STATE UNIVERSITY OF NEW YORK AT STONY BROOK

COLLEGE OF ENGINEERING

REPORT No. 111

THE HILBERT PORT: AN EXTENSION OF THE CONCEPT OF THE N-PORT BASED ON THE CONVOLUTION OF BANACH-SPACE-VALUED DISTRIBUTIONS

by

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JUNE, 1968

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1. Introduction

A significant part of electrical network theory is concerned with the relationships between the idealized physical properties of systems such as linearity, time-invariance, causality, and passivity, on the one hand and certain analytic properties, such as convolution representations and positive-reality, on the other hand. Almost all of this work is concerned with n-ports: that is, with systems whose inputs and outputs are (ordinary or generalized) functions having values in n-dimensional Euclidean space. (See the bibliography in Zemanian [4].) This is despite the fact that many physical phenomena are better represented by systems whose inputs and outputs have values in a Hilbert space. For example, a cavity resonator that is being excited through a wave guide is such a system (Csurgay [1]). What is usually done is to assume that all but a finite number of modes in the wave guide are negligible at any appreciable distance from the cavity resonator so that the system can be viewed approximately as an n-port (Carlin [1]). It may at times be desirable not to make such an approximation, and therefore a realizability theory for systems having Hilbert-space-valued functions on the real line as their inputs and outputs is called for. To coin a phrase, we choose to call such a system a "Hilbert port" since it is such a natural extension of the idea of an n-port.

Previous works that relate the physical and analytic properties of a system having inputs and outputs in an arbitrary Hilbert space are Batt and König [1], Beltrami [2], Dolph [1], Hackenbroch [1], Lax and Phillips [1], Saeks [1], Schwindt [1], and Tonning [1]. Tonning's work deals primarily with frequency-domain properties of electromagnetic propagation systems. The recent report of Saeks possesses great generality but does not attack the problem considered here. The works of Beltrami, Dolph, and Lax-Phillips (see also the references therein) are based on a spectral analysis of the so-called "dissipative operators." For instance, Dolph assumes that the input v and output u are related by (1.1)

$$\frac{du}{dt} - \Lambda u = v \qquad (1.1)$$

where Λ is a dissipative operator. Then, he shows that the resolvent for Λ is positive-real when the system is passive, and conversely. The theories that come closest to this one are those of Batt-König, Schwindt, and Hackenbroch, which extend the classical paper by König and Meixner [1] and obtain time-domain representations for certain linear operators on a Hilbert space.

In contrast to the aforementioned literature, we allow our inputs and outputs to be Hilbert-space-valued distributions and make essential use of the convolution of Banach space-valued distributions. This does not appear to have been done before. Our theory in fact is a generalization of one for an n-port (Zemanian [3] which is the reason we use the phrase "Hilbert port" instead of "Hilbert system." Many of the steps of our analysis, which in the case of n-ports required no proof, must be justified when dealing with Hilbert ports. Furthemore, we have to employ some rather uncommon ideas, such as the convolution and Laplace transformation of Banach-space-valued distributions. The theories for the latter two subjects, which are described in sections 2 and 3 respectively, are we believe rather novel and have not been previously developed in quite theway presented here. In summary, in comparison to n-port theory, the Hilbert port theory is considerably more involved and requires more extensive proofs.

In this work R denotes the real line and C the complex plane. When $a \in R$ and $b \in R$ with $a < b$, $[a,b]$ and (a,b) denote respectively closed and open intervals, and similarly for $(a,b]$, and $[a,b)$. B and later on A will denote Banach spaces. The norm in B is denoted by $\|\cdot\| = \|\cdot\|_B$. H will be a Hilbert space, which need not be separable, and its inner product is denoted by $(\cdot,\cdot)$. The strong kth derivative of any B-valued function $\varphi(\zeta)$ on either R or C is denoted alternatively by

$$\frac{d^k\varphi}{d\zeta^k} = \varphi^{(k)}(\zeta) = D^k\varphi = D_\zeta{}^k\varphi(\zeta)$$

A smooth function is one having continuous derivatives of all orders at all points of its domain. supp φ denotes the support of φ. We at times use the symbol $\underline{\underline{\Delta}}$ to denote an equality when it is a definition.

The symbols $\underset{\sim}{D}$, $\underset{\sim}{D}_K$, $\underset{\sim}{D}'$, $\underset{\sim}{E}$, $\underset{\sim}{E}'$, $\underset{\sim}{D}_{L_1}$, $\underset{\sim}{B}$, $\dot{\underset{\sim}{B}}$, $\underset{\sim}{D}'_{L_1}$ and $\underset{\sim}{B}'$ denote the customary spaces of testing functions and distributions. We will define them subsequently in a more general context.

Next, let $a \in R$, $b \in R$, and

$$\varkappa_{a,b}(t) = \begin{cases} e^{at} & (t \geq 0) \\ e^{bt} & (t < 0) \end{cases}$$

$\underset{\sim}{L}_{a,b}$ in the space of smooth functions φ from R to C such that for each nonnegative integer k,

$$\gamma_k(\varphi) = \sup_{-\infty<t<\infty} |\varkappa_{a,b}(t)\, \varphi^{(k)}(t)| < \infty.$$

It is assigned the topology generated by the sequence of seminorms $\{\gamma_k\}_{k=1}^{\infty}$. $\underset{\sim}{L}'_{a,b}$ is its dual. Now, let $\{a_n\}$ and $\{b_n\}$ be sequences of real numbers such that $a_n \to w+$ and $b_n \to z-$ ($w = -\infty$ and $z = +\infty$ are allowed here). Then $\underset{\sim}{L}(w,z)$ is defined as the countable-union space generated by the $\underset{\sim}{L}_{a_n, b_n}$ spaces. $\underset{\sim}{L}'(w,z)$ is the dual of $\underset{\sim}{L}(w,z)$. These spaces are discussed in greater detail in Zemanian [2], [5].

2. Banach-space-valued Distributions and Their Convolution

The theory of distributions having values in an arbitrary topological vector space has been worked out in great depth by L. Schwartz (see Schwartz [2] and [3]). Sebastiao e Silva [1] also has a theory for such vector-valued distributions based on his axiomatic method of treating distributions as the derivatives of continuous functions. These theories are complicated, but fortunately the present work requires only the much simpler case of Banach-space-valued distributions. J. L. Lions summarizes a few results for this simpler case (see Lions [1], chapter II, section I). Mikusinski [1] has a theory for the convolution of Banach-space-valued distributions based on the Mikusinski-Sikorski [1] sequential theory of distributions.

In this section we shall present that part of the theory of Banach-space-valued distributions that will be used in the subsequent sections. We choose to use Schwartz's approach to such distribution by treating them as continuous linear mappings on certain testing-function spaces.

Let B be a complex Banach space. In subsequent sections, we shall on occasion deal with a real Banach space. But all the results that we state for a complex Banach space carry over to a real Banach space. We now formulate a general type of testing-function space $\underset{\sim}{H}(B)$ of functions from R into B. $\underset{\sim}{H}(B)$ encompasses as special cases all but two of the particular testing-functions spaces that appear in this work.

In the following μ, ν, k, m, n, p, and q denote nonnegative integers. The notation $a \leq k, m, \ldots, q \leq b$ means that $a \leq k \leq b$, $a \leq m \leq b$, ... $a \leq q \leq b$. A sequence $\{K_n\}_{n=0}^{\infty}$ will be called a nested closed-interval cover of R if each K_n is a closed interval, $K_0 \subset K_1 \subset K_2 \subset \ldots$, and $\bigcup_{n=0}^{\infty} K_n = R$. Let $\{K_n\}_{n=0}^{\infty}$ and $\{I_q\}_{q=0}^{\infty}$ be two nested closed interval covers of R. For each pair n and m, let there be given a continuous function $\varkappa_{n,m}(t)$ from R into R such that $\varkappa_{n,m}(t) > 0$ for all t.

(This implies that, for any compact subset Ω of R, there exists an $\epsilon > 0$ such that $\kappa_{n,m}(t) \geq \epsilon$ on Ω). Also, assume that, for each m, $\kappa_{o,m}(t) \geq \kappa_{1,m}(t) \geq \kappa_{2,m}(t) \geq \cdots$. For each pair n and p, we define the functional $\rho_{n,p}$ on suitably restricted smooth functions $\varphi(t)$ from R into B by

$$\rho_{n,p}(\varphi) \triangleq \max_{0 \leq k,m,\, q \leq p} \sup_{t \in I_q} \|\kappa_{n,m}(t)\, \varphi^{(k)}(t)\|_B$$

For each n, we define $\underset{\sim}{H}_n(B)$ as the linear space of all smooth functions $\varphi(t)$ from R into B such that supp $\varphi \subset K_n$ and $\rho_{n,p}(\varphi) < \infty$ for every p. $\underset{\sim}{H}_n(B)$ is assigned the topology generated by the sequence of seminorms $\{\rho_{n,p}\}_{p=o}^{\infty}$. This sequence will be called the multinorm for $\underset{\sim}{H}_n(B)$. $\underset{\sim}{H}_n(B)$ and its topology are independent of the choice of $\{I_q\}_{q=o}^{\infty}$ so long as $\{I_q\}_{q=o}^{\infty}$ is a nested closed interval cover of R. [Note that, if $I_q = R$, then for each $p \geq q$, $\rho_{n,p}$ is a norm on $\underset{\sim}{H}_n(B)$. On the other hand, if $\rho_{n,o}$ is a norm, then $I_o = R$].

Clearly, $\varphi \mapsto \varphi^{(k)}$ is a continuous linear mapping of $\underset{\sim}{H}_n(B)$ into $\underset{\sim}{H}_n(B)$. Moreover, since $\kappa_{n+1,m}(t) \leq \kappa_{n,m}(t)$, we have that $\rho_{n+1,p}(\varphi) \leq \rho_{n,p}(\varphi)$ for every $\varphi \in \underset{\sim}{H}_n(B)$; this implies that, for every n, $\underset{\sim}{H}_n(B) \subset \underset{\sim}{H}_{n+1}(B)$ and the topology of $\underset{\sim}{H}_n(B)$ is stronger then the topology induced on $\underset{\sim}{H}_n(B)$ by $\underset{\sim}{H}_{n+1}(B)$. Also $\underset{\sim}{H}_n(B)$ is a Hausdorff, locally convex, metrizable, topological vector space. A standard argument (see Zemanian [2], the proof of lemma 3.2-1) shows that $\underset{\sim}{H}_n(B)$ is complete. Thus, $\underset{\sim}{H}_n(B)$ is a Fréchet space.

Next, we let $\underset{\sim}{H}(B) = \cup_{n=o}^{\infty} \underset{\sim}{H}_n(B)$, and we supply $\underset{\sim}{H}(B)$ with the following rule for sequential convergence: A sequence $\{\varphi_\nu\}_{\nu=1}^{\infty}$ converges in $\underset{\sim}{H}(B)$ if and only if there exists a $\varphi \in \underset{\sim}{H}(B)$ and a fixed n such that all $\varphi_\nu \in \underset{\sim}{H}_n(B)$, $\varphi \in \underset{\sim}{H}_n(B)$, and $\varphi_\nu \to \varphi$ in $\underset{\sim}{H}_n(B)$. This makes $\underset{\sim}{H}(B)$ a sequential-convergence linear space. (Zemanian [2], section 1.4). Since each $\underset{\sim}{H}_n(B)$ is complete, $\underset{\sim}{H}_n(B)$ is also complete (we will always mean sequentially complete, when we write complete). The proof of this is precisely the same as that for the special case where B = C (Zemanian [2], section 1.7). The fact that we are now dealing with B-valued functions introduces no difficulties; this will always be the situation whenever we make a reference to Zemanian [2], which deals with only C-valued functions.

A special case arises when all $K_n = R$ and simultaneously $\kappa_{n,m}(t) = \kappa_{n+1,m}(t)$ for all n. In this case $\underset{\sim}{H}_n(B) = \underset{\sim}{H}(B)$ for all n so that the sequential-convergence linear space $\underset{\sim}{H}(B)$ is simply a Fréchet space.

A subset Ω of $\underset{\sim}{H}(B)$ is called bounded if, for all $\varphi \in \Omega$, supp $\varphi \subset K_n$ for some fixed n and there exists constants C_p such that $\rho_{n,p}(\varphi) < C_p$ for all $\varphi \in \Omega$. Equivalently, Ω is a bounded set in $\underset{\sim}{H}(B)$ if and only if $\Omega \subset \underset{\sim}{H}_n(B)$ for some n and Ω is a bounded set in $\underset{\sim}{H}_n(B)$. Here again, $\varphi \mapsto \varphi^{(k)}$ is a continuous linear mapping of $\underset{\sim}{H}(B)$ into $\underset{\sim}{H}(B)$, and, as φ traverses a bounded set in $\underset{\sim}{H}(B)$, $\varphi^{(k)}$ also traverses a bounded set in $\underset{\sim}{H}(B)$.

Let Y be an index set. To each $\upsilon \in Y$ we assign a function $\varphi_\upsilon \in \underset{\sim}{H}(B)$ and a sequence $\{\varphi_{\upsilon,\nu}\}_{\nu=1}^{\infty}$ in $\underset{\sim}{H}(B)$. The sequence $\{\varphi_{\upsilon,\nu}\}_{\nu=1}^{\infty}$ is said to converge to φ_υ uniformly in $\underset{\sim}{H}(B)$, if there exists and n such that $\varphi_{\upsilon,\nu} \in \underset{\sim}{H}_n(B)$ for all $\upsilon \in Y$ and all ν and if, for each p, $\rho_{n,p}(\varphi_\upsilon - \varphi_{\upsilon,\nu}) \to 0$ as $\nu \to \infty$ uniformly for all $\upsilon \in Y$.

We now define a terminology that we shall use in this work. <u>Any space of functions that fits into the general formulation for $\underset{\sim}{H}(B)$ will be said to be a ρ-type testing-function space.</u>

In table I, we list the defining quantities for a number of ρ-type testing-function spaces. The symbols in the last two columns of that table will be discussed in a moment. Examples of the special case mentioned in a previous paragraph are given in rows II, III, and VII.

TABLE I

Examples of ρ-type Testing-Functions Spaces and the Notations for Some Corresponding Mappings

	$\underset{\sim}{H}(B)$	K_n	I_q	$\varkappa_{n,m}(t)$	$\rho_{n,p}(\varphi)$
I	$\underset{\sim}{D}(B) = \bigcup_{n=0}^{\infty} \underset{\sim}{D}_{K_n}(B)$	$[-n, n]$	R	$\equiv 1$	$\max_{0 \le k \le p} \sup_{t \in R} \lVert\varphi^{(k)}(t)\rVert_B$
II	$\underset{\sim}{E}(B)$	R	$[-q, q]$	$\equiv 1$	$\max_{0 \le k,\, q \le p} \sup_{-q \le t \le q} \lVert\varphi^{(k)}(t)\rVert_B$
III	$\underset{\sim}{L}_{a,b}(B)$, $a, b \in R$	R	R	$\varkappa_{a,b}(t) = \begin{cases} e^{at} & (t \ge 0) \\ e^{bt} & (t \le 0) \end{cases}$	$\max_{0 \le k \le p} \sup_{t \in R} \lVert\varkappa_{a,b}(t)\varphi^{(k)}(t)\rVert_B$
IV	$\underset{\sim}{L}(w, z)(B) = \bigcup_{n=0}^{\infty} \underset{\sim}{L}_{a_n,b_n}(B)$, $a_n \to w+,\ b_n \to z-$	R	R	$\varkappa_{a_n,b_n}(t) = \begin{cases} e^{a_n t} & (t \ge 0) \\ e^{b_n t} & (t \le 0) \end{cases}$	$\max_{0 \le k \le p} \sup_{t \in R} \lVert\varkappa_{a_n,b_n}(t)\varphi^{(k)}(t)\rVert_B$
V	$D_L(B) = \bigcup_{n=0}^{\infty} \underset{\sim}{D}_{K_n}(B)$	$(-\infty, n]$	$[-q, \infty]$	$\equiv 1$	$\max_{0 \le k,\, q \le p} \sup_{q \le t < \infty} \lVert\varphi^{(k)}(t)\rVert_B$
VI	$\underset{\sim}{D}_R(B) = \bigcup_{n=0}^{\infty} \underset{\sim}{D}_{K_n}(B)$	$[n, \infty)$	$(-\infty, q]$	$\equiv 1$	$\max_{0 \le k,\, q \le p} \sup_{-\infty < t \le q} \lVert\varphi^{(k)}(t)\rVert_B$

When B = C, we denote $\underset{\sim}{H}_n(B)$ simply by $\underset{\sim}{H}_n$ and $\underset{\sim}{H}(B)$ by $\underset{\sim}{H}$. Next, we define $\underset{\sim}{H}'(B)$ as the linear space of all continuous linear mappings of $\underset{\sim}{H}$ into B. Here again we use the notation $\underset{\sim}{H}'(C) = \underset{\sim}{H}'$. $< f, \varphi >$ denotes the member of B assigned by $f \in \underset{\sim}{H}'(B)$ to $\varphi \in \underset{\sim}{H}$. The symbols for $\underset{\sim}{H}'(B)$ in several particular cases are indicated in the next to the last column of Table I. Note that, in rows V and VI, we use a somewhat different notation for $\underset{\sim}{H}'(B)$. For example, the members of $\underset{\sim}{D}_L(B)$ are smooth functions whose supports extend in general infinitely toward the left but are bounded on the right. On the other hand, it can be shown that the continuous linear mappings of $\underset{\sim}{D}_L$ into B have supports that extend in general infinitely to the right but are bounded on the left. To imply this property of the supports we denote the space of such mappings by $\underset{\sim}{D}_R'(B)$, instead of by $\underset{\sim}{D}_L'(B)$. Since $\underset{\sim}{D}$ is a dense subspace of $\underset{\sim}{E}$, $\underset{\sim}{L}(w,z)$, $\underset{\sim}{D}_L$, $\underset{\sim}{D}_R$, and $\underset{\sim}{S}$ and since sequential convergence in $\underset{\sim}{D}$ implies sequential convergence in any of the other spaces, $\underset{\sim}{D}'(B)$ contains as subspaces $\underset{\sim}{E}'(B)$, $\underset{\sim}{L}(w, z)(B)$, $\underset{\sim}{D}_R'(B)$, $\underset{\sim}{D}_L'(B)$, and $\underset{\sim}{S}'(B)$.

We assign to $\underset{\sim}{H}'(B)$ two concepts of sequential convergence. To arrive at the concept of weak convergence, we first define a collection $\{\omega_\varphi\}_{\varphi\in H}$ of seminorms ω_φ on $\underset{\sim}{H}'(B)$ by

$$\omega_\varphi(f) \triangleq \|< f, \varphi >\|_B \qquad f \in \underset{\sim}{H}'(B)$$

Then, a sequence $\{f_\nu\}_{\nu=1}^{\infty}$ is said to converge weakly in $\underset{\sim}{H}'(B)$ of all $f_\nu \in \underset{\sim}{H}'(B)$ and there exists an $f \in \underset{\sim}{H}'(B)$ such that $\omega_\varphi(f_\nu - f) \to 0$ as $\nu \to \infty$.

Similarly upon letting Ω vary through all the bounded sets in $\underset{\sim}{H}$, we define the collection $\{\sigma_\Omega\}_\Omega$ of seminorms σ_Ω on $\underset{\sim}{H}'(B)$ by

$$\sigma_\Omega(f) \triangleq \sup_{\varphi\in\Omega} \|< f, \varphi >\|_B \qquad f \in \underset{\sim}{H}'(B)$$

Then, we say that a sequence $\{f_\nu\}_{\nu=1}^{\infty}$ converges strongly in $\underset{\sim}{H}'(B)$ if all $f_\nu \in \underset{\sim}{H}'(B)$ and there exists an $f \in \underset{\sim}{H}'(B)$ such that $\sigma_\Omega(f_\nu - f) \to 0$ as $\nu \to \infty$. Clearly strong convergence implies weak convergence.

In an analogous way we can define the weak Cauchy sequences in $\underset{\sim}{H}'(B)$. A straightforward modification of a standard argument (Zemanian [2], theorems 1.8-3 and 1.9-2) shows that $\underset{\sim}{H}'(B)$ is weakly (sequentially) complete.

If $\varphi \in \underset{\sim}{H}$ and $a \in B$, then φa is that function from R into B that assigns to each $t \in R$ the value $\varphi(t)a \in B$. Clearly, $\varphi a \in \underset{\sim}{H}(B)$. $\underset{\sim}{H} \otimes B$ is the subspace of H(B) consisting of all finite linear combinations of elements of the form φa.

It is a fact that $\underset{\sim}{D} \otimes B$ is dense in $\underset{\sim}{D}(B)$ (see Schwartz [2], pp. 109-110); we shall need this fact later on.

Note that, if B is real, then it is understood that the members of $\underset{\sim}{H}$ are real-valued functions.

Next, if $g \in H'$ and $a \in B$, ga is defined by the equation;

$$< ga, \varphi > \underset{=}{\Delta} < g, \varphi > a \qquad \varphi \in \underset{\sim}{H}$$

It is easy to show that $ga \in \underset{\sim}{H}'(B)$. $\underset{\sim}{H}' \otimes B$ denotes the subspace of $\underset{\sim}{H}'(B)$ consisting of all finite linear combinations of elements of the form ga.

Next, let B' denote the dual of B, and let [a, a'] be the complex number assigned by $a' \in B'$ to $a \in B$. For any $f \in \underset{\sim}{H}'(B)$ and any $a' \in B'$ we define the symbol [f, a'] through the equation

$$< [f, a'], \varphi > \underset{=}{\Delta} [< f, \varphi >, a'] \quad \varphi \in \underset{\sim}{H}.$$

The right-hand side has a sense as a complex number since $< f, \varphi > \in B$.

We now show that $[f, a'] \in \underset{\sim}{H}'$. Indeed, for $\alpha, \beta \in C$ and $\varphi_1, \varphi_2 \in \underset{\sim}{H}$,

$$< [f, a'], \alpha\varphi_1 + \beta\varphi_2 > = [< f, \alpha\varphi_1 + \beta\varphi_2 >, a'] = [\alpha < f, \varphi_1 > + \beta < f, \varphi_2 >, a']$$

$$= \alpha [< f, \varphi_1 >, a'] + \beta [< f, \varphi_2 >, a'] = \alpha < [f, a'] \varphi_1 > + \beta < [f, a'], \varphi_2 >,$$

so that, $[f, a']$ is a linear functional $\underset{\sim}{H}$. Moreover, if $\varphi_\nu \to 0$ in $\underset{\sim}{H}$, we have that

$$< [f, a'], \varphi_\nu > = [< f, \varphi_\nu >, a'] \to 0 \text{ since } < f, \varphi_\nu > \to 0 \text{ in } B.$$

Hence, $[f, a']$ is also a continuous functional on $\underset{\sim}{H}$.

Moreover, $f \mapsto [f, a']$ is a continuous linear mapping of $\underset{\sim}{H}'(B)$ into $\underset{\sim}{H}'$, where continuity is understood in either of the following two senses: If $f_\nu \to f$ strongly (weakly) in $\underset{\sim}{H}'(B)$, then $[f_\nu, a'] \to [f, a']$ strongly (respectively, weakly) in $\underset{\sim}{H}'$. Indeed, the linearity of $f \mapsto [f, a']$ follows readily from the standard definitions of addition and multiplication by a scalar. On the other hand, $\left| < [f, a'], \varphi > \right| = \left| [< f, \varphi >, a'] \right| \leq \| a' \|_{B'} \| < f, \varphi > \|_B$ from which our assertion concerning continuity follows.

The preceding arguments hold equally well when we assume that B is a Hilbert space H, that a and a′ are members of H, and that $[.,.]$ is replaced by the inner product $(.,.)$ in H. In this case, $f \mapsto (f, a')$ is a continuous linear mapping of $\underset{\sim}{H}'(H)$ into $\underset{\sim}{H}'$.

<u>Proposition 2-1</u>: <u>Let</u> $f \in \underset{\sim}{H}'(B)$, <u>let</u> $\{\rho_{n,p}\}_{p=0}^{\infty}$ <u>be the multinorm for</u> $\underset{\sim}{H}_n$, <u>and assume that, for each</u> n, $\rho_{n,o}$ <u>is a norm on</u> $\underset{\sim}{H}_n$. <u>Then, corresponding to each</u> n, <u>there exist a positive constant</u> M <u>and a nonnegative integer</u> r <u>such that</u>

$$\| < f, \varphi > \|_B \leq M \, \rho_{n,r}(\varphi) \tag{2.1}$$

<u>for all</u> $\varphi \in \underset{\sim}{H}_n$. <u>Both</u> M <u>and</u> r <u>depend in general on</u> f <u>and</u> n.

<u>Proof</u>: Clearly, $\rho_{n,o}(\varphi) \leq \rho_{n,1}(\varphi) \leq \rho_{n,2}(\varphi) \leq \dots$ for each $\varphi \in \underset{\sim}{H}_n$. Since $\rho_{n,o}$ is a norm on $\underset{\sim}{H}_n$, we also have that $\rho_{n,o}(\varphi) > 0$ if $\varphi(t) \not\equiv 0$. Also, note that (2.1) is obviously satisfied if $\varphi(t) \equiv 0$.

Now, suppose that the proposition is not true. Then, for each ν there exists a $\varphi_\nu \in \underset{\sim}{H}_n$ such that $\varphi_\nu(t) \not\equiv 0$ and

$$(2.2) \qquad \|\langle f, \varphi_\nu \rangle\|_B > \nu \, \rho_{n,\nu}(\varphi_\nu) > 0.$$

Set $\theta_\nu = \varphi_\nu / \nu \, \rho_{n,\nu}(\varphi_\nu)$. Hence, θ_ν converges in $\underset{\sim}{H}$ to zero since all $\theta_\nu \in \underset{\sim}{H}_n$ and, for $\mu < \nu$,

$$\rho_{n,\mu}(\theta_\nu) \le \rho_{n,\nu}(\theta_\nu) = \frac{1}{\nu} \to 0 \qquad \nu \to \infty.$$

Therefore, $\langle f, \theta_\nu \rangle \to 0$ in B as $\nu \to \infty$. However, according to (2.2), $\|\langle f, \theta_\nu \rangle\|_B > 1$. This contradiction proves the theorem.

Next, let both A and B be either both complex Banach spaces or both real Banach spaces. L(B,A) denotes the Banach space of all continuous linear mappings of $\underset{\sim}{H}(B)$ into A. We define $\underset{\sim}{H}'[L(B,A)]$ as the linear space of all continuous linear mappings of $\underset{\sim}{H}(B)$ into A. Now, $\langle y, \psi \rangle$ denotes that member of A assigned by $y \in \underset{\sim}{H}'[L(B,A)]$ to $\psi \in \underset{\sim}{H}(B)$. Only one concept of sequential convergence will be assigned to $\underset{\sim}{H}'[L(B,A)]$, the weak one generated by the collection of seminorms $\{\eta_\psi\}_{\psi \in \underset{\sim}{H}(B)}$ where

$$\eta_\psi(y) \triangleq \|\langle y, \psi \rangle\|_A \qquad y \in \underset{\sim}{H}'[L(B,A)]$$

Thus, a sequence $\{y_n\}_{n=1}^{\infty}$ is said to converge in $\underset{\sim}{H}'[L(B,A)]$ if all $y_n \in \underset{\sim}{H}'[L(B,A)]$ and there exists a $y \in \underset{\sim}{H}'[L(B,A)]$ and that $\eta_\psi(y_n - y) \to 0$ as $n \to \infty$ whatever be the choice of $\psi \in \underset{\sim}{H}(B)$.

Again we can define the (weak) Cauchy sequences in $\underset{\sim}{H}'[L(B,A)]$ by using the collection of seminorms $\{\eta_\psi\}_{\psi \in \underset{\sim}{H}(B)}$. Again a modification of a standard argument (Zemanian [2], theorems 1.8-3 and 1.9-2) shows that $\underset{\sim}{H}'(B)$ is (weakly sequentially) complete.

We use the brackets in the symbol $\underset{\sim}{H}'[L(B,A)]$ to distinguish this space from the space $\underset{\sim}{H}'(L(B,A))$ of all continuous linear mappings of $\underset{\sim}{H}$ into L(B,A). We shall show in a moment that to each $y \in \underset{\sim}{H}'[L(B,A)]$ there corresponds a unique $f_y \in \underset{\sim}{H}'(L(B,A))$. But, first we state

Proposition 2-2: Let $y \in \underset{\sim}{H}'[L(B,A)]$, *let* $\{\rho_{n,p}\}_{p=0}^{\infty}$ *be the multinorm for* $\underset{\sim}{H}_n(B)$, *and assume that, for each* n, $\rho_{n,o}$ *is a norm on* $\underset{\sim}{H}_n(B)$. *Then corresponding to each* n, *there exists a positive constant* M *and a nonnegative integer* r *such that*

$$\|\langle y, \varphi\rangle\|_A \leq M \rho_{n,r}(\varphi)$$

for all $\varphi \in \underset{\sim}{H}(B)$. Both M and r depend in general on y and n.

The proof is the same as that of proposition 2-1.

$\underset{\sim}{D}(B)$ is a dense subspace of $\underset{\sim}{E}(B)$, $\underset{\sim}{L}(w,z)(B)$, $\underset{\sim}{D}_L(B)$, $\underset{\sim}{D}_R(B)$, and $\underset{\sim}{S}(B)$, and sequential convergence in $\underset{\sim}{D}(B)$ implies sequential convergence in any of the other spaces. Consequently, $\underset{\sim}{E}'[L(B,A)]$, $\underset{\sim}{L}'(w,z)[L(B,A)]$, $\underset{\sim}{D}_R'[L(B,A)]$, $\underset{\sim}{D}_L'[L(B,A)]$ and $\underset{\sim}{S}'[L(B,A)]$ are all subspaces of $\underset{\sim}{D}'[L(B,A)]$.

We now show that under the hypothesis of properties 2-2, every $y \in \underset{\sim}{H}'[L(B,A)]$ defines a unique continuous linear mapping f_y of $\underset{\sim}{H}$ in $L(B,A)$. Let $\varphi \in H$ and $a \in B$ so that $\varphi a \in \underset{\sim}{H} \otimes B$. Then, $\langle y, \varphi a\rangle \in A$. Thus $a \mapsto \langle y, \varphi a\rangle$ is a mapping of B into A. It is also linear and continuous; its linearity is easily seen and its continuity follows from proposition 2-2 and the inequality:

$$\|\langle y, \varphi a\rangle\|_A \leq M \rho_{n,r}(\varphi a) = M \|a\|_B \max_{0 \leq k,m,q \leq r} \sup_t |\kappa_{n,m}(t) \varphi^{(k)}(t)|$$

where $\underset{\sim}{H}_n$ is chosen to contain φ. We denote the operator $a \mapsto \langle y, \varphi a\rangle$ by $\langle f_y, \varphi\rangle$ and write

$$\langle f_y, \varphi\rangle a \triangleq \langle y, \varphi a\rangle, \qquad \langle f_y, \varphi\rangle \in L(B,A).$$

We have hereby shown that $\varphi \mapsto \langle f_y, \varphi\rangle$ is a mapping of $\underset{\sim}{H}$ into $L(B,A)$, and it is uniquely determined by y. We are naturally lead to denote the latter mapping by f_y. We finally note that the mapping $f_y: \varphi \mapsto \langle f_y, \varphi\rangle$ is clearly linear and is continuous since, by proposition 2-2 again,

$$\|\langle f_y, \varphi\rangle\|_{L(B,A)} = \sup_{\|a\|=1} \|\langle y, \varphi a\rangle\|_A \leq M \max_{0 \leq k,m,q \leq r} \sup_t |\kappa_{n,m}(t) \varphi^{(k)}(t)|.$$

We summarize these results by

Proposition 2-3: *Under the hypothesis of proposition 2-2, every* $y \in \underset{\sim}{H}'[L(B,A)]$ *defines a unique continuous linear mapping* $f_y : \varphi \mapsto \langle f_y, \varphi \rangle$ *of* $\underset{\sim}{H}$ *into* $L(B,A)$ *where* $\langle f_y, \varphi \rangle$ *is defined in turn as the mapping* $a \mapsto \langle y, \varphi a \rangle$.

Note that, if supp $y \subset [o,\infty)$, then supp $f_y \subset [o,\infty)$. We shall use this fact in the proof of theorem 5-1.

We have to generalize the concept of the convolution of distributions in order to arrive at our realizability theory for Hilbert ports. We shall assume that we are dealing with complex spaces. Our discussion of convolution remains the same when all spaces are real.

Let A and B be two complex Banach spaces. Let there also be given three ρ-type testing-function spaces $\underset{\sim}{H}(B)$, $\underset{\sim}{I}$, and $\underset{\sim}{K}$, and the three corresponding spaces $\underset{\sim}{H}'[L(B,A)]$, $\underset{\sim}{I}'(B)$, and $\underset{\sim}{K}'(A)$. We assume that the following four conditions are satisfied:

Conditions E:

E1. If $\varphi \in \underset{\sim}{K}$, then, for each fixed t, $\varphi(t+\tau)$ as a function of τ is a member of $\underset{\sim}{I}$.

E2. If $v \in \underset{\sim}{I}'(B)$, $\varphi \in \underset{\sim}{K}$, and

$$\psi(t) \triangleq \langle v(\tau), \varphi(t+\tau \rangle,$$

then $\varphi \mapsto \psi$ is a continuous linear mapping of K into H(B).

E3. If $\{v_\nu\}_{\nu=1}^{\infty}$ converges strongly in $\underset{\sim}{I}'(B)$ to zero, if $\varphi \in \underset{\sim}{K}$, and if

$$\psi_\nu(t) = \langle v_\nu(\tau), \varphi(t+\tau) \rangle,$$

then $\{\psi_\nu\}_{\nu=1}^{\infty}$ converges in $\underset{\sim}{H}(B)$ to zero uniformly for all φ in any bounded subset $\underset{\sim}{K}$. [This means that, for all φ in a given bounded set Λ in $\underset{\sim}{K}$ and for all ν, the functions $\psi_\nu(t)$ are all contained in $\underset{\sim}{H}_n(B)$ for some n and that $\{\psi_\nu\}$ converges to a limit under every seminorm $\rho_{n,p}$ ($p=o, 1, 2, \ldots$) uniformly for all $\varphi \in \Lambda$.]

E4. Let $\{\rho_{n,p}\}_{p=o}^{\infty}$ be the multinorm for $\underset{\sim}{H}_n(B)$. Given any $y \in \underset{\sim}{H}'[L(B,A)]$ and any integer $n \geq 0$, there exists a constant $M > 0$ and an integer $r \geq 0$ such that

$$\|\langle y, \psi\rangle\|_A \leq M \rho_{n,r}(\psi)$$

for all $\psi \in \underset{\sim}{H}_n(B)$. M and r depend in general on y and n.

We define the convolution products $y * v$ for any $y \in \underset{\sim}{H}'[L(B,A)]$ and $v \in \underset{\sim}{I}'(B)$ as a mapping on $\underset{\sim}{K}$ by

$$(2.3) \qquad \langle y * v, \varphi\rangle = \langle y(t), \langle v(\tau), \varphi(t+\tau)\rangle\rangle \qquad \varphi \in \underset{\sim}{K}.$$

Note that, under conditions E1 and E2, the right-hand side has a sense and is a member of A.

Proposition 2-4: Given the Banach spaces A and B, the ρ-type testing-function spaces $\underset{\sim}{H}(B)$, $\underset{\sim}{I}$, and $\underset{\sim}{K}$ and the corresponding spaces $\underset{\sim}{H}'[L(B,A)]$, $\underset{\sim}{I}'(B)$ and $\underset{\sim}{K}'(A)$, assume that conditions E are satisfied. Define the convolution product $y*v$ of a $y \in \underset{\sim}{H}'[L(B,A)]$ and a $v \in \underset{\sim}{I}'(B)$ by (2.3). Then, the operator $v \mapsto y*v$ is a linear mapping of $\underset{\sim}{I}'(B)$ into $\underset{\sim}{K}'(A)$ that is also continuous in the following sense: If $\{v_\nu\}_{\nu=1}^{\infty}$ converges to v strongly in $\underset{\sim}{I}'(B)$, then $\{y*v_\nu\}_{\nu=1}^{\infty}$ converges to $y*v$ strongly in $\underset{\sim}{K}'(A)$.

Proof: In the following, we define ψ and ψ_ν as in conditions E2 and E3. We have already noted that $y*v$ maps $\underset{\sim}{K}$ into A. $y*v$ is linear on $\underset{\sim}{K}$ since y and $\varphi \mapsto \psi$ are both linear mappings. Moreover, if $\varphi_\nu \to 0$ in $\underset{\sim}{K}$ as $\nu \to \infty$, then $\psi_\nu \to 0$ in $\underset{\sim}{H}(B)$ by condition E2. But then, $\langle y, \psi_\nu\rangle \to 0$ in A. Hence, $y*v$ is also a continuous mapping of $\underset{\sim}{K}$ into A. Thus, $y*v \in \underset{\sim}{K}'(A)$.

Next, the linearity of the mapping $v \mapsto y*v$ follows from the standard definitions of addition and multiplication-by-a-scalar for operators. Finally, let $\{v_\nu\}_{\nu=1}^{\infty}$ converge to v strongly in $\underset{\sim}{I}'(B)$. Then, $\{v_\nu - v\}_\nu$ converges to zero strongly in $\underset{\sim}{I}'(B)$, and, by condition E3, $\{\psi_\nu - \psi\}_\nu$ converges to zero in $\underset{\sim}{H}(B)$ uniformly for all φ in any bounded subset, say, Λ of $\underset{\sim}{K}$. This implies that $\{\psi_\nu - \psi\}_\nu$ is contained in $\underset{\sim}{H}_n(B)$ for some n, whatever be the choice of φ in Λ. Therefore, by condition E4,

$$\|\langle y*(v_\nu - v), \varphi\rangle\|_A = \|\langle y, \psi_\nu - \psi\rangle\|_A \le M \rho_{n,r}(\psi_\nu - \psi).$$

By condition E3 again, as $\nu \to \infty$, the right-hand side converges to zero uniformly for all φ in A. So truly, $y*v_\nu \to y*v$ strongly in $\underset{\sim}{K}'(A)$. The proof is complete.

Each one of the first five columns of table II presents a system of spaces that satisfy the conditions assumed in the hypothesis of properties 2-4. That $\underset{\sim}{H}(B)$, $\underset{\sim}{I}$, and $\underset{\sim}{K}$ are ρ-type testing-function spaces is readily seen by referring to table I. The spaces indicated in the sixth column also satisfy conditions E but now the spaces $\underset{\sim}{H}(B) = \underset{\sim}{D}_{L_1}(B)$, $\underset{\sim}{I} = \underset{\sim}{\dot{B}}$, and $\underset{\sim}{K} = \underset{\sim}{D}_{L_1}$ are not ρ-type testing-function spaces. Nevertheless, our convolution can readily be extended to this case also, as we shall see.

We now discuss in turn the first five cases indicated in table II to show that conditions E are satisfied in each case.

Case I: Condition E1 is obviously satisfied.

To verify condition E2, we have to show that, for any $v \in \underset{\sim}{E}'(B)$ and $\varphi \in \underset{\sim}{D}$, the operator $\varphi \mapsto \psi(t) \triangleq \langle v(\tau), \varphi(t+\tau)\rangle$ is a continuous linear mapping of $\underset{\sim}{D}$ into $\underset{\sim}{D}(B)$. Since the supports of v and φ are both bounded, the support of ψ is also bounded. We can show that $\psi(t)$ is smooth and that

$$\psi^{(k)}(t) = \langle v(\tau), \varphi^{(k)}(t+\tau)\rangle \qquad k = 0, 1, 2, \dots \tag{2.4}$$

through an inductive argument. Assume that (2.4) is true for some k. It is true by definition for $k = 0$. Then, let $\Delta t \neq 0$, let t be fixed, and consider

$$\frac{\psi^{(k)}(t+\Delta t) - \psi^{(k)}(t)}{\Delta t} - \langle v(\tau), \varphi^{(k+1)}(t+\tau)\rangle = \langle v(\tau), \theta_{\Delta t}(\tau)\rangle$$

where

$$\theta_{\Delta t}(\tau) = \frac{\varphi^{(k)}(t+\Delta t+\tau) - \varphi^{(k)}(t+\tau)}{\Delta t} - \varphi^{(k+1)}(t+\tau)$$

$$= \frac{1}{\Delta t}\int_0^{\Delta t} d\xi \int_0^{\xi} \varphi^{(k+2)}(t+\tau+\zeta)\,d\zeta$$

Table II

Systems of Spaces that are Compatible for Our Definition of Convolution

	I	II	III	IV	V $(w < z)$	VI
$H(B)$	$D(B)$	$E(B)$	$D_L(B)$	$D_R(B)$	$L(w, z)(B)$	$D_{L_1}(B)$
$H'[L(B, A)]$	$D'[L(B, A)]$	$E'[L(B, A)]$	$D'_R[L(B, A)]$	$D'_L[L(B, A)]$	$L'(w, z)[L(B, A)]$	$B'[L(B, A)]$
I	E	D	D_L	D_R	$L(w, z)$	$\dot{B}$
$I'(B)$	$E'(B)$	$D'(B)$	$D'_R(B)$	$D'_L(B)$	$L'(w, z)(B)$	$D'_{L_1}(B)$
K	D	D	D_L	D_R	$L(w, z)$	D_{L_1}
$K'(A)$	$D'(A)$	$D'(A)$	$D'_R(A)$	$D'_L(A)$	$L'(w, z)(A)$	$B'(A)$

Note: $D'[L(B, A)]$ contains as proper subspaces all the subsequent spaces indicated in the second row. A similar situation holds when $D'[L(B, A)]$ is replaced by $D'(B)$.

It is not difficult to show that $\theta_{\Delta t}(\tau)$ converges in $\underset{\sim}{D}$ to zero as $|\Delta t| \to 0$. The smoothness of $\psi(t)$ and the validity of (2.4) follow from this fact. Thus, $\psi(t) \in \underset{\sim}{D}(B)$.

Next, let $\lambda \in \underset{\sim}{D}$ be such that $\lambda(\tau) \equiv 1$ on a neighborhood of supp $v(\tau)$. If $K_n = [-n, n]$ is a compact interval containing supp λ, then $\lambda\, \varphi \in \underset{\sim}{D}_{K_n}$ for all $\varphi \in \underset{\sim}{D}$. Consequently, by virtue of proposition 2-1,

$$\|\psi^{(k)}(t)\|_B \leq \|\langle v(\tau), \varphi^{(k)}(t+\tau)\rangle\|_B = \|\langle v(\tau), \lambda(\tau)\, \varphi^{(k)}(t+\tau)\rangle\|_B$$

$$\leq M \max_{0 \leq \mu \leq r} \sup_{\tau} |D_\tau^\mu\, \lambda(\tau)\, \varphi^{(k)}(t+\tau)|$$

$$\leq M \max_{0 \leq \mu \leq r} \sum_{\nu=0}^{\mu} \binom{\mu}{\nu} \sup_{\tau} |\lambda^{(\mu-\nu)}(\tau)| \sup_{\tau} |\varphi^{(k+\nu)}(\tau)| \tag{2.5}$$

Moreover, as φ traverses any convergent sequence in $\underset{\sim}{D}$, the supports of the corresponding ψ are all contained in a fixed bounded interval. This fact and the estimate (2.5) imply that the mapping $\varphi \mapsto \psi$ is continuous, its linearity being obvious. Condition E2 has been verified.

To verify condition E3, assume that $v_\nu \to 0$ strongly in $\underset{\sim}{E}'(B)$ as $\nu \to \infty$. This means that, for every ν, the supp v_ν are all contained in a fixed compact subset, say, G of R and that $\langle v_\nu, \theta\rangle \to 0$ in B uniformly for all θ in any bounded subset of $\underset{\sim}{E}$. Moreover, defining $\lambda \in \underset{\sim}{D}$ to be such that $\lambda(\tau) \equiv 1$ on a neighborhood of G, we may write

$$\psi_\nu(t) = \langle v_\nu(\tau), \lambda(\tau)\, \varphi(t+\tau)\rangle$$

From this it follows that the supp ψ_ν are all contained in a fixed compact subset of R for all ν and all φ in a bounded subset of $\underset{\sim}{D}$.

Furthermore, for any fixed nonnegative integer k, the $\varphi^{(k)}(t+\tau)$ comprise a bounded set in $\underset{\sim}{E}$ as t traverses R and φ traverses any bounded set in $\underset{\sim}{D}$. This implies that, as $\nu \to \infty$, the

$$\psi_\nu^{(k)}(t) = \langle v_\nu(\tau), \varphi^{(k)}(t+\tau)\rangle$$

converge to zero in B uniformly for all $t \in R$ and all φ in any bounded set in $\underset{\sim}{D}$.

We have hereby shown that $\psi_\nu \to 0$ in $\underset{\sim}{D}(B)$ uniformly for all φ in any bounded set in $\underset{\sim}{D}$.

To see that condition E4 is satisfied we need merely invoke proposition 2-2 since $\rho_{n,o}$ is truly a norm on $\underset{\sim}{H}_n(B) = \underset{\sim}{D}_{K_n}(B)$ in case I.

Case II: Here again, condition E1 is obviously fulfilled.

Condition E2 states that, if $v \in \underset{\sim}{D}'(B)$, $\varphi \in \underset{\sim}{D}$, and $\psi(t) = \langle v(\tau), \varphi(t+\tau)\rangle$, then $\varphi \mapsto \psi$ is a continuous linear mapping of $\underset{\sim}{D}$ into $\underset{\sim}{E}(B)$. We can show that $\psi(t) \in \underset{\sim}{E}(B)$ and that (2.4) holds by following the argument used in case I. Moreover, if $K_n = [-n,n]$ and J is another compact interval, then, for all $t \in J$ and all $\varphi \in \underset{\sim}{D}_{K_n}$, all the supports of $\varphi(t+\tau)$ considered as functions of τ are contained in another compact interval, say, I. Hence, by proposition 2-1, there exists a constant $M > 0$ and an integer $r \geq 0$ such that

$$\sup_{t\in J} \|\psi^{(k)}(t)\|_B \leq M \max_{0 \leq \mu \leq r} \sup_{\substack{\tau\in I \\ t\in J}} |\varphi^{(k+\mu)}(t+\tau)|$$

$$= M \max_{0 \leq \mu \leq r} \sup_{-\infty < \tau < \infty} |\varphi^{(k+\mu)}(\tau)|$$

for all $\varphi \in \underset{\sim}{D}_{K_n}$. This inequality implies that $\varphi \mapsto \psi$ is a continuous linear mapping of $\underset{\sim}{D}$ into $\underset{\sim}{E}(B)$.

Turning to condition E3, assume that $\{v_\nu\}$ converges strongly in $\underset{\sim}{D}'(B)$ to zero. This means that $\|\langle v_\nu, \theta\rangle\|_B \to 0$ uniformly for all θ in any bounded set Ω in $\underset{\sim}{D}$. But,

$$\|\psi_\nu^{(k)}(t)\|_B = \|\langle v_\nu(\tau), \varphi^{(k)}(t+\tau)\rangle\|_B$$

and, for any compact set K, the $\varphi^{(k)}(t+\tau)$ comprise a bounded set in $\underset{\sim}{D}$ as t traverses K and φ traverses any bounded subset of $\underset{\sim}{D}$. Consequently, the supremum on $t \in K$ of the last expression converges to zero uniformly for all φ in Ω. This is true whatever be the nonnegative integer k or the compact set K. This proves that $\{\psi_\nu\}$ converges to zero in $\underset{\sim}{E}(B)$ uniformly for all $\varphi \in \Omega$. Condition E3 is verified.

For condition E4, let $y \in \underset{\sim}{E}'[L(B,A)]$. Hence, $y \in \underset{\sim}{D}'[L(B,A)]$ and y has a compact support. Choose $\lambda \in \underset{\sim}{D}$ such that $\lambda(t) \equiv 1$ on a neighborhood of supp y. Therefore, $\langle y, \psi\rangle = \langle y, \lambda\psi\rangle$, and $\lambda\,\psi \in \underset{\sim}{D}_{K_n}(B)$ for all $\psi \in \underset{\sim}{E}(B) = \underset{\sim}{H}_n(B) = \underset{\sim}{H}(B)$, where $K_n \supset \text{supp}\,\lambda$. Invoking proposition 2-2 for $y \in \underset{\sim}{D}'[L(B,A)]$, we may write

$$\|\langle y, \psi\rangle\|_A \le M \max_{0 \le k \le r} \sup_{\tau \in \text{supp}\,\lambda} \|D^{(k)}\lambda(\tau)\,\psi(\tau)\|_B$$

This inequality implies that condition E4 is satisfied.

Case III: Again condition E1 is clearly satisfied.

For condition E2, we first note that the support of $v \in \underset{\sim}{D}'(B)$ is bounded on the left and that the support of $\varphi \in \underset{\sim}{D}_L$ is bounded on the right. It follows that $\psi(t) \overset{\Delta}{=} \langle v(\tau), \varphi(t+\tau)\rangle$ is a B-valued function on R whose support is bounded on the right. That ψ is smooth and that (2.4) holds follows as in case I. Now, however, we have to show that $\theta_{\Delta t}(\tau)$ converges in $\underset{\sim}{D}_L$ to zero for each fixed t. The computation is straightforward. Thus, $\psi \in \underset{\sim}{D}_L(B)$.

Furthermore, let $K_n = (-\infty, n]$, and let n and q be fixed positive integers. Then, for all $\varphi \in \underset{\sim}{D}_{K_n}$ (i.e., for all $\varphi \in \underset{\sim}{E}$ with supp $\varphi \subset K_n$) and for all $t \in [-q, \infty]$, the intersections of supp $v(\tau)$ with the supports of $\varphi(t+\tau)$ considered as functions of τ are all contained in a fixed compact interval, say, J. So, choose $\lambda \in \underset{\sim}{D}$ such that $\lambda(\tau) \equiv 1$ on a neighborhood of J. Then, since v is also a member of $\underset{\sim}{D}'(B)$, we may invoke proposition 2-1 to write

$$\sup_{-q < t < \infty} \|\psi^{(k)}(t)\|_B \le M \max_{0 \le \mu \le r} \sup_{\substack{\tau \in \text{supp}\,\lambda \\ -q < t < \infty}} |D^{\mu}_{\tau}\,\lambda(\tau)\,\varphi^{(k)}(t+\tau)|$$

This implies that $\varphi \mapsto \psi$ is a continuous linear mapping of $\underset{\sim}{D}_L$ into $\underset{\sim}{D}_L(B)$, and therefore condition E2 is verified.

To prove condition E3, let $v_\nu \to 0$ strongly in $\underset{\sim}{D}'_R(B)$ as $\nu \to \infty$, let $\varphi \in \underset{\sim}{D}_L$, and set $\psi_\nu(t) \overset{\Delta}{=} \langle v_\nu(\tau), \varphi(t+\tau)\rangle$. By our assumption, $\|\langle v_\nu, \theta\rangle\|_B \to 0$ uniformly for all θ in any bounded set in $\underset{\sim}{D}_L$. Given any q, as t traverses $[-q, \infty)$ and φ traverses a bounded set Ω in $\underset{\sim}{D}_L$, $\varphi^{(k)}(t+\tau)$ as a function of τ traverses another bounded set in $\underset{\sim}{D}_L$. So, the right-hand side of

$$\sup_{-q < t < \infty} \|\psi_\nu^{(k)}(t)\|_B = \sup_{-q < t < \infty} \|\langle v_\nu(\tau), \varphi^{(k)}(t+\tau)\rangle\|_B$$

converges to zero uniformly for all $\varphi \in \Omega$. This verifies condition E3.

Turning to condition E4, we have $y \in \underset{\sim}{D}'_R[L(B,A)] \subset \underset{\sim}{D}'[L(B,A)]$. Given any n, set $K_n = (-\infty, n]$. Then, for all $\psi \in \underset{\sim}{D}_{K_n}(B) = \underset{\sim}{H}_n(B)$, supp y $\cap$ supp ψ is contained in a fixed compact interval, say, J. Choose $\lambda \in \underset{\sim}{D}$ such that $\lambda(t) \equiv 1$ on J. Then, invoking proposition 2-2, we obtain

$$\|\langle y, \psi\rangle\|_A \le M \max_{0 \le \mu \le r} \sup_{t \in \text{supp}\,\lambda} \|D^\mu \lambda(t)\, \psi(t)\|_B$$

This in turn implies that condition E4 is satisfied.

Case IV: The argument here is the same as that for case III. We merely interchange "left-sided" for "right-sided".

Case V: Condition E1 is again satisfied since $\underset{\sim}{L}(w,z)$ is closed under translation.

That condition E2 is satisfied can be shown by modifying the proofs of lemmas 3.7-1, 3.7-2, and 3.7-3 of Zemanian [2].

Consider now condition E3. That $v_\nu \to 0$ strongly in $\underset{\sim}{L}'(w,z)(B)$ as $\nu \to \infty$ means that $\langle v_\nu, \theta\rangle$ converges in B to zero uniformly for all θ in any bounded set in $\underset{\sim}{L}(w,z)$. Furthermore, Ω is a bounded set in $\underset{\sim}{L}(w,z)$ if and only if there exist an $a > w$ and a $b < z$ such that, for each positive integer m, the values:

$$\gamma_{a,b,m}(\varphi) \overset{\Delta}{=} \sup_{\tau} |\kappa_{a,b}(\tau)\, \varphi^{(m)}(\tau)|$$

are finite and comprise a bounded set as φ traverses Ω. Now, for each fixed k, the functions $\kappa_{a,b}(t)\,\varphi^{(k)}(t+\tau)$ as functions of τ comprise a bounded set in $\underset{\sim}{L}(v,z)$ as t traverses R and φ traverses Ω. Indeed,

$$\kappa_{a,b}(\tau)\; D_\tau^m\,[\kappa_{a,b}(t)\;\varphi^{(k)}(t+\tau)] = \frac{\kappa_{a,b}(t)\kappa_{a,b}(\tau)}{\kappa_{a,b}(t+\tau)}\;\kappa_{a,b}(t+\tau)\;\varphi^{(k+m)}(t+\tau)$$

and $\kappa_{a,b}(t)\;\kappa_{a,b}(\tau)/\kappa_{a,b}(t+\tau)$ is bounded on the (t, τ) plane. (See Zemanian [2], Sec. 3.7). Thus,

$$\sup_t \|\kappa_{a,b}(t)\;\psi_\nu^{(k)}(t)\|_B = \sup_t \|\,\langle v_\nu(\tau),\; \kappa_{a,b}(t)\;\varphi^{(k)}(t+\tau)\rangle\|_B \to 0$$

$$\nu \to \infty$$

uniformly for all $\varphi \in \Omega$. This shows that condition E3 is satisfied.

For condition E4, we need merely invoke proposition 2-2 since $\rho_{n,o}$ is a norm on $\underset{\sim}{L}_{a,b}(B) = \underset{\sim}{H}_n(B)$.

We have so far shown that the systems of spaces indicated in the first five columns of table II satisfy the hypothesis of proposition 2-4 so that the convolution process defined herein can be applied in each case. We shall now show that this convolution process can be applied when we have the system of spaces indicated in the sixth column of table II, even though $\underset{\sim}{D}_{L_1}(B)$ and $\underset{\sim}{\dot{B}}$ are not ρ-type spaces.

We first state what these spaces are; our notation follows that used by Schwartz (see Schwartz [1], vol. II, pp. 55-56). $\underset{\sim}{D}_{L_1}(B)$ is the space of smooth function φ from R into B such that, for each nonnegative integer k,

$$\rho_p(\varphi) = \max_{0 \le k \le p} \int_{-\infty}^{\infty} \|\varphi^{(k)}(t)\|_B\, dt \qquad < \infty$$

$\underset{\sim}{D}_{L_1}(B)$ is supplied the topology generated by the multinorm $\{\rho_k\}_{k=o}^{\infty}$. $\underset{\sim}{D}_{L_1}(B)$ is complete and in fact a Fréchet space. Also, ρ_o is a norm on $\underset{\sim}{D}_{L_1}(B)$. Both the shifting operator σ_x: $\varphi(t) \mapsto \varphi(t+\tau)$ and differentiation are continuous linear mappings of $\underset{\sim}{D}_{L_1}(B)$ into $\underset{\sim}{D}_{L_1}(B)$. As before, when $B = C$, we denote this space simply by $\underset{\sim}{D}_{L_1}$.

Next, $\underset{\sim}{B}'(B)$ is the linear space of all continuous linear mappings of $\underset{\sim}{D}_{L_1}$ into B. The elements of $\underset{\sim}{D}_{L_1}$ are the so-called B-valued bounded distributions. We assign to $\underset{\sim}{B}'(B)$ both a weak and a strong topology in the usual way. Moreover, proposition 2-1 possesses the following analogue which is proven in just the same way.

Proposition 2-5: *Let $f \in \underset{\sim}{B}'(B)$. Then, there exists a positive constant M and a nonnegative integer r such that*

$$\|\langle f, \varphi\rangle\|_B \leq M \rho_r(\varphi)$$

for all $\varphi \in \underset{\sim}{D}_{L_1}$. Both M and r depend on f.

On the other hand, for L(B,A) defined as before, $\underset{\sim}{B}'[L(B,A)]$ is the linear space of all continuous linear mappings of $\underset{\sim}{D}_{L_1}(B)$ into A. We will not need a topology for this space. The analogue to proposition 2-2 reads as follows.

Proposition 2-6: *Let $y \in \underset{\sim}{B}'[L(B,A)]$. Then, there exists a positive constant M and a nonnegative integer r such that*

$$\|\langle y, \psi\rangle\|_A \leq M \rho_r(\psi)$$

for every $\psi \in \underset{\sim}{D}_{L_1}(B)$. M and r depend on y.

When $\underset{\sim}{H}$ denotes $\underset{\sim}{D}_{L_1}$ and $\underset{\sim}{H}'$ denotes $\underset{\sim}{B}' = B'(C)$, the symbols $\underset{\sim}{D}_{L_1} \otimes B$ and $\underset{\sim}{B}' \otimes B$ have the same meanings as before. Moreover, for $f \in \underset{\sim}{B}'(B)$ and $a' \in B'$, $[f, a']$ is also defined as before and is a member of $\underset{\sim}{B}'$, and similarly for (f, a') when B is a Hilbert space H.

$\underset{\sim}{D}(B)$ is a dense subspace of $\underset{\sim}{D}_{L_1}(B)$ (see for example, lemma 8-1) and the topology of $\underset{\sim}{D}(B)$ is stronger then the topology induced on $\underset{\sim}{D}(B)$ by $\underset{\sim}{D}_{L_1}(B)$. Consequently, $\underset{\sim}{B}'(B)$ is a subspace of $\underset{\sim}{D}'(B)$, and $\underset{\sim}{B}'[L(B,A)]$ is a subspace of $\underset{\sim}{D}'[L(B,A)]$.

Finally, B(B) is the space of smooth functions φ from R into B such that φ and each of its derivatives are bounded in B on R. $\underset{\sim}{B}(B)$ is supplied the topology generated by the multinorm $\{\gamma_k\}_{k=0}^{\infty}$ where

$$\gamma_k(\varphi) \triangleq \sup_{-\infty < t < \infty} \|\varphi^{(k)}(t)\|_B$$

$\dot{\underset{\sim}{B}}(B)$ is that subspace of $\underset{\sim}{B}(B)$ such that for $\varphi \in \dot{\underset{\sim}{B}}(B)$ and any nonnegative integer k $\|\varphi^{(k)}(t)\|_B \to 0$ as $|t| \to \infty$. $\dot{\underset{\sim}{B}}(B)$ possesses the topology induced by $\underset{\sim}{B}(B)$. $\underset{\sim L_1}{D'}(B)$ is the linear space of all continuous linear mappings of $\dot{\underset{\sim}{B}}$ into B, and we supply $\underset{\sim L_1}{D'}$ (B) with both a weak and strong topology in the usual way. The members of $\underset{\sim L_1}{D'}$ (B) are the so-called B-valued integrable distributions on R. As before, we drop the argument notation (B) whenever B = C.

Our next objective is to show that the conclusions of proposition 2-4 hold even when we make the identification of spaces indicated in the sixth column of table II. Actually, we need merely verify that the conditions E are satisfied since the proof of theorem 2-4 applies just as well in this case.

That condition E1 is satisfied is again obvious.

Now for condition E2: We first show that, for $v \in \underset{\sim L_1}{D'}(B)$ and $\varphi \in \underset{\sim L_1}{D}$, $\psi(t)$ is smooth and that

$$\psi^{(k)}(t) = \langle v(\tau), \varphi^{(k)}(t+\tau)\rangle \qquad k = 0, 1, 2, \ldots \tag{2.6}$$

by using an inductive argument. Assume Eq. (2.6) is true for some k; it is true by definition for k = 0. Also, assume that $\Delta t \neq 0$ and that t is fixed, and consider

$$\frac{\psi^{(k)}(t+\Delta t) - \psi^{(k)}(\tau)}{\Delta t} - \langle v(\tau), \varphi^{(k+1)}(t+\tau)\rangle = \langle v(\tau), \theta_{\Delta t}(\tau)\rangle$$

where

$$\theta_{\Delta t}(\tau) = \frac{\varphi^{(k)}(t+\Delta t+\tau) - \varphi^{(k)}(t+\tau)}{\Delta t} - \varphi^{(k+1)}(t+\tau)$$

The smoothness of $\psi(t)$ and the validity of Eq. (2.6) will be established once we show that $\theta_{\Delta t}(\tau)$ converges in $\underset{\sim L_1}{D}$ to zero as $|\Delta t| \to 0$.

Some manipulation shows that, for each nonnegative integer m,

$$\int_{-\infty}^{\infty} |\theta_{\Delta t}^{(m)}(\tau)|\, d\tau = \frac{1}{|\Delta t|} \int_{-\infty}^{\infty} d\tau \left| \int_0^{\Delta t} d\xi \int_0^{\xi} \varphi^{(k+m+2)}(t+\tau+\zeta)\, d\zeta \right|$$

By Fubini's theorem, we may change the order of integration. Hence

$$\int_{-\infty}^{\infty} |\theta_{\Delta t}^{(m)}(\tau)|\, d\tau \leq \frac{1}{|\Delta t|} \int_{0}^{|\Delta t|} d\xi \int_{0}^{\xi} d\zeta \int_{-\infty}^{\infty} | \varphi^{(k+m+2)}(t+\tau+\zeta| d\tau$$

$$\leq \frac{|\Delta t|}{2} \int_{-\infty}^{\infty} | \varphi^{(k+m+2)}(\tau)|\, d\tau$$

The right-hand side tends to zero as $|\Delta t| \to 0$, which proves the smoothness of $\psi(t)$ and the validity of Eq. (2.6).

Next, we prove that $\psi \in \underset{\sim}{D}_{L_1}(B)$. This will be accomplished when we show that, for each nonnegative integer k,

$$\int_{-\infty}^{\infty} \|\psi^{(k)}(t)\|_B \, dt \qquad < \infty$$

Since $v \in \underset{\sim}{D}'_{L_1}(B)$, for every $\varphi \in \underset{\sim}{D}_{L_1}$

$$\psi^{(k)}(t) = \langle v(\tau), \varphi^{(k)}(t+\tau)\rangle = \sum_{\mu}^{*} \int_{-\infty}^{\infty} h_{\mu}(\tau)\, \varphi^{(k+\mu)}(t+\tau) d\tau$$

where the μ are nonnegative integers, Σ^{*} denotes a finite sum, and the $h_{\mu}(\tau)$ are B-valued Lebesque-integrable functions on R (i.e., $h_{\nu} \in L_1$ (B)). (See Schwartz [1], vol. II, p. 57). By a standard estimate (see Williamson [1], p. 65),

$$\int_{-\infty}^{\infty} \|\psi^{(k)}(t)\|_B \, dt \leq \sum_{\mu}^{*} \int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} \|h_{\mu}(\tau)\| \, |\varphi^{(k+\mu)}(t+\tau)|\, d\tau$$

$$\leq \sum_{\mu}^{*} \int_{-\infty}^{\infty} \|h_{\mu}(\tau)\| d\tau \int_{-\infty}^{\infty} |\varphi^{(k+\mu)}(t)|\, dt \qquad < \infty \tag{2.7}$$

which is what we wished to show. (Although the references cited here discuss only C-valued functions and distributions, their arguments carry over to B-valued functions and distributions.)

Finally, $\varphi \mapsto \psi$ is clearly a linear mapping of $\underset{\sim}{D}_{L_1}$ into $\underset{\sim}{D}_{L_1}(B)$, and the estimate (2.7) shows that it is also continuous. The completes our verification of condition E2.

We now turn to condition E3. That $v_\nu \to 0$ strongly in $\underset{\sim}{D}'_{L_1}(B)$ means the following. For each μ in a finite set of nonnegative integers, there exists a sequence $\{h_{\mu,\nu}\}_{\nu=1}^{\infty}$ of continuous B-valued functions which converges in $L_1(B)$ to zero (i.e., $\int_{-\infty}^{\infty} \|h_{\mu,\nu}(t)\|_B \, dt \to 0$ as $\nu \to \infty$), and, in addition,

$$(2.8) \qquad \langle v_\nu, \varphi \rangle = \sum_\mu^{*} \int_{-\infty}^{\infty} h_{\mu,\nu}(\tau)\, \varphi^{(\mu)}(\tau)\, d\tau$$

for every φ in $\dot{B}$. (See Schwartz, vol. II, p. 58.) Now, if φ is an arbitrary member in a bounded subset Ω of $\underset{\sim}{D}_{L_1}$, we have that, for each nonnegative integer k,

$$\int_{-\infty}^{\infty} |\varphi^{(k)}(t)|\, dt < K_k$$

where the constants K_k do not depend on $\varphi \in \Omega$. Hence, by (2.8) and the estimate indicated in (2.7),

$$\int_{-\infty}^{\infty} \|\psi_\nu{}^{(k)}(t)\|_B dt \le \sum_\mu^{*} \int_{-\infty}^{\infty} \|h_{\mu,\nu}(\tau)\|_B \, d\tau \int_{-\infty}^{\infty} |\varphi^{(k+\mu)}(t)|\, dt$$

$$\le \sum_\mu^{*} K_{k+\mu} \int_{-\infty}^{\infty} \|h_{\mu,\nu}(\tau)\|_B \, d\tau \to 0 \qquad \nu \to \infty$$

We have established that $\{\psi_\nu\}$ converges in $\underset{\sim}{D}_{L_1}(B)$ to zero uniformly for all φ in any bounded subset of $\underset{\sim}{D}_{L_1}$.

Finally, condition E4 is satisfied by virtue of proposition 2-6.

We have hereby established the following analogue to proposition 2-4.

<u>Proposition 2-7:</u> <u>For any</u> $y \in \underset{\sim}{B}'[L(B,A)]$ <u>and any</u> $v \in \underset{\sim}{D}'_{L_1}(B)$, <u>the convolution product</u> $y * v$ <u>can be defined by</u> (2.3)(<u>with</u> $\underset{\sim}{K} = \underset{\sim}{D}_{L_1}$) <u>as a mapping from</u> $\underset{\sim}{D}_{L_1}$ <u>into</u> A. <u>Moreover,</u> $v \to y * v$ <u>is a linear mapping of</u> $\underset{\sim}{D}'_{L_1}(B)$ <u>into</u> $\underset{\sim}{B}'(A)$ <u>that is also continuous in the following sense: If</u> $\{v_\nu\}_{\nu=1}^{\infty}$ <u>converges to</u> v <u>strongly in</u> $\underset{\sim}{D}'_{L_1}(B)$, <u>then,</u> $\{y * v_\nu\}_{\nu=1}^{\infty}$ <u>converges to</u> $y * v$ <u>strongly in</u> $\underset{\sim}{B}'(A)$.

Another result we shall employ is a regularization process corresponding to the convolution in proposition 2-7. More specifically, we shall need

Proposition 2-8: If $y \in \underset{\sim}{B}'[L(B,A)]$ and $v \in \underset{\sim}{D}_{L_1}(B)$, and if the convolution product $y * v$ is defined by (2.3) for $\underset{\sim}{K} = \underset{\sim}{D}_{L_1}$, then

$$(y * v)(\tau) = \langle y(t),\ v(\tau - t)\rangle \tag{2.9}$$

in the sense of equality in $\underset{\sim}{B}'(A)$. Moreover, $v \mapsto y * v$ is a continuous linear mapping of $\underset{\sim}{D}_{L_1}(B)$ into $\underset{\sim}{B}(A)$.

Proof: The left-hand side of (2.9) has a sense as a convolution in accordance with proposition 2-7 because $\underset{\sim}{D}_{L_1}(B) \subset \underset{\sim}{D}'_{L_1}(B)$. Next, let $\varphi \in \underset{\sim}{D}_{L_1}$. The equality in (2.9) is established through the following manipulations:

$$\begin{aligned}
\langle y * v,\ \varphi\rangle &= \langle y(t),\ \langle v(\tau),\ \varphi(t+\tau)\rangle\ \rangle \\
&= \langle y(t),\ \int_{-\infty}^{\infty} v(\tau)\ \varphi(t+\tau)d\tau\rangle \\
&= \langle y(t),\ \int_{-\infty}^{\infty} v(\tau - t)\ \varphi(\tau)d\tau\rangle && (2.10)\\
&= \int_{-\infty}^{\infty} \langle y(t),\ v(\tau-t)\rangle\varphi(\tau)d\tau && (2.11)\\
&= \langle\langle y(t),\ v(\tau-t)\rangle,\ \varphi(\tau)\rangle
\end{aligned}$$

Note that $\langle v(\tau),\ \varphi(t+\tau)\rangle$ is a Bochner integral and is in fact a member of $\underset{\sim}{D}_{L_1}(B)$. To see this, we need merely refer to the arguments culminating in (2.7). Thus, the equalities down to (2.10) in the above manipulations are valid.

Moreover, the last two expressions in the above manipulations have a sense and are equal because $\langle y(t),\ v(\tau-t)\rangle$ is a member of $\underset{\sim}{B}(A)$, the space of all A-valued bounded smooth functions on R each of whose derivatives are also bounded on R. Indeed, we can show that $\langle y(t),\ v(\tau-t)\rangle$ is smooth in the usual way [see the argument following (2.6)]. That it is a member of $\underset{\sim}{B}(A)$ follows from proposition 2-6 and the following:

$$\begin{aligned}
\|D_\tau^k\ \langle y(t),\ v(\tau-t)\rangle\|_A &= \|\langle y(t),\ v^{(k)}(\tau-t)\rangle\|_A \\
&\le M \max_{0 \le k \le r} \int_{-\infty}^{\infty} \|v^{(k)}(\tau-t)\|_B\ dt && (2.12)\\
&= M \max_{0 \le k \le r} \int_{-\infty}^{\infty} \|v^{(k)}(t)\|_B\ dt \quad < \infty.
\end{aligned}$$

Thus, to establish (2.9) we have only to prove that (2.10) is equal to (2.11). To do this we shall replace the integrals on τ by certain Riemann sums and then will show that the resulting approximations are equal to each other and converge to (2.10) and (2.11).

Let T be (for the moment, an unspecified) positive number. Partition the interval [-T, T] into 2N subintervals of length T/N. Also, let τ_n be a point in the nth subinterval. Since $v \in \underset{\sim}{D}_{L_1}(B)$ and $\varphi \in \underset{\sim}{D}_{L_1}$, we can approximate

$$\int_{-\infty}^{\infty} v(\tau-t)\ \varphi(\tau)\ d\tau \tag{2.13}$$

by

$$\frac{T}{N}\sum_{n=1}^{2N} v(\tau_n-t)\ \varphi(\tau_n) \tag{2.14}$$

Upon applying $y \in \underset{\sim}{B}'[L(B,A)]$ to (2.14), which is a member of $\underset{\sim}{D}_{L_1}(B)$, we obtain the following member of A.

$$\langle y(t), \frac{T}{N}\sum_{n=1}^{2N} v(\tau_n-t)\ \varphi(\tau_n)\rangle = \frac{T}{N}\sum_{n=1}^{2N} \langle y(t), v(\tau_n-t)\rangle\ \varphi(\tau_n) \tag{2.15}$$

We have already noted that $\langle y(t), v(\tau-t)\rangle$ is a member of $\underset{\sim}{B}(A)$, and it therefore follows that, as $N \to \infty$ and then $T \to \infty$, the right-hand side of (2.15) converges to the Bochner integral (2.11). On the other hand, we will have proven that the left-hand side of (2.15) converges to (2.10) as first $N \to \infty$ and then $T \to \infty$ when we prove that

$$A_{N,T}(t) \triangleq \frac{T}{N}\sum_{n=1}^{2N} v(\tau_n-t)\ \varphi(\tau_n) - \int_{-\infty}^{\infty} v(\tau-t)\ \varphi(\tau)\ d\tau.$$

converges in $\underset{\sim}{D}_{L_1}(B)$ to zero.

We first note that (2.13) can be differentiated with respect to t under the integral sign any number of times. Next, for any fixed nonnegative integer k,

$$B_T \overset{\Delta}{=} \int_{-\infty}^{\infty} \left\| \left(\int_{-\infty}^{-T} + \int_{T}^{\infty} \right) v^{(k)}(\tau - t)\, \varphi(\tau)\, d\tau \right\|_B dt$$

$$\leq \int_{-\infty}^{\infty} dt \left(\int_{-\infty}^{-T} + \int_{T}^{\infty} \right) \| v^{(k)}(\tau - t)\|_B \, |\varphi(\tau)| \, d\tau$$

$$\leq \int_{-\infty}^{\infty} \| v^{(k)}(\tau) \|_B \, d\tau \left(\int_{-\infty}^{-T} + \int_{T}^{\infty} \right) |\varphi(\tau)| \, d\tau$$

Given any $\epsilon > 0$, the right-hand side can be made less than $\epsilon/2$ by choosing T large enough. Fix T this way.

Next, using the aforementioned partition of $[-T, T]$ we may write

$$\int_{-\infty}^{\infty} \| A_{N,T}^{(k)}(t) \|_B dt \leq B_T + \int_{-\infty}^{\infty} \left\| \frac{T}{N} \sum_{n=1}^{2N} v^{(k)}(\tau_n - t)\, \varphi(\tau_n) - \int_{-T}^{T} v^{(k)}(\tau - t)\, \varphi(\tau) d\tau \right\|_B dt$$

$$< \frac{\epsilon}{2} + D_{N,T,X} + E_{N,T,X}$$

where

$$D_{N,T,X} = \frac{T}{N} \sum_{n=1}^{2N} |\varphi(\tau_n)| \left(\int_{-\infty}^{-X} + \int_{X}^{\infty} \right) \| v^{(k)}(\tau_n - t) \|_B dt$$

$$+ \int_{-T}^{T} |\varphi(\tau)| \left(\int_{-\infty}^{-X} + \int_{X}^{\infty} \right) \| v^{(k)}(\tau - t) \|_B dt \, d\tau$$

and

$$E_{N,T,X} = \int_{-X}^{X} dt \left\| \frac{T}{N} \sum_{n=1}^{2N} v^{(k)}(\tau_n - t)\, \varphi(\tau_n) - \int_{-T}^{T} v^{(k)}(\tau - t)\, \varphi(\tau) d\tau \right\|_B$$

(We have used Fubini's theorem to change the order of integration in the last term of $D_{N,T}$.) Now,

$$\left(\int_{-\infty}^{-X} + \int_{X}^{\infty} \right) \| v^{(k)}(\tau - t) \|_B \, dt$$

can be made as small as desired uniformly for all $|\tau| \leq T$ by choosing X large enough. Moreover,

$$\frac{T}{N} \sum_{n=1}^{2N} |\varphi(\tau_n)| \to \int_{-T}^{T} |\varphi(\tau)| \, d\tau$$

as $N \to \infty$. It follows that $D_{N,T}$ can be made less than $\epsilon/4$ uniformly for all $N \geq 1$ by choosing X large enough. Fix X this way.

Finally, $E_{N,T,X}$ becomes less than $\epsilon/4$ for all sufficiently large N because the integrand of the integral on t tends to zero uniformly on $-X < t < X$ as $N \to \infty$ by virtue of the fact that $v^{(k)}(\tau - t)$ and $\varphi(\tau)$ are smooth on the (t, τ) plane. This completes the proof of (2.9).

The last statement of proposition 2-8 is now implied by the inequality (2.12). Q.E.D.

We also have a regularization for column I of table II. We shall need it in section 4.

Proposition 2-9: If $y \in \underset{\sim}{D}'[L(B,A)]$ and $v \in \underset{\sim}{D}(B)$, and if the convolution product $y * v$ is defined by (2.3) for $\underset{\sim}{K} = \underset{\sim}{D}$, then

$$(y * v)\ (\tau) = \langle y(t),\ v(\tau - t)\rangle$$

in the sense of equality in $\underset{\sim}{D}'(A)$. Moreover, $v \mapsto y * v$ is a continuous linear mapping of $\underset{\sim}{D}(B)$ into $\underset{\sim}{E}(A)$.

The proof of this is just like that of proposition 2-8 and in fact simpler. We therefore omit it.

We shall need still another property of our convolution process. Let x be any real number, and let σ_x be the shifting operator. That is, for any function $\varphi(t)$ from R into B, $\sigma_x\, \varphi(t) \triangleq \varphi(t+x)$. It is a fact that σ_x is a continuous linear operator mapping $\underset{\sim}{H}(B)$ into $\underset{\sim}{H}(B)$, where $\underset{\sim}{H}(B)$ denotes either $\underset{\sim}{D}_{L_1}(B)$, $\underset{\sim}{B}(B)$, $\dot{\underset{\sim}{B}}(B)$, or anyone of the testing-function spaces indicated in the first column of table I. This is readily seen from the definitions of these spaces. In these cases we define σ_x on $\underset{\sim}{H}'(B)$ by

$$\langle \sigma_x\, f,\ \varphi\rangle = \langle f,\ \sigma_{-x}\, \varphi\rangle \qquad f \in \underset{\sim}{H}'(B),\ \varphi \in \underset{\sim}{H},$$

and it follows that σ_x is a continuous linear mapping of $\underset{\sim}{H}'(B)$ into $\underset{\sim}{H}'(B)$.

We can now prove

Proposition 2-10: If, for any of the systems of spaces indicated in table II, $y \in \underset{\sim}{H}'[L(B,A)]$ and $v \in \underset{\sim}{I}'(B)$, then $\sigma_x(y * v) = y * (\sigma_x v)$; in other words, σ_x commutes with the operator $v \mapsto y * v$.

Proof: For $\varphi \in \underset{\sim}{H}$,

$$\langle \sigma_x(y * v), \varphi \rangle = \langle y * v, \sigma_{-x} \varphi \rangle = \langle y(t), \langle v(\tau), \sigma_{-x}\varphi(t + \tau) \rangle \rangle$$

$$= \langle y(t), \langle \sigma_x v(\tau), \varphi(t+\tau) \rangle \rangle = \langle y * (\sigma_x v), \varphi \rangle$$

Q.E.D.

Later on, we will identify t as the time variable. In such a case, we will refer to the property that σ_x commutes with the operator $v \mapsto y * v$ by saying that the operator $v \mapsto y * v$ is time-invariant.

3. The Laplace Transformation

We shall say that f is a Laplace-transformable member of $\underset{\sim}{D}'(B)$ if there exists two members σ_1 and σ_2 of the extended real line $[-\infty, \infty]$ such that $\sigma_1 < \sigma_2$, $f \in \underset{\sim}{L}'(\sigma_1, \sigma_2)(B)$, and in addition, $f \notin \underset{\sim}{L}'(w,z)(B)$, if either $w < \sigma_1$ or $z > \sigma_2$. The strip in the complex s plane:

$$\Omega_f = \{s: \sigma_1 < \text{Re } s < \sigma_2\}$$

will be called the strip of definition for the Laplace transform $\mathfrak{L}f$ of f. Noting that $e^{-st} \in \underset{\sim}{L}(\sigma_1, \sigma_2)$ for $s \in \Omega_f$, we define $\mathfrak{L}f$ by

$$F(s) \triangleq (\mathfrak{L}f)(s) \triangleq \langle f(t), e^{-st}\rangle \qquad s \in \Omega_f$$

It is easily shown that F(s) is a B-valued analytic function on Ω_f. (One need merely modify slightly the proof of theorem 3.3-1 of Zemanian [2].)

Similarly, if there exist two members η_1 and η_2 of the extended real line $[-\infty, \infty]$ such that $\eta_1 < \eta_2$ and if $y \in \underset{\sim}{L}'(\eta_1, \eta_2)[L(B,A)]$ and $y \notin \underset{\sim}{L}'(w,z)[L(B,A)]$ whenever either $w < \eta_1$ or $z > \eta_2$, then y will be called a Laplace-transformable member of $\underset{\sim}{D}'[L(B,A)]$. Also, corresponding to y, we have the unique mapping f_y of $\underset{\sim}{L}(\eta_1, \eta_2)$ into L(B,A) specified in proposition 2-3. We define the Laplace transform $\mathfrak{L}y$ of y as simply the Laplace transform $\mathfrak{L}f_y$ of f_y, and we take $\Omega_y \triangleq \{s: \eta_1 < \text{Re } s < \eta_2\}$ as the corresponding strip of definition. Thus,

$$Y(s) \triangleq (\mathfrak{L}y)(s) \triangleq (\mathfrak{L}f_y)(s) \triangleq \langle f_y(t), e^{-st}\rangle \qquad s \in \Omega_y.$$

In this case, Y(s) is an L(B,A)-valued analytic function on Ω_y. This is proven by modifying the proof of theorem 3.3-1 in Zemanian [2] in an obvious way. Whenever we write "$f \in \underset{\sim}{D}'(B)$ and $\mathfrak{L}f = F(s)$ for $s \in \Omega_f$", it is understood that f is a Laplace-transformable member of $\underset{\sim}{D}'(B)$ and that Ω_f is the strip of definition for $\mathfrak{L}f$. A similar convention is followed whenever we write "$y \in \underset{\sim}{D}'[L(B,A)]$ and $\mathfrak{L}y = Y(s)$ for $s \in \Omega_y$."

We are now ready to state the exchange formula which is given by (3.1) below. This formula states how convolution is converted by the Laplace transformation.

3. The Laplace Transformation

We shall say that f is a Laplace-transformable member of $\underset{\sim}{D}'(B)$ if there exists two members σ_1 and σ_2 of the extended real line $[-\infty, \infty]$ such that $\sigma_1 < \sigma_2$, $f \in \underset{\sim}{L}'(\sigma_1, \sigma_2)(B)$, and in addition, $f \notin \underset{\sim}{L}'(w,z)(B)$, if either $w < \sigma_1$ or $z > \sigma_2$. The strip in the complex s plane:

$$\Omega_f = \{s: \sigma_1 < \text{Re } s < \sigma_2\}$$

will be called the strip of definition for the Laplace transform $\mathfrak{L}f$ of f. Noting that $e^{-st} \in \underset{\sim}{L}(\sigma_1, \sigma_2)$ for $s \in \Omega_f$, we define $\mathfrak{L}f$ by

$$F(s) \triangleq (\mathfrak{L}f)(s) \triangleq \langle f(t), e^{-st} \rangle \qquad s \in \Omega_f$$

It is easily shown that F(s) is a B-valued analytic function on Ω_f. (One need merely modify slightly the proof of theorem 3.3-1 of Zemanian [2].)

Similarly, if there exist two members η_1 and η_2 of the extended real line $[-\infty, \infty]$ such that $\eta_1 < \eta_2$ and if $y \in \underset{\sim}{L}'(\eta_1, \eta_2)\,[L(B,A)]$ and $y \notin \underset{\sim}{L}'(w,z)\,[L(B,A)]$ whenever either $w < \eta_1$ or $z > \eta_2$, then y will be called a Laplace-transformable member of $\underset{\sim}{D}'[L(B,A)]$. Also, corresponding to y, we have the unique mapping f_y of $\underset{\sim}{L}(\eta_1, \eta_2)$ into L(B,A) specified in proposition 2-3. We define the Laplace transform $\mathfrak{L}y$ of y as simply the Laplace transform $\mathfrak{L}f_y$ of f_y, and we take $\Omega_y \triangleq \{s: \eta_1 < \text{Re } s < \eta_2\}$ as the corresponding strip of definition. Thus,

$$Y(s) \triangleq (\mathfrak{L}y)(s) \triangleq (\mathfrak{L}f_y)(s) \triangleq \langle f_y(t), e^{-st} \rangle \qquad s \in \Omega_y.$$

In this case, Y(s) is an L(B,A)-valued analytic function on Ω_y. This is proven by modifying the proof of theorem 3.3-1 in Zemanian [2] in an obvious way. Whenever we write "$f \in \underset{\sim}{D}'(B)$ and $\mathfrak{L}f = F(s)$ for $s \in \Omega_f$", it is understood that f is a Laplace-transformable member of $\underset{\sim}{D}'(B)$ and that Ω_f is the strip of definition for $\mathfrak{L}f$. A similar convention is followed whenever we write "$y \in \underset{\sim}{D}'[L(B,A)]$ and $\mathfrak{L}y = Y(s)$ for $s \in \Omega_y$."

We are now ready to state the exchange formula which is given by (3.1) below. This formula states how convolution is converted by the Laplace transformation.

Proposition 3-1: If $\mathfrak{L}y = Y(s)$ for $s \in \Omega_y$ and $y \in \underset{\sim}{D}'[L(B,A)]$, if $\mathfrak{L}v = V(s)$ for $s \in \Omega_v$ and $v \in \underset{\sim}{D}'(B)$, and if $\Omega_y \cap \Omega_v$ is not empty, then $y * v$ exists in the sense of the convolution of $y \in \underset{\sim}{L}'(w,z)\ [L(B,A)]$ with $v \in \underset{\sim}{L}'(w,z)\ (B)$ where the open interval (w,z) is the intersection of $\Omega_y \cap \Omega_y$ with the real axis. Moreover, for $s \in \Omega_y \cap \Omega_v$,

$$\mathfrak{L}(y * v) = Y(s)\ V(s) \tag{3.1}$$

and for each fixed $s \in \Omega_y \cap \Omega_v$, $Y(s)\ V(s) \in A$.

Proof: That $y * v$ exists in the stated sense follows from proposition 2-4 and the facts that, if $v \in \underset{\sim}{L}'(\sigma_1, \sigma_2)\ (B)$ and $\sigma_1 \leq w < z \leq \sigma_2$, then $v \in \underset{\sim}{L}'(w,z)\ (B)$, and similarly for y. Moreover, for each fixed s such that $w < \operatorname{Re} s < z$

$$\begin{aligned} \mathfrak{L}(y * v) &= \langle y(t), \langle v(\tau), e^{-s(t+\tau)} \rangle\rangle \\ &= \langle y(t), e^{-st} \langle v(\tau), e^{-s\tau} \rangle\rangle \end{aligned}$$

Here, $e^{-st} \in \underset{\sim}{L}(w,z)$ and $\langle v(\tau), e^{-s\tau} \rangle \in B$. Hence,

$$e^{-st} \langle v(\tau), e^{-s\tau} \rangle \in \underset{\sim}{L}(w,z) \otimes B$$

and by proposition 2-3 we get

$$\mathfrak{L}(y * v) = \langle f_y(t), e^{-st} \rangle \langle v(\tau), e^{-s\tau} \rangle = Y(s)\ V(s)$$

Finally, since $Y(s) \in L(B,A)$ and $V(s) \in B$, we have that $Y(s)\ V(s) \in A$ for any fixed $s \in \Omega_y \cap \Omega_v$. Q.E.D.

We now state two lemmas which will be used in the proof of an inversion theorem for the Laplace transformation.

Lemma 3-1: Let $y \in \underset{\sim}{L}'(\sigma_1, \sigma_2)\ [L(B,A)]$, and let f_y be the corresponding member of $\underset{\sim}{L}'(\sigma_1, \sigma_2)\ (L(B,A))$ in accordance with proposition 2-3. Also, let $\varphi \in \underset{\sim}{D}(B)$, $\varphi' \in \underset{\sim}{D}$, $\psi(s) = \int_{-\infty}^{\infty} \varphi(t)\ e^{-st} dt$, and $\psi'(s) = \int_{-\infty}^{\infty} \varphi'(t)\ e^{-st} dt$. Then, for any two fixed real numbers r and σ such that $0 < r < \infty$ and $\sigma_1 < \sigma < \sigma_2$ and for $s = \sigma + i\,\omega$, (3.2) $\int_{-r}^{r} \langle f_y(\tau), e^{-s\tau}\rangle \psi(s)\, d\omega = \langle y(\tau), \int_{-r}^{r} e^{-s\tau} \psi(s)\, d\omega\rangle$

and

(3.3) $\int_{-r}^{r} \langle f_y(\tau), e^{-s\tau}\rangle \psi'(s)\, d\omega = \langle f_y(\tau), \int_{-r}^{r} e^{-s\tau} \psi'(s)\, d\omega\rangle$

Lemma 3-2: *Let a, b, σ, and r be real numbers with $a < \sigma < b$. Also, let $\varphi \in \mathcal{D}(B)$. Then, as $r \to \infty$,*

$$\frac{1}{\pi} \int_{-\infty}^{\infty} \varphi(t+\tau)\, e^{\sigma t} \frac{\sin rt}{t}\, dt$$

converges in $\mathcal{L}_{a,b}(B)$ to $\varphi(\tau)$.

Except for some obvious changes, the proofs of these two lemmas are respectively the same as those of lemmas 3.5-1 and 3.5-2 in Zemanian [2]. The integrand on the left-hand side (3.2) is the result of applying the L(B,A)-valued function $\langle f_y(\tau), e^{-s\tau}\rangle$ to the B-valued function $\psi(s)$. The integral on the left-hand side can be defined from the corresponding Riemann sums in the usual way, and possesses the usual properties (see Sec. 7).

Proposition 3-2 (An Inversion Formula): *If $y \in \mathcal{D}'[L(B,A)]$ and $\mathfrak{L}y = Y(s)$ for $s \in \Omega_y = \{s: \sigma_1 < \text{Re } s < \sigma_2\}$, then, in the sense of weak convergence in $\mathcal{D}'[L(B,A)]$*

(3.4) $$y(t) = \lim_{r \to \infty} \frac{1}{2\pi} \int_{-r}^{r} Y(s)\, e^{st}\, d\omega$$

where $s = \sigma + i\omega$ and σ is a fixed real number satisfying $\sigma_1 < \sigma < \sigma_2$. Similarly, if $f \in \mathcal{D}'(B)$ and $\mathfrak{L}f = F(s)$ for $s \in \Omega_f = \{s: \sigma_1 < \text{Re } s < \sigma_2\}$, then, in the sense of weak convergence in $D'(L(B,A))$,

(3.5) $$f(t) = \lim_{r \to \infty} \frac{1}{2\pi} \int_{-r}^{r} F(s)\, e^{st}\, d\omega$$

where s and σ are restricted as before.

Proof: We prove the first statement only since the proof of the second one is almost the same. Let $\varphi \in D(B)$, and choose a and b such that $\sigma_1 < a < \sigma < b < \sigma_2$. We want to show that

$$\lim_{r \to \infty} \langle \frac{1}{2\pi} \int_{-r}^{r} Y(s)\, e^{st}\, d\omega, \varphi(t)\rangle = \langle y, \varphi\rangle$$

We note that the integral on ω within the left-hand side is an $L(B,A)$-valued continuous function of t. Thus, the passage to the limit here is taking place in the Banach space A. The left-hand side without the limit notation is equal to

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-r}^{r} Y(s)\, e^{st}\, d\omega\, \varphi(t)\, dt$$

and, by a Fubini theorem (Hille and Phillips [1], p. 84), the order of integration may be changed. Then, upon invoking lemma 3-1, we find the last expression to be equal to

$$\langle y(\tau),\ \frac{1}{2\pi} \int_{-r}^{r} e^{-s\tau} \int_{-\infty}^{\infty} \varphi(t)\, e^{st}\, dt\, d\omega \rangle$$

which, by the same Fubini theorem, is the same as

$$\langle y(\tau),\ \frac{1}{\pi} \int_{-\infty}^{\infty} \varphi(t+\tau)\, e^{\sigma t}\, \frac{\sin rt}{t}\, dt \rangle$$

Lemma 3-2 now completes the proof.

Corollary 3-2a (The Uniqueness of the Laplace Transformation): Let y_1 *and* y_2 *be members of* $\mathcal{D}'[L(B,A)]$, *let* $\mathfrak{L} y_\nu = Y_\nu(s)$ *for* $s \in \Omega_{y_\nu}$ *and* $\nu = 1, 2$, *and let* $Y_1(s) = Y_2(s)$ *on* $\Omega_{y_1} \cap \Omega_{y_2}$. *Then,* $y_1 = y_2$ *in the sense of equality in* $\mathcal{L}'(w,z)[L(B,A)]$ *where the interval* (w,z) *is the intersection of* $\Omega_{y_1} \cap \Omega_{y_2}$ *with the real axis. A similar statement holds true when the spaces* $\mathcal{D}'[L(B,A)$ *and* $\mathcal{L}'(w,z)[L(B,A)]$ *are replaced by* $\mathcal{D}'(L(B,A))$ *and* $\mathcal{L}'(w,z)(L(B,A))$ *respectively.*

Proof: That y_1 and y_2 are identical on $\mathcal{D}(B)$ follows from proposition 3-2. Furthermore, $\mathcal{D}(B)$ is dense in $\mathcal{L}(w,z)(B)$, and y_1 and y_2 are both members of $\mathcal{L}'(w,z)[L(B,A)]$. Therefore, y_1 and y_2 are identical on $\mathcal{L}(w,z)(B)$ as well. The second conclusion is proven in the same way.

We can characterize Laplace transforms as follows:

Proposition 3-3: Necessary and sufficient conditions for $Y(s)$ *to be the Laplace transform of a* $y \in \mathcal{D}'[L(B,A)]$ *and for the corresponding strip of definition to be* $\Omega_y = \{s\colon \sigma_1 < \mathrm{Re}\, s < \sigma_2\}$ *are that* $Y(s)$ *be an* $L(B,A)$*-valued analytic function on* Ω_y *and, for each closed substrip* $\{s\colon a \le \mathrm{Re}\, s \le b\}$ *of* Ω $(\sigma_1 < a < b < \sigma_2)$, *there* be a polynomial P such that $\|Y(s)\|_{L(B,A)} \leq P(|s|)$ for $a \leq \text{Re } s \leq b$.

Note: It is a fact that P depends in general on the choices of a and b.

Proof: Necessity: It has already been pointed out that Y(s) is an L(B,A)-valued analytic function on Ω_y. Moreover,

$$\mathfrak{L}y = \langle f_y(\tau), e^{-st} \rangle \qquad s \in \Omega_y$$

where $f_y \in \underset{\sim}{L}_{a,b}(L(B,A))$ for every a and b such that $\sigma_1 < a < b < \sigma_2$. Therefore, by proposition 2-1, there exist a positive number M and a nonnegative integer r such that, for all s in the strip $\{s: a \leq \text{Re } s \leq b\}$,

$$\|Y(s)\|_{L(B,A)} = \|\langle f_y(t), e^{-st}\rangle\|_{L(B,A)} \leq M \max_{0 \leq k \leq r} \sup_t |\kappa_{a,b}(t)\, D_t^k\, e^{-st}|$$

$$= M \max_{0 \leq k \leq r} |s|^{(k)} \sup_t |\kappa_{a,b}(t)\, e^{-st}| \leq P(|s|)$$

Sufficiency: We first prove

Lemma 3-3: If, on the strip $\{s: a < \text{Re } s < b\}$, G(s) is an L(B,A)-valued analytic function that satisfies $\|G(s)\|_{L(B,A)} \leq K/|s|^2$, where K is a constant, and if

(3.6) $$g(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} G(s)\, e^{st}\, d\omega$$

where $s = \sigma + i\,\omega$ and σ is fixed with $a < \sigma < b$, then g(t) is an L(B,A)-valued continuous function that does not depend on the choice of σ, and also g(t) generates a member g of $\underset{\sim}{L}'(a,b)$ (L(B,A)) through the definition:

$$\langle g, \varphi \rangle = \int_{-\infty}^{\infty} g(t)\, \varphi(t)\, dt \qquad \varphi \in \underset{\sim}{L}(a,b)$$

Moreover, $\mathfrak{L}g = G(s)$ for at least $a < \text{Re } s < b$.

Proof of lemma 3-3: That g(t) does not depend on σ follows from Cauchy's theorem (Hille and Phillips [1], p. 95). Also,

$$e^{-\sigma t}\, g(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} G(\sigma + i\,\omega)\, e^{i\omega t}\, d\omega$$

is a continuous function by the uniform convergence of the infinite integral. Furthermore, for at least $a < \text{Re } s < b$, we have that

$$G(s) = \int_{-\infty}^{\infty} g(t)\ e^{-st}\ dt$$

This can be established by extending the proof of theorem 8.2-3 of Zemanian [1] to the case of two-sided L(B,A)-valued functions in accordance with the discussion of Secs. 6.2 and 6.3 of Hille and Phillips [1]. Finally, for each σ such that $a < \sigma < b$, $\|e^{-\sigma t}\ g(t)\|_{L(B,A)}$ is bounded on $-\infty < t < \infty$. Therefore, $g(t) \in \underset{\sim}{L}'(a,b)\ (L(B,A))$, and

$$\mathfrak{L}g = \langle g(t),\ e^{-st}\rangle = \int_{-\infty}^{\infty} g(t)\ e^{-st}\ dt = G(s)$$

for at least $a < \text{Re } s < b$.

Turning now to the proof of sufficiency, we choose a numerically-valued polynomial $Q(s)$ that is different from zero on the given strip $\{s: a \leq \text{Re } s \leq b\}$ and satisfies

$$\left\|\frac{Y(s)}{Q(s)}\right\|_{L(B,A)} \leq \frac{k}{|s|^2} \qquad a \leq \text{Re } s \leq b.$$

Set $G(s) = Y(s)/Q(s)$. Then $g(t)$, as determined by (3.6), possesses the properties stated in the conclusion of lemma 3.3. In the sense of differentiation D in the space $\underset{\sim}{L}'(a,b)\ (L(B,A))$, we set $f_y(t) = Q(D)\ g(t)$, so that, for all $\varphi \in \underset{\sim}{L}_{c,d}$ where $\sigma_1 < a < c < d < b < \sigma_2$, we have

(3.7) $$\langle f_y,\ \varphi\rangle = \int_{-\infty}^{\infty} g(t)\ Q(-D)\ \varphi(t)\ dt \in L(B,A)$$

Furthermore, we define y on $\underset{\sim}{L}_{c,d}(B)$ by

(3.8) $$\langle y,\ \psi\rangle \overset{\Delta}{=} \int_{-\infty}^{\infty} g(t)\ Q(-D)\ \psi(t)\ dt \ \in A \qquad \psi \in \underset{\sim}{L}_{c,d}(B)$$

For each t, the last integrand is a member of A. Also, the integral converges; indee

$$\|\langle y,\ \psi\rangle\|_A \leq \int_{-\infty}^{\infty} \left\|\frac{g(t)}{\varkappa_{c,d}(t)}\right\|_{L(B,A)} \|\varkappa_{c,d}(t)\ Q(-D)\ \psi(t)\|_B\ dt$$

$$\leq \int_{-\infty}^{\infty} \left\|\frac{g(t)}{\varkappa_{c,d}(t)}\right\|_{L(B,A)} dt\ \sup_t \|\varkappa_{c,d}(t)\ Q(-D)\ \psi(t)\|_B$$

This also shows that $y \in \underset{\sim}{L}'_{c,d}$ [L(B,A)].

Now, by the standard formula:

$$(\mathfrak{L}Df)\ (s) = s(\mathfrak{L}f)\ (s) \qquad s \in \Omega_f.$$

we have that, for at least $c < \text{Re}\ s < d$,

$$(\mathfrak{L}y)\ (s) = \langle f_y(t),\ e^{-st} \rangle = Q(s)\ G(s) \doteq Y(s)$$

since y and f_y correspond in accordance with proposition 2-3. Since a and c can be chosen arbitrarily close to σ_1, and b and d arbitrarily close to σ_2, we have that $y \in \underset{\sim}{L}'(\sigma_1, \sigma_2)$ [L(B,A)] and $\mathfrak{L}y = Y(s)$ for $\sigma_1 < \text{Re}\ s < \sigma_2$ by the uniqueness of the Laplace transformation (corollary 3-2a). This completes the proof of proposition 3-3.

Proposition 3-4: For $w < z$, there is a one-to-one correspondence $y \mapsto f_y$ from all of the space $\underset{\sim}{L}'(w,z)$ [L(B,A)] onto the space $\underset{\sim}{L}'(w,z)$ (L(B,A)) defined by the equation:

$$(3.9) \qquad \langle y,\ \varphi a \rangle = \langle f_y,\ \varphi \rangle\ a$$

where $y \in \underset{\sim}{L}'(w,z)$ [L(B,A)], $f \in \underset{\sim}{L}'(w,z)$ (L(B,A)), $\varphi \in \underset{\sim}{L}(w,z)$, and $a \in B$.

Proof: Proposition 2-3 asserts that each $y \in \underset{\sim}{L}'(w,z)$ [L(B,A)] determines through (3.9) a unique $f_y \in \underset{\sim}{L}'(w,z)$ (L(B,A)). Conversely, we saw in the preceding proof that f_y defines through (3.7) and (3.8) a $y \in \underset{\sim}{L}'(w,z)$ [L(B,A)]. We wish to show that y is unique. Set $\psi = \varphi a$, $\varphi \in L(w,z)$, $a \in B$. It follows from (3.9) that each $f_y \in \underset{\sim}{L}'(w,z)$ (L(B,A)) defines y uniquely on $\underset{\sim}{L}(w,z) \otimes B$. Now, $\underset{\sim}{L}(w,z) \otimes B$ contains $\underset{\sim}{D} \otimes B$, and $\underset{\sim}{D} \otimes B$ is a dense subspace of $\underset{\sim}{D}(B)$ (see Schwartz [2], pp. 109-110). Also, $\underset{\sim}{D}(B)$ is dense in $\underset{\sim}{L}(w,z)$ (B). Therefore, $\underset{\sim}{L}(w,z) \otimes B$ is a dense subspace of $\underset{\sim}{L}(w,z)$ (B). Since y is a continuous mapping of $\underset{\sim}{L}(w,z)$ (B) into A, it follows that y is uniquely determined on $\underset{\sim}{L}(w,z)$ (B). That is, two different members y_1 and y_2 of $\underset{\sim}{L}'(w,z)$ [L(B,A)] cannot simultaneously satisfy (3.9) for a given $f_y \in \underset{\sim}{L}'(w,z)$ (L(,B,A)). This ends the proof.

4. Input-Output Systems with Convolution Representations

Almost invariably the input-output systems considered in network theory are finite-dimensional. That is, in the time-domain the input v to a system is assumed to be an m x 1 vector whose components are functions or, more generally, distributions on the time axis, whereas the output u is assumed to be an n x 1 vector having similar components. When such a system possesses certain physical properties, namely, single-valuedness, linearity, time-invariance, and a certain type of continuity, the system possesses a convolution representation (Zemanian [3]:

$$u = y * v \tag{4.1}$$

where y is an n x m matrix whose components are distributions. The rules of matrix multiplication are followed when computing the convolution on the right-hand side of (4.1) from the components of y and v.

This kind of a representation is used not only for lumped networks, but also as an approximation to distributed systems. As an example, consider a resonant cavity to which is connected an input wave guide and an output wave guide. These wave guides carry electromagnetic energy to and from the cavity. Now, at a fixed transverse plane somewhere in the input wave guide, both the transverse electric and magnetic fields are time-dependent elements of a Hilbert space. The same thing is true for the transverse electric and magnetic fields at a fixed transverse plane somewhere in the output wave guide. Then, fixing our attention on, say, just the electric fields, we see that the system acts as an operator mapping a time-dependent element in one Hilbert space into a time-dependent element in another Hilbert space. Furthermore, the inner-product of the transverse electric and magnetic fields at a transverse plane in a wave guide is the electromagnetic power flowing past that plane.

Thus, the input and output variables of such distributed systems are naturally elements of infinite-dimensional Hilbert spaces. Nevertheless, the common approach at least in network theory is to decompose any elements of a Hilbert space into components with respect to some orthonormal basis and then to assume that all components with indices higher than some integer can be neglected. Thus, the infinite-dimensional system is replaced by a finite-dimensional one (see, for example, Carlin [1]). Although this technique may be adequate in most cases, it may be desirable at times to avoid such an approximation. This is the motivation of the present work.

In this section we consider a system whose input v is a member of some subspace of $\underset{\sim}{D}'(B)$ and whose output u is a member of some subspace of $\underset{\sim}{D}'(A)$, where A and B are, in general, different Banach spaces. Our discussion is based on the convolution process developed in the previous sections, and, in particular, we assume that the system $\mathfrak{N}$ has a convolution representation:

$$u = \mathfrak{N} v = y * v$$

where y is a suitably restricted fixed member of $\underset{\sim}{D}'[L(B,A)]$. We shall show that $\mathfrak{N}$ is causal if and only if supp $y \subset [0, \infty]$, which is certainly no surprise.

In sections 5, 6, and 8, we restrict outselves to the special case when the input and output are both members of $\underset{\sim}{D}'(H)$, where H is a Hilbert space, and we take up energy considerations using the inner product (u, v) as the power into $\mathfrak{N}$. In this way we extend the concept of an n-port to an infinite-dimensional system, which we choose to call a "Hilbert port."

Throughout the following, it is understood that $\underset{\sim}{H}'[L(B,A)]$, $\underset{\sim}{I}'(B)$, and $\underset{\sim}{K}'(A)$ are the three spaces in any given column of table II. Let y be a fixed member of $\underset{\sim}{H}'[L(B,A)]$. Also, assume that t, the independent variable of the testing functions, represents time. Thus, $\underset{\sim}{I}'(B)$ and $\underset{\sim}{K}'(A)$ are spaces of Banach-space-valued distributions on the real time axis. Now, we can define a system

$\mathfrak{N} = y\,*$, which maps any $v \in \underset{\sim}{I}'(B)$ into a $u \in \underset{\sim}{K}'(A)$ through the equation:

$$u = \mathfrak{N}v = y * v. \tag{4.1}$$

By virtue of propositions 2-4, 2-8, and 2-10, $\mathfrak{N}$ is a continuous, linear time-invariant mapping of $\underset{\sim}{I}'(B)$ into $\underset{\sim}{K}'(A)$.

Definition 4-1: Let $\underset{\sim}{X}(B)$ be a subset of $\underset{\sim}{I}'(B)$, which may in fact be all of $\underset{\sim}{I}'(B)$. $\mathfrak{N}$ is said to be causal on $\underset{\sim}{X}(B)$ if, whenever $v_1 \in \underset{\sim}{X}(B)$, $v_2 \in \underset{\sim}{X}(B)$, and $v_1(t) = v_2(t)$ for $-\infty < t < t_0$, we also have that $\mathfrak{N}v_1 = \mathfrak{N}v_2$ for $-\infty < t < t_0$ and if this condition holds for every t_0.

Proposition 4-1: Let $\mathfrak{N} = y\,*$, when $y \in \underset{\sim}{H}'[L(B,A)]$. If $\mathfrak{N}$ is causal on $\underset{\sim}{D}(B)$, then supp $y \subset [o, \infty)$. On the other hand, if supp $y \subset [o, \infty)$, then $\mathfrak{N}$ is causal on $\underset{\sim}{I}'(B)$.

Proof: Assume that $\mathfrak{N}$ is causal on $\underset{\sim}{D}(B)$. This statement has a sense since $\underset{\sim}{D}(B)$ is a subspace of every $\underset{\sim}{I}'(B)$ indicated in table II. By virtue of proposition 2-9 and the fact that every $\underset{\sim}{H}'[L(B,A)]$ indicated in table II is a subspace of $\underset{\sim}{D}'[L(B,A)]$, we have that, for all $v \in \underset{\sim}{D}(B)$,

$$(\mathfrak{N}v)\ (t) = (y * v)\ (t) = \langle y(\tau),\ v(t-\tau)\rangle \in \underset{\sim}{E}(A). \tag{4.2}$$

Moreover, by hypothesis, for all $v \in \underset{\sim}{D}(B)$ such that supp $v \subset [t_0, \infty)$, we have that $\langle y(\tau),\ v(t-\tau)\rangle = 0$ for $-\infty < t < t_0$. But, treating $v(t-\tau)$ as a function of τ, we see that supp $v(t-\tau) \subset (-\infty,\ t - t_0)$. Moreover, by choosing $t < t_0$ and v appropriately we can set $v(t-\tau)$ equal to any $\varphi(\tau) \in \underset{\sim}{D}(B)$ such that supp $\varphi \subset (-\infty, o)$. Thus, $\langle y, \varphi\rangle = 0$ for all such φ; that is, supp $y \subset [o, \infty)$.

Now, assume that supp $y \subset [o, \infty)$ and that $v \in \underset{\sim}{I}'(B)$. We first show that supp $y * v \subset [t_0, \infty)$ assuming that supp $v \subset [t_0, \infty)$. Let $\varphi \in \underset{\sim}{K}$ with supp $\varphi \subset (-\infty, t_0)$. Then, $\langle v(\tau),\ \varphi(t+\tau)\rangle \in \underset{\sim}{H}(B)$, and supp $\langle v(\tau),\ \varphi(t+\tau)\rangle \subset (-\infty, o)$. But then,

$$\langle y * v,\ \varphi\rangle = \langle y(t),\ \langle v(\tau),\ \varphi(t+\tau)\rangle\rangle = 0$$

Since φ is arbitrary, this shows that supp $y * v \subset [t_0, \infty)$. By virtue of the linearity of $\mathfrak{N} = y\,*$, we can now conclude that $\mathfrak{N}$ is causal on $\underset{\sim}{I}'(B)$. Q.E.D.

5. Passive Hilbert Ports with Convolution Representations

Let H be a Hilbert space; H need not be separable. We henceforth assume that both of the Banach spaces B and A considered before are the space H. Moreover, we assume that the inner product $(\cdot\ ,\ \cdot)$ in H has the significance of power in the following way:

If the input $v \in \underset{\sim}{I}'(H)$ and the output $u \in K'(H)$ happen to be ordinary functions at some point t and therefore map t into H, then $\mathrm{Re}(u(t),\ v(t))$ is the power p(t) flowing into the system. [In what follows, the reader must bear in mind that the inner product $(u(t),\ v(t))$ is with respect to H and does not denote an integration with respect to t. Thus, $(u(t),\ v(t))$ varies in general as t varies.] Moreover, if p(t) happens to be Lebesgue integrable on any open interval of the form $(-\infty,\ x)$, where x is finite, then

$$e(x) \triangleq \mathrm{Re} \int_{-\infty}^{x} (u(t),\ v(t))\, dt \tag{5.1}$$

is the energy absorbed by the system $\mathfrak{N}$ during the time interval $-\infty < t < x$.

We have hereby arrived at a "Hilbert port," an infinite-dimensional generalization of an n-port. As an example, consider again a resonant cavity having this time only one wave guide leading to it, through which electromagnetic energy flows. In this case u and v can be respectively related to the transverse magnetic field and transverse electric field at some fixed transverse plane in the wave guide in such a way that (5.1) truly represents the total energy absorbed by the cavity and wave guide beyond the fixed plane.

We can now introduce the concept of passivity for a Hilbert port.

<u>Definition 5-1: Let</u> $\mathfrak{N} = y*$ <u>with</u> $y \in H'[L(H,\ H)]$. <u>Also, let</u> $\underset{\sim}{W}(H)$ be a subset of $\underset{\sim}{I}'(H)$ <u>consisting of H-valued ordinary functions.</u> $\mathfrak{N}$ <u>is said to be passive on</u> $\underset{\sim}{W}(H)$ <u>if, for every</u> $v \in \underset{\sim}{W}(H)$, <u>for</u> $u = \mathfrak{N}v$, <u>and for every finite real number</u> x, <u>we have that</u> $(u(t),\ v(t))$ <u>is Lebesgue integrable on</u> $-\infty < t < x$ <u>and</u>

$$e(x) \triangleq \text{Re} \int_{-\infty}^{x} (u(t), v(t))\, dt \geq 0.$$

Lemma 5-1: If $u \in \underset{\sim}{B}(H)$ and $v \in \underset{\sim}{D}_{L_1}(H)$, then $(u, v) \in \underset{\sim}{D}_{L_1}$. Similarly, if $u \in \underset{\sim}{E}(H)$ and $v \in \underset{\sim}{D}(H)$, then $(u, v) \in \underset{\sim}{D}$.

Proof: We first show that $(u, v) \in \underset{\sim}{E}$ whenever u and v are both members of $\underset{\sim}{E}(H)$. Consider the equation, wherein $\Delta t \neq 0$:

$$\frac{(u(t + \Delta t)), v(t + \Delta t)) - (u(t), v(t))}{\Delta t}$$

$$= \left(\frac{u(t + \Delta t) - u(t)}{\Delta t},\ v(t + \Delta t)\right) + \left(u(t),\ \frac{v(t + \Delta t) - v(t)}{\Delta t}\right)$$

Since the inner product is a continuous function of its arguments, we may pass to the limit as $\Delta t \to 0$ to obtain

$$D_t\, (u(t), v(t)) = (u^{(1)}(t), v(t)) + (u(t), v^{(1)}(t)).$$

Repeatedly applying this result, we get

$$D_t^k(u(t), v(t)) = \sum_{p=o}^{k} \binom{k}{p} (u^{(p)}(t), v^{(k-p)}(t)) \qquad k = 0, 1, 2, \ldots$$

Thus, (u, v) is truly in $\underset{\sim}{E}$.

Next, we apply Schwarz's inequality:

$$|D_t^k(u(t), v(t))| \leq \sum_{p=o}^{k} \binom{k}{p} \|u^{(p)}(t)\|\ \|v^{(k-p)}(t)\|$$

The first statement of the lemma now follows from the facts that $\|u^{(p)}(t)\| \in \underset{\sim}{B}$ and $\|v^{(k-p)}(t)\| \in \underset{\sim}{D}_{L_1}$. The second statement follows in a similar way. Q.E.D.

Proposition 5-1: Let $\mathfrak{N} = y*$, where $y \in H'\,[L(H, H)]$. Also, let $\mathfrak{N}$ be passive on $\underset{\sim}{D}(H)$. Then, supp $y \subset [o, \infty)$ so that $\mathfrak{N}$ is causal on $\underset{\sim}{I}'(H)$.

Proof: Note that $\underset{\sim}{D}(H) \subset \underset{\sim}{I}'(H)$ for every space in the fourth row of Table II, and therefore D(H) is truly in the domain of $\mathfrak{N} = y*$. By lemma 5-1 and proposition 2-9, if $v \in \underset{\sim}{D}(H)$ and $u = \mathfrak{N}\, v$, then $(u, v) \in \underset{\sim}{D}$, in which case

$$\int_{-\infty}^{x} (u, v)\, dt$$

exists for all x. Now, assume in addition that, for some real number t_o, $v(t) = 0$ for $t < t_o$. Also, let $v_1 \in \underset{\sim}{D}(H)$ and set

$$v_2 = v_1 + bv$$

where b is an arbitrary real number. Therefore, $v_2(t) = v_1(t)$ for $t < t_o$. We may write

$$u_2 = \mathfrak{N}v_2 = \mathfrak{N}v_1 + b\mathfrak{N}v$$

For any fixed $x < t_o$, we have by the passivity of $\mathfrak{N}$ that

$$0 \leq \operatorname{Re} \int_{-\infty}^{x} (u_2, v_2)\, dt = \operatorname{Re} \int_{-\infty}^{x} (\mathfrak{N}v_1 + b\mathfrak{N}v, v_1)\, dt$$

$$= \operatorname{Re} \int_{-\infty}^{x} (\mathfrak{N}v_1, v_1)\, dt + b \operatorname{Re} \int_{-\infty}^{x} (\mathfrak{N}v, v_1)\, dt$$

Since b is arbitrary, we must have that

$$\operatorname{Re} \int_{-\infty}^{x} (\mathfrak{N}v, v_1)\, dt = 0$$

for all $x < t_o$. In view of lemma 5-1, this implies that $(\mathfrak{N}v, v_1) \equiv 0$ on $-\infty < t < t_o$. But, since $v_1(t) \in \underset{\sim}{D}(H)$ can be chosen arbitrarily, we see that $(\mathfrak{N}v)(t) \equiv 0$ on $-\infty < t < t_o$. It now follows from the linearity of $\mathfrak{N}$ that $\mathfrak{N}$ is causal on $\underset{\sim}{D}(H)$. By proposition 4-1, supp $y \subset [o, \infty)$, and $\mathfrak{N}$ is causal on $\underset{\sim}{I}'(H)$.

<u>Lemma 5-2</u>: <u>If $y \in \underset{\sim}{B}'[L(B, A)]$, where B and A are Banach spaces, and if supp $y \subset [o, \infty)$, then $y \in \underset{\sim}{L}'[o, \infty)\ [L(B, A)]$.</u>

<u>Proof</u>: Let a and b be any real numbers satisfying $0 < a < b < \infty$, and let $\varphi \in \underset{\sim}{L}_{a,b}(B)$. Also, let $\lambda(t)$ be a smooth function such that $\lambda(t) \equiv 0$ for $-\infty < t < -1$ and $\lambda(t) \equiv 1$ for $-\frac{1}{2} < t < \infty$. Then, $\lambda\, \varphi \in \underset{\sim}{D}_{L_1}(B)$. Indeed,

$$(5.2) \qquad \int_{-\infty}^{\infty} \| D^k \lambda(t) \varphi(t) \|_B dt \leq \sum_{\mu=o}^{k} \binom{k}{\mu} \int_{-\infty}^{\infty} |\lambda^{(k-\mu)}(t)|\ \| \varphi^{(\mu)}(t) \|_B dt$$

Since $\| \varphi^{(\mu)}(t) \| \leq Ke^{-at}$ on $-1 < t < \infty$ for some constant K, the right-hand side of (5.2) is finite. Since supp $y \subset [o, \infty)$ and $y \in \underset{\sim}{B}[L(B, A)]$, we can extend the definition of y onto $\underset{\sim}{L}_{a,b}(B)$ through the equation $\langle y, \varphi \rangle \triangleq \langle y, \lambda\, \varphi \rangle$, $\varphi \in \underset{\sim}{L}_{a,b}(B)$.

From (5.2) we also obtain

$$(5.3) \qquad \int_{-\infty}^{\infty} \|D^k \lambda(t)\varphi(t)\|_B dt \leq \sum_{\mu=0}^{k} \binom{k}{\mu} M_{k-\mu} \sup_{-\infty < t < \infty} \|\varkappa_{a,b}(t)\varphi^{(\mu)}(t)\|_B$$

where the $M_{k-\mu}$ are the real numbers:

$$M_{k-\mu} = \int_{-\infty}^{\infty} \frac{|\lambda^{(k-\mu)}(t)|}{\varkappa_{a,b}(t)} dt < \infty .$$

Upon invoking proposition 2-6, we obtain

$$\|< y, \varphi >\|_A = \|< y, \lambda \varphi >\|_A \leq M \max_{o \leq k \leq r} \int_{-\infty}^{\infty} \|D^k \lambda(t)\varphi(t)\|_B dt$$

which, in view of (5.3), implies that y is a continuous linear mapping of $\underset{\sim}{L}_{a,b}(B)$ into A. But this is true for every a and b such that $o < a < b < \infty$. Hence, $y \in \underset{\sim}{L}'(o,\infty)[L(B, A)]$. Q.E.D.

We now state a basic definition:

Definition 5-2: Y(s) is called a positive mapping of H into H if the following three conditions are satisfied on the half-plane $\{s: \text{Re } s > o\}$.

1. For each fixed s, Y(s) is a continuous linear mapping of H into H [i.e., $Y(s) \in L(H, H)$].

2. Y(s) is analytic.

3. For every $a \in H$, Re $(\underset{\sim}{Y}(s)\ a, a) \geq 0$.

Assuming that condition 1 holds true, we can interpret condition 2 in either of two equivalent ways: First, Y(s) is analytic on $\{s: \text{Re } s > 0\}$ if and only if as $|\Delta s| \to 0$,

$$(5.4) \qquad \frac{Y(s + \Delta s) - Y(s)}{\Delta s}$$

converges under the norm of the space L(H, H). Secondly, Y(s) is analytic on $\{s: \text{Re } s > 0\}$ if and only if, for every choice of $a \in H$ and $b \in H$, $(Y(s)\ a, b)$

is an analytic function on $\{s: \text{Re } s > 0\}$. That these two criterion are equivalent is a standard result. It is also a fact that, in this case, the convergence of (5.4) is automatically uniform with respect to s in any compact subset of $\{s: \text{Re } s > 0\}$. (See Hille and Phillips [1], pp. 92-94 or Taylor [1], pp. 653-654.)

One of the principal results of this work is

Theorem 5-1: If $\mathfrak{N} = y*$, where $y \in \underset{\sim}{B}[L(H, H)]$ and if $\mathfrak{N}$ is passive on $\underset{\sim}{D}_{L_1}(H)$, then $Y(s) = (\mathfrak{L}y)(s)$ exists for $\{s: \text{Re } s > 0\}$ and $Y(s)$ is positive.

Proof: By proposition 5-1 and lemma 5-2, supp $y \subset [o, \infty)$ and $y \in \underset{\sim}{L}'(o,\infty)[L(H, H)]$. Therefore, y does possess a Laplace transform whose region of definition contains $\{s: \text{Re } s > 0\}$; namely, for at least Re s > 0,

$$Y(s) \triangleq (\mathfrak{L}y)\ (s) = (\mathfrak{L}f_y)(s) = \langle f_y(t),\ e^{-st}\rangle$$

Here, $f_y(t)$ is the member of $\underset{\sim}{L}'(o,\infty)(L(H, H))$ generated by y as indicated in proposition 2-3. It follows that Y(s) is an L(H, H)-valued analytic function for Re s > 0. Thus, the first two conditions of definition 5-2 are satisfied.

To prove the third condition, choose any two real numbers x and x_1 such that $-\infty < x < x_1 < \infty$. Given any s with Re s > 0, choose $\varphi(t) \in \underset{\sim}{E}$ such that $\varphi(t) = e^{st}$ on $-\infty < t < x_1$ and $\varphi(t) = 0$ on $x_1 + 1 < t < \infty$. Hence, $\varphi(t) \in \underset{\sim}{L}(-\infty, \text{Re } s)$, and it follows that we may convolve y with $v = \varphi\, a$, $a \in H$, in accordance with column 5 of Table II wherein w = 0 and z = Re s. Moreover, since $y \in \underset{\sim}{B}'[L(H,H)]$ and $v = \varphi\, a \in \underset{\sim}{D}_{L_1}(H)$, we may invoke proposition 2-8 to write

$$u = \mathfrak{N}v = y* (\varphi\ a) = \langle y(\tau),\ \varphi(t-\tau)\ a\rangle = \langle f_y(\tau),\ \varphi(t-\tau)\rangle\ a \in \underset{\sim}{B}\ (H).$$

By lemma 5-1, $(u, v) \in \underset{\sim}{D}_{L_1}$. So, by the passivity of $\mathfrak{N}$ on $\underset{\sim}{D}_{L_1}(H)$ we have

$$0 \leq \text{Re} \int_{-\infty}^{x} (u,\ v)\ dt = \text{Re} \int_{-\infty}^{x} (\langle f_y(\tau),\ \varphi(t-\tau)\rangle\ a,\ \varphi(t)\ a)\ dt$$

Since supp $y \subset [0,\infty)$, supp $f_y \subset [0,\infty)$. Therefore, because $-\infty < t < x$, the last inequality may be rewritten as

$$0 \le \text{Re} \int_{-\infty}^{x} (\langle f_y(\tau),\ e^{s(t-\tau)} \rangle\ a,\ e^{st} a)\ dt$$

$$= \int_{-\infty}^{x} e^{(s+\bar{s})t}\ dt\ \ \text{Re}\ (\langle f_y(\tau),\ e^{-s\tau} \rangle\ a,\ a)$$

$$= \int_{-\infty}^{x} e^{2\ \text{Re}\ st} dt\ \ \text{Re}\ (Y(s)\ a,\ a)$$

Q.E.D.

We now wish to introduce the idea of a real mapping of H into H. In order to do this we shall assume that H has been generated from a real Hilbert space H_r through complexification (Bachman and Narici [1]). That is, given a real Hilbert space H_r, the elements of H are defined to be $a_1 + ia_2$ where a_1 and a_2 are arbitrary members of H_r. Also, the inner product $(\cdot\ ,\ \cdot)$ on H_r is extended onto H through the definition

(5.5) $$(a_1 + i\,a_2,\ b_1 + i\,b_2) \triangleq (a_1,\ b_1) + (a_2,\ b_2) + i\,(a_2,\ b_1) - i\,(a_1, b_2)$$

where a_1, a_2, b_1, and b_2 are arbitrary numbers of H_r. It is easy to show that H is a complex Hilbert space whenever it is generated in this way from a given H_r. Also, any orthonormal basis for H_r is also an orthonormal basis for H. Throughout the remainder of this section it is understood that H has this structure.

Next, a linear mapping Z of H into H is said to be real if Z maps H_r into H_r. Note that, if the (not necessarily real) linear mapping Z is defined only on H_r, it can be extended into a real linear mapping of H into H through the definition:

$$Z\ (a + ib) = Z\,a + i\,Z\,b \qquad a,\ b \in H_r$$

This convention is adopted in the following.

On the other hand, if Z is a (not necessarily real) mapping of H into H, we can decompose it into real and imaginary parts as follows. Z maps any $a \in H_r$ into a member $b_1 + i\, b_2$ of H where $b_1 \in H_r$ and $b_2 \in H_r$. So, we define the real part Z_1 and the imaginary part Z_2 of Z by

$$b_1 = Z_1 a, \; b_2 = Z_2 a$$

and we write $Z = Z_1 + i\, Z_2$. The complex conjugate of Z is by definition $\bar{Z} = Z_1 - i\, Z_2$.

Definition 5-3: *Y(s) is called a positive-real mapping of H into H if it is a positive mapping of H into H and, for each real positive number σ, Y(σ) is a real mapping of H into H.*

Corollary 5-1a: *If $\mathfrak{N} = y*$, where $y \in \underset{\sim}{B}'[L(H_r, H_r)]$, and if $\mathfrak{N}$ is passive on $\underset{\sim}{D}_{L_1}(H_r)$, then $\mathfrak{N}$ satisfies the hypothesis of theorem 5-1 and Y(s) is positive-real.*

Proof: Any $\varphi \in \underset{\sim}{D}_{L_1}(H)$ can be decomposed into $\varphi = \varphi_1 + i\, \varphi_2$ where φ_1 and φ_2 are unique members of $\underset{\sim}{D}_{L_1}(H_r)$. So, following the aforementioned convention, we define y on $\underset{\sim}{D}_{L_1}(H)$ by

$$\langle y, \varphi \rangle \triangleq \langle y, \varphi_1 \rangle + i\, \langle y, \varphi_2 \rangle \in H.$$

It follows readily that $y \in \underset{\sim}{B}'[L(H, H)]$.

Moreover, for $v \in \underset{\sim}{D}_{L_1}(H)$, we have that $u \in \underset{\sim}{B}(H)$ by virtue of proposition 2-8. Upon making the usual decompositions $v = v_1 + i\, v_2$, $v_1 \in \underset{\sim}{D}_{L_1}(H_r)$, $v_2 \in \underset{\sim}{D}_{L_1}(H_r)$ and $u = u_1 + i\, u_2$, $u_1 = \underset{\sim}{B}(H_r)$, $u_2 \in \underset{\sim}{B}(H_r)$, we see from (5.5) that $\mathfrak{N}$ is passive on $\underset{\sim}{D}_{L_1}(H)$ whenever it is passive on $\underset{\sim}{D}_{L_1}(H_r)$. Thus $\mathfrak{N}$ satisfies the hypothesis of theorem 5-1.

Finally, since $y \in \underset{\sim}{B}'[L(H_r, H_r)]$, it follows that $f_y \in \underset{\sim}{B}'(L(H_r, H_r))$. Consequently, for any real positive σ, $Y(\sigma) \stackrel{\Delta}{=} \langle f_y(t), e^{-\sigma t}\rangle$ is a member of $L(H_r, H_r)$. The proof is complete.

6. Representations for Positive-real Mappings.

Our final objective is to develop the converse to theorem 5-1 and its corollary. We accomplish this by using some representations of positive-real mappings, which is the subject of the present section. We start with a representation due to Beltrami [1].

Beltrami's Representation: Y(s) is a positive mapping of H into H if and only if, for Re $s > 0$, Y(s) admits the following representation: For every a and b in H,

$$(Y(s)\, a,\, b) = (Aa,\, b)\, s + (Q\, a,\, b) + \int_{-\infty}^{\infty} \frac{i\eta s - 1}{i\eta - s}\, d\, (\psi(\eta) a,\, b) \tag{6.1}$$

where the following conditions are satisfied. A is constant self-adjoint non-negative-definite continuous linear mapping of H into H. Q is a constant skew-self-adjoint (i.e., $Q = -Q^*$ where Q^* denotes the adjoint of Q) continuous linear mapping of H into H. For each $\eta \in R$, $\psi(\eta)$ is a self-adjoint continuous linear mapping of H into H; for each $a \in H$, $(\psi(\eta)a,\, a)$ is a real nondecreasing bounded function on $-\infty < \eta < \infty$; for every a and b in H, $(\psi(\eta)a,\, b)$ is of bounded variation on $-\infty < \eta < \infty$; with $(\psi(-\infty)a,\, b) \triangleq \lim_{\eta \to -\infty} (\psi(\eta)a,\, b) = 0$.

We extend Beltrami's representation as follows:

Proposition 6-1: Y(s) is a positive-real mapping of H into H if and only if the representation (6.1) possesses the following additional properties: A and Q are real mappings, and $\psi(\eta) = -\overline{\psi(-\eta)}$.

We first prove

Lemma 6-1: Let H and H_r be as before. An operator Z mapping H into H is real if and only if (Z a, b) is real whenever a and b are in H_r.

Proof: Let $c = c_1 + i\, c_2 = Za$ where c_1, c_2 and a are in H_r. If (Z a, b) is real for all a and b in H_r, then $(c_2, b) = 0$ because

$$(Z\, a,\, b) = (c_1,\, b) + i\, (c_2,\, b)$$

Since b is arbitrary, $c_2 = 0$. Hence, Z is real. The converse is obvious.

Proof of Proposition 6-1: Assume that A, Q, and ψ have the stated properties. Setting $\psi = \psi_1 + i\,\psi_2$ where ψ_1 and ψ_2 are real operators, we see that $\psi_1(\eta)$ is an odd mapping of the real line into L(H, H) and that $\psi_2(\eta)$ is an even mapping. Moreover, for σ real and positive, the imaginary part of the last term on the right-hand side of (6.1) is

$$\int_{-\infty}^{\infty} \frac{\eta(1-\sigma^2)}{\sigma^2 + \eta^2}\, d\,(\psi_1(\eta)a,\, b) \int_{-\infty}^{\infty} \frac{\sigma(1+\eta^2)}{\sigma^2 + \eta^2}\, d\,(\psi_2(\eta)a,\, b)$$

This expression is equal to zero for σ real and positive by virtue of the oddness and evenness of the two integrands and of $(\psi_2(\eta)a, b)$ and $(\psi_1(\eta)a, b)$ with respect to η. By lemma 6-1, $Y(\sigma)$ is a real operator for σ real and positive.

To prove the "only if" part of proposition 6-1 we shall need two more lemmas.

Lemma 6-2: Let Q be a linear operator mapping the (complex) Hilbert space H into itself. If Q is skew-self-adjoint, then (Qa, a) is imaginary for all $a \in H$.

Proof: If Q is skew-self-adjoint, then $(Qa, a) = (a, Q^*a) = -(a, Qa) = -\overline{(Qa, a)}$ so that (Qa, a) is imaginary.

Lemma 6-3: Let Q be a skew-self-adjoint continuous linear mapping of the complex Hilbert space H which is generated from the real Hilbert space H_r through complexification. Then, Q is real if and only if (Qa, a) = 0 for all $a \in H_r$.

Proof: If Q is real, then $Q\,a \in H_r$ for all $a \in H_r$, and (Qa, a) is real. But since Q is skew-self-adjoint, (Qa, a) is imaginary according to lemma 6-2. Therefore, (Qa, a) = 0.

Conversely, assume that (Qa, a) = 0 for all $a \in H_r$, and set $Q = Q_1 + i\,Q_2$ where Q_1 and Q_2 are real. Then, $Q^* = Q_1^* - i\,Q_2^*$. Since $Q = -Q^*$, we have $Q_1 = Q_1^*$ and $Q_2 = Q_2^*$. Moreover,

$$(Qa, a) = (Q_1 a, a) + i\,(Q_2 a, a).$$

Since Q_1 is skew-self-adjoint, $(Q_1 a, a)$ is imaginary by lemma 6-2. But $(Q_1 a, a)$ is also real since Q_1 is real and $a \in H_r$. Hence, $(Q_1 a, a) = 0$ for all $a \in H_r$. By assumption, $(Qa, a) = 0$ for all $a \in H_r$ also. Thus, $(Q_2 a, a) = 0$ for all $a \in H_r$. But, if Q_2 were not the zero operator, there would be at least one $a \neq 0$ in H_r for which $(Q_2 a, a) \neq 0$ because

$$\|Q_2\| = \sup_{\|a\|=1, a \in H_r} |(Q_2 a, a)|$$

(See Riesz and Nagy [1], pp. 229-230.) Thus, Q_2 must be the zero operator, and Q therefore real.

Returning to the proof of proposition 6-1, assume that Y(s) is positive-real. Since $(\psi(\eta)a, b)$ is of bounded variation on $-\infty < \eta < \infty$, it follows that for σ real,

$$\lim_{\sigma \to \infty} \frac{(Y(\sigma)a, b)}{\sigma} = (A\ a, b).$$

(See Shohat and Tamarkin [1], pp. 23-24.) Thus, (Aa, b) is real for every a and b in H_r, and therefore A is a real operator by lemma 6-1.

Next, since Q is skew-self-adjoint, (Qa, a) is imaginary for all $a \in H$, According to lemma 6-2. Now, set s = 1 and a = b in (6.1) to get

$$(Y(1)a, a) = (Aa, a) + (Qa, a) + \int_{-\infty}^{\infty} d\ (\psi(\eta)a, a)$$

The left-hand side is real for all $a \in H_r$ because Y(1) is a real operator. Also, the first and last terms are real since A and $\psi(\eta)$ are self-adjoint (see Riesz and Nagy [1], p. 229). Consequently, (Qa, a) is equal to zero for all $a \in H_1$. By lemma 6-3, Q is a real operator.

We can now conclude that, for all a and b in H_r and all real positive s, (Y(s) a, b), (Aa, b), and (Qa, b) are real. Therefore,

$$K(s) \triangleq \int_{-\infty}^{\infty} \frac{i\eta s-1}{i\eta-s}\, d\ (\psi(\eta)\ a, b) \tag{6.2}$$

is real under the same restrictions on a, b, and s. Since K(s) is analytic for

Re s > 0, we have from the reflection principle that

$$(6.3) \qquad K(s) = \overline{K(\bar{s})} \qquad \text{Re } s > 0$$

Moreover, upon making the change of variable $\xi = -\eta$, we obtain

$$\overline{K(\bar{s})} = \int_{-\infty}^{\infty} \frac{i\xi s-1}{i\xi-s} \, d\,(-\overline{\psi(-\xi)}a,b \qquad a,\ b \in H_r$$

We now replace ξ by η in this expression and invoke (6.3) to get

$$\int_{-\infty}^{\infty} \frac{i\eta s-1}{i\eta-s} \, d\,([\psi(\eta) + \overline{\psi(-\eta)}]\ a,\ b) = 0 \qquad \text{Re } s > 0$$

for all a and b in H_r. By a known inversion formula (see Greenstein [1], theorem 2.1), $([\psi(\eta) + \overline{\psi(-\eta)}]\ a,\ b)$ is a constant on $-\infty < \eta < \infty$, and this constant is equal to zero since $(\psi(-\infty)\ a,\ b) = 0$. Since this is true for all a and b in H_r, $\psi(\eta) = -\overline{\psi(-\eta)}$ for all η. This completes the proof of proposition 6-1.

For our purposes we shall have to strengthen Beltrami's representation. The next proposition is a step in this direction, but we will eventually assume a still stronger representation. We first note that, if we set s = 1 in Beltrami's representation, we obtain, for all a and b in H,

$$(Y(1)\ a,\ b) = (Aa,\ b) + (Qa,\ b) + \lim_{\eta \to \infty} (\psi(\eta)\ a,\ b)$$

or

$$\lim_{\eta \to \infty} (\psi(\eta)\ a,\ b) = ([Y(1) - A - Q]\ a,\ b)$$

Since Y(1), A, and Q are members of L(H, H), it follows that Y(1) - A - Q, which we denote by $\psi(\infty)$, is in L(H, H) also. Moreover, we have that

$$(6.4) \qquad \lim_{\eta \to \infty} (\psi(\eta)\ a,\ b) = (\psi(\infty)\ a,\ b)$$

for all a and b in H. Thus, we can view $\psi(\eta)$ as being defined on the extended real line $-\infty \leq \eta \leq \infty$ with $\psi(-\infty) = 0$ and $\psi(\infty)$ satisfying (6.4).

Next, we show that that $\psi(\eta)$ is of bounded variation on $-\infty \leq \eta \leq \infty$ in the sense of definition 3.2.4(2) of Hille and Phillips [1]. Let $\{(\alpha_i, \beta_i)\}$ be any finite collection of nonoverlapping open intervals on the extended real line $[-\infty, \infty]$. Then, $\Sigma_i \psi(\beta_i) - \psi(\alpha_i)$ is a member of L(H, H) because $\psi(\eta) \in L(H, H)$ for each η in $[-\infty, \infty]$. Moreover, for each fixed $a \in H$ and each fixed $\eta \in [-\infty, \infty]$, $(\psi(\eta)\, a, b)$ defines a continuous linear functional on all $b \in H$. Also, for fixed a and b, $(\psi(\eta)\, a, b)$ is of bounded variation on $-\infty \leq \eta \leq \infty$ because it is of bounded variation on $-\infty < \eta < \infty$ and continuous from the right and the left at $\eta = -\infty$ and $\eta = \infty$ respectively. Thus

$$|([\sum_i(\psi(\beta_i) - \psi(\alpha_i)]\, a, b)| = |\sum_i(\psi(\beta_i)\, a, b) - (\psi(\alpha_i)\, a, b)| \leq \text{Var.}(\psi(\eta)\, a, b)$$

where the right-hand side denotes the total variation of $(\psi(\eta)\, a, b)$ on $-\infty \leq \eta \leq \infty$. Consequently, by the principle of uniform boundedness (Hille and Phillips, p. 26), for any fixed $a \in H$

$$\sup \|[\sum_i \psi(\beta_i) - \psi(\alpha_i)]\, a\|_H < \infty$$

where the supremum is taken over all choices of the collection $\{(\alpha_i, \beta_i)\}$. Since this is true for every $a \in H$, we may apply the principle of uniform boundedness again to write

$$\sup \|\sum_i \psi(\beta_i) - \psi(\alpha_i)\|_{L(H, H)} < \infty \tag{6.5}$$

where the supremum is again taken over all choices of $\{(\alpha_i, \beta_i)\}$.

So truly, $\psi(\eta)$ is of bounded variation on $-\infty \leq \eta \leq \infty$. Henceforth, we denote the left-hand side of (6.5) by $\text{Var}_{-\infty \leq \eta \leq \infty}\, \psi(\eta)$.

We now note that for each fixed s with a positive real part,

$$\frac{i\eta s-1}{i\eta-s}$$

is a continuous function on $-\infty < \eta < \infty$ which converges to s as either $\eta \to -\infty$ or $\eta \to \infty$. Therefore, by theorem 3.3.2 of Hille and Phillips [1], the integral:

$$\int_n^m \frac{i\eta s-1}{i\eta-s}\, d\psi(\eta)$$

exists in the norm topology of L(H, H) for every finite n and m, and we can take $n \to \infty$ and $m \to \infty$ independently to get the integral

$$J(s) \triangleq \int_{-\infty}^{\infty} \frac{i\eta s-1}{i\eta-s}\, d\psi(\eta)$$

which also exists in the norm topology of L(H, H). It follows that for any a and b in H

$$(J(s)\ a,\ b) = \int_{-\infty}^{\infty} \frac{i\eta s-1}{i\eta-s}\, d\ (\psi(\eta)\ a,\ b)$$

It now follows from Beltrami's representation that

$$J(s)\ a = Y(s)\ a - Aas - Qa$$

for every $a \in H$, and that

$$J(s) = Y(s) - As - Q.$$

We have thus arrived at the following stronger representation:

Proposition 6-2: Y(s) is a positive mapping of H into H if and only if, for Re s > 0, Y(s) admits the representation:

(6.6) $$Y(s) = As + Q + \int_{-\infty}^{\infty} \frac{i\eta s-1}{i\eta-s}\, d\psi(\eta)$$

where A, Q, and $\psi(\eta)$ satisfy the conditions stated in Beltrami's representation and in addition $\psi(\eta)$ is of bounded variation on $-\infty \leq \eta \leq \infty$ with $\psi(-\infty) = 0$ and $\psi(\infty) = Y(1) - A - Q$. The integral in (6.6) exists in the norm topology of L(H, H). Furthermore, Y(s) is positive-real if and only if, in addition to the preceding conditions, A and Q are real and $\psi(\eta) = -\overline{\psi(-\eta)}$.

7. The Riemann-Stieltjes Integral of an H-valued Integrand with Respect to an L(H, H)-valued Integrator.

Definition 7-1: Let B be a Banach space. A B-valued function $\psi(\eta)$ on the real line $-\infty < \eta < \infty$ is said to be of strong bounded variation on the real line if the supremum of the set of strong total variations on all compact intervals is finite. (See Hille and Phillips [1], definition 3.2.4(3) for the definition of strong total variation on a compact interval.) This supremum is denoted by

$$\underset{-\infty < \eta < \infty}{\text{S. Var.}} \; \psi(\eta)$$

Since

$$\underset{0 \leq \eta \leq \xi}{\text{S. Var.}} \; \psi(\eta)$$

is a monotonically increasing bounded function of ξ as $\xi \to \infty$, it follows that

(7.1) $$\underset{\xi \leq \eta < \infty}{\text{S. Var}} \; \psi(\eta) \to 0 \quad \xi \to \infty .$$

But, by the definition of strong bounded variation on compact intervals

$$\|\psi(\eta) - \psi(\xi)\| \to 0$$

as η and ξ tend to infinity independently. By the completeness of B, there exists a $\psi(\infty) \in B$ to which $\psi(\eta)$ converges in B as $\eta \to \infty$. Similarly, there exists a $\psi(-\infty) \in B$ to which $\psi(\eta)$ converges in B as $\eta \to -\infty$. Thus, when $\psi(\eta)$ is of strong bounded variation on $-\infty < \eta < \infty$, we can define it through continuity on the extended real line $-\infty \leq \eta \leq \infty$. By formally applying the definition 3.2.4(3) of Hille and Phillips [1] for the interval $[-\infty, \infty]$, we obtain a definition for

$$\underset{-\infty \leq \eta \leq \infty}{\text{S. Var.}} \; \psi(\eta)$$

and it follows easily that

$$\underset{-\infty \leq \eta \leq \infty}{\text{S. Var.}} \; \psi(\eta) = \underset{-\infty < \eta < \infty}{\text{S. Var.}} \; \psi(\eta)$$

We henceforth follow these conventions in extending the definition of $\psi(\eta)$ onto $[-\infty, \infty]$.

We have defined $\psi(\infty)$ in two ways, as a limit under the operator norm and as a weak limit according to (6.4). These two definitions are equivalent whenever $\psi(\eta)$ is of strong bounded variation on $-\infty < \eta < \infty$. The same situation holds for $\psi(-\infty)$.

Next, we let $\underset{\sim}{B}_o(B)$ denote the space of B-valued (strongly) continuous bounded functions on $-\infty < \eta < \infty$. Our first objective in this section is to show that we can apply a sense to the Riemann-Stieltjes integral:

$$(7.2) \qquad \int_{-\infty}^{\infty} h(\eta) \, d\, \psi(\eta)$$

where $h(\eta) \in \underset{\sim}{B}_o(B)$ and $\psi(\eta)$ is an $L(B, A)$-valued function of strong bounded variation on $-\infty < \eta < \infty$. Here, A is another Banach space. We shall show in particular that (7.2) is a member of A and will obtain an estimate for it.

Let $[\eta_L, \eta_R]$ denote a compact interval, and let π denote a finite partition of it:

$$\eta_L = \eta_o \le \eta_1 \le \cdots \le \eta_n = \eta_R$$

Set $|\pi| \overset{\Delta}{=} \max_i |\eta_i - \eta_{i-1}|$. Also, we associate with π the points ξ_i such that $\eta_{i-1} \le \xi_i \le \eta_i$. The Riemann-Stieltjes sum:

$$(7.3) \qquad S_\pi(h, \psi) \overset{\Delta}{=} \sum_{i=1}^{n} h(\xi_i)[\psi(\eta_i) - \psi(\eta_{i-1})]$$

is a member of A, and we shall show that it converges as $|\pi| \to 0$. In (7.3) it is understood that $h\psi$ is just another way of writing ψh.

Next, let π' be another partition of $[\eta_L, \eta_R]$ of n' segments and partition points η_i'; ξ_i' denotes the associated points as above. Thus,

$$S_{\pi'}(h, \psi) \overset{\Delta}{=} \sum_{i=1}^{n'} h(\xi_i')[\psi(\eta_i') - \psi(\eta_{i-1}')].$$

We now take the union of $\{\eta_i\}$ and $\{\eta_i'\}$ to obtain the partition points $\{\delta_i\}$ of the partition:

$$\eta_L = \delta_o \leq \ldots \leq \delta_m = \eta_R$$

This yields

$$S_\pi(h, \psi) - S_{\pi'}(h, \psi) = \sum_{i=1}^{m} [h(\zeta_i) - h(\zeta_i')][\psi(\delta_i) - \psi(\delta_{i-1})]$$

where each ζ_i is one of the points ζ_i and each ζ_i' is one of the points ζ_i' and $|\zeta_i - \zeta_i'| \leq 2 \max (|\pi|, |\pi'|)$.

Now, $h(\eta)$ is uniformly continuous on $[\eta_L, \eta_R]$. Hence, given an $\epsilon > 0$, there exists a δ such that, for $\max (|\pi|, |\pi'|) < \delta$,

$$\|h(\zeta_i) - h(\zeta_i')\|_B < \frac{\epsilon}{\underset{-\infty < \eta < \infty}{\text{S. Var.}} \psi(\eta)}$$

for all i. Therefore,

$$\|S_\pi(h,\psi) - S_{\pi'}(h,\psi)\|_A < \frac{\epsilon}{\underset{-\infty < \eta < \infty}{\text{S. Var.}} \psi(\eta)} \sum_{i=1}^{m} \|\psi(\delta_i) - \psi(\delta_{i-1})\|_{L(H, H)} \leq \epsilon$$

We can now invoke the completeness of A to conclude that there exists a member of A, which we denote by

$$(7.4) \qquad \int_{\eta_L}^{\eta_R} h(\eta)\, d\, \psi(\eta)$$

and to which $S_\pi(h, \psi)$ converges as $|\pi| \to 0$. Moreover,

$$\|S_\pi(h, \psi)\| \leq \sum_{i=1}^{n} \|h(\zeta_i)\|_B \|\psi(\eta_i) - \psi(\eta_{i-1})\|_{L(H, H)}$$

$$\leq \sup_{\eta_L \leq \eta \leq \eta_R} \|h(\eta)\|_B \underset{\eta_L \leq \eta \leq \eta_R}{\text{S. Var}} \psi(\eta)$$

Since convergence in A implies convergence of the norms, we can conclude that

$$(7.5) \qquad \left\|\int_{\eta_L}^{\eta_R} h(\eta)\, d\, \psi(\eta)\right\|_A \leq \sup_{\eta_L \leq \eta \leq \eta_R} \|h(\eta)\|_B \underset{\eta_L \leq \eta \leq \eta_R}{\text{S. Var}} \psi(\eta)$$

We now turn to the improper integral (7.2). This is defined as the limit in A of (7.4) as $\eta_L \to -\infty$ and $\eta_R \to \infty$ independently. This limit exists by virtue of (7.1) and (7.5). Thus, we have established

Proposition 7-1: If $h \in \underset{\sim}{B}_o(B)$ and ψ is an $L(B, A)$-valued function of strong bounded variation on the real line $-\infty < \eta < \infty$, then (7.2) exists in the norm topology of A and

$$(7.6) \qquad \left\|\int_{-\infty}^{\infty} h(\eta)\, d\, \psi(\eta)\right\|_A \leq \sup_{-\infty < \eta < \infty} \|h(\eta)\|_B \underset{-\infty < \eta < \infty}{\text{S. Var.}} \psi(\eta)$$

If, in addition, $h(\eta) = g(\eta)\, a$, where $g \in \underset{\sim}{B}_o(C)$ and $a \in B$, then

$$(7.7) \qquad \int_{-\infty}^{\infty} g(\eta)\, a\, d\, \psi(\eta) = \int_{-\infty}^{\infty} g(\eta)\, d\, [\psi(\eta)\, a]$$

where $\psi(\eta)\, a$ is an A-valued function of strong bounded variation on $-\infty < \eta < \infty$. This result is established by working with the Riemann-Stieltjes sum approximations of both sides of (7.7).

In the very same way we can show that, if A = B = H where H is a Hilbert space and if $b \in H$ also, then

$$(7.8) \qquad \left(\int_{-\infty}^{\infty} g(\eta)\, a\, d\psi(\eta),\, b\right) = \int_{-\infty}^{\infty} g(\eta)\, d\, (\psi(\eta)a,\, b)$$

where $(\psi(\eta)\, a,\, b)$ is a complex-valued function of bounded variation in the usual sense on $-\infty < \eta < \infty$. We can also assign a sense to the expression:

$$(7.9) \qquad \int_{-\infty}^{\infty} (h(\eta)\, d\, \psi(\eta)\,,\, q(\eta))$$

where $h(\eta)$ and $q(\eta)$ are both members of $\underset{\sim}{B}_o(H)$ and $\psi(\eta)$ is an $L(H, H)$-valued function of strong bounded variation on $-\infty < \eta < \infty$. To do this we start with a compact interval $[\eta_L, \eta_R]$ and a partition π of it as above. Then, working with the Riemann-Stieltjes sum:

$$\Sigma_\pi(h, \psi, q) \triangleq \sum_{i=1}^{n} (h(\xi_i)[\psi(\eta_i) - \psi(\eta_{i-1})]\,,\, q(\xi_i))$$

much as before, we can show that, as $|\pi| \to 0$, this converges to a complex number which we denote by

(7.10) $$\int_{\eta_L}^{\eta_R} (h(\eta)\ d\ \psi(\eta),\ q(\eta).$$

In doing so, we use the boundedness and uniform continuity of $h(\eta)$ and $q(\eta)$ on $\eta_L \leq \eta \leq \eta_R$. We also obtain the estimate:

$$\left|\int_{\eta_L}^{\eta_R} (h(\eta)\ d\ \psi(\eta),\ q(\eta))\right| \leq \sup_{\eta_L \leq \eta \leq \eta_R} \|h(\eta)\|\ \|g(\eta)\| \underset{\eta_L \leq \eta \leq \eta_R}{\text{S. Var.}} \psi(\eta).$$

We can then show that (7.10) converges to a complex number, which we denote by (7.9), as $\eta_L \to -\infty$ and $\eta_R \to \infty$ independently. In addition, we get the estimate:

(7.11) $$\left|\int_{-\infty}^{\infty} (h(\eta)\ d\ \psi(\eta),\ q(\eta)\right| \leq \sup_{-\infty < \eta < \infty} \|h(\eta)\|\|q(\eta)\| \underset{-\infty < \eta < \infty}{\text{S. Var.}} \psi(\eta)$$

We conclude this section with

Lemma 7-1: If $\psi(\eta)$ is an $L(H, H)$-valued function of strong bounded variation on $-\infty < \eta < \infty$ and if $(\psi(\eta)\ a,\ a)$ is a nondecreasing function of η for each $a \in H$, then, for each $h(\eta) \in \underset{\sim}{B}_0(H)$,

$$\int_{-\infty}^{\infty} (h(\eta)\ d\ \psi(\eta),\ h(\eta)) \geq 0$$

Proof:

$$\Sigma_\pi(h,\ \psi,\ h) \overset{\Delta}{=} \sum_{i=1} (h(\xi_i)[\psi(\eta_i) - \psi(\eta_{i-1})) \geq 0$$

We obtain our conclusion from this by passing to the limit as $|\pi| \to 0$.

8. The Realizability of Every Strongly Positive Mapping

In this final section we develop a partial converse to theorem 5-1 and its corollary 5-1a. To do so, we impose an additional condition on the representation (6.6) for a positive mapping.

Definition 8-1: Y(s) is called a stongly positive (positive-real) mapping of H into H if it is a positive (positive-real) mapping of H into H such that the L(H, H)-valued function $\psi(\eta)$ in the representation (6.6) is of strong bounded variation on the real line $-\infty < \eta < \infty$.

Before stating our converse to theorem 5-1, we first prove

Proposition 8-1: If Y(s) is a strongly positive (positive-real) mapping of H into H, then Y(s) is the Laplace transform of a $y \in \underset{\sim}{B}'[L(H, H)]$ (respectively, $y \in \underset{\sim}{B}'[L(H_r, H_r)]$) which is defined on any $\varphi \in \underset{\sim}{D}_{L_1}(H)$ by

$$\langle y, \varphi\rangle = -A\, \varphi^{(1)}(0) + Q\, \varphi\, (0)$$

$$+ \int_{-\infty}^{\infty} d\, \psi(\eta)\Big[\int_0^{\infty} e^{i\eta t}\varphi(t)dt + \int_0^{\infty} (1 - e^{i\eta t})\varphi^{(2)}(t)dt\Big] \tag{8.1}$$

Here, A, Q, and $\psi(\eta)$ are the quantities indicated in the representation (6.6) with $\psi(\eta)$ being of strong bounded variation on $-\infty < \eta < \infty$.

Proof: First it is easily seen that, as a function of η, the Bochner integral:

$$\int_0^{\infty} [e^{i\eta t}\varphi(t) + (1 - e^{i\eta t})\, \varphi^{(2)}(t)]\, dt$$

is a member of $\underset{\sim}{B}_0(H)$. Consequently, by virtue of proposition 7-1, the last term on the right hand side of (8.1) exists in the norm topology of H. Obviously, the other two terms on the right-hand side of (8.1) are also members of H. Moreover, by the estimate (7.6),

$$\|\langle y, \varphi\rangle\|_H \leq \|A\|_{L(H,H)}\|\varphi^{(1)}(o)\|_H + \|Q\|_{L(H,H)}\|\varphi(o)\|_H$$

$$+ \Big[\int_0^{\infty}\|\varphi(t)\|_H\, dt + 2\int_0^{\infty}\|\varphi^{(2)}(t)\|_H\, dt\Big] \underset{-\infty < \eta < \infty}{\text{S. Var.}}\ \psi(\eta)$$

This proves that y is a continuous mapping of $\underset{\sim}{D}_{L_1}(H)$ into H, its linearity on $\underset{\sim}{D}_{L_1}(H)$ being obvious. In other words, $y \in \underset{\sim}{B}'[L(H), H)]$.

It is also obvious that supp $y \subset [0, \infty)$. Therefore, by lemma 5.2, $y \in \underset{\sim}{L}'(0, \infty)[L(H, H)]$. So, we can take its Laplace transform to get, for at least Re $s > 0$,

$$Y(s) = \langle f_y(t), e^{-st}\rangle = As + Q + \int_{-\infty}^{\infty} d\psi(\eta)[\int_0^{\infty} e^{i\eta t}e^{-st}dt + \int_0^{\infty}(1 - e^{i\eta t})D_t^2 e^{-st} dt]$$

$$= As + Q + \int_{-\infty}^{\infty} \frac{1 - i\eta s}{s - i\eta} d\psi(\eta)$$

This verifies that Y(s) is the Laplace transform of y.

Finally, assume that Y(s) is strongly positive-real. By the second statement of proposition 6-2, A and Q are members of $L(H_r, H_r)$. Consequently, both of the mappings $\varphi \mapsto -A\varphi^{(1)}(0)$ and $\varphi \mapsto Q\,\varphi(0)$ are members of $\underset{\sim}{B}'[L(H_r, H_r)]$.

Moreover, for any $\varphi \in \underset{\sim}{D}_{L_1}(H_r)$, the real and imaginary parts of $\int_0^{\infty} e^{i\eta t}\varphi(t)dt$ are respectively even and odd mappings of the real line into H_r. Also, since $\psi(\eta) = -\overline{\psi(-\eta)}$, the real and imaginary parts of $\psi(\eta)$ are respectively odd and even mappings of the real line into $L(H_r, H_r)$. It follows that the imaginary part of

$$(8.2) \qquad \int_{-\infty}^{\infty} d\psi(\eta) \int_0^{\infty} e^{i\eta t}\varphi(t)dt$$

is equal to zero. In other words, the mapping that assigns to each $\varphi \in \underset{\sim}{D}_{L_1}(H_r)$ the member (8.2) of H is also a member of $\underset{\sim}{B}'[L(H_r, H_r)]$.

A similar argument leads to the same conclusion for

$$\varphi \mapsto \int_{-\infty}^{\infty} d\psi(\eta) \int_0^{\infty} (1 - e^{i\eta t})\,\varphi^{(2)}(t)\,dt.$$

Hence, $y \in \underset{\sim}{B}'[L(H_r, H_r)]$, and the proof of proposition 8-1 is complete.

We need two lemmas.

Lemma 8-1: $\underset{\sim}{D} \otimes H$ is dense in $\underset{\sim}{D}_{L_1}(H)$.

Proof: It is a fact that $\underset{\sim}{D} \otimes H$ is dense in $\underset{\sim}{D}(H)$ (see Schwartz [2], pp. 109-110). Moreover, $\underset{\sim}{D}(H)$ is dense in $\underset{\sim}{D}_{L_1}(H)$. Indeed, let $\lambda_\nu(t) = \lambda(t/\nu)$ where $\lambda \in \underset{\sim}{D}$, $\lambda(t) = 1$ for $|t| < 1$ and $\lambda(t) = 0$ for $|t| > 2$. Also, let $\varphi \in \underset{\sim}{D}_{L_1}(H)$. Hence, $\lambda_\nu \varphi \in \underset{\sim}{D}(H)$. A simple computation shows that $\lambda_\nu \varphi \to \varphi$ in $\underset{\sim}{D}_{L_1}(H)$ as $\nu \to \infty$, which justifies our assertion. Lemma 8-1 now follows from the first two sentences of this proof and the fact that sequential convergence in $\underset{\sim}{D}(H)$ implies sequential convergence in $\underset{\sim}{D}_{L_1}(H)$.

Lemma 8-2: If $\mathfrak{N}$ is a continuous linear mapping of $D_{L_1}(H)$ into $B(H)$ and if $\mathfrak{N}$ is passive on $D \otimes H$, then $\mathfrak{N}$ is passive on $D_{L_1}(H)$.

Proof: In view of lemma 8-1 and the fact that $D_{L_1}(H)$ has a countable basis of neighborhoods at each of its points, for any given $v \in D_{L_1}(H)$ we can choose a sequence $\{v_n\}_{n=1}^{\infty}$ such that $v_n \in D \otimes H$ and $v_n \to v$ in $D_{L_1}(H)$ as $n \to \infty$. Upon setting $u_n = \mathfrak{N} v_n$ and $u = \mathfrak{N} v$, we then obtain that $u_n \to u$ in $B(H)$. Our proof will be complete when we show that, for each finite real number x,

$$(8.3) \qquad \int_{-\infty}^{x} (u_n, v_n)\, dt \to \int_{-\infty}^{x} (u, v)\, dt \qquad n \to \infty$$

because the real part of the left-hand integral is nonnegative by hypothesis. In (8.3) it must be remembered that (u_n, v_n) is a function of t.

Now,

$$\left|\int_{-\infty}^{x} [(u_n, v_n) - (u, v)]\, dt\right| \leq \int_{-\infty}^{x} \|u_n\|\, \|v_n - v\|\, dt + \int_{-\infty}^{x} \|u_n - u\|\, \|v\|\, dt$$

Since $u_n \to u$ in $\underset{\sim}{B}(H)$, there exists a constant M such that $\|u_n\| < M$ and $\|u\| < M$ for all n and all t. So,

$$\int_{-\infty}^{x} \|u_n\|\, \|v_n - v\|\, dt \leq M \int_{-\infty}^{x} \|v_n - v\|\, dt \to 0 \qquad n \to \infty$$

because $v_n \to v$ in $\underset{\sim}{D}_{L_1}(H)$.

Also, given any $\epsilon > 0$, there exists a T such that, for all $x \leq T$,

$$\int_{-\infty}^{x} \|u_n - u\| \, \|v\| \, dt \leq 2M \int_{-\infty}^{x} \|v\| \, dt < \epsilon/2$$

Fix T this way. Then, for $x > T$,

$$\int_{T}^{x} \|u_n - u\| \, \|v\| \, dt \leq (\sup_{-\infty < t < \infty} \|v\|) \int_{T}^{x} \|u_n - u\| \, dt < \epsilon/2$$

for all sufficiently large n because $u_n \to u$ in $\underset{\sim}{B}(H)$. Thus,

$$\int_{-\infty}^{x} \|u_n - u\| \, \|v\| \, dt \to 0 \qquad n \to \infty$$

and our proof is complete.

We are finally ready to state the last major conclusion of this work.

<u>Theorem 8-1: For every strongly positive (positive-real) mapping</u> Y(s) <u>of</u> H <u>into</u> H, <u>there exists a unique convolution operator</u> $\mathfrak{N} = y *$ <u>which is passive on</u> $\underset{\sim}{D}_{L_1}(H)$ <u>and is such that</u> $y \in \underset{\sim}{B}'[L(H, H)]$ (<u>respectively</u>, $y \in \underset{\sim}{B}'[L(H_r, H_r)]$) <u>with the Laplace transform of</u> y <u>being</u> Y(s).

<u>Proof:</u> In view of propositions 2-8 and 8-1 and lemma 8-2, we need merely prove that $\mathfrak{N} = y *$ is passive on $\underset{\sim}{D} \times H$ where y is defined on any $\varphi \in \underset{\sim}{D}_{L_1}(H)$ by (8.1).

Corresponding to the second term on the right-hand side of (8.1) we have the convolution operator $Q \, \delta \, *$, which maps each $v \in \underset{\sim}{D} \otimes H$ into $Qv \in \underset{\sim}{D} \otimes H$. Moreover, for each fixed t, (Qv, v) is purely imaginary according to lemma 6-2. Hence, $Q \, \delta \, *$ is certainly passive on $\underset{\sim}{D} \otimes H$.

Also, corresponding to the first term on the right-hand side of (8.1) we have the convolution operator $A \, \delta^{(1)} *$, which maps each $v \in \underset{\sim}{D} \otimes H$ into $A \, v^{(1)} \in \underset{\sim}{D} \otimes H$. Now, an arbitrary $v \in \underset{\sim}{D} \otimes H$ is equal to the finite sum $\Sigma_{j=1}^{n} a_j f_j$ where $a_j \in H$ and $f_j \in \underset{\sim}{D}$. Hence,

$$u = (A\ \delta^{(1)})* v = \sum_{j=1}^{n} (A\ a_j)\ f_j^{(1)}$$

and

$$(u,\ v) = \sum_{j=1}^{n} \sum_{k=1}^{n} (A\ a_j,\ a_k)\ f_j^{(1)} \overline{f}_k$$

Consider a main diagonal term in this quadratic form. Since $(A\ a_j,\ a_j)$ is real,

$$\text{Re} \int_{-\infty}^{x} (A\ a_j,\ a_j)\ f_j^{(1)} \overline{f}_k\ dt = \tfrac{1}{2}(A\ a_j,\ a_j)\ f(x)\ \overline{f}(x)$$

On the other hand, some straightforward manipulation shows that, for a pair of terms symmetrically placed around the main diagonal we have from the fact that A is self-adjoint that

$$\text{Re} \int_{-\infty}^{x} [(A\ a_j,\ a_k)\ f_j^{(1)} \overline{f}_k + (A\ a_k,\ a_j)\ f_k^{(1)} \overline{f}_j]\ dt$$

$$= \tfrac{1}{2} \int_{-\infty}^{x} [(A\ a_j,\ a_k)\ D(f_j \overline{f}_k) + (A\ a_k,\ a_j)\ D\ (\overline{f}_j f_k)]\ dt$$

$$= \tfrac{1}{2}[(A\ a_j,\ a_k)\ f_j\ (x)\ \overline{f}_k\ (x) + (A\ a_k,\ a_j)\ f_k\ (x)\ \overline{f}_j\ (x)]$$

Thus, for every x,

$$\text{Re} \int_{-\infty}^{x} (A\ \delta^{(1)} * v,\ v)\ dt = \tfrac{1}{2}\ (A \sum_{j=1}^{n} f_j\ (x)\ a_j,\ \sum_{j=1}^{n} f_j\ (x)\ a_j) \geq 0$$

because A is a nonnegative-definite member of $L(H;\ H)$. This shows that $A\ \delta^{(1)}*$ is also passive on $\underset{\sim}{D} \otimes H$.

Finally, corresponding to third term on the right-hand side of (8.1) we have the convolution operator y_3* where for any $v \in \underset{\sim}{D} \otimes H$

$$u_3 = y_3 * v = \langle y_3(\tau),\ v(t-\tau)\rangle = \int_{-\infty}^{\infty} d\psi(\eta)[\int_0^{\infty} e^{i\eta\tau} v(t-\tau)d\tau + \int_0^{\infty}(1 - e^{i\eta\tau})v^{(2)}(t-\tau)d\tau]$$

The quantity within the brackets on the right-hand side is, as a function of η (t fixed), a member of $\underset{\sim}{B}_0(H)$ so that the right-hand side exists according to proposition 7-1. Moreover, by proposition 2-8, $u_3 \in \underset{\sim}{B}(H)$, and, by lemma 5-1, $(u_3,\ v) \in \underset{\sim}{D}$.

Next, set

$$v = \sum_{j=1}^{n} a_j f_j \qquad a_j \in H,\ f_j \in \underset{\sim}{D}$$

as before. Making use of (7.7) and (7.8), we may write

$$e_3(x) = \mathrm{Re} \int_{-\infty}^{x} (u_3, v)\, dt$$

$$= \mathrm{Re} \int_{-\infty}^{x} dt \sum_{j=1}^{n} \sum_{k=1}^{n} \overline{f_k(t)} \int_{-\infty}^{\infty} d(\psi(\eta)a_j, a_k) \int_{0}^{\infty} [e^{i\eta\tau} f_j(t-\tau) + (1 - e^{i\eta\tau}) f_j^{(2)}(t-\tau)]\, d\tau$$

Now, $(\psi(\eta)a_j, a_k)$ is of bounded variation in the usual sense on $-\infty < \eta < \infty$. Moreover, $f_k \in \underset{\sim}{D}$ and the integral on τ inside the right-hand side is a continuous bounded function on the (t, η) plane. It follows that we may invoke Fubini's theorem to change the order of integration on t and η. Some integrations by parts then yield

$$e_3(x) = \mathrm{Re} \sum_{j=1}^{n} \sum_{k=1}^{n} \int_{-\infty}^{\infty} d(\psi(\eta)a_j, a_k)(1 + \eta^2) \int_{-\infty}^{x} dt\, \overline{f_k(t)}\, e^{i\eta t} \int_{-\infty}^{t} e^{-i\eta\tau} f_j(\tau)\, dt$$

$$- \mathrm{Re}\, i \sum_{j=1}^{n} \sum_{k=1}^{n} \int_{-\infty}^{\infty} \eta\, d(\psi(\eta)a_j, a_k) \int_{-\infty}^{x} f_j(t)\, \overline{f_k(t)}\, dt$$

Since ψ is self-adjoint, the double integral inside the last double summation is real, and hence the last expression on the right-hand side is equal to zero. By breaking the first double summation into real and imaginary parts and then noting the symmetry of the integrand around the $t = \tau$ line in the (t, τ) plane, we see that we may write

$$e_3(x) = \tfrac{1}{2}\, \mathrm{Re} \sum_{j=1}^{n} \sum_{k=1}^{n} \int_{-\infty}^{\infty} (1 + \eta^2) J_j(x, \eta)\, \overline{J_k(x, \eta)}\, d(\psi(\eta)a_j, a_k)$$

where

$$J_\nu(x, \eta) = \int_{-\infty}^{x} f_\nu(\tau)\, e^{-i\eta\tau}\, dt$$

An integration by ports shows that, for each fixed x, $J_\nu(x, \eta) = O(\eta^{-1})$ as $|\eta| \to \infty$.

Then, by (7.8) again,

$$e_3(x) = \tfrac{1}{2}\,\mathrm{Re} \sum_{j=1}^{n} \sum_{k=1}^{n} \int_{-\infty}^{\infty} (1+\eta^2)(a_j\, J_j(x,\eta)\, d\psi(\eta),\, a_k\, J_k(x,\eta))$$

$$= \tfrac{1}{2}\,\mathrm{Re} \int_{-\infty}^{\infty} (1+\eta^2)\Big(\sum_{j=1}^{n} a_j\, J_j(x,\eta)\, d\psi(\eta)\ ,\ \sum_{j=1}^{n} a_j\, J_j(x,\eta)\Big)$$

Lemma 7-1 now shows that $e_3(x) \geq 0$ so that $y_3{*}$ is also passive on $\underset{\sim}{D} \otimes H$.

This completes the proof of theorem 8-1.

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