STATE UNIVERSITY OF NEW YORK AT STONY BROOK

CEAS Technical Report 626

THE EXISTENCE AND UNIQUENESS OF NODE VOLTAGES IN A NONLINEAR RESISTIVE TRANSFINITE ELECTRICAL NETWORK

A.H.Zemanian

This work was supported by the National Science Foundation under Grants MIP 8822774 AND DMS-9200738

May 7, 1992

THE EXISTENCE AND UNIQUENESS OF NODE VOLTAGES IN A NONLINEAR RESISTIVE TRANSFINITE ELECTRICAL NETWORK

A. H. Zemanian
State University of New York at Stony Brook
Stony Brook, N. Y. 11794-2350

Abstract — A nonlinear resistive transfinite network is shown to possess a unique set of node voltages with respect to any arbitrarily chosen node so long as the nonlinear resistors satisfy a certain set of conditions related to modular sequence spaces. A consequence of this is that sourceless reduced networks satisfy the maximum principle for node voltages.

1 Introduction

This paper is written as a sequel to [5]; we use some of the definitions of that work without repeating them here.

In general, unique node voltages for all the nodes of all ranks in a transfinite resistive electrical network may not exist even when there is a perceptible path connecting every node to the chosen ground node n_g. This is because the voltage at a node n_0 may depend upon the choice of the perceptible path between n_0 and n_g along which that node voltage is determined. This inconsistency can arise when two tips are perceptible and nondisconnectable but are not shorted together. An example of this is given in [5, Section 7]. However, when every such pair of nodes is shorted and when there is at least one perceptible path from the chosen ground node n_g to every other node, then a unique set of node voltages will exist. This was shown for linear networks in [5].

The objective of this work is to establish this result for nonlinear networks whose nonlinearities are of a form related to modular sequence spaces [2], [3, Sections 4.3 - 4.6]. This requires a nonlinear version of the idea of "perceptibility" of a transfinite path, a concept which up-to-now has been defined only for linear networks [3, pages 92 and 154].

Another consequence of all this is that the maximum principle for node voltages continues to hold for any sourceless, reduced part of such a nonlinear transfinite network. The linear version of this result was established in [4, Section 11].

2 A Class of Nonlinear Transfinite Networks

Let ω denote the first transfinite ordinal. Let N^ν ($\nu \le \omega$) denote a nonlinear ν-network [3], every branch of which is in the Norton form, as shown in Figure 1. Thus, the jth branch is a parallel connection of a pure current source h_j and a nonlinear resistor $R_j(\cdot)$ carrying a current f_j. In accordance with the polarity conventions shown in Figure 1, the branch voltage v_j equals $R_j(f_j) = R_j(i_j + h_j)$, where i_j is the branch current. We take the branch's orientation to be that indicated by the arrow for i_j in Figure 1. In the special case of a linear resistor, the constant branch resistance r_j is the slope of the straight line $f_j \mapsto R_j(f_j)$, i.e., $r_j = dR_j/df_j$.

Conditions 2.1. $\mathbf{N}^\nu$ *($\nu \le \omega$) is a nonlinear ν-network whose countably many branches are in the Norton form, shown in Figure 1. There is no coupling between branches. Every R_j is a continuous, strictly increasing, odd mapping of the real line R^1 into R^1 with $R_j(f) \to \infty$ as $f \to \infty$.*

For $f, v \in R^1$, we set $M_j(f) = \int_0^f R_j(x)\,dx$ and $M_j^*(v) = \int_0^v G_j(y)\,dy$, where $G_j : v \mapsto G_j(v) = f$ is the inverse function of $R_j : f \mapsto R_j(f) = v$. Given that $\mathbf{N}^\nu$ satisfies Conditions 2.1, the modular sequence space $l_{\mathbf{M}}$ is defined as the set of all sequences $\mathbf{f} = (f_1, f_2, \ldots)$ of real numbers f_j such that $\sum M_j(f_j/t) < \infty$ for some $t > 0$, and $c_{\mathbf{M}}$ is the subspace of $l_{\mathbf{M}}$ for which $\sum M_j(f_j/t) < \infty$ for all $t > 0$. (Here, $\sum$ denotes a summation over all branch indices j.) $l_{\mathbf{M}}$ and $c_{\mathbf{M}}$ are Banach spaces with the norm

$$\|\mathbf{f}\| \;=\; \inf\{t\colon t > 0, \sum M_j\left(\frac{f_j}{t}\right) \le 1\}.$$

We assume henceforth

Conditions 2.2. *The nonlinear ν-network $\mathbf{N}^\nu$ satisfies Conditions 2.1. Moreover, $\mathbf{h} = (h_1, h_2, \ldots) \in l_{\mathbf{M}}$. Furthermore, there are a positive integer j_0 and two positive numbers α and β, both greater than 1, such that, for every $j \ge j_0$,*

(i) *$fR_j(f) \le \alpha M_j(f)$ for $0 \le f \le a_j$, where a_j is the unique positive number for which*

$M_j(a_j) = 1$, *and*

(ii) $uG_j(u) \leq \beta M_j^*(u)$ *for* $0 \leq u \leq d_j$, *where* d_j *is the unique positive number for which* $M_j^*(d_j) = 1$

Let $l_{\mathbf{M}}^*$ be the dual of $l_{\mathbf{M}}$ and $c_{\mathbf{M}}^*$ be the dual of $c_{\mathbf{M}}$. Also, let $\cong$ denote an isomorphism between Banach spaces. Under Conditions 2.2, we have the following relations [3, pages 129-130].

$$l_{\mathbf{M}} = c_{\mathbf{M}} \tag{1}$$

$$l_{\mathbf{M}}^* = c_{\mathbf{M}}^* \cong l_{\mathbf{M}^*} \tag{2}$$

$$(l_{\mathbf{M}}^*)^* \cong (l_{\mathbf{M}^*})^* \cong l_{(\mathbf{M}^*)^*} = l_{\mathbf{M}} \tag{3}$$

Thus, $l_{\mathbf{M}}$ is reflexive; that is, the dual of the dual of $l_{\mathbf{M}}$ is isomorphic to $l_{\mathbf{M}}$.

A *basic current* in the transfinite network $\mathbf{N}^\nu$ is defined on [3, page 154]. It is a countable superposition of (generally transfinite) loop currents satisfying certain conditions. $\mathcal{L}^0$ is the span of all basic currents in $l_{\mathbf{M}}$, and $\mathcal{L}$ is the closure of $\mathcal{L}^0$ in $l_{\mathbf{M}}$. $\mathcal{L}$ is a linear subspace of $l_{\mathbf{M}}$ and is a Banach space in itself when supplied with the norm of $l_{\mathbf{M}}$.

The next theorem [3, Theorem 5.5-7] is an adaptation to transfinite networks of a part of the fundamental theory established by DeMichele and Soardi [1] for ordinary infinite nonlinear networks. Here, R denotes the resistance operator $R(\mathbf{f}) = (R_1(f_1), R_2(f_2), \ldots)$, where $\mathbf{f} = (f_1, f_2, \ldots)$. R maps $l_{\mathbf{M}}$ into $l_{\mathbf{M}^*}$ [3, Lemma 4.7-2]. According to the polarity conventions of Figure 1, the branch voltage vector $\mathbf{v} = (v_1, v_2, \ldots)$ is related to the branch current vector $\mathbf{i} = (i_1, i_2, \ldots)$ and the branch current source vector $\mathbf{h} = (h_1, h_2, \ldots)$ through the equation $\mathbf{v} = R(\mathbf{i} + \mathbf{h})$. Furthermore, for $\mathbf{v} \in l_{\mathbf{M}^*}$ and $\mathbf{s} \in l_{\mathbf{M}}$, $\langle \mathbf{v}, \mathbf{s} \rangle$ will denote the pairing $\langle \mathbf{v}, \mathbf{s} \rangle = \sum v_j s_j \in R^1$ [3, page 126].

Theorem 2.3. *Under Conditions 2.2, there exists a unique* $\mathbf{i} \in \mathcal{L}$ *such that*

$$\langle R(\mathbf{i} + \mathbf{h}), \mathbf{s} \rangle = 0 \tag{4}$$

for all $\mathbf{s} \in \mathcal{L}$. *Furthermore,* $\mathbf{i}$ *satisfies Kirchhoff's current law at every finite ordinary 0-node, and* $\mathbf{v} = R(\mathbf{i} + \mathbf{h})$ *satisfies Kirchhoff's voltage law around every 0-loop and also around every* ζ*-loop* $(1 \leq \zeta \leq \nu)$ *having a unit flow in* $\mathcal{L}$.

We assume henceforth that the voltage-current regime in $\mathbf{N}^\nu$ is the one dictated by Theorem 2.3.

3 Node Voltages

In this section we establish sufficient conditions for the nonlinear ν-network $\mathbf{N}^\nu$ to have node voltages. Our first task is to extend the idea of "perceptibility" to the paths, loops, and tips in our nonlinear network $\mathbf{N}^\nu$. In the following, p and q are real numbers such that $p^{-1}+q^{-1}=1$ and $1<p<\infty$. Thus, $\infty>q>1$. The inequality (5) below is illustrated in Figure 2.

Lemma 3.1. *Assume that a branch resistance function R_j is bounded according to*

$$\left(\frac{f}{\gamma_j}\right)^{p-1} \;\leq\; R_j(f) \;\leq\; \rho_j f^{p-1} \tag{5}$$

for all $f>0$, where γ_j and ρ_j are positive numbers with $1/\gamma_j^{p-1} \leq \rho_j$. Then, both restrictions (i) and (ii) of Conditions 2.2. are satisfied by that R_j.

Proof. By (5) and for all $f \geq 0$,

$$f R_j(f) \;\leq\; \rho_j f^p \;=\; p\gamma_j^{p-1}\rho_j \int_0^f \left(\frac{x}{\gamma_j}\right)^{p-1} dx$$

$$\leq\; p\gamma_j^{p-1}\rho_j \int_0^f R_j(x)\,dx \;=\; p\gamma_j^{p-1}\rho_j M_j(f). \tag{6}$$

Since $p\gamma_j^{p-1}\rho_j > 1$, we have obtained a sufficient condition for the satisfaction of (i) in Conditions 2.2.

As for (ii), first note that $q-1=(p-1)^{-1}$. Hence, with $u=R_j(f)$ and with G_j being the inverse function of R_j, we have from (5) that

$$\left(\frac{u}{\rho_j}\right)^{q-1} \;\leq\; G_j(u) \;\leq\; \gamma_j u^{q-1}$$

for all $u \geq 0$. Consequently,

$$uG_j(u) \;\leq\; \gamma_j u^q \;=\; q\gamma_j\rho_j^{q-1} \int_0^u \left(\frac{y}{\rho_j}\right)^{q-1} dy$$

$$\leq\; q\gamma_j\rho_j^{q-1} \int_0^u G_j(y)\,dy \;=\; q\gamma_j\rho_j^{q-1} M_j^*(u).$$

Now, $1 < q < \infty$ and $\gamma_j \rho_j^{q-1} \geq 1$. Thus, here too we have a sufficient condition for the satisfaction of (ii) of Conditions 2.2. ♣

Definition 3.2. Let P denote either a ζ-path or a ζ-loop ($\zeta \leq \nu$) and let Π be the index set for the branches embraced by P except for possibly finitely many of them. P is called *strongly perceptible* if Π can be so chosen that, for every $j \in \Pi$, there are two positive numbers γ_j and ρ_j such that the following hold:

(i) $c = \gamma_j^{p-1} \rho_j$, where c is independent of $j \in \Pi$ and $1 \leq c < \infty$.

(ii) The bounds (5) hold for all $f \geq 0$ and $j \in \Pi$.

(iii) $\sum_{j \in \Pi} \rho_j < \infty$.

If P is a representative of a ζ-tip t^ζ, then t^ζ is also called *strongly perceptible.*

The *voltage rise* along the oriented ζ-path or ζ-loop P is

$$\sum_{j \in J} \mp v_j \tag{7}$$

where J is the index set for all the branches embraced by P and the minus (plus) sign is used if the orientations of P and the jth branch agree (respectively, disagree). Kirchhoff's voltage law, when it holds for a ζ-loop P, asserts that (7) equals 0.

Lemma 3.3. *Let P be a strongly perceptible ζ-path or ζ-loop. Then, (7) converges absolutely.*

Proof. We need merely show absolute convergence for all but possibly finitely many of the terms in (7). So, with Π as before and with $\sum$ denoting a summation over the $j \in \Pi$, we have by Holder's inequality and the relationship $q(p-1) = p$ that

$$\sum |u_j| \;=\; \sum |R_j(f_j)| \;\leq\; \sum \rho_j |f_j|^{p-1}$$

$$=\; \sum \rho_j^{1/q} |f_j|^{p-1} \rho_j^{1/p} \;\leq\; \left(\sum \rho_j |f_j|^p\right)^{1/q} \left(\sum \rho_j\right)^{1/p}.$$

The right-hand side is finite. Indeed, $\sum \rho_j < \infty$ by (iii) of Definition 3.2. Also, by (6), (i) of Definition 3.2, and the evenness of the M_j, we have

$$\sum \rho_j |f_j|^p \;\leq\; pc \sum M_j(f_j).$$

Since every branch is in the Norton form, we have $u_j = v_j$. It follows from Theorem 2.3 and Equation (1) that $\mathbf{f} = \mathbf{i} + \mathbf{h} \in l_{\mathbf{M}} = c_{\mathbf{M}}$. This insures that $\sum M_j(f_j) < \infty$. ♣

Theorem 3.4. *Under Conditions 2.2 and the voltage-current regime dictated by Theorem 2.3, Kirchhoff's voltage law is satisfied around every strongly perceptible ζ-loop ($\zeta \leq \nu$), and for such a loop (7) converges absolutely.*

Proof. By virtue of Theorem 2.3 and Lemma 3.3, we need merely show that every strongly perceptible ζ-loop P, where $\zeta \geq 1$, has a unit flow $\mathbf{s}$ in $\mathcal{L}$. So, let J be the index set for all the branches embraced by P. With Π defined as in Definition 3.2, Π is an infinite set and $J \backslash \Pi$ is a finite set. Let $|s_j| = 1$ for all $j \in J$ and let $s_j = 0$ for all $j \notin J$. Then,

$$\sum_{\forall j} M_j(s_j) \; = \; \sum_{j \in J \backslash \Pi} M_j(s_j) + \sum_{j \in \Pi} M_j(s_j).$$

The first summation on the right-hand side has finitely many terms and is therefore finite. For the second summation we have from Definition 3.2 and the evenness of the M_j that

$$\sum_{j \in \Pi} M_j(s_j) \; = \; \sum_{j \in \Pi} \int_0^1 R_j(x)\, dx$$

$$\leq \; \sum_{j \in \Pi} \rho_j \int_0^1 x^{p-1}\, dx \; = \; \frac{1}{p} \sum_{j \in \Pi} \rho_j \; < \; \infty.$$

Hence, $\mathbf{s} \in l_{\mathbf{M}}$. Moreover, since $\mathbf{s}$ is a loop current and since a loop current is a special case of a basic current, we have $\mathbf{s} \in \mathcal{L}$. So truly, P has a unit flow in $\mathcal{L}$. ♣

Let m and n be two nonembracing nodes, whose ranks need not be the same. Also, let P be a ζ-path ($\zeta \leq \nu$) that meets m and n terminally. If P is strongly perceptible, we say that n is strongly perceptible from m along P. In this case, (7) converges absolutely according to Lemma 3.3, and we define(7) to be the *node voltage of n with respect to m along P*. Let P and Q be two strongly perceptible paths (with possibly differing ranks) that meet m and n terminally. If the ranks of P and Q are both 0, then (7) is the same along both paths. However, if at least one of them has a rank larger than 1, then (7) need not be the same for both. An example of this discrepancy was given in [5, section 7]. An additional condition is needed to insure that the choice of the strongly perceptible path between m and n does not affect (7). The one used in [5] (namely, Condition 8.1 therein) for a linear network also works for the nonlinear network considered herein. It is given by Condition 3.5 below.

We again need the definition of "nondisconnectable tips". Recall that a representative of a μ-tip, where μ is any natural number, is a one-ended μ-path which in turn is a one-way infinite alternating sequence of μ-nodes n_i^μ and $(\mu-1)$-paths $P_i^{\mu-1}$ of the form:

$$P^\mu = \{n_0^\eta, P_0^{\mu-1}, n_1^\mu, P_1^{\mu-1}, n_2^\mu, P_2^{\mu-1}, \ldots\} \tag{8}$$

where the first node n_0^η has a rank $\eta \leq \mu$ and certain conditions are satisfied [3, page 144]. (If $\mu = 0$, the $(\mu-1)$-paths embraced in (8) are replaced by branches.) Similarly, a representative of an $\vec{\omega}$-tip is a one-ended $\vec{\omega}$-path which in turn is an alternating sequence of the form:

$$P^{\vec{\omega}} = \{n_0^\eta, P_0^{\mu_0-1}, n_1^{\mu_1}, P_1^{\mu_1-1}, n_2^{\mu_2}, P_2^{\mu_2-1}, \ldots\} \tag{9}$$

where $\eta \leq \mu_0 < \mu_1 < \mu_2 < \cdots$ and again certain conditions are satisfied [3, page 147]

Now consider an infinite sequence $\{m_1, m_2, m_3, \ldots\}$ of nodes m_l with possibly differing ranks. We shall say that the m_l *approach* a μ-tip t^μ (alternatively, an $\vec{\omega}$-tip $t^{\vec{\omega}}$) if there is a representative (8) for t^μ (respectively, (9) for $t^{\vec{\omega}}$) such that, for each natural number i, all but finitely many of the m_l are shorted to nodes embraced by the members of (8) (respectively, (9)) lying to the right of n_i^μ (respectively, $n_i^{\mu_i}$).

Let t_a and t_b be two tips, not necessarily of the same rank. We say that t_a and t_b are *nondisconnectable* if there is an infinite sequence of nodes that approach both t_a and t_b.

Condition 3.5. *If two tips are strongly perceptible and nondisconnectable, then those tips are shorted together.*

Theorem 3.6. *Under the hypothesis of Theorem 3.4, assume that the tips of ranks no larger than $\vec{\omega}$ in the ν-network $\mathbf{N}^\nu$ ($\nu \leq \omega$) satisfy Condition 3.5. Let n_g and n_0 be two nodes (of possibly different ranks), and let there be at least one strongly perceptible path connecting n_g and n_0. Then, n_0 has a unique node voltage with respect to n_g; that is, n_0 obtains the same node voltage with respect to n_g along all strongly perceptible paths between n_g and n_0.*

Using now our definition of strong perceptibility and the foregoing results of this section, we can prove Theorem 3.6 in virtually the same way as [5, Theorem 8.2] was proven. As before, the proof is based upon Kirchhoff's voltage law, which for our nonlinear networks is asserted by Theorem 3.4.

Corollary 3.7. *Under the hypothesis of Theorem 3.4, assume that the tips of all ranks no larger than $\vec{\omega}$ in the ν-network $\mathbf{N}^\nu$ ($\nu \leq \omega$) satisfy Condition 3.5. Also, assume that every two nodes of $\mathbf{N}^\nu$ are connected through at least one strongly perceptible path. Choose a ground node n_g in $\mathbf{N}^\nu$ arbitrarly. Then, every node of $\mathbf{N}^\nu$ has a unique node voltage with respect to n_g.*

4 A Maximum Principle for Node Voltages

Again let μ be any natural number. A maximum principle for the node voltages in a purely resistive, μ-connected part $\mathbf{K}^\mu$ of $\mathbf{N}^\nu$ can now be established if $\mathbf{K}^\mu$ is "finitely structured" in a certain sense. This has already been established for linear networks in [4]. The argument for nonlinear networks is virtually the same. So, let us merely state the result.

The "finite structure" we are now invoking is stated explicitly by Conditions 9.4 of [4] when ν is any natural number and by Conditions 3.2 of [6] when $\nu = \omega$. One consequence of the finite structure of $\mathbf{N}^\nu$ is that it has only finitely many ν-nodes. Also, every μ-section ($\mu < \nu$) of $\mathbf{N}^\nu$ has only finitely many incident ($\mu + 1$)-nodes. Hence, the next theorem will hold in particular for any μ-section. With regard to the perceptibility of certain paths as asserted in the aforementioned conditions, that perceptibility is now understood in the strong sense in accordance with Definition 3.2 of this paper. Another consequence of the finite structure is that every node of whatever rank can be isolated from all other nodes of the same or higher ranks by a finite cut. Moreover, Kirchhoff's current law will hold at that cut [4, Lemma 10.2], [6, Section 4]; this fact is an essential part of the proof of the next theorem.

Theorem 4.1. *Let $\mathbf{N}^\nu$ be finitely structured and let the hypothesis of Theorem 3.4 be satisfied. Let $\mathbf{K}^\mu$ be any purely resistive (i.e., sourceless) reduced network of $\mathbf{N}^\nu$ such that $\mathbf{K}^\mu$ is of rank μ, where $\mu < \nu$, and $\mathbf{K}^\mu$ has at most finitely many incident ($\mu + 1$)-nodes. Then, there exists exactly two possibilities:*

(i) *All the node voltages in $\mathbf{K}^\mu$ are the same.*

(ii) *All the node voltages in $\mathbf{K}^\mu$ are strictly less (and strictly larger) than the largest (respectively, least) voltage for the finitely many ($\mu + 1$)-nodes that are incident to $\mathbf{K}^\mu$.*

References

[1] L.DeMichele and P.M.Soardi, A Thomson's principle for infinite, nonlinear, resistive networks, *Proc. Amer. Math. Soc.*, **109** (1990), 461-468.

[2] J.Y.T.Woo, On modular sequence spaces, *Studia Mathematica*, **48** (1973), 271-289.

[3] A.H.Zemanian *Infinite Electrical Networks*, Cambridge University Press, Cambridge, England, 1991.

[4] A.H.Zemanian, *Transfinite Random Walks Based on Electrical Networks: II*, CEAS Technical Report 604, State University of New York at Stony Brook, June 4, 1991.

[5] A.H.Zemanian, *Connectedness in Transfinite Graphs and the Existence and Uniqueness of Node Voltages*, CEAS Technical Report 622, State University of New York at Stony Brook, March 12, 1992.

[6] A.H.Zemanian, Random walks on ω-networks, in *Harmonic Analysis and Discrete Potential Theory* (M.Picardello, Editor), Plenum Publishing Co., London, 1992, in press.

Figure Captions

Figure 1. The Norton form of a nonlinear resistive branch with an independent current source. Here $f_j = i_j + h_j$ and $v_j = u_j = R_j(f_j)$.

Figure 2. The bounds on R_j and G_j illustrated for the case where $p > 2$.

FOOTNOTES

Received by the Editors

1980 Mathematics Subject Classification (1985 Revision). Primary 05C99, 94C05; Secondary 05C40, 94C15.

Key Words an Phrases. Transfinite graphs, node voltages, nonlinear electrical networks, modular sequence spaces.

Research on this paper was supported by the National Science Foundation under Grants MIP-8822774 and DMS-9200738.

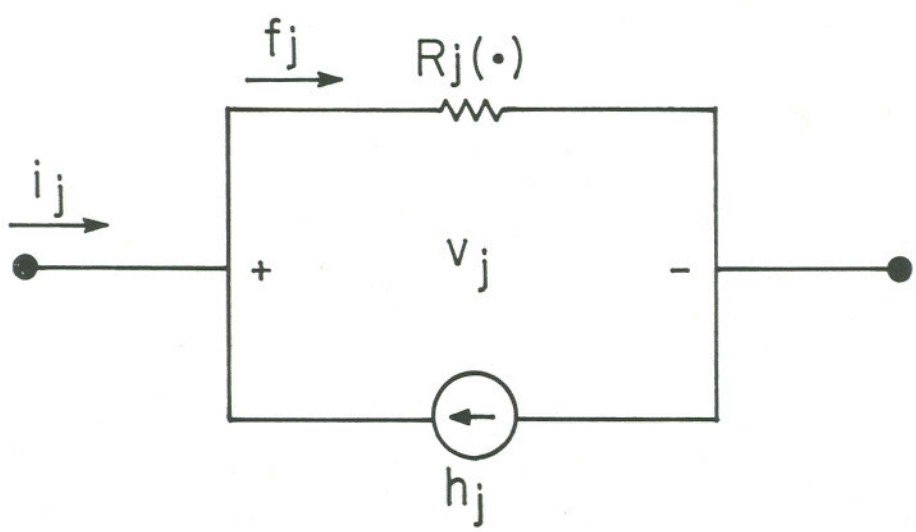

FIG. 1

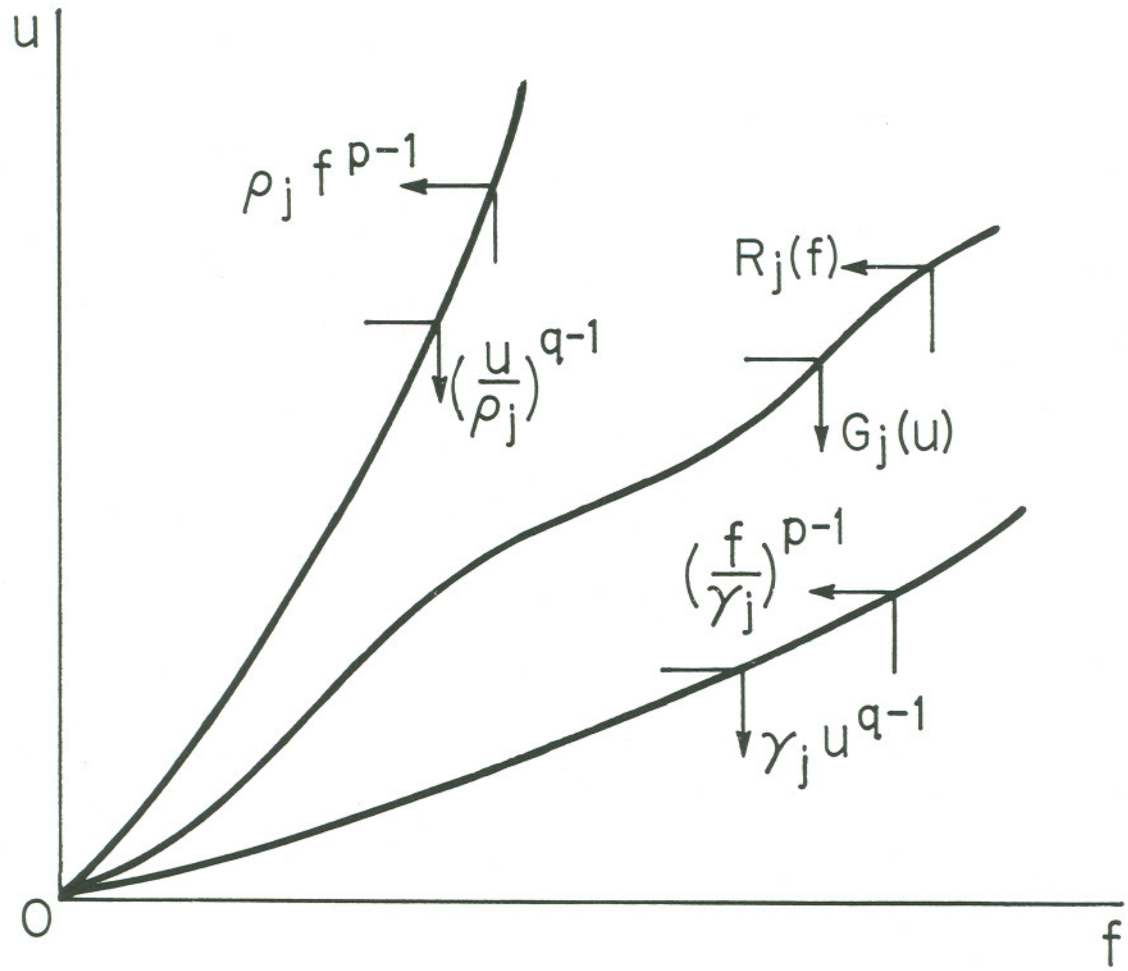

FIG. 2