STATE UNIVERSITY OF NEW YORK AT

STONY BROOK

College of Engineering Technical Report

No. 351

THE DYNAMIC BEHAVIOR OF TWO FIRMS SHARING A JOINT VENTURE

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This work was supported by the National Science Foundation under Grant MCS 78-01992

October 15, 1980

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Abstract. Two firms with constant capital run their separate operations and in addition cooperate in a joint venture. These three operations are periodic in that an investment at the beginning of a period receives a payoff at the end of the period. Between two periods each firm adjusts its capital allocation toward that operation that payed it a greater rate of return in the preceding period. Thus, each firm has a rule for capital allocation, but we do not specify the precise values of that rule in order to maintain the generality of our analysis. Given an initial capital allocation and the production functions of the three operations, the rules generate time series in the capital allocations. It is shown that, if the firms do not react too strongly to differences in their rates of return and if the rates of return do not vary too strongly with changes in capital input, then these time series converge to a unique globally stable equilibrium point. The location of the equilibrium point shows which operation will succeed and which will fail. It also indicates how the two firms will eventually interact.

*This work was supported by the National Science Foundation Grant MCS 78-01992.

Introduction

We consider herein the behavior of two firms, which in addition to their own operations, engage in a joint venture. We assume that the venture as well as the individual operations are periodic in the sense that investments made at the beginning of a period receive a payoff at the end of that period. At the latter time, each firm is free to decide how much of its capital it will invest into the joint venture for the forthcoming period. An example of this model might be two farmers who jointly own a cider mill and whose yearly capital investments are the quantities of apples they convert to cider, the alternative being to sell the apples whole. (This assumes that all apples are of uniform quality, no rotten or bruised ones that are good only for cider.) Another example might be two oil companies that jointly drill offshore. At the end of each exploration, each company is free to decide how much, if anything, it will invest into the next joint exploration.

We ask whether the joint venture will succeed, and, if it does, what share of the joint venture will each firm eventually settle upon. Actually, nine cases can arise because each firm may either abandon the joint venture, abandon its own enterprise, or support both. The case where a firm abandons both the joint venture and its own enterprise does not arise in our analysis since we assume that the firm has a fixed amount of capital which it must invest in either or both operations. We seek and find precise conditions under which the nine possibilities occur.

We attack our problem by examining the dynamic behavior of each firm and how it adjusts its capital allocations between its own operation and the joint venture. In this regard, our approach is a behaviorist one in the spirit of the Cyert-March behavioral theory of the firm [1], but the behavior we attribute to the firms is simply that they seek to improve the overall rates of return to their investments when adjusting their capital allocations. It is not a marginalist approach - at least at this point - in that the amount of that adjustment is left unspecified. We merely assume that at the end of each period the firm examines the two rates of return it has just received and then shifts its next capital allocation toward that enterprise that provided the better rate of return. These rates of return are determined by the production functions of the operations and the capital investments. We also assume consistency of behavior: Under the same circumstances (i.e., faced with the same received rates of return and the same prior capital allocation) the firm will make the same capital adjustment. However, the two firms are allowed to behave differently. Thus, each firm is characterized by its own rule for adjusting capital allocations. This rule will appear as a function g whose precise values we leave unspecified. We only impose some assumptions concerning the general shape of the g functions.

The adjustment rules generate time series in the capital allocations, and the question we are led to is whether those time series converge. The answer is yes if two conditions hold: The firms must not react too sharply to imbalances in their rates of return (they should not overreact), and the rates of return must not vary too radically with variations in the capital inputs (the operations should be relatively stable). These ideas are made precise by Conditions H below.

Under these circumstances we show that there is one and only one equilibrium point E toward which the time series converge. When none of the three operations are ever abandoned, E occurs where the three rates of return are equal - a classical marginalist result. When one or two of the operations are eventually abandoned, the location of E is determined by certain inequalities between the rates of return, which we explicate in Figure 4. It's noteworthy that the location of E does not depend upon the adjustment rules g. Those g merely determine how quickly convergence to E occurs.

It is also worth pointing out that we do not assume that the firms are in conflict, as we would were we to take a game-theoretic approach to their interaction. Nor is the purpose of the joint venture to coalesce interests or inhibit competition. Its purpose is taken to be profit, and the firms take part in it perhaps because they do not have enough capital to maintain both it and the initially desired levels of operation for their own enterprises or perhaps because the joint venture may initially appear to be too risky for any company to undertake alone.

A remark or two about notation: We use the symbol (a, b) to denote both an open interval in the real line and a point in the real plane. Which meaning is intended in any particular case will be clear from the context. A closed interval is denoted by [a, b], whereas (a, b] is an interval open on the left and closed on the right. $\mathring{\Theta}$ denotes the interior of a set Θ, and $\bar{\Theta}$ denotes its closure.

CAPITAL THETA "small circle"

The Model

Let there be two firms, one with capital N, the other with capital M available for investment. So as not to introduce too many symbols, we'll refer to these firms as "firm N" and "firm M". Also, let there be three ventures having the production functions X, Y, and Z, each of the latter being a function of its total investment. We'll call these "venture X", "venture Y", and "venture Z". Only firm N invests in venture X, only firm M invests in venture Z, whereas firms N and M invest jointly in venture Y. These investments are made at the times $t = 1, 3, 5, \ldots$. The returns to the firms occur at times $t+1 = 2, 4, 6, \ldots$. This is illustrated in Figure 1.

At time t, firm N invests the capital n_t into venture Y and the capital $N - n_t$ into venture X, and firm M invests the capital m_t into venture Y and the capital $M - m_t$ into venture Z. Throughout this work, M and N are positive numbers, and n_t and m_t are restricted to the intervals $0 \leq n_t \leq N$ and $0 \leq m_t \leq M$. These investments result in the profits $X(N - n_t)$ and $Z(M - m_t)$, which are returned to firms N and M respectively at time $t + 1$. At the same time, the profit $Y(n_t + m_t)$ is divided between firms N and M according to the customary rule that each firm shares that profit in proportion to its investment. Thus, firm N receives

$$u_{t+1} = \frac{n_t}{n_t + m_t} Y(n_t + m_t)$$

and firm M receives

$$v_{t+1} = \frac{m_t}{n_t + m_t} Y(n_t + m_t)$$

from the venture Y. Thus, at time t + 1, firm N receives the rates of return

$$(1) \qquad r_{xt} = \frac{X(N - n_t)}{N - n_t} = r_x(N - n_t)$$

and

$$(2) \qquad r_{yt} = \frac{Y(n_t + m_t)}{n_t + m_t} = r_y(n_t + m_t)$$

from ventures X and Y respectively. Also, firm M receives the rate of return (2) from venture Y and the rate of return

$$(3) \qquad r_{zt} = \frac{Z(M - m_t)}{M - m_t} = r_z(M - m_t)$$

from venture Z. We shall assume

<u>Condition R.</u> <u>Every rate of return (i.e., r_x, r_y, or r_z) is a continuous, positive-valued, strictly monotonic decreasing function defined on a closed interval [0, C] of values for its capital inputs.</u>

This implies that the production function X is a continuous function on the interval [0, N] with X(0) = 0 and positive on (0, N]. Moreover, X is such that every straight line passing through the origin and intersecting the graph of X at least once over (0, N), has exactly one intersection over (0, N) and passes from below that graph to above that graph as the independent variable increases. Similar conditions hold for Y and Z.

The Behavior of the Firms

How each firm decides to allocate its capital between its two ventures at the beginning of each investment period (at t = 1, 3, 5, ...) is yet to assumed. To this end, we use the procedure introduced in [2] and [3]. Consider firm N. Let

(4) $$\alpha_t = \frac{r_{yt}}{r_{xt}}$$

and

(5) $$\gamma_t = \frac{n_t}{N - n_t}.$$

We assume that at time t + 2 firm N adjusts its investments into ventures X and Y in accordance with the relative rates of return from these ventures at time t + 1. It does so by increasing (decreasing) its investment to that venture that provided the greater (respectively, lesser) rate of return. Greater discrepancies in the rates of return induce greater adjusments. These ideas are formulated by

(6) $$\gamma_{t+2} = g_N(\alpha_t, \gamma_t)$$

where the function g_N is assumed to satisfy the following conditions on the first quadrant. Here, g_N is represented by g, α_t by ξ, and γ_t by ζ.

<u>Conditions G</u>. <u>For</u> $0 \leq \xi < \infty$ <u>and</u> $0 \leq \zeta < \infty$,

<u>(i)</u> $g(\xi, \zeta)$ <u>is nonnegative and continuous</u>,

<u>(ii)</u> <u>for fixed</u> $\zeta > 0$ (<u>or fixed</u> $\xi > 0$), $g(\xi, \zeta)$ <u>is a strictly monotonically increasing function of</u> ξ (<u>respectively</u>, ζ), <u>and</u>,

<u>(iii)</u> <u>for some fixed</u> $\xi_0 \geq 1$, $g(\xi_0, \zeta) = \zeta$ <u>for all</u> ζ.

Similarly, the behavior of firm M is determined by the following three equations:

(7) $$\beta_t = \frac{r_{yt}}{r_{zt}},$$

(8) $$\delta_t = \frac{m_t}{M - m_t},$$

(9) $$\delta_{t+2} = g_M(\beta_t, \delta_t),$$

where g_M satisfies Conditions G with g replaced by g_M, ξ by β_t, and ζ by δ_t.

We set $\alpha_0 = \xi_0$, when g is replaced by g_N, and set $\beta_0 = \xi_0$, when g is replaced by g_M. α_0 need not be equal to β_0.

The meaning of Conditions G is the following. Consider (6), let $\xi = \alpha_t$, $\zeta = \gamma_t$, and $g = g_N$, and assume for the moment that $\xi_0 = 1$. Conditions G, especially its monotonicity requirement with respect to ξ and the condition $g(1, \zeta) = \zeta$, insure that $g(\xi, \zeta) < \zeta$ for $\xi < 1$. But, according to (4), the condition $\xi < 1$ means that for firm N the rate of return from the joint venture Y is not as good as the rate of return from its own venture X. The fact that $g(\xi, \zeta) < \zeta$ when $\xi < 1$ insures through (6) that firm N will cut back on its investment into the joint venture Y and will increase its investment into venture X. Similarly, $\xi > 1$ yields $g(\xi, \zeta) > \zeta$ and the result that firm N increases its investment into venture Y and decreases its investment into venture X. This is just the behavior we wish to model. The monotonicity requirement with respect to ζ insures that, when the ratio of returns (4) remains constant, adjustments from higher relative investments into the joint venture yield higher relative investments into the joint venture again.

When $\alpha_0 > 1$, firm N has a bias toward its own venture X. That is, the rate of return r_{yt} from Y must be larger than the r_{xt} from X in order for N to treat both X and Y as equally attractive. When $\alpha_0 = 1$, there is no such preference toward X. Analogous meanings attach to the behavior of firm M and ventures Y and Z when $\beta_0 > 1$ and $\beta_0 = 1$.

Equations (6) and (9) taken together comprise the rules by which the trajectory of the joint capital allocation $E_t = (n_t, m_t)$ can be determined for $t = 3, 5, 7, \ldots$ from the initial capital allocation (n_1, m_1). Indeed, (6) and (9) can be rearranged as follows.

$$(10) \qquad n_{t+2} = \frac{N}{1 + \dfrac{1}{g_N\left(\dfrac{r_{yt}}{r_{xt}}, \dfrac{n_t}{N-n_t}\right)}} = f_N(n_t, m_t)$$

$$(11) \qquad m_{t+2} = \frac{M}{1 + \dfrac{1}{g_M\left(\dfrac{r_{yt}}{r_{zt}}, \dfrac{m_t}{M-m_t}\right)}} = f_M(n_t, m_t)$$

For the sake of a simpler notation, we shall at times discard the subscript t on the variables n_t and m_t. Let

$$\Omega = \{(n, m): 0 \leq n \leq N,\ 0 \leq m \leq M\}.$$

Since g_N, g_M, and the rates of return are continuous functions, f_N and f_M are continuous on $\mathring{\Omega}$. Moreover, Conditions G insure that f_N and f_M have continuous extensions onto the boundary of Ω, which we take to be the definitions of f_N and f_M on that boundary. In essence, we are imposing another assumptions on the behavior of the firms. For instance, we are requiring that, at the boundary point $n = N$ and $m \in [0, M]$, firm N remains aware of what its return r_{xt} would be if it made a very small investment into venture X.

Equilibrium Points

Equations (10) and (11) are equivalent to (6) and (9) respectively. From any given initial values n_1 and m_1, they determine time series in the n_t and m_t. We say that $E = (n_0, m_0)$ is an equilibrium point for this system if the equations $n_1 = n_0$ and $m_1 = m_0$ imply that $n_t = n_0$ and $m_t = m_0$ for all $t = 3, 5, 7, \ldots$. A principal result of this paper is the following theorem. Its proof is given in the Appendix.

Theorem 1. Let there be two firms, which, in addition to their own exclusive ventures, may share a joint venture as indicated in Figure 1. Let the time sequence $E_1, E_3, E_5, \ldots$ of their joint capital allocations be determined by (10) and (11), where $E_t = (n_t, m_t)$. Assume that Conditions R and G are satisfied. Then, there exists at most one equilibrium point in the interior $\mathring{\Omega}$ of Ω. When there is no equilibrium point in $\mathring{\Omega}$, the boundary of Ω will have at least one equilibrium point.

Finding the equilibrium point within $\mathring{\Omega}$, if one exists therein, requires a closer examination of (10) and (11). First of all, the nonnegativity of g_N and g_M imply that f_N maps Ω into $[0, N]$ and f_M maps Ω into $[0, M]$. This is indicated in Figure 2, where the point $E_t = (n_t, m_t) \in \Omega$ is mapped under the simultaneous application of (10) and (11) into $E_{t+2} = (n_{t+2}, m_{t+2}) \in \Omega$. The direction of the shift from E_t to E_{t+2} is determined by two curves C_n and C_m, whose derivation is also given in the Appendix.

C_m is a continuous curve within Ω. It terminates on the lines $n = 0$ and $n = N$. With respect to the n axis, it is a continuous monotonic decreasing curve. If any portion of it lies within $\mathring{\Omega}$, the chords of that portion will have slopes strictly less than 0 and strictly greater than -1. Those parts of C_m that lie on the boundary of Ω will be contained within the $m = 0$ or $m = M$ lines. C_n has similar properties except that the roles of n and m are reversed. It follows that there is one and only one intersection point E between C_n and C_m. E may be anywhere within Ω including its boundary. All this is illustrated in Figure 2. Other configurations for C_n and C_m are indicated in the first column of Figure 4.

We show in the Appendix that the shift from E_t to E_{t+2}, as dictated by (10) and (11), is toward the curves C_n and C_m; that is, $n_{t+2} - n_t$ is a shift toward C_n, which may or may not pass beyond C_n, and $m_{t+2} - m_t$ is a shift toward C_m, which also may or may not pass beyond C_m. We illustrate such shifts in Figure 2 for the case where the individual shifts (i.e., the solid arrows) do not reach the said curves whereas the combined shift passes over C_m. It is also shown in the Appendix that when the initial point $E_t = (n_t, m_t)$ lies on one of the curves (say, C_m), then there is no shift in the corresponding variable (that is, $m_{t+2} = m_t$). It follows that C_m is a locus of equilibrium points for the m_t variable alone. Similarly, C_n is such a locus for n_t alone. Consequently, E is an equilibrium point for the system (10) and (11) taken together. These ideas comprise the essence of our first theorem's proof, and they lead in addition

Correction for page 13:

Replace the last paragraph on page 13

The only places in Ω where equilibriu

distinct from E may appear are the four co
the four points where C_n meets the edges m
C_m meets the edges n = 0 and n = N.

to the following corollary.

Corollary 1a. *Under the hypothesis of Theorem 1, the unique intersection point E between C_n and C_m is an equilibrium point for the system (10) and (11). When $E \in \mathring{\Omega}$, E is the only equilibrium point within $\mathring{\Omega}$. When $E \notin \mathring{\Omega}$, there is no equilibrium point within $\mathring{\Omega}$.*

Actually, the only places on the boundary that equilibrium points other than E can occur are the four corners of Ω. This is so because Conditions G allow the following possibilities. $f_M(n, 0)$ can equal zero along an interval $[\omega, N]$, where $0 \leq \omega \leq N$, in the n axis. $f_N(0, m)$ can equal zero along an interval $[\omega, M]$, where $0 \leq \omega \leq M$, in the m axis. $f_M(n, M)$ can equal M along an interval $[0, \omega]$, where $0 \leq \omega \leq N$, in the line $m = M$. $f_N(N, m)$ can equal N along an interval $[0, \omega]$, where $0 \leq \omega \leq M$, in the line $n = N$. At each of the four corners, a pair of such extreme values for f_N and f_M can occur simultaneously. When this happens, the corner can be an equilibrium point distinct from E.

Whether or not E lies within $\mathring{\Omega}$, it is the only possible limit of any convergent sequence arising from (10) and (11) and starting from within $\mathring{\Omega}$. More precisely, we have

Lemma 1. *Under the hypothesis of Theorem 1, let* $\{E_t: t = 1, 3, 5, \ldots\}$ *be a sequence of points* $E_t = (n_t, m_t)$ *determined by (10) and (11). Assume that* $(n_1, m_1) \in \mathring{\Omega}$. *If* $\{E_t\}$ *converges, it converges to* E.

See the Appendix for the proof of this lemma.

Another Restriction on the Behavior of the Firms.

The intersection point E between C_n and C_m may have another property besides being the only possible equilibrium point within $\mathring{\Omega}$. It may be the only stable equilibrium point anywhere in Ω (including Ω's boundary). We are able to prove this syability property in the case where the shift in n_t does not cross over the curve C_n and the shift in m_t does not cross over the curve C_m. A crossing of one or both of these curves may still occur when both shifts are considered together, as is indicated in Figure 2, but the shift in n_t alone or in m_t alone is to be prohibited from such a crossing.

The meaning of this condition is that the system reacts moderately; that is, the shifts in n_t and m_t are never too large.

Our purpose in this section is to impose some precise conditions on the functions g_N and g_M and the rates of return r_x, r_y, and r_z that insure the prohibition against crossings.

We need to define some more symbols. Set

(12) $$\beta(n, m) = \frac{r_y(n + m)}{r_z(M - m)}$$

and

(13) $$\delta(m) = \frac{m}{M - m}.$$

Next, let n be any fixed number in $[0, N]$. Then, define $m(n)$ as that value of $m \in (0, M)$ for which $r_y(n + m) = \beta_0 r_z(M - m)$, if such a value exists in $(0, M)$. (There will be at most one such $m(n)$ because $r_y(n + m)$ is a strictly monotonically decreasing function of m whereas $r_z(M - m)$ is a strictly monotonically increasing function of m.) In this case, we have $\delta(m(n)) = m(n)/(M - m(n))$. On the other hand, if $r_y(n) \leq \beta_0 r_z(M)$, we set $m(n) = 0$ and $\delta(m(n)) = 0$. Moreover, if $r_y(n + M) \geq \beta_0 r_z(0)$, we set $m(n) = M$ and $\delta(m(n)) = \infty$. Thus, in all cases $m(n)$ is that value of m for which $(n, m) \in C_m$.

Condition H (for firm M). Let $n \in [0, N]$ be given. Then, for every $\Delta m \neq 0$ such that $(n, m(n) + \Delta m) \in \Omega$, it is required that

$$g_M(\beta(n, m(n) + \Delta m), \delta(m(n) + \Delta m))$$

be greater than $\delta(m(n))$ if $\Delta m > 0$ and less than $\delta(m(n))$ if $\Delta m < 0$.

The meaning of this condition can be appreciated by examining the function g_M on the first quadrant. Upon fixing $n \in [0, N]$ and letting m vary from $m = 0$ to $m = M$, we obtain a locus of corresponding values for (12) and (13). This is illustrated in Figure 3. For example, consider the locus for $n = n_2$; $m(n_2)$ is that value of m for which the locus intersects the vertical line at $\beta = \beta_0$. On the other hand, there may be no such intersection, as is indicated for $n = N$. The actual shapes of these loci depend of course on just what the functions r_y and r_z are.

Consider any locus L that intersects the vertical line $\beta = \beta_0$ at some point I. Condition H for firm M states that, on the portion of L to the left and higher than I, the function g_M takes on values greater than its vlaue at I and, on the portion of L to the right and lower than I, the function g_M takes on values less than its value at I. No restriction is imposed if L does not intersect the line $\beta = \beta_0$.

Remember that $g_M(\beta, \delta)$ is a monotonic increasing function of β for fixed δ and a monotonic increasing function of δ for fixed β. Thus, our condition requires indirectly that g_M does not vary too strongly on either side of the line $\beta = \beta_0$ as β varies. This is equivalent to saying that firm M does not adjust its capital investments m_t too radically.

Note also that the flatter r_y and r_z are, the steeper will be the loci of Figure 3, and the more likely will be the fulfillment of Condition H for a given g_M; here too, flatter rates of return signify a more moderate system so far as the effects of investment variations are concerned.

This discussion can also be appled to firm N so long as its behavior function g_N satisfies an analog to the above Condition H. To state the appropriate analog, we now set

$$(14) \qquad \alpha(n, m) = \frac{r_y(n + m)}{r_x(N - n)}$$

and

$$(15) \qquad \gamma(n) = \frac{n}{N - n} .$$

For a given $m \in (0, M)$, we define $n(m)$ as that value of $n \in (0, N)$ for which $r_y(n + m) = \alpha_0 r_x(N - n)$ if such a value exists. If however $r_y(m) \leq \alpha_0 r_x(N)$, we set $n(m) = 0$ and $\gamma(n(m)) = 0$. If $r_y(N + m) \geq \alpha_0 r_x(0)$, we set $n(m) = N$ and $\gamma(n(m)) = \infty$. Thus, in every case $n(m)$ is that value of n for which $(n, m) \in C_n$.

Condition H (for firm N). Let $m \in [0, M]$ be given. Then, for every $\Delta n \neq 0$ such that $(n(m) + \Delta n, m) \in \Omega$, it is required that

$$g_N(\alpha(n(m) + \Delta n, m), \gamma(n(m) + \Delta n))$$

be greater than $\gamma(n(m))$ if $\Delta n > 0$ and less than $\gamma(n(m))$ if $\Delta n < 0$.

Our discussion for this case now continues exactly as before.

We prove the next lemma in the Appendix. It states that the aforementioned crossings of C_n and C_m do not occur.

Lemma 2. Let Conditions R, G, and H be fulfilled.

(i) Assume that $m = m_t$ is held fixed and that n_{t+2} is determined by (10). Let $n(m)$ be that unique point of $[0, N]$ such that $(n(m), m) \in C_n$. If $0 < n_t < n(m)$, then $n_t < n_{t+2} < n(m)$. If $n(m) < n_t < N$, then $n(m) < n_{t+2} < n_t$.

(ii) Assume that $n = n_t$ is held fixed and that m_{t+2} is determined by (11). Let $m(n)$ be that unique point of $[0, M]$ such that $(n, m(n)) \in C_m$. If $0 < m_t < m(n)$, then $m_t < m_{t+2} < m(n)$. If $m(n) < m_t < M$, then $m(n) < m_{t+2} < m_t$.

Convergence and Stability

Our second principal conclusion, given by the next theorem, asserts that under certain conditions the equilibrium point E is globally stable and is indeed the limit toward which the dynamic behavior of the two firms tends. This too is proven in the Appendix.

Theorem 2. Let there be two firms, which, in addition to their own exclusive ventures, may share a joint venture as indicated in Figure 1. Let the time sequence E_1, E_3, E_5, ... of their joint capital allocations be determined by (10) and (11), where $E_t = (n_t, m_t)$, and assume that $E_1 \in \mathring{\Omega}$. Also, assume that Conditions R, G, and H are satisfied. Then, the sequence E_1, E_3, E_5, ... converges. Moreover, there is a unique point $E \in \Omega$ (namely, the intersection between curves C_n and C_m) that is the limit of every sequence E_1, E_3, E_5, ... that starts in $\mathring{\Omega}$ (i.e., for which $E_1 \in \mathring{\Omega}$). Thus, E is the one and only globally stable equilibrium point for the system of Figure 1 under the assumption that the initial flows at t = 1 along the four legs of that figure are all nonzero.

Note that neither this theorem nor Theorem 1 requires a specification of particular functions for g_N and g_M, and in this way we maintain the generality of our model. The rate of convergence to E will depend of course on just what functions g_N and g_M may be. But, Conditions R and G alone insure the existence of the equilibrium point E, and Conditions R, G, and H alone insure that convergence to E will occur - regardless of its rate.

Special Cases: The Abandonment of Ventures

It can occur that, in the limit as $t \to \infty$, firms N and M together or singly abandon venture Y, or firm N abandons venture X, or firm M abandons venture Z. These situations arise precisely when the equilibrium point E occurs on the boundary of Ω. In fact, eight different possibilities arise, as indicated in Figure 4: E may be at one of the corner points of Ω, or it may be somewhere within the straight line segments between those corner points. For example, when E is at the origin, firms N and M eventually abandon venture Y and concentrate their investments on ventures X and Z respectively. On the other hand, when E is somewhere on the n axis, say, at $(n_0, 0)$ where $0 < n_0 < N$, then eventually firm M abandons Y and concentrates on Z, whereas firm N splits its investments by putting n_0 into Y and $N - n_0$ into X.

The necessary and sufficient conditions for the occurrence of each of the eight possible cases can easily be determined by examining how the surfaces $\alpha_0 r_x$, r_y, and $\beta_0 r_z$ must be situated with respect to each other in order for the intersection between C_n and C_m to occur at the various parts of Ω's boundary. The results are tabulated in Figure 4, where typical (but not all possible shapes)for C_n and C_m are also indicated. Since $E = C_n \cap C_m$ is unique, the listed necessary and sufficient conditions are mutually exclusive. For the sake of completeness, we also exhibit in Figure 4 a ninth case: No venture is abandoned; now, $E \in \mathring{\Omega}$. We are assuming once again that Conditions R, G, and H are fulfilled and that all dynamic processes start within $\mathring{\Omega}$.

The first eight degenerate cases are of interest because they indicate the different ways the three ventures may eventually become successes or failures and how the joint venture may eventually fall under the exclusive proprietorship of one of the firms. Let's examine each of the cases of Figure 4 from this point of view. (As a compromise between clarity and brevity, we will merely use the phrases "is less than" and "is greater than" when in fact we should always add the words "or equal to"; the phrases "is no greater than" and "is no less than" seem awkward. Also, by a "weighted rate of return" we mean r_x multiplied by α_0 or r_z multiplied by β_0.)

Case 1. The indicated necessary and sufficient conditions mean that the worst weighted rates of return from venture X and from venture Z are greater than the best rate of return from venture Y. In this case, venture Y ends in failure; both firms eventually abandon it.

Case 5. This is the extreme opposite of Case 1. Now, the worst rate of return from venture Y is greater than the best weighted rates of return from ventures X and Z. In this case, venture Y becomes a success, so much so that it preempts all the capital from firms N and M, and ventures X and Z end in failure.

Case 3. Now, the best weighted rate of return from venture X is less than the worst weighted rate of return from venture Z, and lying somewhere between these two values is the rate of return from venture Y for the case where venture Y has maximum capital input from firm N and no capital input from firm M. Our analysis has shown that in this case venture X fails whereas ventures Y and Z succeed with firm N exclusively taking over venture Y and firm M devoting all its capital to venture Z.

Case 7. This is the same as Case 3 except that firms N and M interchange their roles and ventures X and Z also interchange their roles.

Case 2. Consider for the moment the special circumstance where firm N, being the only firm to invest in venture Y, has some nonzero capital allocations n_0 and $N - n_0$ to ventures Y and X for which the rate of return from venture Y equals the weighted rate of return from venture X, Assume in addition that the worst weighted rate of return from venture Z is greater than this rate of return from venture Y. It follows in this case that, under any dynamic process, firm M eventually concentrates on venture Z whereas firm N eventually operates with the aforementioned capital allocations n_0 and $N - n_0$.

Case 8. This is the same as Case 2 except that the roles of firms N and M and the roles of ventures X and Z are again interchanged.

Case 4. Now, consider the special case where firm N puts all of its capital into venture Y. Assume that in this situation firm M has some nonzero capital allocations m_0 and $M - m_0$ such that its rate of return from venture Y equals its weighted rate of return from venture Z. Assume also that the best weighted rate of return from venture X is less than the indicated rate of return from venture Y. Then, any dynamic process will tend toward the situation where firm N abandons venture X and firm M operates with the capital allocations m_0 and $M - m_0$.

Case 6. This is Case 4 with the usual interchange between the roles of the firms and ventures.

Case 9. This is the nondegenerate case. It occurs when all three (weighted) rates of return are equal at some point in the interior of Ω. We have shown that there is at most only one such point.

Acknowledgement

The author is indebted to Professor Edward Ames who suggested the problem considered herein and provided a number of valuable comments.

Appendix

Proof of Theorem 1 and Corollary 1a. The nonnegativity of g_N and g_M implies that f_N maps Ω into [0, N] and f_M maps Ω into [0, M]. Equation (10) coupled with Conditions G indicates that $n_{t+2} - n_t$ and $r_{yt} - \alpha_0 r_{xt}$ will have the same sign, or they will both be zero together, so long as (n_t, m_t) lies in $\mathring{\Omega}$. Similarly, under the same condition on (n_t, m_t), $m_{t+2} - m_t$ and $r_{yt} - \beta_0 r_{zt}$ will have the same sign or they will both be zero together. What happens when (n_t, m_t) lies on the boundary of Ω requires a special consideration, which we undertake later on.

The sign of $r_{yt} - \beta_0 r_{zt}$ can be ascertained by comparing the surfaces (2) and (3) in the three-dimensional (n, m, r) space over the rectangle Ω, as indicated in Figure 5. Under Condition R, the surface $r_y(n + m)$ peaks over the origin and is monotonically decreasing over every straight line in Ω that passes through the origin. Each of its equielevation lines is a straight line whose projection onto Ω intersects the n and m axes at the same value. On the other hand, the surface $\beta_0 r_z(M - m)$ is a monotonically increasing function of m, and its equielevation lines are parallel to the n axis.

Since both surfaces are continuous and have positive values over $\mathring{\Omega}$, their intersection I_{yz} (if it exists) is a continuous curve above $\mathring{\Omega}$, and the projection of I_{yz} onto $\mathring{\Omega}$ is a continuous curve all of whose chords have slopes (with respect to the n axis) larger than -1 and less than 0. The latter fact follows directly from the assumption that r_y and r_z are continuous, strictly monotonically decreasing functions. (To see this, examine the point of intersection in m between $r_y(n + m)$ and $r_z(M - m)$. Because of the continuous and strict monotonicities, a shift to the left of the function $m \mapsto r_y(n + m)$ by $\Delta n > 0$ to get the function $m \mapsto r_y(n + \Delta n + m)$ yields a shift to the left in the point of intersection by the amount Δm, where $0 < \Delta m < \Delta n$.)

It may happen that, for some value of $n \in [0, N]$, $\beta_0 r_z(M - m)$ is larger (smaller) than $r_y(n + m)$ for all $m \in [0, M]$. In this case, the corresponding point on the I_{yz} curve is taken to be $(n, 0)$ (respectively, (n, M)) to reflect the fact that firm M would want to decrease (respectively, increase) its investment m into venture Y as much as possible. As a consequence, the projection of I_{yz} onto Ω is a continuous, monotonically decreasing curve C_m (with respect to the n axis) that starts somewhere on the line $n = 0$ and ends somewhere on the line $n = N$. It satisfies the aforementioned restriction on its chords except that it may coincide with the line $m = M$ for some lowest values or all values of n and with the line $m = 0$ for some highest values or all values of n. The curve C_m corresponding to the situation depicted in Figure 5 is illustrated again in Figure 3.

We mentioned before that, in view of (11) and Condition G, we can say that $m_{t+2} - m_t$ has the same sign as $r_{yt} - \beta_0 r_{zt}$ or the two quantities are both zero together. Equivalently, $f_M(n, m)$ is less than m (or equal to zero or larger than m) if and only if $r_y(n + m)$ is less than (respectively, equal to or larger than) $\beta_0 r_z(M - m)$. These two assertions require the restriction that m be larger than 0 and less than M. This is illustrated in Figure 6 for the situation corresponding to Figure 5. It should be noted however that $f_M(n, m)$ may tend to zero as $m \to 0+$ (n being held fixed) even though $r_y(n + m)/\beta_0 r_z(M - m)$ tends to a number larger than one; this will be the case if $g_m(\beta, 0) = 0$ for $\beta \in (\beta_0, \infty)$. Similarly, $f_M(n, m)$ may tend to M as $m \to M-$ even though $r_y(n + m)/\beta_0 r_z(M - m)$ tends to a number smaller than one; this happens when $g_M(\beta, \delta) \to \infty$ as $\delta \to \infty$ for fixed $\beta \in (0, \beta_0)$.

We can conclude that, when $(n_t, m_t) \in \mathring{\Omega}$, the shift $m_{t+2} - m_t$ is toward the curve C_m, as indicated in Figure 3. (That shift in m_t may extend beyond curve C_m if g_M varies strongly enough.) When (n_t, m_t) is on the curve C_m, there is no shift in m_t; that is, the curve C_m is a locus of equilibrium points for the variable m_t. Finally, when (n_t, m_t) is on the boundary lines $m = 0$ or $m = M$, $f_M(n, m)$ may be 0 or M and there may be no shift in m_t even when (n_t, m_t) is not on C_m; this situation corresponds to an unstable equilibrium for m_t. It occurs when $f_M(n, m)$ approaches the origin from above the line m or approaches M from below the line m, as is indicated in Figure 6 for $n = 0$ for example. There are no other equilibrium points for m_t.

It should also be noted that the boundary points of Ω may not be equilibrium points. For example, $g_M(\beta, 0)$ may be greater than 0 for $\beta \in (\beta_0, \infty)$, in which case $f_M(n, 0)$ will be greater than 0 and will dictate a shift away from $m = 0$ according to (11). Similarly, for $\beta \in (0, \beta_0)$, $g_M(\beta, \delta)$ may tend to a finite limit as $\delta \rightarrow \infty$, thereby yielding $f_M(n, M) < M$.

This analysis can be applied to the variable n_t too. In this case, we compare (1) and (2) by examining the intersection I_{xy} of the surface $\alpha_0 r_x(N - n)$ with the surface $r_y(n + m)$. With regard to the axes of Figure 2, $\alpha_0 r_x(N - n)$ is a continuous surface above Ω that increases with n. Its equielevation lines are straight lines parallel to the m axis. It follows that I_{xy}, if it exists, is a continuous curve above Ω whose projection onto Ω is such that all its chords have slopes (with respect to the n axis) less than -1 and greater than $-\infty$. If, for some m, $\alpha_0 r_x(N - n)$ is larger than (smaller than) $r_y(n + m)$ for all n, set I_{xy} equal to $(0, m)$ (respectively, equal to (N, m)) to reflect the fact that firm N would want to decrease (respectively, increase) its investment n into venture Y as much as possible. With this extension of I_{xy}, the projection C_n of I_{xy} onto Ω has the same properties as C_m when the roles of n and m are reversed. In particular, C_n starts somewhere on the line $m = 0$ and ends somewhere on the line $m = M$.

Upon considering $f_N(n, m)$ in the same way as we did $f_M(n, m)$, we can conclude that C_n is a locus of equilibrium points for the variable n_t, and, when $(n_t, m_t) \in \mathring{\Omega}$, the shift from n_t to n_{t+2} is toward (and perhaps crosses over) C_n. This too is illustrated in Figure 3. As before, the points on the lines $n = 0$ and $n = N$ which are not on C_n may be points of unstable equilibrium for n_t. There are no other equilibrium points for n_t.

In view of the properties of C_n and C_m, we can conclude that these two curves intersect at exactly one point E, as is indicated in Figure 2. E may be anywhere in Ω, including its boundary. We can also conclude that E is an equilibrium point for the variable $E_t = (n_t, m_t)$ where the sequences in n_t and m_t are determined by (10) and (11). Although E will be the only equilibrium point in $\mathring{\Omega}$ if it truly appears therein, there may be other equilibrium points on the boundary of Ω. Each of the four corners of Ω may or may not be an equilibrium point depending upon whether f_M and f_N simultaneously assume the corresponding edge values at the corner in question. For similar reasons, each of the four points where C_n meets the edges $m = 0$ and $m = M$ and C_m meets the edges $n = 0$ and $n = N$ may or may not be equilibrium points.

<u>Proof of Lemma 1.</u> Let $E_\infty = \lim_{t\to\infty} E_t$. We wish to show that $E_\infty = E$. Now, $m_{t+2} - f_M(n_t, m_t)$ is the increment through which m_t changes to m_{t+2}. It is also the distance between the surface $(n, m) \mapsto m$, which we denote by $\hat{m}$, and the surface f_M in the (n, m, r) space above the point (n_t, m_t). Thus, we have a sequence of points $\{(n_t, m_t, m_t): t = 1, 3, 5, \ldots\}$ in the surface $\hat{m}$ and a sequence of points $\{(n_t, m_t, f_M(n_t, m_t): t = 1, 3, 5, \ldots\}$ in the surface f_M. $\{$Since $E_t: t = 1, 3, 5, \ldots\}$ converges, the increments in m_t tend to zero, which means that the two surfaces tend toward each other along the points E_t. Since these surfaces are continuous, it follows that they meet at E_∞. Therefore, E_∞ is on C_m or perhaps on the line $m = 0$ or on the line $m = M$ since these are the only places in Ω over which the two surfaces can intersect. But, E_∞ cannot be at any point P of those parts of the lines $m = 0$ or $m = M$ that do not coincide with C_m because $0 < f_M(n_t, m_t) < M$ whenever $(n_t, m_t) \in \mathring{\Omega}$ and because, for all points in $\mathring{\Omega}$ sufficiently close to P, the increments in m_t are directed away from P. (P is an unstable equilibrium point.) Thus, E_∞ is on C_m.

By the same argument applied to $f_N(n_t, m_t)$, we can conclude that E_∞ is also on C_n. But, C_n and C_m have one and only one intersection point, namely, E. Therefore $E_\infty = E$.

Proof of Lemma 2. We prove assertion (i), the proof of assertion (ii) being the same except for notational changes. Consider one of the loci of Figure 3 that intersects the $\beta = \beta_0$ line. With n held fixed, an increase (decrease) in the value of m results in a shift along a locus upward and to the left (respectively, downward and to the right) because of Condition R. Now Condition H asserts that the values of $g_M(\beta(n, m), \delta(m))$ on that part of a given locus to the left (right) of the vertical line at β_0 is strictly larger (respectively, strictly smaller) than its value at the intercept of that locus and vertical line, if such an intercept exists. In view of (11), this implies that, with n fixed, the curve $f_M(n, m)$ in Figure 6 intersects the straight line $m \mapsto m$ at $m(n) \in [0, M]$ and remains larger than $m(n)$ for $m(n) < m \leq M$ (if $m(n) < M$) and remains smaller than $m(n)$ for $0 \leq m < m(n)$ (if $m(n) > 0$). Furthermore, the strict monotonicity asserted in Condition G(ii) implies that $g_M(\beta(n, m), \delta(m))$ is less than (greater than) $\delta(m)$ when $(\beta(n, m), \delta(m))$ is a point to the left (respectively, right) of the vertical line at β_0. Consequently, $f_M(n, m) < m$ for $m(n) < m < M$ if $m(n) < M$ (respectively, $f_M(n, m) > m$ for $0 < m < m(n)$ if $m(n) > 0$); that is, as m increases, $f_M(n, m)$ crosses at $m(n)$ the line $m \mapsto m$ from above that line to below it. Since for fixed n the increment $m_{t+2} - m_t$ is the difference from $f_M(n, m)$ to the straight line $m \mapsto m$ at the point $m = m_t$, it follows that, if $m_t \neq m(n)$, then m_{t+2} is closer to $m(n)$ than is m_t; more specifically, we have $m_t < m_{t+2} < m(n)$ if $0 < m_t < m(n)$, and $m(n) < m_{t+2} < m_t$ if $m(n) < m_t < M$.

Proof of Theorem 2.

We now set about proving that $E = (n_0, m_0)$ is the limit point toward which every sequence $\{E_t\} = \{(n_t, m_t)\}$ that starts in $\mathring{\Omega}$ and is governed by (10) and (11) converges. We establish this fact under the assumption that $E \in \mathring{\Omega}$; later on, we point out how our arguments still apply in the case where E is on the boundary of Ω. We consider several cases depending on where in $\mathring{\Omega}$ a point E_t of the sequence happens to be. Partition Ω into four open regions bounded by segments of C_n, C_m, and the boundary of Ω, as indicated in Figure 3. Recall that C_n and C_m are continuous curves in Ω such that within $\mathring{\Omega}$ the chords of C_n have slopes more negative than -1 and the chords of C_m have slopes larger than -1 but less than 0. (We are measuring these slopes with respect to the n axis.)

Case 1. Assume that $E_t = (n_t, m_t) \in \Omega_2$ for some t. Then, by (10) and Lemma 2, $n_{t+2} - n_t$ is a positive (nonzero) shift to the right such that $(n_{t+2}, m_t) \in \Omega_2$; that is, the shift from n_t to n_{t+2}, with m_t kept fixed, does not reach C_n or pass beyond C_n into Ω_3. In view of the chord restrictions on C_n, $n_{t+2} < n_0$. Similarly, $m_{t+2} - m_t$ is a negative (nonzero) shift downward such that $(n_t, m_{t+2}) \in \Omega_2$. The chord restrictions on C_m insure that $m_{t+2} > m_0$ and also $(n_{t+2}, m_{t+2}) \in \Omega_2$. By induction, it now follows that $n_t, n_{t+2}, n_{t+4}, \ldots$ is a monotonically increasing sequence bounded above by n_0; therefore, it converges. Also, $m_t, m_{t+2}, m_{t+4}, \ldots$ is a monotonically decreasing sequence, bounded below by m_0, and therefore it too converges. Thus, the sequence $E_t, E_{t+2}, E_{t+4}, \ldots$ remains within Ω_2, converges, and by Lemma 1 has the limit $E = (n_0, m_0)$.

A similar argument shows that, if $E_t \in \Omega_4$, then $E_t, E_{t+2}, E_{t+4}, \ldots$ remains within Ω_4 and converges monotonically upward and to the left toward E.

Case 2. Assume $E_t = (n_t, m_t) \in C_n$. If $E_t = E$, then the subsequent points in the series remain fixed at E. If $E_t \neq E$, then $n_{t+2} = n_t$, and, by Lemma 2, m_{t+2} is such that $E_{t+2} = (n_{t+2}, m_{t+2})$ is in $\Omega_2 \cup \Omega_4$. Thus, the subsequent points $E_{t+2}, E_{t+4}, \ldots$ converge to E as in Case 1.

A similar argument shows that, if $E_t \in C_m$, then $E_{t+2}, E_{t+4}, \ldots$ converges to E as in Case 1 again. Upon combining Cases 1 and 2, we have that, if $E_t \in \bar{\Omega}_2 \cup \bar{\Omega}_4$, then $E_t, E_{t+2}, \ldots$ converges to E.

Case 3a. Assume that, the sequence $E_1, E_3, \ldots$ remains within Ω_1. Then, $n_1, n_3, \ldots$ and $m_1, m_3, \ldots$ are both monotonically increasing sequences that are bounded above. Thus, they converge, and therefore $E_1, E_3, \ldots$ converges. By Lemma 1 again, $E_1, E_3, \ldots$ converges to E. The same result occurs if the sequence remains within Ω_3.

Case 3b. Assume that $E_1 \in \Omega_1$ and there is some $t > 1$ such that $E_t \in \bar{\Omega}_2 \cup \bar{\Omega}_4$. Then, $E_1, E_3, \ldots$ converges to E, as was shown in Cases 1 and 2.

Case 3c. Assume that there exists an increasing infinite sequence $t_1 = 1, t_2, t_3, \ldots$ of odd integer time values such that $E_{t_k}, \ldots E_{t_{k+1}-1}$ is in Ω_1 for k odd and is in Ω_3 for k even. That is, the sequence $E_1, E_3, \ldots$ starts off in Ω_1 for one or more terms, then shifts to Ω_3 for one or more terms, then goes back to Ω_1 for still more terms, and so forth, alternating between Ω_1 and Ω_3. (The following argument does not change in any essential way for the case where the sequence $E_1, E_3, \ldots$ starts off in Ω_3.)

Let $E_1 = (n_1, m_1)$ and $E = (n_0, m_0)$. If $m_1 < m_0$, let G_1 be that unique point on C_n having m_1 as its m coordinate; if $m_1 \geq m_0$, set $G_1 = E$. (See Figure 7.) Similarly, if $n_1 < n_0$, let F_1 be that unique point on C_m having n_1 as its n coordinate; if $n_1 \geq n_0$, set $F_1 = E$. Note that G_1 and F_1 cannot both be equal to E Let H_1 be that point in Ω_1 having the same n coordinate as F_1 and the same m coordinate as G_1. (If $m_1 < m_0$ and $n_1 < n_0$, we have $H_1 = E_1$, as indicated in Figure 7.) Finally, let H_2 be that point having the same n coordinate as G_1 and the same m coordinate as F_1. Since the chords of C_n and C_m have negative (neither zero nor infinite) slopes within Ω, we have that $H_2 \in \Omega_3$. Let R_1 be the closed rectangle $H_1F_1H_2G_1$. Because of Lemma 2 and the chord restrictions on C_n and C_m, every point in $R_1 \cap \Omega_1$ will, under the displacement dictated by (10) and (11), remain in R_1. This fact coupled with our assumption concerning the subsequences of $\{E_t\}$ implies that $E_1, \ldots, E_{t_2-1}$ are in $R_1 \cap \Omega_1$ and E_{t_2} is in $R_1 \cap \Omega_3$.

Set $H_2 = (\eta_2, \mu_2)$. It follows from the above construction that $\eta_2 \geq n_0$ and $\mu_2 \geq m_0$. Now, let G_2 be that unique point on C_m having the same n coordinate as H_2, namely, η_2. Also, let F_2 be that unique point on C_n having the same m coordinate as H_2, namely, μ_2. Then, let H_3 be the point having the same n coordinate as F_2 and the same m coordinate as G_2. Let R_2 be the closed rectangle $H_2F_2H_3G_2$. As before, the displacement dictated by (10) and (11) carries every point of $R_2 \cap \Omega_3$ into

R_2. Also, $E_{t_2}, \ldots, E_{t_3-1}$ are in $R_2 \cap \Omega_3$, and E_{t_3} is in $R_2 \cap \Omega_1$.

Set $H_3 = (\eta_3, \mu_3)$. As before, $\eta_3 \leq n_0$ and $\mu_3 \leq m_0$. Let G_3 be that unique point on C_n having μ_3 as its m coordinate. Let F_3 be that unique point on C_m having η_3 as its n coordinate. Then, H_4 is the point having the same n coordinate as G_3 and the same m coordinate as F_3. Let R_3 be the closed rectangle $H_3F_3H_4G_3$. As before, H_4 is in Ω_3, $E_{t_3}, \ldots, E_{t_4-1}$ are in $R_3 \cap \Omega_1$, and E_{t_4} is in $R_3 \cap \Omega_3$.

Continuing this process, we obtain a nested sequence $R_1, R_2, R_3, \ldots$ of closed rectangles whose intersection is E. This is because the G_k alternate between C_n and C_m and tend toward E from below, whereas the F_k do the same from above E. Since each segment $E_{t_k}, \ldots, E_{t_{k+1}-1}$ of the sequence $\{E_t\}$ is in R_k, it follows that $E_t \to E$ as $t \to \infty$.

(If $m_1 > m_0$ or $n_1 < n_0$ and if $E_1 \in \Omega_1$, it can be shown that Case 3b ensues, but we don't have to argue this case.)

All the cases considered above exhaust the various ways a sequence generated by (10) and (11) can occur in Ω under Conditions R, G, and H and the assumption that both E_1 and E are in $\mathring{\Omega}$.

The possibility that E occurs on the boundary of Ω has yet to be considered. In this situation, one or more of the regions Ω_1, Ω_2, Ω_3, and Ω_4 are void, and so only some of the cases listed above need be considered. For those cases that do arise, the arguments presented above apply virtually word for word. Now, however, it should be noted in particular that, since $E_1 \in \mathring{\Omega}$, the sequence $E_1, E_3, E_5, \ldots$ will remain within $\mathring{\Omega}$ and will thereby avoid any of the unstable limit points that might occur on the boundary of Ω.

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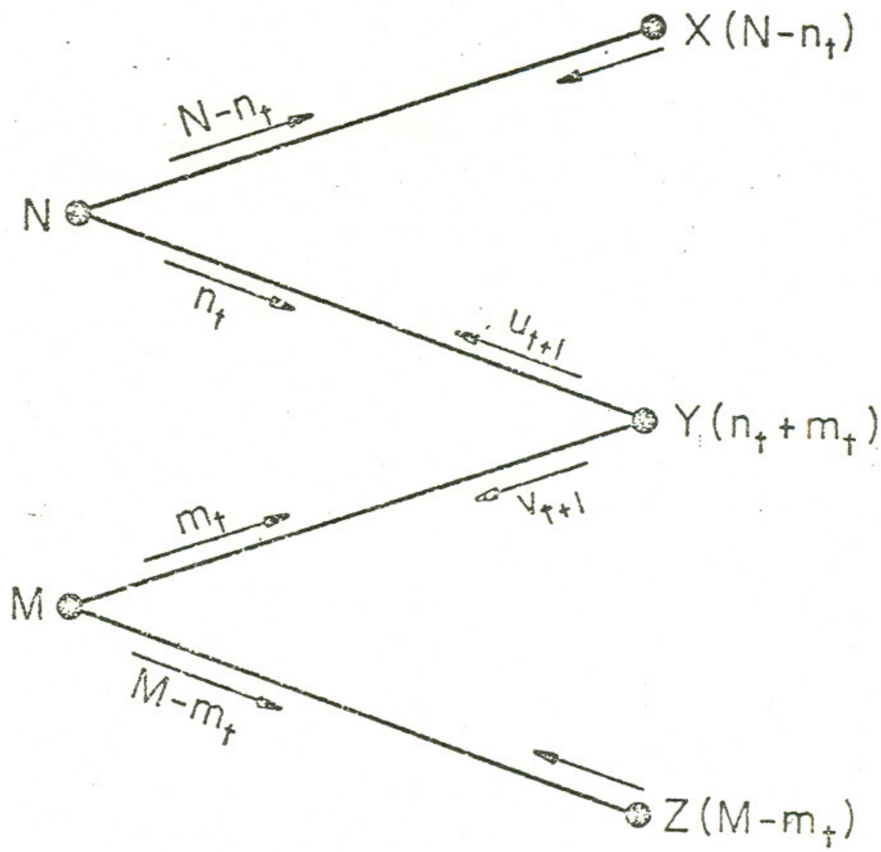


Figure 1.

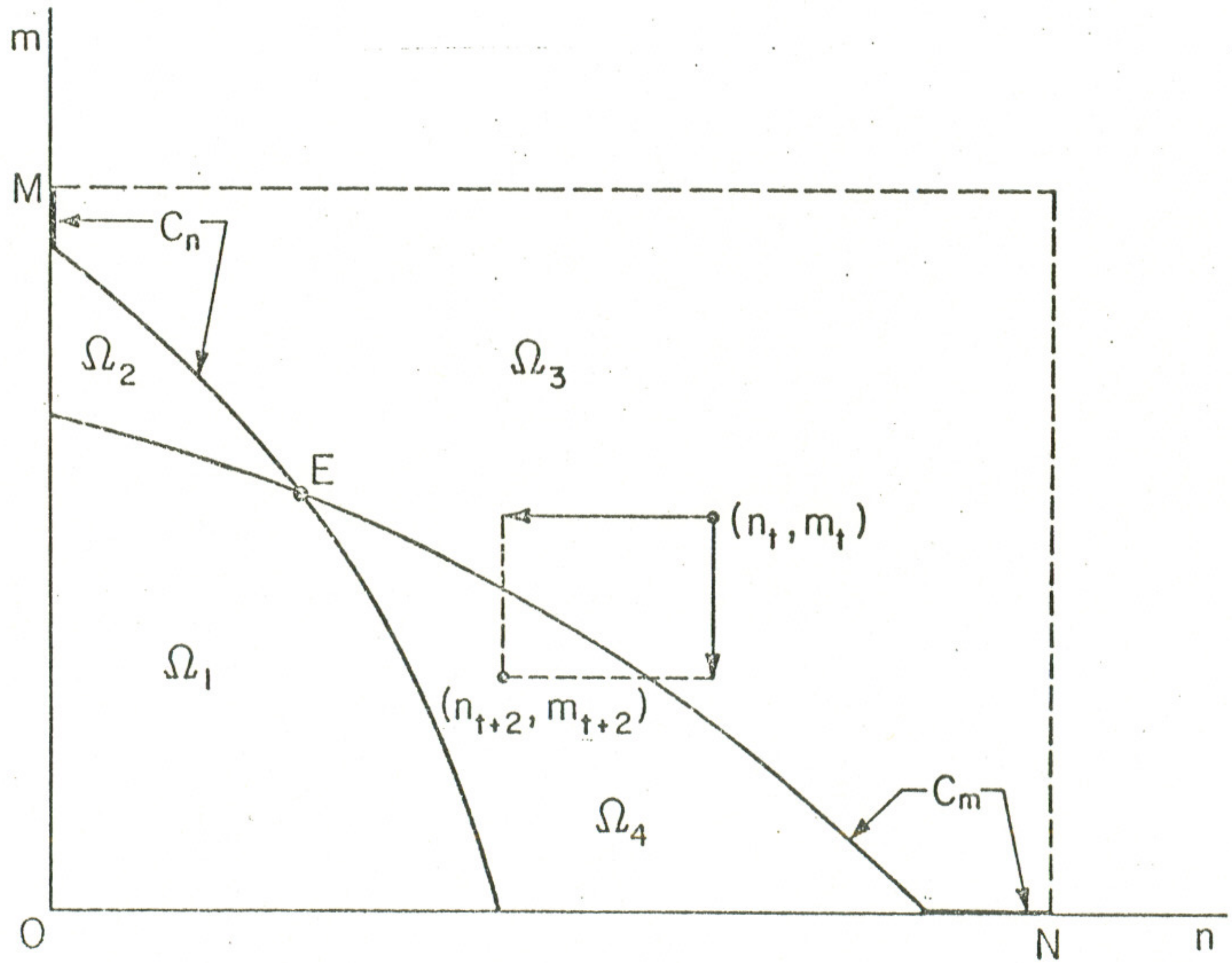


Figure 2.

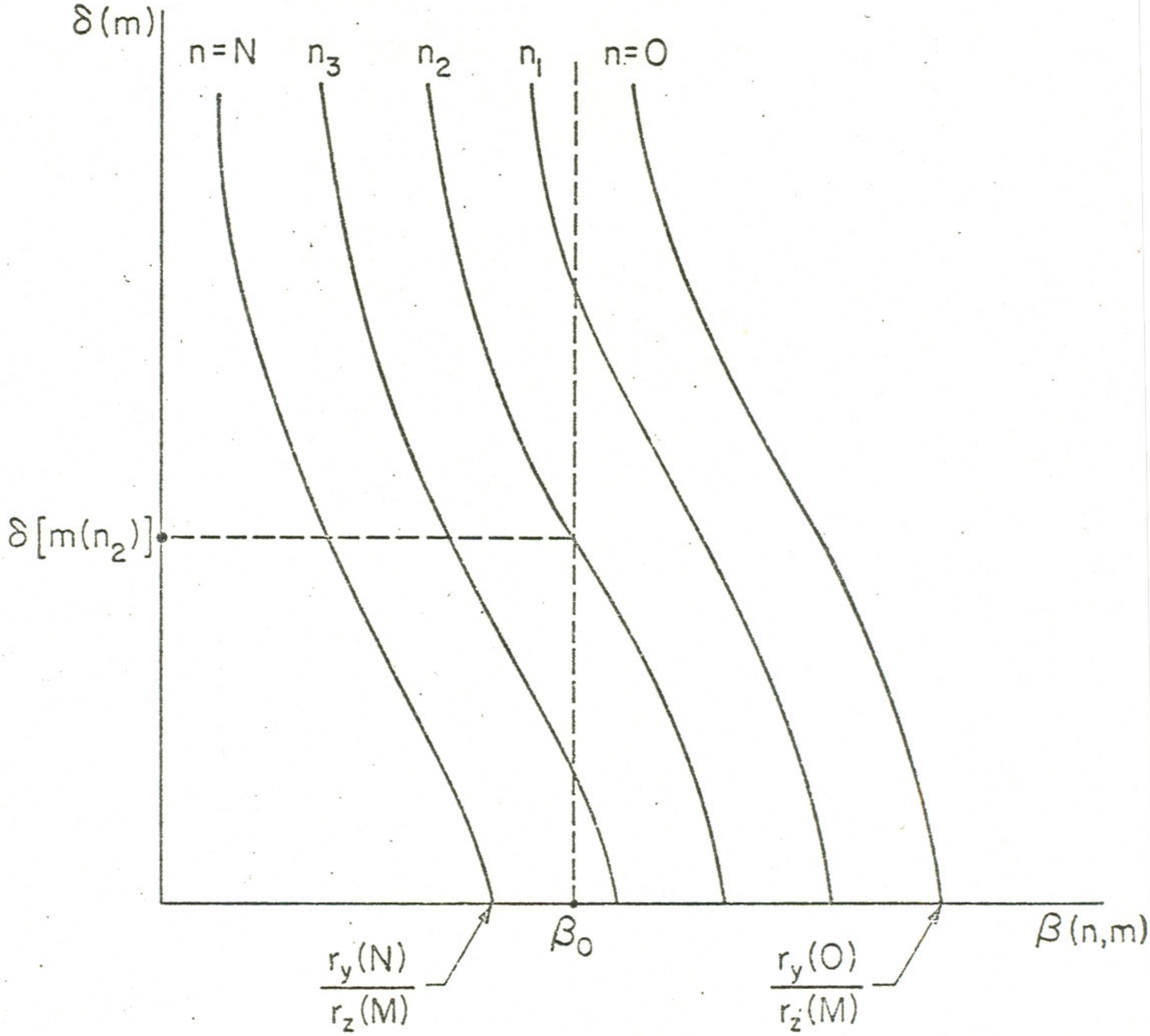


Figure 3.

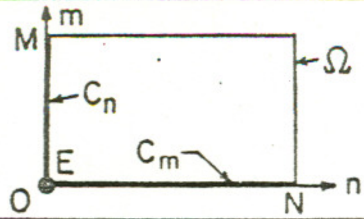
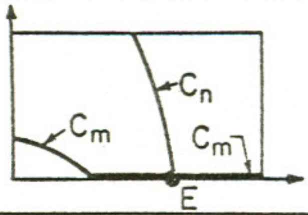
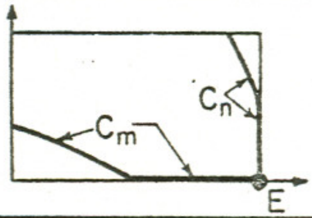
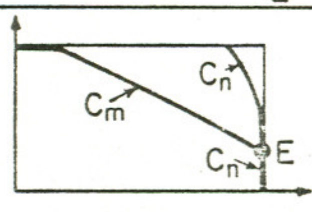
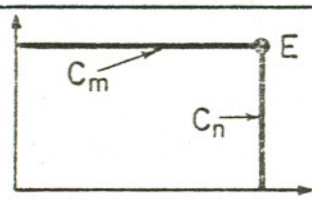
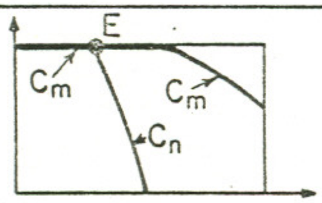
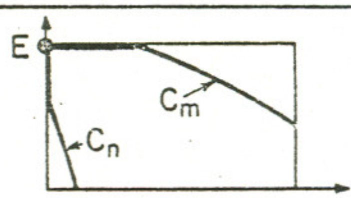
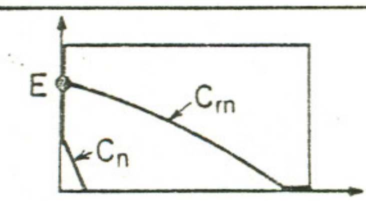
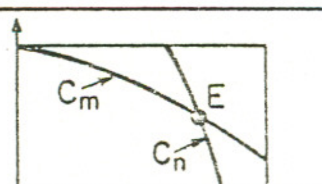
LOCATION OF E IN Ω	NECESSARY AND SUFFICIENT CONDITIONS (AND INTERPRETATIONS)
	CASE 1: $\alpha_o r_x(N) \ge r_y(0)$, $\beta_o r_z(M) \ge r_y(0)$. (N and M both abandon Y.)
	CASE 2: $\alpha_o r_x(N-n) = r_y(n)$ has a solution $n = n_o$ where $0 < n_o < N$, and $\beta_o r_z(M) \ge r_y(n_o)$. (M abandons Y.)
	CASE 3: $\alpha_o r_x(0) \le r_y(N) \le \beta_o r_z(M)$. (N abandons X, M abandons Y.)
	CASE 4: $\beta_o r_z(M-m) = r_y(N+m)$ has a solution $m = m_o$ where $0 < m_o < M$, and $\alpha_o r_x(0) \le r_y(N+m_o)$. (N abandons X.)
	CASE 5: $\alpha_o r_x(0) \le r_y(N+M)$, $\beta_o r_z(0) \le r_y(N+M)$. (N abandons X, M abandons Z.)
	CASE 6: $\alpha_o r_x(N-n) = r_y(n+M)$ has a solution $n = n_o$ where $0 < n_o < N$, and $\beta_o r_z(0) \le r_y(n_o + M)$. (M abandons Z.)
	CASE 7: $\beta_o r_z(0) \le r_y(M) \le \alpha_o r_x(N)$. (N abandons Y, M abandons Z.)
	CASE 8: $\beta_o r_z(M-m) = r_y(m)$ has a solution $m = m_o$ where $0 < m_o < M$, and $\alpha_o r_x(N) \ge r_y(m_o)$. (N abandons Y.)
	CASE 9: $\alpha_o r_x(N-n) = r_y(n+m) = \beta_o r_z(M-m)$ has a solution $n = n_o$, $m = m_o$ where $0 < n_o < N$ and $0 < n_o < N$. (No venture is abandoned.)

Figure 4.

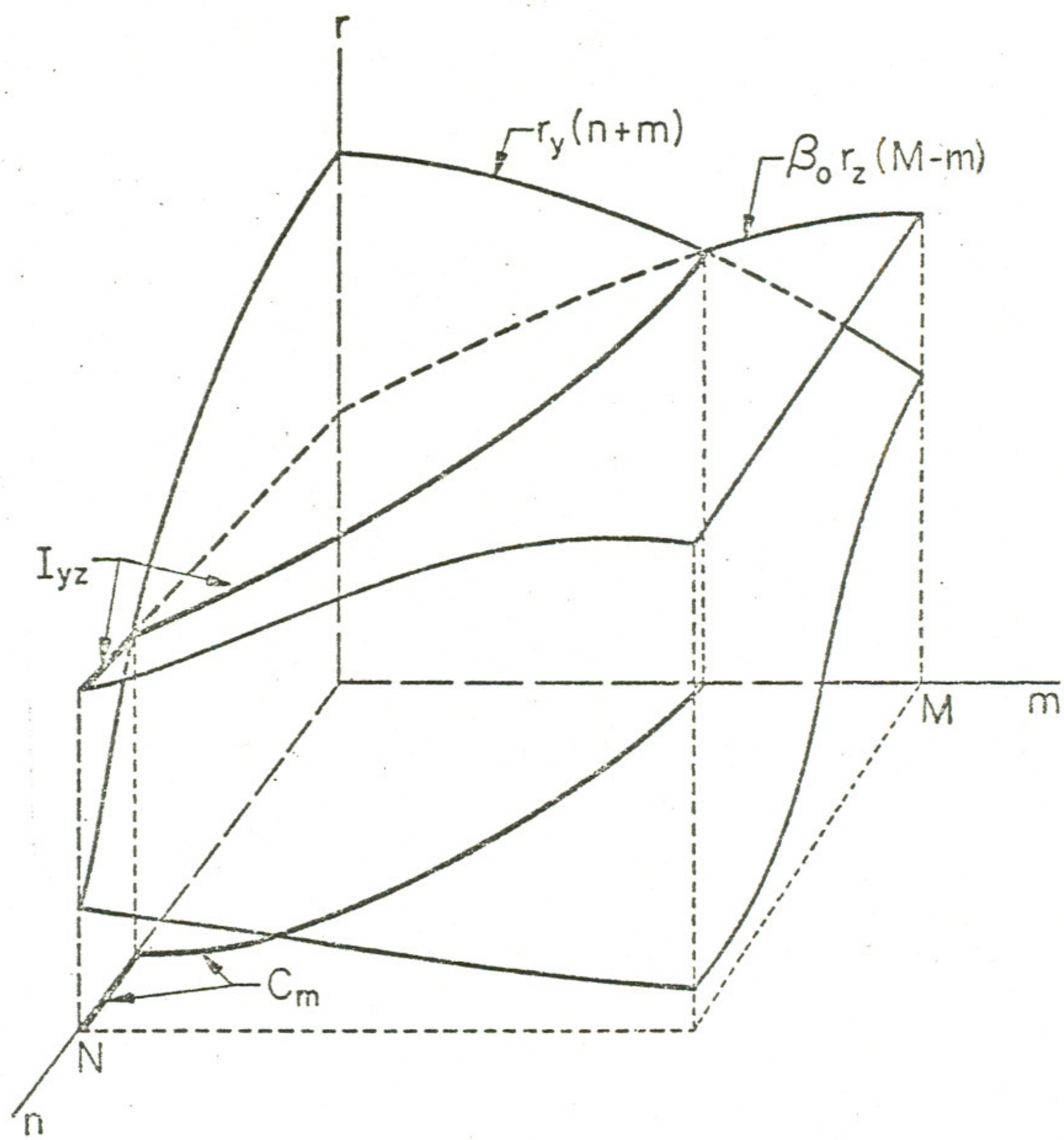


Figure 5

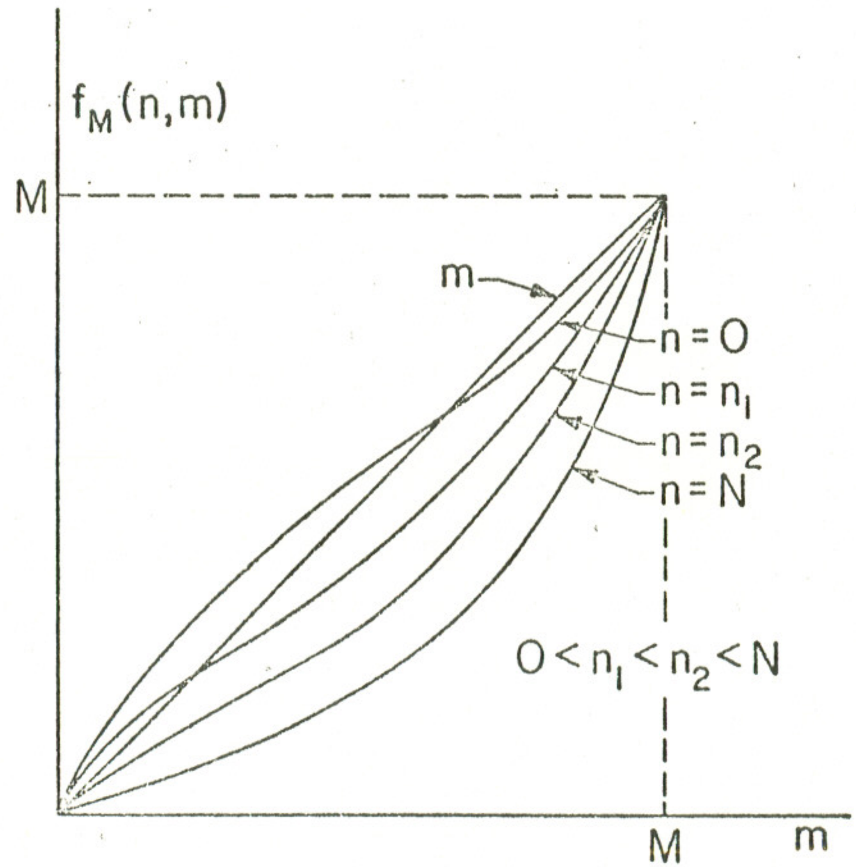
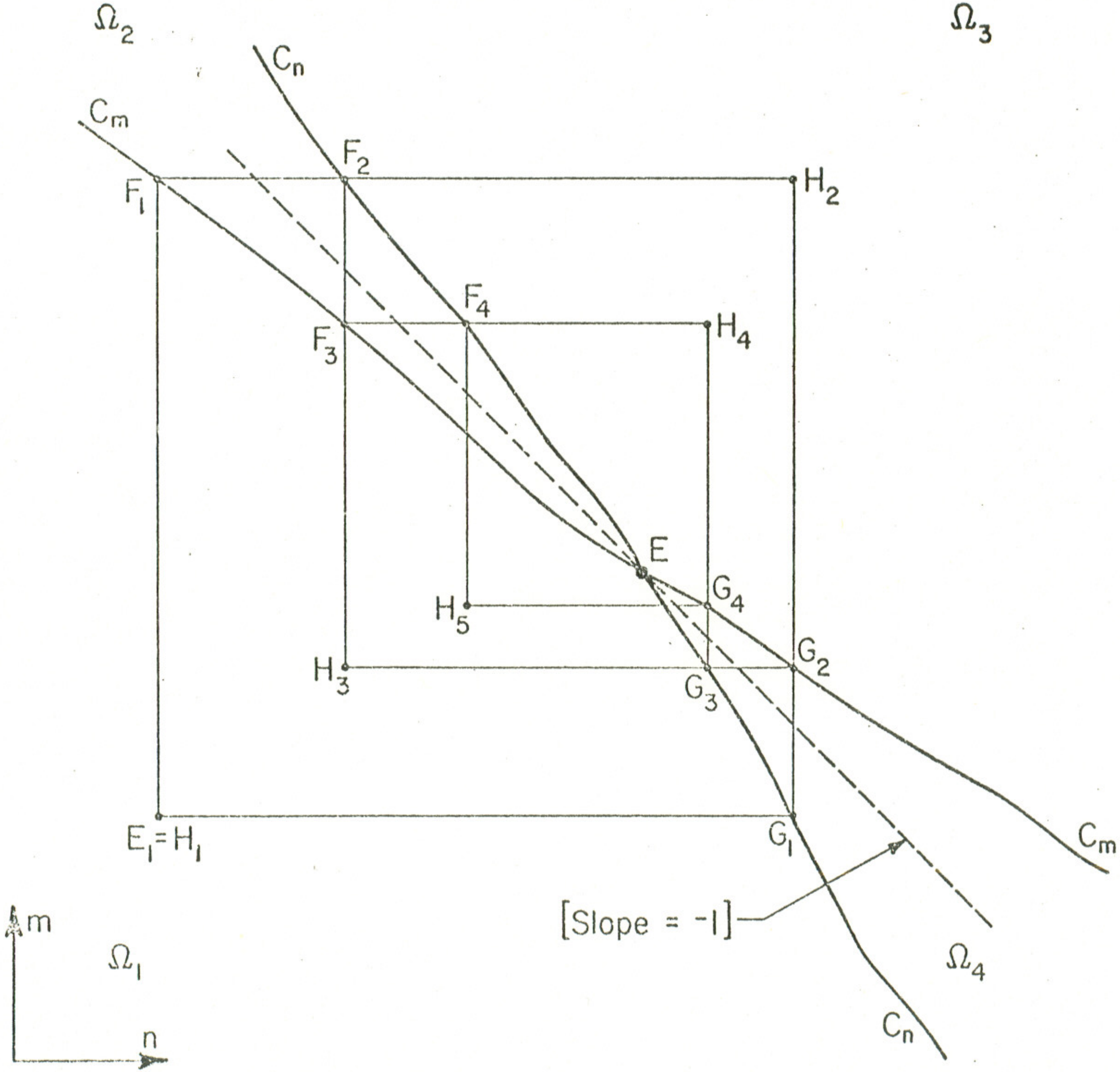


Figure 6.



Ω2
Ω3
Cn
Cm
F2
H2
F1
F4
H4
F3
E
G4
H5
G2
H3
G3
E1=H1
G1
Cm
[Slope = -1]
m
Ω1
Ω4
Cn
n

Figure 7