian estimators. Bayesian analog of the CRLB on the estimator $\hat{\boldsymbol{\theta}}$ is usually called posterior CRLB (PCRLB). PCRLB is the lower bound on the Bayesian MSE and has the form

$$E_{\mathbf{y},\boldsymbol{\theta}}[(\hat{\boldsymbol{\theta}} - \boldsymbol{\theta})(\hat{\boldsymbol{\theta}} - \boldsymbol{\theta})^{\top}] \geq \mathbf{I}_T^{-1} \tag{4}$$

where $\mathbf{I}_T$ is an information matrix with ij-th element defined as

$$\{\mathbf{I}_T\}_{ij} = -E_{\mathbf{y},\boldsymbol{\theta}}[\frac{\partial^2 \ln p(\mathbf{y},\boldsymbol{\theta})}{\partial\theta_i\partial\theta_j}]. \tag{5}$$

Notice that the expectation here is with respect to both the likelihood $p(\mathbf{y}|\boldsymbol{\theta})$ and the prior $p(\boldsymbol{\theta})$, hence PCRLB is independent on $\boldsymbol{\theta}$.

3. THE LOWER BOUNDS OF BAYESIAN ESTIMATIOR FOR DETERMINISTIC PARAMETERS

In practice, Bayesian approaches are often used to estimate the deterministic parameters. Although in such cases, parameters are treated as if they are random, the only randomness of the problem comes from the data. Hence, the variance rather than the Bayesian MSE of the estimators would be the characteristic for evaluating the performance of the estimators. However, since CRLB does not consider the contribution from the prior distribution $p(\boldsymbol{\theta})$, it cannot be used as the lower bound for the Bayesian estimators. In the following sections of the paper, the theory of CRLB is extended, under the Bayesian framework, to provide the lower bound of the Bayesian estimators for the deterministic parameters.

3.1. The lower bounds for the scalar parameter

Let θ and $\hat{\theta}_B$ represent the unknown scalar parameter and its Bayesian estimate. Since θ is deterministic, the bounds that we are about to discuss are imposed on $E_{\mathbf{y}}[(\hat{\theta}_B - \theta)^2]$ which defines the variance of the estimate $\hat{\theta}_B$ when $\hat{\theta}$ is unbiased, or MSE when $\hat{\theta}$ is biased. Before finding the lower bounds, we would like to first introduce the following proposition.

Proposition 1

$$E_{\mathbf{y}}[\frac{\partial \ln p(\mathbf{y},\theta)}{\partial \theta}]^2 = -E_{\mathbf{y}}[\frac{\partial^2 \ln p(\mathbf{y}|\theta)}{\partial \theta^2}] + (\frac{\partial \ln p(\theta)}{\partial \theta})^2 \tag{6}$$

where expectation is with respect to the likelihood distribution $p(\mathbf{y}|\boldsymbol{\theta})$.

Proof: Consider a trivial equality

$$\int p(\mathbf{y},\theta)d\mathbf{y} = p(\theta) \tag{7}$$

Differentiating the both side of (7) over θ, and considering the transformations that $\frac{\partial p(\mathbf{y},\theta)}{\partial \theta} = \frac{\partial \ln p(\mathbf{y},\theta)}{\partial \theta} p(\mathbf{y},\theta)$ and $\frac{\partial p(\theta)}{\partial \theta} = \frac{\partial \ln p(\theta)}{\partial \theta} p(\theta)$, we have

$$\int \frac{\partial \ln p(\mathbf{y},\theta)}{\partial \theta} p(\mathbf{y}|\theta) d\mathbf{y} = \frac{\partial \ln p(\theta)}{\partial \theta}. \tag{8}$$

Now, differentiate (8) again over θ, we have

$$\int \frac{\partial^2 \ln p(\mathbf{y},\theta)}{\partial \theta^2} p(\mathbf{y}|\theta) d\mathbf{y} + \int \frac{\partial \ln p(\mathbf{y},\theta)}{\partial \theta} \frac{\partial \ln p(\mathbf{y}|\theta)}{\partial \theta} p(\mathbf{y}|\theta) d\mathbf{y} = \frac{\partial^2 \ln p(\theta)}{\partial \theta^2} \tag{9}$$

which may be further written as

$$\int \frac{\partial^2 \ln p(\mathbf{y}|\theta)}{\partial \theta^2} p(\mathbf{y}|\theta) d\mathbf{y} + \int \frac{\partial^2 \ln p(\theta)}{\partial \theta^2} p(\mathbf{y}|\theta) d\mathbf{y} + \int (\frac{\partial \ln p(\mathbf{y},\theta)}{\partial \theta})^2 p(\mathbf{y}|\theta) d\mathbf{y} - \frac{\partial \ln p(\theta)}{\partial \theta} \int \frac{\partial \ln p(\mathbf{y},\theta)}{\partial \theta} p(\mathbf{y}|\theta) d\mathbf{y} = \frac{\partial^2 \ln p(\theta)}{\partial \theta^2}. \tag{10}$$

In review of (8), we have

$$E_{\mathbf{y}}[\frac{\partial^2 \ln p(\mathbf{y}|\theta)}{\partial \theta^2}] + E_{\mathbf{y}}[\frac{\partial \ln p(\mathbf{y},\theta)}{\partial \theta}]^2 - (\frac{\partial \ln p(\theta)}{\partial \theta})^2 = 0 \tag{11}$$

or

$$E_{\mathbf{y}}[\frac{\partial \ln p(\mathbf{y},\theta)}{\partial \theta}]^2 = -E_{\mathbf{y}}[\frac{\partial^2 \ln p(\mathbf{y}|\theta)}{\partial \theta^2}] + (\frac{\partial \ln p(\theta)}{\partial \theta})^2. \tag{12}$$

$\diamond$

Now let us first consider an unbiased Bayesian estimator $\hat{\theta}$. The following theorem provides a lower bound on the variance of $\hat{\theta}$.

Theorem 1 (Lower Bound of Unbiased Bayesian Estimator for the Deterministic Scalar Parameter)

It is assumed that $p(\mathbf{y}|\theta)$ satisfies the regularity condition

$$E_{\mathbf{y}}[\frac{\partial \ln p(\mathbf{y}|\theta)}{\partial \theta}] = 0 \quad \textit{for all } \theta. \tag{13}$$

Then, the variance of any unbiased Bayesian estimator must satisfy

$$var(\hat{\theta}) \geq \left(-E_{\mathbf{y}}[\frac{\partial^2 \ln p(\mathbf{y}|\theta)}{\partial \theta^2}] + (\frac{\partial \ln p(\theta)}{\partial \theta})^2\right)^{-1}. \tag{14}$$

Furthermore, an unbiased Bayesian estimator may be found that attains the bound for all θ if and only if

$$\frac{\partial \ln p(\mathbf{y},\theta)}{\partial \theta} = k(\theta)(g(\theta) - \theta) \tag{15}$$

for some functions $k(\cdot)$ and $g(\cdot)$. That estimator is $\hat{\theta} = g(\theta)$ and the minimum variance is $1/k(\theta)$.

Proof. Since the estimator is unbiased, we have

$$E_{\mathbf{y}}(\hat{\theta} - \theta) = 0 \tag{16}$$

or

$$\int (\hat{\theta} - \theta))p(\mathbf{y}|\theta)d\mathbf{y} = 0 \tag{17}$$

After multiplying the prior $p(\theta)$ to both side of (17) and differentiating over θ, we have

$$-\int p(\mathbf{y},\theta)d\mathbf{y} + \int (\hat{\theta} - \theta)\frac{\partial p(\mathbf{y},\theta)}{\partial \theta}d\mathbf{y} = 0. \tag{18}$$

Notice that $\int p(\mathbf{y}|\theta)dy = 1$, the equation (18) can be further expressed as

$$\int (\hat{\theta} - \theta)\frac{\partial \ln p(\mathbf{y},\theta)}{\partial \theta}p(\mathbf{y},\theta)d\mathbf{y} = p(\theta). \tag{19}$$

By applying the Schwarz inequality on (19), we have

$$\int (\hat{\theta} - \theta)^2 p(\mathbf{y},\theta)d\mathbf{y} \cdot \int (\frac{\partial \ln p(\mathbf{y},\theta)}{\partial \theta})^2 p(\mathbf{y},\theta)d\mathbf{y} \geq p^2(\theta). \tag{20}$$

or

$$
\begin{aligned}
var(\hat{\theta}) &\geq \left(E_{\mathbf{y}}[(\frac{\partial \ln p(\mathbf{y},\theta)}{\partial \theta})^2]\right)^{-1} \\
&= \left(-E_{\mathbf{y}}[\frac{\partial^2 \ln p(\mathbf{y}|\theta)}{\partial \theta^2}] + (\frac{\partial \ln p(\theta)}{\partial \theta})^2\right)^{-1}.
\end{aligned}
\tag{21}
$$

The last equality is the direct result of Proposition 1. Note that the condition for equality in (20) is

$$
\frac{\partial \ln p(\mathbf{y},\theta)}{\partial \theta} = k(\theta)(\hat{\theta} - \theta) \tag{22}
$$

for some function $k(\cdot)$, which is the condition for $\hat{\theta}$ to attain the lower bounds.

◇

Comparing the lower bound (14) with the CRLB, we can see that the first term in the denominator (14) is the same as that in the CRLB, wherea, the second term $(\frac{\partial \ln p(\theta)}{\partial \theta})^2 \geq 0$. Therefore we have the following corollary.

Corollary 1 *Let $\hat{\theta}_B$ denote a unbiased Bayesian estimator who attains the lower bound (14) and $\hat{\theta}_E$ denote a efficient estimator who attains the CRLB. Then*

$$
var(\hat{\theta}_B) \leq var(\hat{\theta}_E). \tag{23}
$$

The above corollary indicates that as long as we can obtain an unbiased Bayesian estimator which attains the lower bound (14), it will have smaller variance than the efficient estimator. However, in practice the Bayesian estimators are often biased. Next the lower bound for the biased estimator is provided.

Theorem 2 (Lower Bound of Biased Bayesian Estimator for the Deterministic Scalar) *It is assumed that $p(\mathbf{y}|\theta)$ satisfies the regularity condition*

$$
E_{\mathbf{y}}[\frac{\partial \ln p(\mathbf{y}|\theta)}{\partial \theta}] = 0 \quad for\ all\ \theta. \tag{24}
$$

Then, the MSE of any biased Bayesian estimator must satisfy

$$
mse(\hat{\theta_B}) \geq \frac{(1 + \frac{\partial \ln(b(\theta)p(\theta))}{\partial \theta} b(\theta))^2}{-E_{\mathbf{y}}[\frac{\partial^2 \ln p(\mathbf{y}|\theta)}{\partial \theta^2}] + (\frac{\partial \ln p(\theta)}{\partial \theta})^2}. \tag{25}
$$

where $b(\theta)$ is the bias of the estimator $\hat{\theta}$.

Apparently, this is a bound on MSE of the estimator.

3.2. The lower bounds for the vector parameters

In this section, we extend the above discussion to provide corresponding bounds of the Bayesian estimators for the deterministic vector parameters. The theorems that follow are presented without proof due to lack of space.

Theorem 3 **(Lower Bound of Uniased Bayesian Estimators for the Deterministic Vector Parameter)** *It is assumed that* $p(\mathbf{y}|\boldsymbol{\theta})$ *satisfies the regularity condition*

$$E_{\mathbf{y}}[\frac{\partial \ln p(\mathbf{y}|\boldsymbol{\theta})}{\partial \boldsymbol{\theta}}] = 0 \quad \text{for all } \boldsymbol{\theta}. \tag{26}$$

Then, the covariance matrix of any unbiased Bayesian estimator $\hat{\boldsymbol{\theta}}_B$ *must satisfy*

$$\begin{aligned} \mathbf{C}(\hat{\boldsymbol{\theta}}_B) &\geq \mathbf{I}_B^{-1}(\boldsymbol{\theta}) \\ &= (\mathbf{I}(\boldsymbol{\theta}) + \frac{\partial \ln p(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}} \frac{\partial \ln p(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}}^{\mathsf{T}})^{-1} \end{aligned} \tag{27}$$

where $\mathbf{I}$ *is the Fisher information matrix. Furthermore, the unbiased Bayesian estimators may be found that attain the bound for all* $\boldsymbol{\theta}$ *if and only if*

$$\frac{\partial \ln p(\mathbf{y}|\boldsymbol{\theta})}{\partial \boldsymbol{\theta}} = \mathbf{I}_b(\boldsymbol{\theta})(\mathbf{g}(\boldsymbol{\theta}) - \boldsymbol{\theta}) \tag{28}$$

for some function $\mathbf{g}(\cdot)$. *That estimator is* $\hat{\boldsymbol{\theta}}_B = \mathbf{g}(\theta)$ *and its covariance matrix is* $\mathbf{I}_B^{-1}(\boldsymbol{\theta})$.

◇

Note that the positive semidefinite of $\mathbf{I}_B(\boldsymbol{\theta})$, a direct implication from the Theorem 3 by the lower bound on the variance of each component of $\hat{\boldsymbol{\theta}}_B$ is

$$var([\hat{\boldsymbol{\theta}}_B]_i) \geq [\mathbf{I}_B^{-1}]_{ii} \qquad i = 1, 2, \cdots, N \tag{29}$$

Theorem 4 **(Lower Bound of Biased Bayesian Estimators for the Deterministic Vector Parameter)** *It is assumed that* $p(\mathbf{y}|\boldsymbol{\theta})$ *satisfies the regularity condition*

$$E_{\mathbf{y}}[\frac{\partial \ln p(\mathbf{y}|\boldsymbol{\theta})}{\partial \boldsymbol{\theta}}] = 0 \quad \text{for all } \boldsymbol{\theta}. \tag{30}$$

Consider a biased Bayesian estimator $\hat{\boldsymbol{\theta}}_B$ *with bias* $\mathbf{b}(\boldsymbol{\theta})$, *then*

$$E_{\mathbf{y}}[(\hat{\boldsymbol{\theta}}_B - \boldsymbol{\theta})(\hat{\boldsymbol{\theta}}_B - \boldsymbol{\theta})^{\mathsf{T}}] \geq \mathbf{H}(\boldsymbol{\theta})\mathbf{I}_B^{-1}(\boldsymbol{\theta})\mathbf{H}(\boldsymbol{\theta}) \tag{31}$$

where $\mathbf{H} = \mathbf{1} + \frac{\partial \mathbf{b}(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}} + \mathbf{b}(\boldsymbol{\theta}) \frac{\partial \ln p(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}}^{\mathsf{T}}$. *Since* $\mathbf{H}(\boldsymbol{\theta})\mathbf{I}_B^{-1}(\boldsymbol{\theta})\mathbf{H}(\boldsymbol{\theta})$ *is at least positive semidefinite, therefore we have the bound for the mse of each element of* $\hat{\boldsymbol{\theta}}_B$ *as*

$$mse([\hat{\boldsymbol{\theta}}_B]_i) \geq \left[\mathbf{H}(\boldsymbol{\theta})\mathbf{I}_B^{-1}(\boldsymbol{\theta})\mathbf{H}(\boldsymbol{\theta})\right]_{ii} \qquad i = 1, 2, \cdots, N. \tag{32}$$

3.3. Lower bounds in the presence of the nuisance parameters

In this section, we extend our results to the case where part of the parameters are nuisance parameters. In the Bayesian inference, the effect of the nuisance parameter on the final estimator is usually eliminated by integrating the nuisance parameter out from the posterior distribution. This results in the marginalized Bayesian estimator. Apparently the variance of maginalized Bayesian estimator is different from the unmarginalized estimator. Hence, we should have a different lower bound for the variance of the marginalized estimators. Now we partition $\boldsymbol{\theta}$ as $\boldsymbol{\theta}^{\mathsf{T}} = [\Phi^{\mathsf{T}}, \Psi^{\mathsf{T}}]$, where Φ and Ψ represent the subvector of desired parameters and nuisance parameters respectively. Suppose the prior distributions for nuisance parameters are $p(\Psi)$. The marginalized likelihood function can be defined as

$$p(\mathbf{y}|\Phi) = \int p(\mathbf{y}|\boldsymbol{\theta})p(\Psi)d\Psi. \tag{33}$$

Then all the discussion about the lower bounds in the previous sections can be directly applied to Bayesian estimators for Φ by simply replacing all related expectation operations by those with respect to the marginalized likelihood function.

4. EXAMPLE

Let us consider the example of estimation of DC level in the white Gaussian noise. The problem can be model as

$$y_t = A + w_t \qquad t = 0, 1, \cdots, N-1 \tag{34}$$

where y_t denotes the observation at time t, A is the DC level that is desired to estimate, and $w_t \sim \mathcal{N}(0, \sigma^2)$ with σ^2 assuming known. It is well known that the CRLB for the estimate of A is

$$var(\hat{A}) \geq \frac{\sigma^2}{N} \tag{35}$$

and the unbiased estimator which attains the bound is the sample mean estimator, being mathmatically expressed as $\hat{A}_{ML} = \frac{1}{N}\sum_{t=0}^{N-1} y_t$.

Now, in order to study the problem from a Bayesian aspect, we adopt a conjugate prior for A, i.e., $p(A) = \mathcal{N}(\mu_A, \sigma_A^2)$ with μ_A and σ_A^2 preassigned. We know that the choice of μ_A and σ_A^2 will affect the accuracy of the Bayesian estimator. Typically, a more accurate estimator than $\hat{A}_{ML}$ could be obtained when μ_A is chosen around the true value of A, and performance of the Bayesian estimator would be more resemble to $\hat{A}_{ML}$ when $\sigma_A^2 \to \infty$. We should expect that the lower bound of the Bayesian estimator embody above phenomenon. First, the PCRLB can be shown as

$$Bmse(\hat{A}) \geq (\frac{N}{\sigma^2} + \frac{1}{\sigma_A^2})^{-1}. \tag{36}$$

Obviously this bound is unable to reflect the effect of the prior $p(A)$ on the accuracy of the estimator. This is because that the PCRLB is acquired for the random parameters. For this problem, the lower bound on the variance of any unbiased Bayesian esitmator can be obtained easily from Theorem 1 as

$$var(\hat{A}_B) \geq (\frac{N}{\sigma^2} + \frac{(A-\mu_A)^2}{\sigma_A^4})^{-1} \tag{37}$$

Several observations arise on the bound. First, the bound depends on A. Second, no matter what are the choices for prior mean μ_A and variance σ_A^2, the bound is always lower or at least equal to the CRLB. However, the bound still can not correctly reflect the the different choice of $p(A)$ on the accuracy of the estimator. Actually, one can show that there does not exist an unbiased Bayesian estimator which attains the bound (37). Quite often the Bayesian estimators are biased. One such instance is the well known MMSE estimator which, for this problem, can be shown as

$$\hat{A}_{MMSE} = \alpha \hat{A}_{ML} + (1-\alpha)\mu_A \tag{38}$$

where $\alpha = \sigma_A^2/(\sigma_A^2 + \sigma^2/N)$. This estimator is biased with an bias of $b(\hat{A}_{MMSE}) = (1-\alpha)(\mu_A - A)$. Besides, the variance of $\hat{A}_{MMSE}$ is $\alpha^2\sigma^2/N$. Now let us determine the lower bound on the MSE of all the Bayesian estimator which has the bias $b(A)$. It follows from Theorem 2 that this bound can be written as

$$\begin{aligned} mse(\hat{A}) &\geq \frac{(\alpha + (1-\alpha)(\mu_A - A)^2/\sigma_A^2)^2}{N/\sigma^2 + (A-\mu_A)^2/\sigma_A^4)} \\ &= \alpha^2\sigma^2/N + (1-\alpha)^2(A-\mu_A)^2 \\ &= b(\hat{A}_{MMSE}) + var(\hat{A}_{MMSE}) \\ &= mse(\hat{A}_{MMSE}) \end{aligned} \tag{39}$$

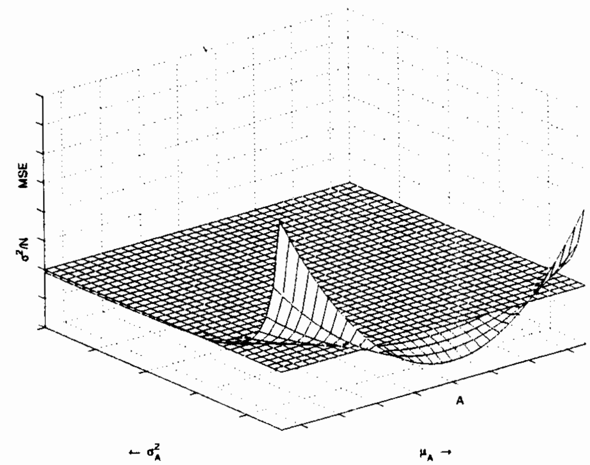

Figure 1: Plot of CRLB and proposed bound for the example as a function of μ_A and σ_A.

The last equality demonstrates that among all Bayesian estimators which bears the bias $b(\hat{A}_{MMSE})$, the MMSE estimator is the one which attains the bound. In Figure 1, the bound (39) is plotted together with the CRLB as a function of μ_A and σ_A. It shows that the bound (39) correctly reflects the effect of the choice of the prior on the accuracy of the estimator. Therefore the bound (39) is the right bound which can reflect the true state of the Bayesian estimators for the deterministic parameters.

5. CONCLUSION

The lower bounds have developed for both unbiased and biased Bayesian estimators for the deterministic parameters. We have shown that the variance of any unbiased Bayesian estimator which attains the proposed bound is always smaller than CRLB. In addition, the corresponding lower bounds in the presence of nuisance parameters were also discussed. An example demonstrated that our proposed bounds are the appropriate benchmark to performance of the Bayesian esitmators on the deterministic parameters.

6. REFERENCES

[1] S. M. Kay, *Fundamentals of Statistical Signal Processing: Estimation Theory*, Prentice-Hall, Englewood Cliffs, NJ, 1993.

[2] H. L. van Trees, *Detection, Estimation and Modulation Theory*, John Wiley & Sons, New York, 1968.

[3] Gart J. J., "An extension of the cramér-rao inequality," in *Ann. Math. Atatist.*, vol. 30, pp. 367–380.

[4] C-M. Cho and P. M. Djurić, "Bayesian detection and estimation of cisoids in colored noise," in *IEEE Trans. on Signal Processing*, Dec 1995, vol. 43, pp. 2943–2952.

[5] Christophe Andrieu and Arnaud Doucer, "A bayesian approach to hamonic retrieval with clipped data," in *Signal Processing*, 1999, vol. 74, pp. 239–252.