THE COMPLETE BEHAVIOR OF CERTAIN INFINITE NETWORKS UNDER KIRCHHOFF'S NODE AND LOOP LAWS

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Abstract. The objective of this work is the investigation of all possible current distributions in an infinite electrical network subject only to Kirchhoff's node and loop laws. These laws are in general not strong enough to yield a unique set of branch currents, All prior investigations of infinite networks imposed additional requirements, such as finiteness of the total power dissipation, in order to force uniqueness, but in doing so the other possible responses of a network were discarded,

The main result of this work holds not for all countably infinite, locally finite networks but for some fairly general classes of such networks. It states that, if the currents in certain branches, called "joints", are arbitrarily chosen, then all other branch currents are uniquely determined, Moreover, all possible sets of branch currents satisfying only the node and loop laws are encompassed in this result; one need merely choose the joint currents properly in order to obtain any given permissible set of branch currents. Another result concerns the idea of a homogeneous current flow; this is a set of branch currents satisfying the node and loop laws when all sources are set equal to zero, The dimension of the linear space of all homogeneous current flows is shown to be equal to the cardinal number of the set of joints. Finally, it is worth noting that our analysis provides a method for calculating the current in any given

branch through a finite number of computational steps.

~~1. Introduction~~. In addition to the fairly extensive literature on lumped infinite transmission lines, the idea of an infinite electrical network has been cropping up in the literature for quite some time now. See, for example, [5], [8], [9], [11] - [17]. In fact, some of these works employ infinite networks to model phenomena governed by partial differential equations [5], [11] - [16]. However, it was only until the recent works of H. Flanders [6], [7] that a general method for analyzing a fairly unrestricted class of infinite networks was devised. Alternative methods of still more recent vintage are given in [3], [4], [18] - [20].

Actually, Kirchhoff's node and loop laws coupled with the relationships between voltages and currents imposed by the elements of the network are not in general strong enough to yield a unique current distribution in the infinite network. For example, assuming that each branch in the endless series circuit of Figure 1 has a complex impedance and that there is no mutual coupling between branches, we see that any complex value is allowed for the current i flowing therein. Indeed, Kirchhoff's node law is obviously satisfied and Kirchhoff's loop law, which is a requirement only for finite loops, is automatically satisfied since the network has no finite loops. In the aforementioned methods for analyzing an infinite network, a unique current distribution in the network is acheived by imposing additional requirements such as the finiteness of the total power dissipation throughout the network. Thus, if every branch resistance in Figure 1 were one ohm, then i would have to be zero for the total power dissipation to be finite.

The objective of this work is to investigate the class of all current distributions in a given infinite network. We impose only Kirchhoff's node and loop laws, and therefore many different current distributions are possible. Actually, in addition to the node and loop laws we also have the connections between voltages and currents # resulting from the elements of the network. That the latter are also operative will be tacitly understood throughout this work, and we will usually not mention them explicitly. Unlike finite networks, infinite networks with dissipation in all branches can carry a current flow under the node and loop laws even when there are no voltage or current sources present.

For the sake of a more concise terminology, whenever a graph of a network has a certain property, we shall say that the network itself has that property. We shall assume that our networks are connected, countably infinite, and locally finite; that is, the set of all nodes is countable and every node has a finite number of incident branches. We allow parallel branches but do not permit isolated nodes or self loops. (A self-loop is a single branch both of whose ends coincide at a single node.) Voltages, currents, and impedances are complex numbers, and mutual coupling between branches is allowed. We shall also assume that the network possesses only voltage sources; this is really no restriction since current sources can always be converted into voltage sources.

The main result of this work is an existence theorem for the current distribution in the network. It states that, if the currents in certain branches, called "joints", are arbitrarily chosen, then the currents in all other branches are uniquely determined, and indeed every current distribution satisfying Kirchhoff's node and loop laws can be uniquely specified by assigning the joint currents properly. This result has been established not for all infinite networks but rather for what appears to be a fairly broad class of infinite networks. A variety of examples for which our method works and one for which it doesn't appear to work will be given. Our existence theorem contains a step-by-step procedure for computing any given branch current. In particular, the current in any branch can be computed by solving a finite number of linear simultaneous equations having only a finite number of unknowns.

2. Some examples. To motivate our subsequent discussions we shall now present a number of examples showing how the branch currents can be computed in a step-by-step procedure. As in Figure 1, branches will be indicated by lines; it is understood that each branch is a series connection of a complex impedance and a complex voltage source, either or both of which may be zero.

Example 2.1. In Figure 2 let us assign arbitrarily the currents in branches 1, 2, and 3. This determines the currents in branches 0 and -1 by Kirchhoff's node law and the current in branch 4 by Kirchhoff's loop law. Then, the current in branch -2 is obtained by the loop law and the currents in branches 5 and 6 by the node law. Continuing in this way, we can obtain the current in any branch. Thus, we cannot arbitrarily assign a current to any other branch once the currents in branches 1, 2, and 3 are assigned. Another choice of currents in branches 1, 2, and 3 will yield a different current distribution throughout the network. Moreover, it can be seen that an initial choice of the currents in only two or less branches will not determine the branch currents in the rest of the network.

A heuristic approach to this situation is to assume that there are energy sources out at infinity feeding currents into the network. As with a common approach to finite networks, let us choose a spanning tree T for our network and assume that the current distribution consists of a superposition of fundamental mesh currents. Let us further assume that the currents coming from the sources out at infinity are fundamental mesh currents. We choose T as the subgraph consisting of the two endless horizontal paths and the single vertical branch numbered 1. Thus, T is the graph induced by all branches other than ..., -5, -2, 4, 7, ... Now, the sources at infinity cannot feed current exclusively through the tree if branches 1, 2, and 3 are removed. In this fashion the specification of the currents in branches 1, 2, and 3 indirectly specifies the sources at infinity. We could of course have started with another spanning tree but that tree would have to have the property that exactly three branches in it must be removed in order to stop all currents flowing exclusively in the tree. This is because exactly three independent sources can be connected out at infinity between the four infinitely long legs of T. For example, the tree consisting of the branches ..., -5, -3, -2, -1, 1, 3, 4, 5, 7, ... does not possess the stated property and would not therefore be a proper choice. These admittedly nebulous ideas provide nevertheless a clue for our subsequent discussion, Example 2.2, In Figure 3 we have an infinite grid that covers a half-plane. Let us assume that the currents in all the branches labeled a_k, where $k = \ldots, -1, 0, 1, \ldots$, are given. Then, the node law determines the currents in all the branches b_k; next, the loop law determines the currents in all the c_k; then, the node law determines the currents in the d_k, and so forth. Thus, the specification of the currents in the a_k determine the currents in the rest of the network.

Here again, the a_k can be viewed as branches whose removal disrupts all endless paths in a certain spanning tree, That tree can be taken to be the subgraph induced by the a_k and all horizontal branches b_k, d_k, f_k, ...

Example 2.3. The one-ended lattice structure of Figure 4 can be viewed as an infinite sequence of bridges. Branch 1 is the central branch of the bridge whose arms comprise the set B_1 of branches 2, 3, 4, and 5; branches 6, 7, 8, and 9 are the arms of the next bridge, and we denote that set by B_2; etc. If we specify the current in branch 1, then Kirchhoff's node law applied to the two nodes of branch 1 and Kirchhoff's loop law applied to the loops 1, 2, 4 and 1, 3, 5 yield four equations in the four unknown currents in branches 2, 3, 4, and 5. If the bridge is unbalanced, the equations can be solved. Once this is done, the node and loop laws determine in the same fashion the currents in branches 6, 7, 8, and 9 so long as this bridge is also unbalanced. Continuing this step-by-step procedure, we obtain the currents in all branches. To identify branch 1 as a branch whose removal opens up all endless paths in a certain tree, we choose that tree as the one induced by the branches 1, 2, 5, 6, 9, 10, 13, ... ▪

At this point it is worth pointing out a fact concerning the homogeneous current flows in the examples presented so far. By a homogeneous current flow in a given network we mean a set of branch currents which satisfy Kirchhoff's node and loop laws when all sources in the network are set equal to zero. A homogeneous current flow is called nonzero if at least one of its branch currents is nonzero. The network of Figure 1 obviously possesses a nonzero homogeneous current flow whatever be its branch impedances. The same is true for the networks of Figures 2 and 3 if all branches have nonzero impedances. We need merely specify nonzero values for the currents in branches 1, 2, and 3 for Figure 2 and nonzero values for the currents in the branches a_k for Figure 3. Upon setting all sources equal to zero and then computing as before, we will obtain the corresponding homogeneous current flows. This procedure will work no matter what nonzero values the branch impedances assume,

This is no longer the case for the network of Figure 4. It will not have a nonzero homogeneous current flow if every bridge is balanced. For then, the current in branch 1 must be zero since it is the central branch of the first bridge. Similarly, the current passing through the first bridge is the central current for the second bridge, and so It too must be zero. This in turn implies that the current in the arms 2, 3, 4, and 5 must be zero. Continuing in this way, we see that every brsnch current is zero. Actually, the network of Figure 4 will not have a homogeneous current flow if there exists an infinite subsequence $\{B_{k_i}\}_{i=1}^{\infty}$ of the arm sets B_k such that the arms of every B_{k_i} are balanced with regard to the corresponding bridge.

We conclude that not every infinite network 'containing an endless path possesses a noniero homogeneous current flow.

3. Some concepts concerning infinite graphs. Our subsequent discussion will employ certain notions concerning infinite graphs. We shall define them here and develop some preliminary results. Actually, we have already used some of the terms defined below, but nevertheless we give their definitions here to avoid subsequent ambiguity.

We assume that the reader is already familiar with the idea of a connected, countably infinite, locally finite graph G in which parallel brsnches are allowed but no isloated vertices or self-loops are permitted. In general, we denote branches by b_k and nodes by n_k, but other symbols are also used. A *walk* W in G is an alternating sequence of branches and nodes such that every branch is immediately preceded and succeeded by the two nodes to which it is incident. W is said to be *finite*, *one-ended*, or *endless* if the sequence is finite, one-way infinite, or two-way infinite respectively. A *path* P in G is a walk such that no node appears more than once in P. A *loop* L is a finite walk such that no node appears more than once except for the first and last nodes; these nodes are identical and appear nowhere else in L.

If we remove the branches b_1, b_2, b_3, ... from G but do not remove the nodes to which b_1, b_2, b_3, ... are incident, we denote the resulting graph by $G_1 - b_1 - b_2 - \dots$.

A tree T in G is any connected subgraph of G that contains no loops.' A tree may be finite or infinite. T is said to be *spanning* if it contains all the nodes of G; in this case, T must be infinite since G is infinite.

The foregoing are common definitions. We now introduce some new concepts and terminology which will be useful in our subsequent development. Let T be a spanning tree in G. If T contains an endless path P_1, choose any brench b_1 in P_1 and remove it from T. $T - b_1$ will have precisely two components, both of which are infinite trees. If $T - b_1$ contains an endless path P_2, choose any branch b_2 in P_2 and remove it. $T - b_1 - b_2$ will have exactly 3 components each of which is an infinite tree. We continue this procedure of removing a branch b_k from an endless path P_k contained in $T - b_1 - b_2 - \cdots - b_{k-1}$ until there no longer exists any endless paths, If the process terminates after a finite number, say, m of branches are removed, then $T - b_1 - \cdots - b_m$ will have m + 1 components. Otherwise, an infinity of branches must be removed in order to break all endless paths and the resulting graph $T - b_1 - b_2 - \cdots$ will have an infinity of components. We shall refer to the b_k as joints. The finite or infinite set $J = \{b_k: k = 1, 2, \ldots\}$ of all joints will be called a **full set of Joints**. The components of $T - b_1 - b_2 \cdots$ will be called limbs.

Lemma 3.1 (The structure of a limb).

(i) Every limb is an infinite tree containing no endless paths.

(ii) Every node of G is contained in a unique limb,

(iii) Given any node of G there exists a unique one-ended path P starting at the node and contained in the limb that contains the node.

(iv) All other paths starting at that node and contained exclusively in the limb are finite.

(v) Only a finite number of the latter finite paths have no branches in common with P.

Proof. By its definition every limb must be a tree containing no endless paths. Let us assume that one of the limbs, say, F is a finite tree. Because of our local-finiteness condition, there can be only a finite number of joints connected to F in the original tree T. This means that in the process of removing the joints one by one there will be one stage at which there exists exactly one joint connected to F. But this is impossible because that joint will not then be in an endless path. This contradiction establishes (i).

(ii) follows from the fact that at no point in the process of generating limbs is a node removed.

To establish (iii), we first observe that every infinite tree must contain a one-ended path [2; p. 17]. By the connectedness of each limb we can conclude that there exists at least one one-ended path starting at any given node of a limb and contained in that limb. We cannot hsve two different one-ended paths in the limb starting from the same node for, otherwise, the limb would contain either a loop or an endless path. This would contradict the fact that each limb is a tree possessing no endless paths.

(iv) follows immediately from (iii).

Finally, for (v), let n_0 be a node in a limb L and let us consider the subgraph H of L consisting of all finite paths in L starting at n_0 and having no branches in common with the unique one-ended path P in L that starts at n_0. If there are an infinity of such paths, then H must be an infinite connected graph and therefore must contain a one-ended path Q [2; p. 17]. Since P and Q are distinct, this contradicts (iv). Lemma 3.1 is hereby established.

Another concept we shall introduce is that of a "twig". Consider a tree containing at least one endless path. Then consider the set of all branchss d_1, d_2, ... that are not contained in endless paths, A twig is a component of the subgraph induced by the set of all d_k. T with all its twigs removed (i.e., $T - d_1 - d_2 - \cdots$) will be called a stripped tree. Similarly, a stripped limb is a component of the subgraph obtained by removing from T a full set of joints and all the d_k. (Isolated nodes are ignored.)

Our next objective is to show that the cardinality of any full set J of joints is an intrinsic property of the tree T and does not vary for different choices of J. Assume for the moment that there are only a finite nonzero number of endless paths in T and consider the corresponding stripped tree T_s. These endless paths lie exclusively in T_s and every branch of T_s lies in at least one endless path. Given any choice of a full set J of joints we can choose a finite subtree T_f of T_s that contains every joint because there are only a finite number of joints. Then, T_s is the union [1; p. 76] of T_f and a finite number of subtrees P_1, P_2, ... , P_n such that the nodes of P_k and the nodes of P_j form disjoint sets if $k \neq j$. Each end-node of P_k must also lie in T_f since T_s has no twigs.

Actually, each P_k is a one-ended path that is cannected to T_f only at its end node. Indeed, upon applying Lemma 3.1(i) to T_s and noting that T_f is finite, we see first of all that P_k is an infinite tree containing no endless paths, Now, assume that P_k is not a one-ended path. Then, P_k contains a node n_0 whose degree is greater than two. By (iii) and (iv) of Lemma 3.1, there exists a finite path Q passing through n_0 and terminating at two end nodes n_1 and n_2 of P_k. But then, n_1 and n_2 must also lie in T_f, and so Q is part of a loop in T_s. This cnntradicts the fact that T_s is a tree. Thus, P_k is a one-ended path. It is connected to T_f only at its unique end node because, were it to be connected to T_f at two or more nodes, there would again be a loop in T_s.

Since T_f is finite, it now follows that every limb resulting from the given J contains one and only one P_k; that is, the number n of the P_k is equal to the number of limbs.

Every infinite path in T ccnsists of the union of two one-ended paths, say, P_k and P_j, where $k \neq j$, and a finite path in T_f; conversely, every such union yields an endless path in T. Since n is the number of P_k, it follows that there exist $(n - 1)!$ endless paths in T. But, the number of endless paths in T is an intrinsic property of T and does not depend upon the choice of J. Therefore, the same can be said of the number n.

Now, the choice of a joint in T_s partitions the set $\{P_k\}_{k=1}^{n}$ into two subsets, two of the P_k being in the same subset if and only if there exists an endless path passing through both of them and not through the joint. Upon choosing two joints in T_s, we will partition P_k into three subsets under the same rule. This process can be continued until each subset contains exactly one P_k, at which point we will have chosen $n - 1$ joints. It follows now that the number of joints in a full set J is independent of the choice of the joints,

Let us turn now to the case where T possesses an infinity of endless paths. Then, J must have an infinity of joints. Indeed, if only a finite number, say, k of joints sufficed to disrupt all endless paths, then the removal of these k joints from T_s would yield exacly k+1 components, none of which contains an endless path. But this implies that T_s would have exactly k! endless paths, and therefore so too would T. This is a contradiction. Thus, the cardinality of J is $\aleph_0$, the cardinal number for a countably infinite set.

In this case we will also have an infinity of limbs. Consequently, the cardinality of the set of limbs must also be $\aleph_0$ since it cannot be greater than the cardinality of the set of branches.

We summarize our foregoing discussion as follows.

Lemma 3.2. If a tree T contains at least one endless path, then the cardinality of a full set J of joints is independent of the choice of the joints and is equal to one less than the cardinality of the set of limbs resulting from the removal of J from T.

Let us return now to our infinite graph G and assume that a spanning tree T and a full set J of joints in T have been chosen. Thus, the limbs in G are the ccmponents of the subgraph of T obtained by removing all joints in J from T. As is commonly done, every branch not in T will be called a chord. Given a chord, there exists a unique loop consisting of the chord and unique tree path connecting the nodes of the chord; this loop will be called the chord-tree loop. Moreover, if both nodes of the chord are contained in a single limb, we shall also refer to that loop as the chord-limb loop. On the other hand, if the two nodes are contained in different limbs, it follows from Lemma 3(iii) that there exists a unique endless path consisting of the chord and the two unique one-ended paths starting at the nodes of the chord and remaining within their respective limbs; this endless path will be called the chord-limb path. Later on, it will be convenient not to have to refer to chord-limb loops and chord-limb paths with two separate names. We will refer to these two entities as chord-limb orbs. Sometimes we designate a chord by a symbol, say, b and then refer to the b-tree loop and b-limb orb.

Similarly, the nodes of any given joint must lie in two different limbs. By Lemma 3.1(iii) again, there exists a unique endless path passing through the joint and the two limbs containing the joint's nodes. This endless path will be called the Joint-limb path, or the b-limb path if the joint is denoted by b.

In summary then we can state the following.

Lemma 3.3. Every chord lies in a unique chord-limb orb, and every joint lies in a unique joint-limb path.

4. Current flow. From now on, we shall assume that every branch in G has an orientation. The current in any branch is a complex number measured with respect to that branch's orientation. As before, we assume that we have fixed upon a spanning tree and a full set J of joints in our graph G.

Lemma 4.1. A specification of all the chord currents and joint currents uniquely determines all the branch currents if Kirchhoff's node law is satisfied at every node.

Proof. We need to determine the currents in all the limb branches. Let L be any limb. Then, L possesses at least one node n_1 having exactly one limb b_1 incident to it. For, if this were not so, L would have to contain an endless path. If n_1 has other branches incident to it, the latter branches will all be chords or joints. Thus, Kirchhoff's node law applied to

n_3 uniquely determines the current in b_1.

Next we consider either $L - b_1$ or any other limb and determine the current in another limb branch by applying Kirchhoff's node law to a node, only one of whose incident branches is contained in either $L - b_1$ or that other limb. Continuing in this fashion, we determine all branch currents.

Assumption 4.1. We assume that each chord current flows along its chord-limb orb and that each joint current flows along its corresponding joint-limb path. The direction of each such current flow agrees with the orientation of the corresponding chord or joint.

That we are free to adopt Assumption 4.1 without violating Kirchhoff's node law can be shown as follows. Given any node n_0 in some limb L, let F denote the union of all the finite paths specified in (v) of Lemma 3.1. Thus, F is a finite tree. Now, every chord current or joint current that passes through n_0 under Assumption 4.1 passes into or out of L at one of the nodes of F. Because G is locally finite, it follows that only a finite number of chord currents and joint currents flow through n_0. But, every such chord or joint current by itself satisfies Kirchhoff's node law at n_0, and therefore so too do the branch currents incident at n_0. Since every node lies in some limb according to Lemma 3.1(ii), Kirchhoff's node law is satisfied everywhere.

Since the current in a branch is a combination of some or all of the chord and joint currents impinging on one of its nodes through adjacent branches, the argument of the preceding paragraph also shows that tach limb branch carries only a finite number of chord currents or joint currents. We calculate the current in any limb branch b_0 by summing the chord currents and joint currents in b_0 after affixing a plus (minus) sign to such a current it its flow agrees (respectively, disagrees) with the orientation of the branch.

In view of Lemma 4.1, we have now obtained the following result.

Lemma 4.2. Upon specifying all chord currents and joint currents and then computing the remaining branch currents in accordance with Assumption 4.1, we obtain the unique set of branch currents dictated by Kirchhoff's node law.

5. Partitioning into finite subnetworks. Given an infinite network $\mathfrak{N}$, we define the union $\cup \mathfrak{N}_k$ of a finite or infinte collection of subnetworks $\mathfrak{N}_k$ of $\mathfrak{N}$ in the same way as is the union of subgraphs of a given finite graph [1; p. 76]. $\cup \mathfrak{N}_k$ is that subnetwork of $\mathfrak{N}$ whose node set is the union $\cup N(\mathfrak{N}_k)$ of the node sets $N(\mathfrak{N}_k)$ of the $\mathfrak{N}_k$ and whose branches are those branches of $\mathfrak{N}$ having both nodes in $\cup N(\mathfrak{N}_k)$. On the other hand, a partition of a subnetwork $\mathfrak{M}$ of $\mathfrak{N}$ is a collection $\{\mathfrak{N}_k\}$ of rubnetworks $\mathfrak{N}_k$ of $\mathfrak{M}$ such that every branch of $\mathfrak{M}$ occurs in one and only one of the $\mathfrak{N}_k$ and $\mathfrak{M} = \cup \mathfrak{N}_k$. Finally, $\mathfrak{N} - \mathfrak{M}$ denotes the subnetwork of $\mathfrak{N}$ induced by all branches that are not in $\mathfrak{M}$.

We are now ready to state our main theorem.

Theorem 5.1. Let $\mathfrak{N}$ be an infinite network satisfying the conditions stated in the penultimate paragraph of the Introduction and possessing at least one endless path. Assume there exists a spanning tree T, a full set J of joints, and a partition $\{\mathfrak{N}_k\}$ of a subnetwork $\mathfrak{M}$ of $\mathfrak{N}$ (possibly $\mathfrak{M} = \mathfrak{N}$) into finite subnetworks $\mathfrak{N}_k$ such that the following conditions are satisfied.

(i) $\mathcal{M}$ contains all the chords of $\mathcal{N}$, and, if a chord is in $\mathcal{N}_n$, then the corresponding chord-tree loop lies entirely within $\bigcup_{k \le n} \mathcal{N}_k$ and the corresponding chord-limb orb lies entirely within $\mathcal{N} - \bigcup_{k \le n-1} \mathcal{N}_k$.

(ii) Arbitrary mutual coupling between the branches of a single $\mathcal{N}_k$ is allowed, but mutual coupling between a branch b_1 in $\mathcal{N}_k$ and a branch b_2 in $\mathcal{N}_m$, where $k < m$, is allowed only if the following unilateral property holds: The current in b_1 may produce a nonzero voltage in b_2, but a current in b_2 produces zero voltage in b_1.

(iii) Write down the equations dictated by Kirchhoff's loop law around the chord-tree loops for ell the chords in $\mathcal{N}_k$, treating the corresponding chord currents as unknowns but treating as known all the joint currents and all the other chord currents (that is, the chord currents for the chords in all $\mathcal{N}_m$, where $m \neq k$). In doing so, let the current in each limb-branch be computed in accordance with Assumption 4.1. Let Z_k be the square impedance matrix obtained by writing these equations In the matrix form $Z_k \underset{\sim}{c}_k = \underset{\sim}{g}_k$, where $\underset{\sim}{c}_k$ is the vector of unknown chord currents for the chords in $\mathcal{N}_k$ and $\underset{\sim}{g}_k$ is a vector of known quantities. Assume that Z_k is nonsingular for every k.

Then, upon assigning an arbitrary current to each joint, we have that Kirchhoff's node and loop laws uniquely determine the current in every branch of $\mathcal{N}$; moreover, any set of branch currents that satisfy Kirchhoff's node and loop laws corresponds in this way to a particular choice of joint currents.

<u>Proof.</u> First assume that an arbitrary current has been

assigned to each joint. The vector $\underset{\sim}{g}_k$ is a function of the voltage sources, impedances, and joint currents, all of which are known quantities. Because of conditions (i) and (ii), $\underset{\sim}{g}_k$ is also a function of the chord currents for the chords in $\mathcal{N}_1, \ldots, \mathcal{N}_{k-1}$, but it does not depend on the chord currents for the chords in $\mathcal{N}_{k+1}, \mathcal{N}_{k+2}, \ldots$ (When $k = 1$, there are no chord currents in $\underset{\sim}{g}_1$.) Since each Z_k is nonsingular, we can solve in turn each of the equations $Z_k \underset{\sim}{c}_k = \underset{\sim}{g}_k$ for $k = 1, 2, \ldots$ to determine the $\underset{\sim}{c}_k$ and thereby any chord current.

Since all the joint currents have been specified, we have from Lemma 4.2 that all the branch currents are uniquely determined under the imposition of Kirchhoff's node law. Moreover, Kirchhoff's loop law is satisfied around each chord-tree loop since this is precisely what the equations $Z_k \underset{\sim}{c}_k = \underset{\sim}{g}_k$ require. Also, it can be shown in just the same way as is done for finite networks that the sum of the voltage drops around an arbitrarily chosen loop L is equal to the sum of the voltage drops around a finite number of chord-tree loops [10; Theorem 4.5-1]. Thus, Kirchhoff's loop law is satisfied around L as well.

Finally, let us assume that we are given any set of branch currents that satisfy Kirchhoff's node and loop laws. This means that we have specified the joint currents as well. Since the loop law is satisfied and each Z_k is nonsingular, the equations $Z_k \underset{\sim}{c}_k = \underset{\sim}{g}_k$ must yield precisely the same chord currents as the given ones. On the other hand, the limb-branch currents resulting from Assumption 4.1 must also agree with those given because of the uniqueness assertion of Lemma 4.2. This completes the proof.

Note that, since each branch carries only a 'finite number of chord currents, we need compute only a finite number of the $\underline{c}_k$ and then apply Assumption 4.1 in order to determine the current in any given branch. This involves no more than a finite number of computational steps.

It is also worth mentioning that, if does not possess an endless path (in violation of the hypothesis of Theorem 5.1), it must consist exclusively of an infinity of finite blocks. Thus, its behavior can be analyzed by applying the techniques for finite networks to each of the blocks.

Example 5.1. In the network of Figure 3 let T be the tree induced by all the branches a_k and all the horizontal branches b_k, d_k, , . . . Then, the a_k will comprise a full set J of joints. For $\mathcal{N}_1$ we may choose the subnetwork induced by branches a_0, b_0, b_{-1}, and c_0; in $\mathcal{N}_1$ there is but one chord, namely, c_0. For $\mathcal{N}_2$ use a_1, b_1, c_1, the chord being c_1. For $\mathcal{N}_3$ use a_{-1}, b_{-2}, c_{-1}, the chord being c_{-1}. For $\mathcal{N}_4$ use d_0, d_{-1}, e_0, the chord being e_0. For $\mathcal{N}_5$ use a_2, b_2, c_2, the chord being c_2. For $\mathcal{N}_6$ use d_1 end e_1 with e_1 being the chord. For $\mathcal{N}_7$ use a_{-2}, b_{-3}, and c-2 with c_{-2} being the chord, For $\mathcal{N}_8$ use d_{-2} and e_{-1} with e_{-1} as the chord. For $\mathcal{N}_9$ use f_0, f_{-1}, and g_0, with g_0 as the chord. Continuing in this triangular fashion, we can choose our $\mathcal{N}_k$ such that condition (i) of Theorem 5.1 is satisfied. (Note that each chord-tree loop lies to the left of' its chord and each chord-limb pnth lies to the right of its chord.) Assume there is no mutual coupling so that (ii) is trivially satisfied. Now each $\mathcal{N}_k$ has only one chord, and so Z_k is a 1X 1 matrix. Z_k will be nonsingular if each chord impedance is nonzero; in this case condition (iii) will also be satisfied. Upon assigning a current to each a_k, we can conclude from Theorem 5.1 that the network has a unique current in every branch.

Example 5.2. In the network of Figure 4 we choose as our tree T the subgraph induced by the branches 1, 2, 5, 6, 9, 10, 13, ... The chords are now the diagonal branches, Since T is simply an endless path, only one joint is needed for our full set J; let this be branch 1. Choose for $\mathcal{N}_1$ the subgraph induced by branches 1, 2, 3, 4, 5, for $\mathcal{N}_2$ that induced by branches 6, 7, 8, 9, for $\mathcal{N}_3$ that induced by branches 10, 11, 12, 13, and so forth. Condition (i) can now be seen to be satisfied. With no mutual coupling, condition (ii) is also satisfied. Finally, if the bridge corresponding to each $\mathcal{N}_k$ is unbslenced (i.e., letting z_k be the impedance in branch k, we require that $z_2z_5 \neq z_3z_4$, $z_6z_9 \neq z_7z_8$, ...), then each z_k is a 2% 2 nonsingular matrix so that condition (iii) is fulfilled. Thus, upon choosing the current in branch 1, we determine the currents in all other branches,

6. The set of branch-current vectors. Our objective in this section is to develop an expression for the branch currents and then to discuss the dimensionality of the space of homogeneous current flows. We assume throughout this section that the hypothesis of Theorem 5.1 holds,

Let us number all the branches of our network using the positive integers. $\mathbf{i} = [i_1, i_2, i_3, \ldots]^T$ will denote the branch-current vector, where i_m is the current in the mth branch measured with respect to the orientation of that branch. Here as well as elsewhere, the superscript T denotes matrix transpose. Thus, $\mathbf{i}$ is an $\aleph_0 \times 1$ vector, Similarly, $\underset{\sim}{e} = [e_1, e_2, \ldots]^T$ will denote the branch-voltage source vector, where e_m is the voltage rise in the mth branch measured in the direction of the branch's orientation.

Next, we number all the joints consecutively using again the positive integers until all joints are numbered. Thus, each joint will have two numbers assigned to it, its number as a branch and its number as a joint. $\underset{\sim}{j} = [j_1, j_2, \ldots]^T$ will be the joint-current vector, where j_m is the current in the mth joint measured in the direction of the joint's orientation.

Similarly, in addition to a chord's branch number, we assign a chord number to every chord, but now we do so in a special way. We consecutively number the chords 1, 2, 3, ..., starting first with the chords in $\mathcal{n}_1$, then proceeding to the chords in $\mathcal{n}_2$, then to the chords in $\mathcal{n}_3$, and so forth. $\underset{\sim}{c} = [c_1, c_2, \ldots]^T$ is the chord-current vector, where c_m is the current in the mth chord measured in accordance with the chord's orientation. Next, we let

$$\underset{\sim}{c}_k = [c_{n_1+\cdots+n_{k-1}+1}, \ldots, c_{n_1+\cdots+n_k}]^T$$

be the vector of chord currents for the chords in $\mathcal{n}_k$, where $k = 1, 2, \ldots$ and n_k is the number of chords in $\mathcal{n}_k$. Thus, $\underset{\sim}{c}$ can be written in partitioned form as

$$\underset{\sim}{c} = \begin{bmatrix} \underset{\sim}{c}_1 \\ \hline \underset{\sim}{c}_2 \\ \hline \cdot \\ \cdot \\ \cdot \end{bmatrix}$$

Let us now write the Kirchhoff-loop-law equations for the chord-tree loops corresponding to the chords in $\mathcal{n}_1$. In matrix form we obtain

$$Z_1 \underset{\sim}{c}_1 = E_1 \underset{\sim}{e} + J_1 \underset{\sim}{j}$$

where the matrices E_1 and J_1 have n_1 rows and each of their rows has only a finite number of nonzero entries. The same equations for $\mathcal{N}_2, \ldots, \mathcal{N}_k, \ldots$ yield

$$Z_2 \underset{\sim}{c}_2 = E_2 \underset{\sim}{e} + J_2 \underset{\sim}{j} + X_{2,1} \underset{\sim}{c}_1$$

$$\begin{aligned} Z_k \underset{\sim}{c}_k &= E_k \underset{\sim}{e} + J_k \underset{\sim}{j} + X_{k,1} \underset{\sim}{c}_1 + X_{k,2} \underset{\sim}{c}_2 + \cdots \\ &\vdots \qquad\qquad + X_{k,k-1} \underset{\sim}{c}_{k-1} \end{aligned}$$

As before, E_k and J_k have n_k rows and each row has only a finite number of nonzero entries. Moreover, $X_{k,m}$ is an $n_k \times n_m$ matrix. Since by assumption each Z_k is nonsingular, we may eliminate the $\underset{\sim}{c}_1, \ldots, \underset{\sim}{c}_{k-1}$ from the equation for $Z_k \underset{\sim}{c}_k$ by using the prior equations. Then, solving that equation for $\underset{\sim}{c}_k$, we get

$$\underset{\sim}{c}_k = W_{E,k} \underset{\sim}{e} + W_{J,k} \underset{\sim}{j}$$

where again $W_{E,k}$ and $W_{J,k}$ have n_k rows and each row has only a finite number of nonzero entries. Upon setting

$$W_E = \begin{bmatrix} W_{E,1} \\ \hline W_{E,2} \\ \hline \vdots \end{bmatrix}, \qquad W_J = \begin{bmatrix} W_{J,1} \\ \hline W_{J,2} \\ \hline \vdots \end{bmatrix},$$

we obtain

$$\underset{\sim}{c} = W_E \underset{\sim}{e} + W_J \underset{\sim}{j} .$$

We now define some incidence matrices. Let C_C be the matrix whose k,m entry is 1 or -1 if branch m is in the chord-limb orb for chord k and the orientations of the branch and the orb agree or, respectively, disagree. (We orient the orb in accordance with the orientation of its chord, and similarly for chord-tree loops and joint-limb paths.) That k,m entry is zero if branch m is not in the chord-limb orb for chord k. Similarly, let C_J be the matrix whose k,m entry is 1 or -1 if branch m is in the joint-limb path for joint k with agreement or respectively disagreement in the orientations and whose k,m entry is zero if branch m is not in the joint-limb path for joint k. Since only a finite number of joint-limb paths and chord-limb orbs pass through any given branch, each column of C_c or C_J contains only a finite number of nonzero entries. As a result, we have the following equations relating the branch-current vector $\underset{\sim}{i}$ to the branch-voltage-source vector $\underset{\sim}{e}$ and the joint-current vector $\underset{\sim}{j}$.

$$\underset{\sim}{i} = C_C{}^T \underset{\sim}{c} + C_J{}^T \underset{\sim}{j}$$

$$= C_C{}^T W_E \underset{\sim}{e} + (C_C{}^T W_J + C_J{}^T)\underset{\sim}{j}$$

$$= K\underset{\sim}{e} + H\underset{\sim}{j}$$

Here, K has an infinity of columns and the number n_J of columns of H equals the number of joints. Thus, $n_J = \aleph_0$ if there are an infinity of joints. On the other hand, each row of K and H has only a finite number of nonzero entries. This expression shows that each branch current is a function not only of the branch voltage sources but also of the assumed joint currents. In other words, even when all branch voltage sources are fixed, there is an infinity of possible branch-current vectors so long as there exists at least one joint. The latter occurs if the network $\mathfrak{N}$

contains at least one endless path.

Let us now discuss the branch currents occurring when every branch voltage source is zero. We shall say that the branch-current vector $\mathbf{i}$ is homogeneous whenever $\mathbf{e} = \mathbf{0}$. In this case, $\mathbf{i} = H\mathbf{j}$, and it follows that the set $\mathcal{J}$ of all homogeneous branch-current vectors is a linear space under componentwise addition.

Next, let $\mathbf{1}_k$ denote the $n_J \times 1$ vector whose kth entry is 1 and whose other entries are all 0. The, with j_k being the kth joint current, we have

$$\mathbf{i} = \sum_{k=1}^{n_J} j_k H \mathbf{1}_k \quad .$$

Observe that   is a linearly independent set in the space $\mathcal{J}$. Indeed, each joint carries no joint current other that its own and no chord current; consequently, the entry in $E\mathbf{1}_k$ corresponding to the branch number for, say, the mth joint if 1 for $k = m$ and is 0 for $k \neq m$. We can conclude that, when n_J is finite, $\{H\mathbf{1}_k\}$ is a basis for $\mathcal{J}$, and $\mathcal{J}$ is n_J-dimensional. When n_J is $\aleph_0$, $\mathcal{J}$ is infinite-dimensional.

Theorem 6.1. Under the hypothesis of Theorem 5.1, the cardinality of a full set J of joints is equal to the dimension of the linear apace $\mathcal{J}$ of all homogeneous. branch-current vectors.

From this theorem and Lemma 3.2 it follows that all trees, for which the hypothesis of Theorem 5.1 is satisfied, possess the same cardinal number of joints and therefore the same cardinal number of limbs,

7. A general class of networks satisfying the conditions of Theorem 5.1. We shall now describe a fairly broad class of networks that satisfy the hypothesis of Theorem 5.1, One condition we shall impose is that there be no parallel branches, However,

in those cases were mutual coupling does not exist this is no restriction because parallel branches can then be combined into a single branch through Thevenin's theorem.

We make use of the following standard terminology. If N is a subset of the set of nodes of our network $\mathcal{N}$, the subgraph induced by N is the graph whose node set is N and whose branch set consists of those branches in $\mathcal{N}$ having all their nodes in N. This subgraph will be denoted by $\langle N \rangle$. Here are the restrictions on the graphs of those networks we wish to consider.

Conditions 7.1. Let $\mathcal{N}$ be a connected, countably infinite, locally finite network having no parallel branches, isolated nodes, or self loops and satisfying the roll-owing conditions: Its set of nodes can be partitioned into (finite or infinite) subsets N_1, N_2, N_3, ... such that at least one N_i has two or more nodes and such that

(i) each branch either lies entirely within $\langle N_i \rangle$ or has one node in $\langle N_i \rangle$ and the other node in $\langle N_{i+1} \rangle$ for some i,

(ii) each node in N_i is adjacent to one or more nodes in N_{i+1}, and finally

(iii) no two distinct nodes in N_i are adjacent to the same node in N_{i+1}.

The networks of Figures 1 through 3 satisfy Conditions 7.1. For example, in Figure 3 we may take all the nodes lying on a vertical line as comprising one of the N_i.

We will now indicate how appropriate choices of a tree T, a full set J of joints, and the subnetworks $\mathcal{N}_k$ can be made. Before doing so however it will be convenient to introduce still more terminology. By a radial branch we mean a branch with one node in N_i and the other node in N_{i+1} for some i. A tangential branch is a branch with both nodes in N_i for some i.

The choice of T. For T we choose a subgraph in $\mathcal{N}$ such that every radial branch is in T, and, for every i, the branches that are both in T and in $\langle \bigcup_{k=1}^{i} N_k \rangle$ induce a spanning tree in $\langle \bigcup_{k=1}^{i} N_k \rangle$.

Actually, T can always be chosen by first choosing a tree in $\langle N_1 \rangle$, then extending the tree by adding all radial branches between N_1 and N_2, then appending enough branches in $\langle N_2 \rangle$ to extend the tree to all nodes in $\langle N_2 \rangle$, then adding all the radial branches between N_2 and N_3, and so forth. Clearly the resulting subgraph T contains all nodes of $\mathcal{N}$ but no finite loops; that is, T is a spanning tree in $\mathcal{N}$.

The choice of J. Every tangential T-branch is chosen to be a joint, and, in addition, for each i and ehch node n_0 in $\langle N_i \rangle$, all but one of the branches that are incident to n_0 and connect to nodes in N_{i+1} are also chosen as joints.

We hsve to show that these choices comprise a full set J of joints, We will use the fact that all radial brsnches are in T. By Condition 7.1(ii), from every node n_0 there exists a one-ended path starting at n_0 and consisting entirely of radial,branches. By Condition 7.1 (iii), any one-ended entirely radial path starting at one node of a tangential T-branch does not meet any of the nodes of the one-ended entirely radial paths 'starting at the node of that T-branch. Thus, that tangential T-branch lies in at least one endless path in T and can therefore be chosen as a joint, Indeed, every tangential T-branch can be chosen as a joint since the removal of any such branch does not disturb the aforementioned endless paths passing through the other tangential T-branches, Now, consider the subgraph induced by all radial branches. If, for some node n_0 in N_i, there are two or more radial brenches incident to n_0 and to nodes in N_{i+1}, then there exists at least one endless path consisting of two one-ended entirely radial paths starting at n_0. The removal of any branch incident to n_0 and a node in N_{i+1} breaks at least one such endless T-path, We may continue removing radial branches incident tc n_0 and nodes in N_{i+1} until all but one of them are removed. If we do this at every node, we will obtain the remaining joints as indicated in our choice cf J.

Since at least one N_i contains two or more nodes, $\mathcal{J}$ is not empty; moreover, T and $\mathcal{T}$ posses:; at least one endless path.

The graph remaining sfter the removal of all these joints from T does not contain an endless path since every endless T-path must either pass through a tangential T-branch or pass through two adjacent radial branches incident at a node in N_i and two distinct nodes in N_{i+1}. So truly, J Is a full set of joints.

For later use some observations are worth making at this point.

<u>Lemma 7.1</u>. If we remove all joints from T, we obtain a set of limbs. each of which is a cne-ended path consisting entirely of radial branches. Moreover, every chord-limb orb in $\mathcal{T}$ is an endless path whose only tangential branch is the chord itself'. If thnt chord lies in $\langle N_i \rangle$, then its chord-limb path is contained in $\langle \bigcup_{k=i}^{\infty} N_k \rangle$.

This lemma follows directly from our choices of T and J.

<u>Lemma 7.2.</u> Let $\mathcal{T}$ satisfy Conditions 7.1 and let T be chosen as indicated above. Choose a chord b in, say, $\langle N_i \rangle$. Then, the b-tree loop is contained in $\langle \bigcup_{k=1}^{i} N_k \rangle$. Moreover, the b-tree loop and b-limb path have only one branch in common, b itself. Moreover, the other chord-limb paths that have branches in common with the

b-tree loop are finite in number and correspond tc chords in those $\langle N_k \rangle$ for which $k < i$.

This lemma follows easily from the preceding one.

<u>The choice of the partition</u> $\{\mathfrak{N}_k\}$. In the following procedure, each $\mathfrak{N}_k$ is ccnstructed by first choosing a certain specified chord. If that chord does not exist, it is understood thet the corresponding step is simply skipped.

Choose any chord b_1 in $\langle N_1 \rangle$ and let $\mathfrak{N}_1$ be the unique b_1-tree loop.

Next, choose a chord c in $\langle N_2 \rangle$ and examine the c-tree lcop. By Lemma 7.2, there will be at most a finite number of chord currents flowing in the various branches of that loop. Denote the corresponding chords by b_2, b_3, ... , b_α. By Lemma 7.1, b_2, ... , b_α will all lie in $\langle N_1 \rangle$. Choosing $k = 2, \ldots, \alpha$ in turn, we let $\mathfrak{N}_k$ be induced by b_k and those branches of the b_k-tree loop that have not been assigned to prior subnetworks. We then let $\mathfrak{N}_{\alpha+1}$ be induced by $c = b_{\alpha+1}$ and the other unassigned branches of the $b_{\alpha+1}$-tree loop.

Next, we choose any chord $b_{\alpha+2}$ in $\langle N_1 \rangle$ and construct $\mathfrak{N}_{\alpha+2}$ as before. Following this, we choose a chord in $\langle N_2 \rangle$ and proceed as in the preceding paragraph to construct several more subnetworks, say, $\mathfrak{N}_{\alpha+3}, \ldots, \mathfrak{N}_\beta$. Then, we choose a chord d in $\langle N_3 \rangle$ and examine the d-tree loop ascertaining which chord-limb paths meet it for chords that have not as yet been used to generate subnetworks. With these latter chords, we generate as before subnetworks first using those chords in $\langle N_1 \rangle$ and then using those chords in $\langle N_2 \rangle$ and any other needed chords in N_A, obtaining thereby $\mathfrak{N}_{\beta+1}, \ldots, \mathfrak{N}_\gamma$. Finally, we let $\mathfrak{N}_{\gamma+1}$ be induced by $d = b_{\gamma+1}$ and those branches in the d-tree loop that have not as yet been assigned to subnetworks.

Proceeding in this triangular fashion, we can choose our subnetworks in such a way that each chord. b_k Lies in some $\mathcal{N}_k$ and every $\mathcal{N}_k$ contains only one chord b_k. Note also that condition (i) of Theorem 5.1 is automatically satisfied by this procedure. Indeed, given any chord b_n, all branches of the b_n-tree loop that do not lie in $\mathcal{N}_k$ for some $k < n$ induce $\mathcal{N}_n$ so that that loop lies in $\bigcup_{k \leq n} \mathcal{N}_k$. On the other hand, no branch of the b_n-limb path can lie in $\bigcup_{k \leq n-1} \mathcal{N}_k$, for; otherwise our procedure would have required us to choose b_n as *the*.chord for one of the prior $\mathcal{N}_j$, where $j \leq n - 1$. Thus, the b_n-limb path lies in $\mathcal{N} - \bigcup_{k \leq n-1} \mathcal{N}_k$.

The objective of this section is the following result.

~~Theorem~~ <u>7.1</u>. Let $\mathcal{N}$ be a network satisfying Conditions 7.1. Assume that each branch is a series connection of a complex impedance and a complex voltage source, either or both of which may be zero. Choose T, J, and $\{\mathcal{N}_k\}$ as indicated above, and assume that each chord impedance is nonzero. Also, assume that there are neither current sources nor mutual coupling. Then, $\mathcal{N}$ satisfies the hypothesis of Theorem 5.1.

<u>Proof</u>. We have already noted that $\mathcal{N}$ satisfies condition (i) of Theorem 5.1. Since there is no mutual coupling, condition (ii) is trivially satisfied.

Now, each $\mathcal{N}_k$ possesses exactly one chord b_k, and so the matrix Z_k described in condition (iii) is a 1×1 matrix whose element is the impedance In chord b_k. Since this is nonzer[illegible], [illegible]$_k$ nonsingular. It follows that $\mathcal{N}$ does satisfy all the assumptions in Theorem 5.1.

<u>Corollary 7.1</u>. If $\mathfrak{N}$ satisfies the hypothesis of Theorem 7.1, then, $\mathfrak{N}$ possesses a nonzero homogeneous current flow.

<u>Proof</u>, Since J is not empty, we can choose at least one joint current as nonzero, This yields a nonzero homogeneous current flow when all sources are set equal to zero,

<u>8. More examples</u>, We have already mentioned that the networks of Figures 1, 2, and 3 satisfy Conditions 7.1. In fact, they also satisfy the hypothesis of Theorem 7.1 if suitable choices of T, J, and $\{\mathfrak{N}_k\}$ and appropriate assumptions on the element values are male. On the other hand, the network of Figure 4 does not appear to satisfy Conditions 7.1 since condition (iii) seems unattainable; in fact, there does not appear to be any way of choosing T and the $\{\mathfrak{N}_k\}$ such that each $\mathfrak{N}_k$ has only one chord.

We shell now present a number of other examples. Some of the following networks can satisfy the hypotheses of both Theorems 5.1 and 7.1. Others do not appear to do so. Henceforth, the branches in the tree T will be denoted by solid lines, with the joints being indicated by the thickest lines, The chords will be denoted by dotted lines. We indicate in this way our choices of T and J. In all but the penultimate example we will also indicate what our choices of the $\mathfrak{N}_k$ are. In these cases appropriate element values can then be 'assigned to insure that the corresponding matrices Z_k are nonsingular and that mutual coupling either satisfies consition (ii) of Theorem 5.1 or is nonexistent for Theorem 7.1,

<u>Example</u> 8.1. Figure 5 shows a network that can satisfy the hypotheses of both Theorems 5.1 and 7.1. An appropriate set of

$\mathcal{N}_k$ can be obtained by modifying the triangular procedure of Example 5.1. For Conditions 7.1, N_1 is the vertical line of joints, and, for $i = 2, 3, \ldots$, each N_i is taken to be a pair of vertical lines of branches equally distant from N_1.

Example 8.2. The network of Figure 6 is another network that can satisfy the hypotheses of Theorems 5.1 and 7.1. An appropriate set of $\mathcal{N}_k$ is indicated by the numbers within the faces of this plane network. This numbering can be continued by extending the indicated triangular array. Each $\mathcal{N}_k$ consists of those branches that border the face k and another face m, where m > k. For the N_i of Conditions 7.1 we can choose pairs of lines paralleling and equally distant from the boldface line (i.e., the set of joints) in Figure 6.

Example... The network of Figure 7 can satisfy the hypothesis of Theorem 5.1 when T, J, and $\mathcal{N}_k$ are chosen as shown. The $\mathcal{N}_k$ are indicated as in the preceding example. However, the network does not seem to satisfy the hypothesis of Theorem 7.1 because there does not appear to be any choice of the N_i for which Condition 7.1 (iii) is satisfied. Nevertheless, every $\mathcal{N}_k$ in Figure 7 ccntains exactly one chord.

Example 8.4. We have already remarked that there does not seem to be any way of choosing the $\mathcal{N}_k$ in Figure 4 such that every $\mathcal{N}_k$ contains only one chord. Figure 8 is another such network. Thus, the hypothesis of Theorem 7.1 does not appear to be satisfied,. A set of $\mathcal{N}_k$ such that the hypothesis of Theorem 5.1 can be satisfied. is indicated as follows. $\mathcal{N}_1$ is induced by the five branches labeled 1, $\mathcal{N}_2$ by the branches labeled 2, $\mathcal{N}_3$ by the branches labeled 3, etc.

Example 8.5. Figure 9 shows a network for which there does not seem to be any way of choosing the T, J, and $\mathcal{N}_k$ such that the $\mathcal{N}_k$ are finite subnetworks. For example, if we choose T and J as shown and then write Kirchhoff's voltage law around the chord-tree loop for chord 1, we find that the equation contains the current for chord 2. So, if $\mathcal{N}_1$ contains chord 1, it must also contain chord 2 if Z_1 is to be a square matrix. But, by the same reasoning, it must also contain chord 3. Continuing in this fashion, we see that $\mathcal{N}_1$ must contain an infinity of chords and hence cannot be a finite subnetwork. Thus, it appears that Theorems 5.1 and 7.1 are not applicable to this network,

Example 8.6. Another fairly general class of networks for which Theorem 5.1 holds is indicated in Figure 10. The nodes of the network are indicated by heavy dots, the limb branches by the thin solid lines, and the joints by the heavy solid lines. The chords are not shown. Instead, the dotted lines now delineate portions of the network to which the chords are restricted; in particular, the two nodes of any given chord are required to lie within a region enclosed by one of the dotted lines. This condition restricts the confluences between the chord currents in such a way that a proper sequence of finite subnetworks $\mathcal{N}_k$ can be chosen; each $\mathcal{N}_k$ will be a subnetwork contained within one of the dotted lines together with those limb branches that intersect the lower horizontal portion of that dotted line.

9. Countably infinite networks that are not locally finite, Under certain circumstances, a countably infinite network $\mathfrak{N}$ that; is not locally finite may also be analyzed by our method as follows. $\mathfrak{N}$ is expanded into a locally finite network $\mathfrak{N}_e$ by replacing every node n_0 of infinite degree with a one-ended or endless path P of short circuits. The branches b_k incident to n_0 are then connected to the nodes in P in such a fashion that the degree of each node in P is finite, By a proper arrangement of the branches b_k it may occur that $\mathfrak{N}_e$ satisfies the hypothesis of Theorem 5.1. A choice of joint currents then yields a current distribution in $\mathfrak{N}_e$, and this in turn yields a current distribution in $\mathfrak{N}$, which is obtained by assigning to any branch in $\mathfrak{N}$ the same current that it has in $\mathfrak{N}_e$ Kirchhoff's loop law will be satisfied around every loop in $\mathfrak{N}$, and Kirchhoff's node law will be satisfied at the nodes of finite degree but not necessarily at the nodes of infinite degree, Moreover, it is not difficult to show that every current distribution in $\mathfrak{N}$ that satisfies the loop and node laws, in this way can be obtained from an appropriate choice of joint currents in $\mathfrak{N}_e$.

10. Infinite operator networks. The results of this paper apply just as well to those infinite networks whose source values and impedance values, rather than being complex numbers, are respectively elements of a Hilbert space H and bounded linear operators on that space. (See [3], [4], [18], and [19].) Upon

applying our method to such a network, we find that the impedance matrix Z_k indicated in condition (iii) of Theorem 5.1 Is now a bounded linear operator on the direct sum $S = H \oplus \cdots \oplus H$, where H occurs as many times as there are chords in $\mathfrak{n}_k$. The only alteration we need make in Theorem 5.1 is to assume that Z_k is invertible on S. Similarly, to keep Theorem 7.1 in force we need merely revise the assumpticn on each chord impedance by assuming that it is an invertible operator on H.

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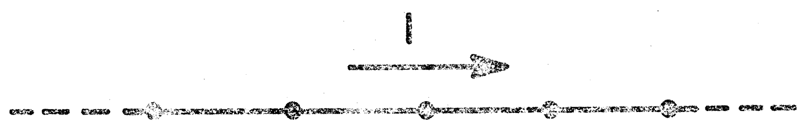


Fig*l

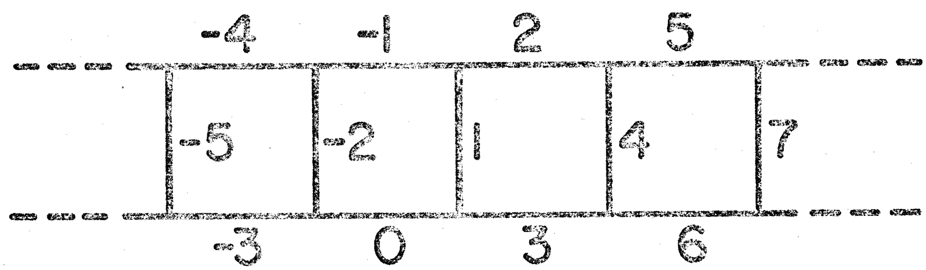
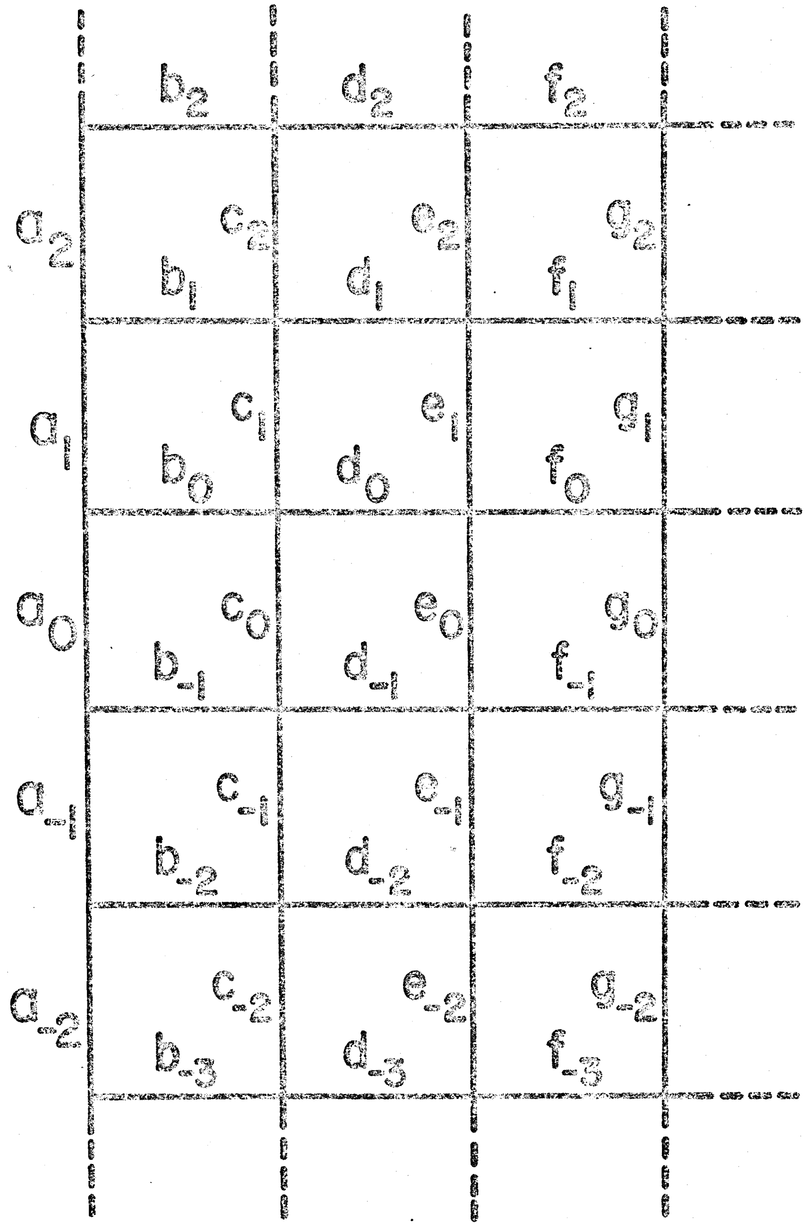


Fig. 2



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a2
c2
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g2
b1
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f1
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c1
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f0
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g0
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f-3

Fig. 3

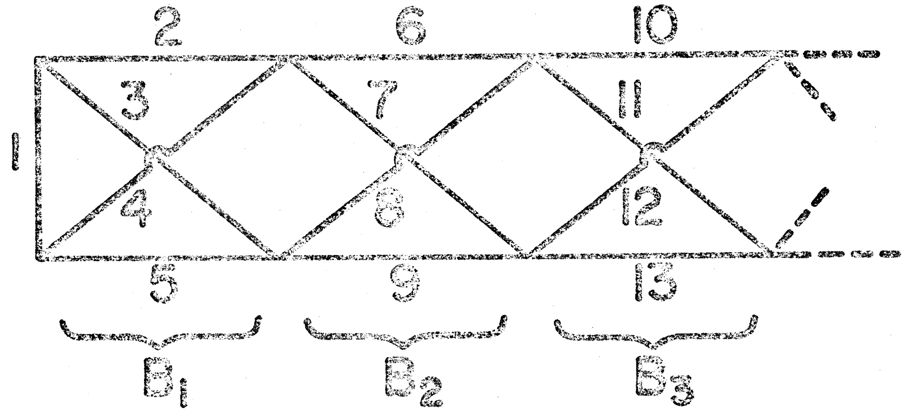

Fig. 4.

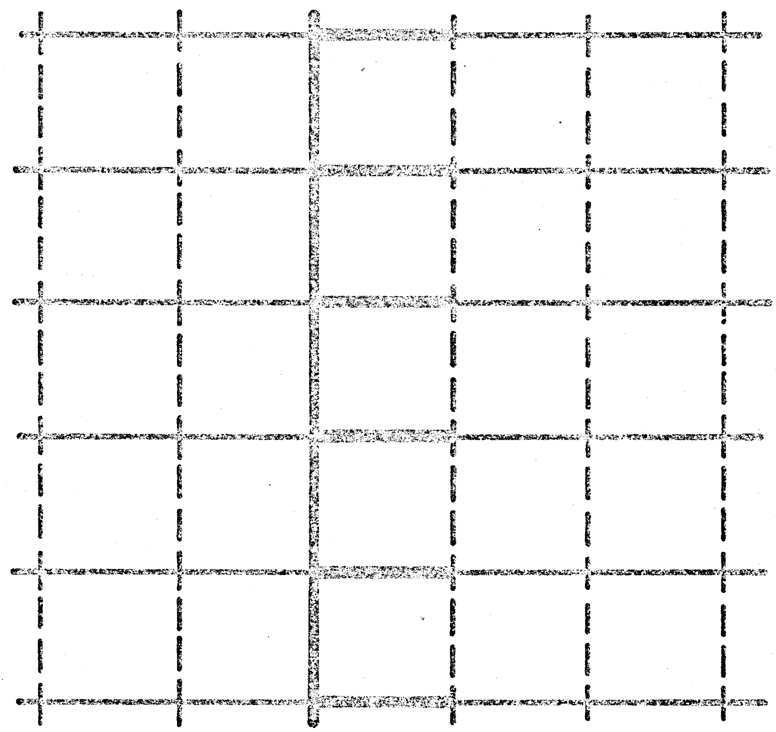

Fig.5

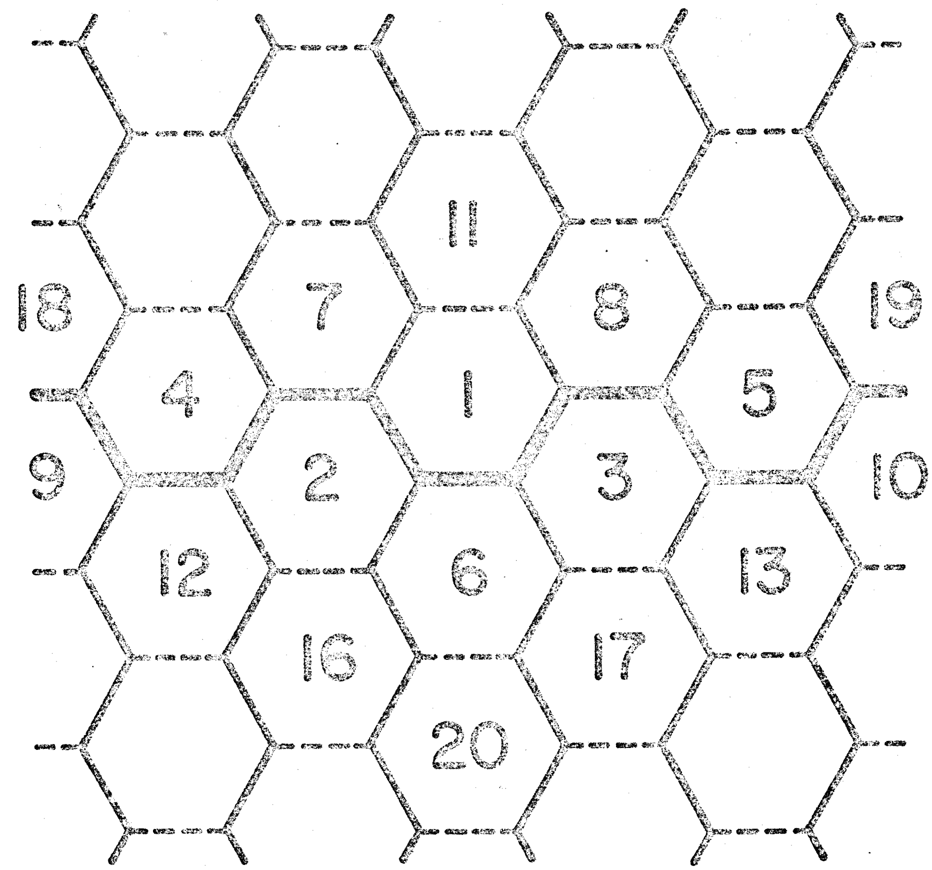

Fig.6

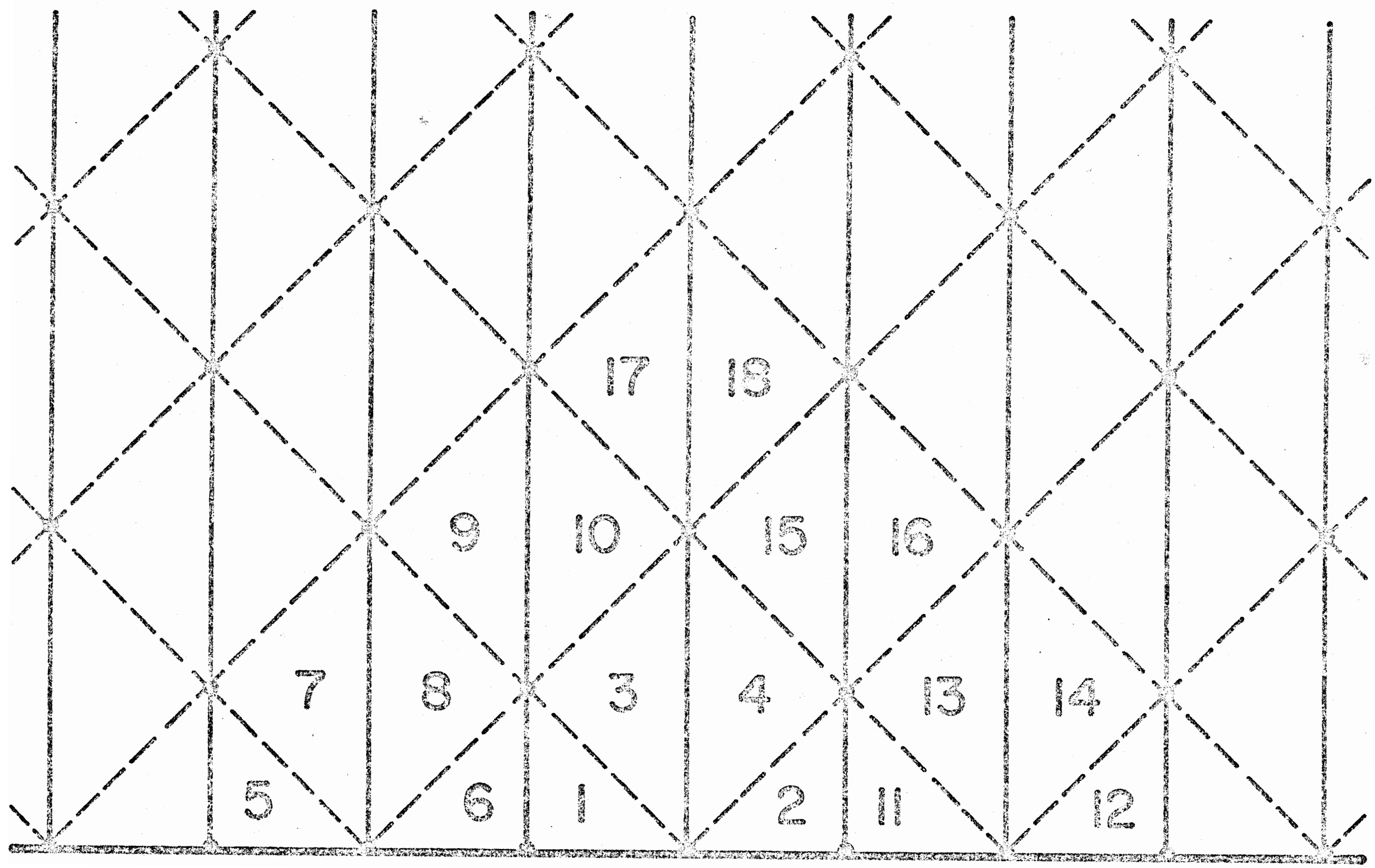

Fig.7

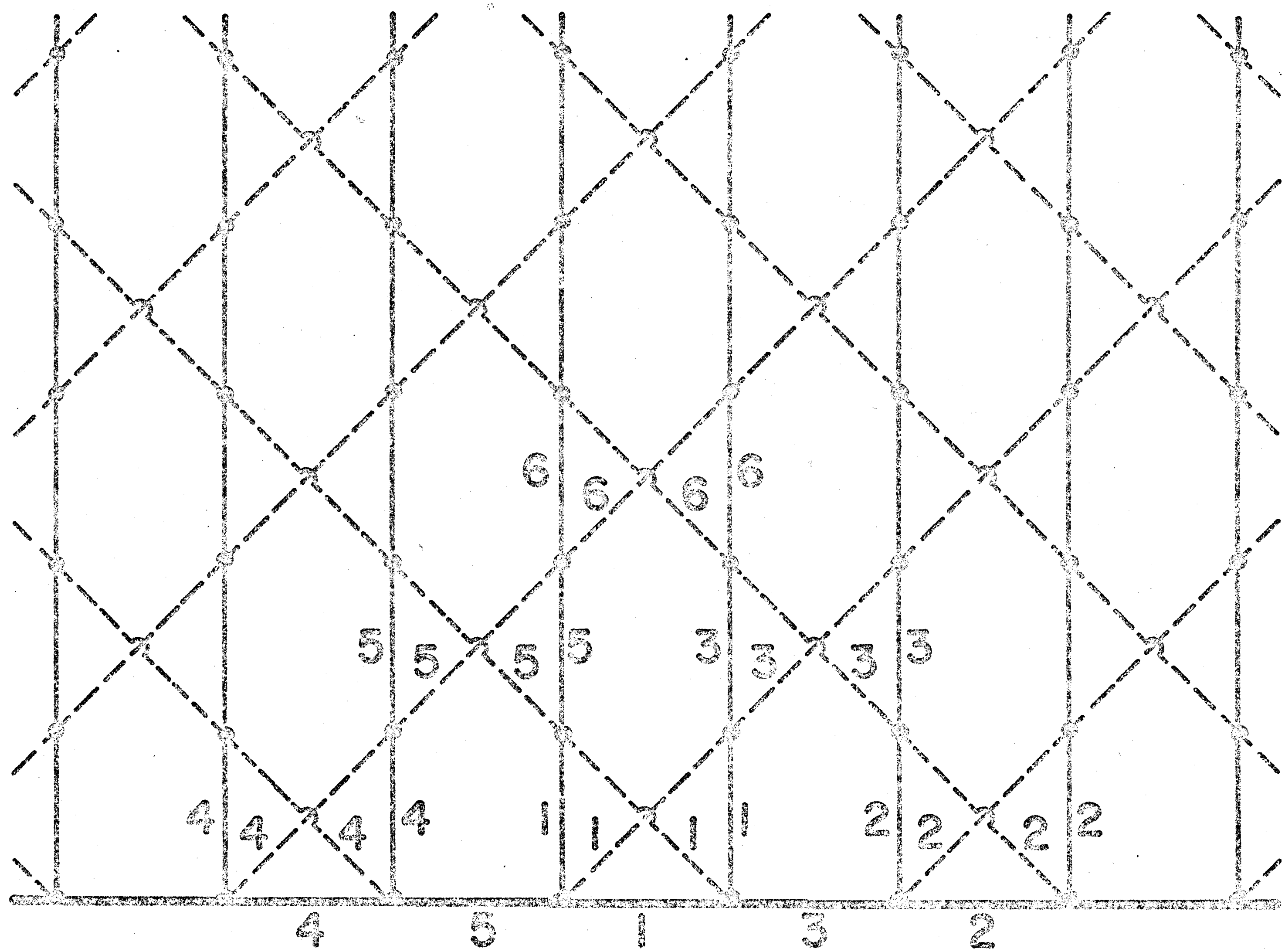

Fig.8

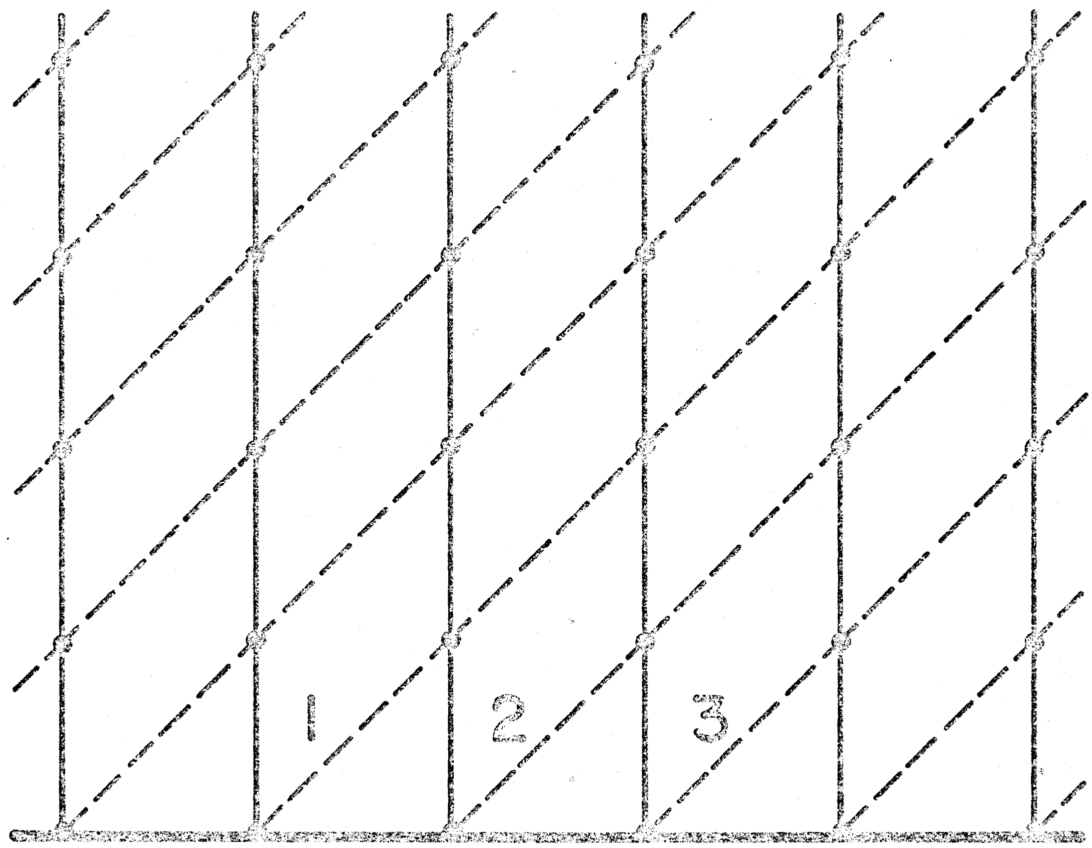

Fig.9

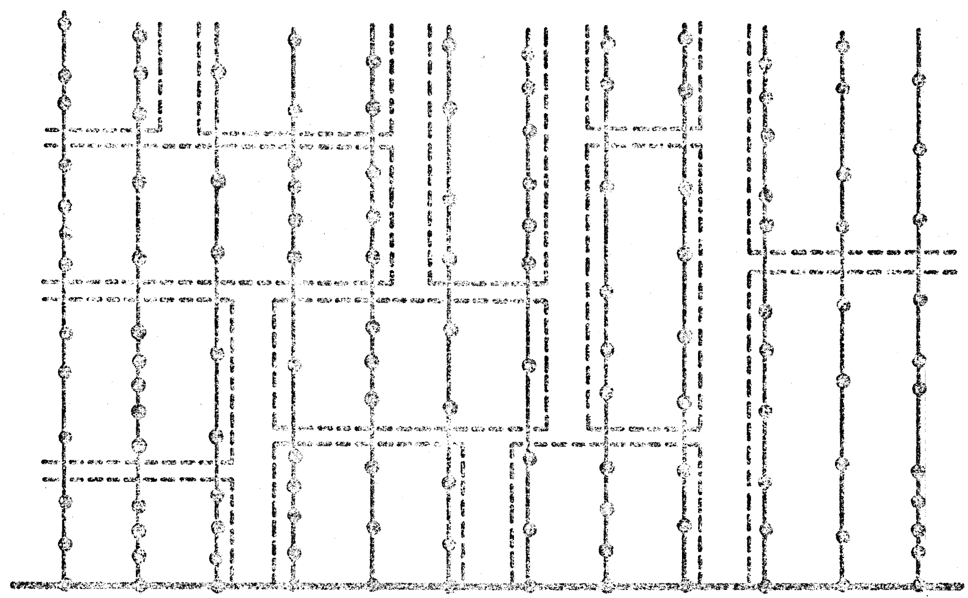

Fig. 10