UNIVERSITY AT STONY BROOK

CEAS Technical Report 714

EVERY FINITELY STRUCTURED ν-GRAPH HAS A SPANNING TREE

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This work was supported by the National Science Foundation under Grants DMS-9200738 and MIP-9423732.

June 21, 1995

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Abstract — It is shown by example that not every transfinite graph has a spanning tree. However, if the graph is finitely structured, it does have a spanning tree. This is established by induction on subsections and on ranks of transfiniteness.

1 Introduction

Every conventional connected graph (whether finite or infinite) has a spanning tree. Does very connected transfinite graph have one? The answer is "no." We will show this by example. We will then prove that, if the transfinite graph is finitely structured, it will have a spanning tree. (See [1, Section 4.5] for the definition of finite structuring.) This is established inductively by working from lower ranks for subsections to higher ranks and by building a spanning tree for each subsection by expanding a tree through subsections of lower ranks.

Most of the definitions and terminology used herein are explicated in [1]. We cannot repeat them here without excessively lengthening this paper, and so we refer the reader to [1] and its Index for the special terms that are not defined below. As with a conventional graph, a *spanning tree* in a ν-connected ν-graph $\mathcal{G}^\nu$ is a ν-connected subgraph that meets every node and has no loops. As a standard definition for a conventional graph, the *distance* between two 0-nodes of a 0-graph is the number of branches in a shortest path between them.

2 Transfinite Graphs Without Spanning Trees

Perhaps the simplest such graph is the one shown in Figure 1. Therein we find an infinite series circuit of parallel circuits consisting of the branches a_k and b_k. That infinite series

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circuit has infinitely many 0-tips. Two of those 0-tips, one with a representative along the a_k branches and the other with a representative along the b_k branches are connected through two 1-nodes n_a^1 and n_b^1 respectively to two branches β_a and β_b. The latter are incident to two 0-nodes n_c^0 and n_d^0 respectively. The other 0-tips of that series circuit are also embraced by 1-nodes, one for each, but those 1-nodes are singletons and are not shown in the figure. There is no spanning tree for this graph because no tree can meet any two of its 1-nodes without containing the branches a_k and b_k for some sufficiently large k — that is, without containing a loop. Indeed, if the tree is to meet any two 1-nodes, it must traverse a representative for each, and, since those representatives must differ, they will differ for an a_k and b_k for some k.

The simple graph of Figure 1 can appear imbedded in more complicated graphs, making it impossible for the latter to have spanning trees as well. An example of this is a one-ended infinite ladder. A more complicated example is shown in Figure 2. One might speculate that an "essentially spanning" tree might be found, one that meets every nonsingleton node, is ν-connected, and contains no loops. This is so for the ladder. But, the graphs of Figures 1 and 2 dispel that idea.

What we shall do in the rest of this paper is establish that all finitely structured transfinite graphs $\mathcal{G}^\nu$ have spanning trees, this being true for all ranks $\nu \leq \omega$ and certainly for many higher ranks as well.

3 1-Graphs

Theorem 3.1. *Every finitely structured 1-graph $\mathcal{G}^1$ has a spanning tree.*

Proof. Choose any 0-subsection of $\mathcal{G}^1$. For a subsequent purpose, we shall denote it by $\mathcal{S}_{b,0,1}^0$. $\mathcal{S}_{b,0,1}^0$ is either a 0-graph of a 1-graph, but its internal rank is 0. So, let us consider at first the 0-graph of $\mathcal{S}_{b,0,1}^0$, that is, the graph consisting of all the branches and 0-nodes of $\mathcal{S}_{b,0,1}^0$. (The 1-nodes of $\mathcal{S}_{b,0,1}^0$ will all be bordering nodes and are only finitely many in number.) In that 0-graph we can choose a spanning tree T_0^0 as follows. Choose any internal 0-node n^0 in $\mathcal{S}_{b,0,1}^0$ along with its finitely many incident branches. This is a star graph. We now proceed inductively as follows. Assume that a spanning tree has been chosen for the subgraph induced by all the 0-nodes at distances from n^0 less than m $(m \geq 1)$. Let $n^0_{m,k}$ $(k = 1, \ldots, K_m)$ be the finitely many 0-nodes that are at a distance m from n^0. Set $\mathcal{N}^0_m = \{n_{m,k} : k = 1, \ldots, K_m\}$. Choose any branch connecting $n^0_{m,1}$ to a 0-node in $\mathcal{N}^0_{m-1}$; there will be such a branch. Proceed along the nodes of $\mathcal{N}^0_m$ as follows. Having chosen one such branch for each $n^0_{m,l}$ $(1 \leq l < k \leq K_m)$, choose one branch connecting $n^0_{m,k}$ to a node in $\mathcal{N}^0_{m-1}$. At each k there will be at least one such branch, and — whatever be that choice — no loop will be formed with the previously chosen branches because that last branch will be an end branch. Proceeding in this fashion along the nodes of each $\mathcal{N}^0_m$ and then successively for $m = 2, 3, \ldots$, we obtain a spanning tree T^0_0 for the 0-graph of $\mathcal{S}^0_{b,0,1}$.

Now, consider any bordering node n^1 of $\mathcal{S}^0_{b,0,1}$. If n^1 contains a 0-node n^0, then the tree T^0_0 will meet n^1 through a branch incident to n_0. Otherwise, the tree will reach n^1 through one or more 0-tips. However, the arms $\mathcal{A}_p$ $(p = 1, 2, \ldots)$ for any contraction to n^1 within $\mathcal{S}^0_{b,0,1}$ will have bases $\mathcal{V}_p$ whose cardinalities are all bounded by some natural number q. It follows that the tree T^0_0 can reach n^1 through no more than q 0-tips. Since $\mathcal{S}^0_{b,0,1}$ has only finitely many bordering nodes, the tree T^0_0 will have only finitely many 0-tips.

Next, consider the 1-loops and 0-loops created by the shorts imposed by the bordering nodes of $\mathcal{S}^0_{b,0,1}$ upon the 0-tips and finitely many of the elementary tips of T^0_0. There can be only finitely many such 1-loops and 0-loops. We can break all of them one-by-one by opening finitely many branches without disconnecting the resulting graphs. This yields a spanning tree T for $\mathcal{S}^0_{b,0,1}$. (T may have 1-nodes.)

Repeat this procedure to get such a spanning tree for every 0-subsection of $\mathcal{G}^1$. Let us denote the 0-subsections by $\mathcal{S}^0_{b,m,l}$ $(m = 0, 1, 2, \ldots; l = 1, \ldots, L_m; L_0 = 1)$, where, for each $m \geq 1$ and each l, $\mathcal{S}^0_{b,m,;l}$ is 1-adjacent to at least one subsection $\mathcal{S}^0_{b,m-1,l}$ but not 1-adjacent to any $\mathcal{S}^0_{b,p,l}$, where $p < m - 1$. The union of any finite number of spanning trees for the $\mathcal{S}^0_{b,m,l}$ will contain no more than finitely many 0-loops or 1-loops. These too can be broken one-by-one by opening branches without disconnecting the resulting graphs. We do so by proceeding sequentially starting with a spanning tree for $\mathcal{S}^0_{b,0,1}$, then appending all the spanning trees for the $\mathcal{S}^0_{b,1,l}$ and breaking loops, then appending all the spanning trees for the $\mathcal{S}^0_{b,2,l}$ and breaking loops, and so on. Since $\mathcal{G}^1$ is finitely structured, it has only finitely many 0-subsections. Thus, the procedure ends, and the result is a spanning tree for $\mathcal{G}^1$. ♣

Corollary 3.2. *Let $\mathcal{S}_b^1$ be a 1-subsection in a finitely structured ν-graph ($\nu \geq 2$). Then, $\mathcal{S}_b^1$ has a spanning tree.*

Note. Here, the superscript 1 indicates that the internal rank of the subsection is 1. Thus, $\mathcal{S}_b^1$ is a $(2-)$-subsection.

Proof. By working with the 0-subsections of $\mathcal{S}_b^1$ as in the preceding proof, we can construct a spanning tree T for the 1-graph of $\mathcal{S}_b^1$, but in this case the procedure may continue indefinitely for $m = 1, 2, \ldots$ since $\mathcal{S}_b^1$ may have countably many 0-subsections. Nonetheless, the said spanning tree will be achieved.

Every bordering node of $\mathcal{S}_b^1$ will impose shorts among some of the tips (i.e., elementary tips, 0-tips, and 1-tips) of T. But, here again $\mathcal{S}_b^1$ has only finitely many bordering nodes, and T will reach them through no more than finitely many 0-tips and 1-tips because of the finite structuring. Therefore, only finitely many loops will be created by shorts imposed by the bordering nodes of $\mathcal{S}_b^1$. So, here again we can break them one-by-one by opening branches without disconnecting the resulting graphs. In the end we will have a spanning tree for $\mathcal{S}_b^1$. ♣

4 Higher Ranks

This procedure for constructing a spanning tree can be extended inductively and without much alteration to graphs with higher ranks. The induction is on the internal ranks of subsections. We start with the natural-number ranks.

Theorem 4.1. *Every finitely structured μ-graph $\mathcal{G}^\mu$, where μ is a natural number, has a spanning tree.*

Proof. Consider any α-subsection $\mathcal{S}_b^\alpha$ of $\mathcal{G}^\mu$, where α is the internal rank of $\mathcal{S}_b^\alpha$ ($2 \leq \alpha \leq \mu$). $\mathcal{S}_b^\alpha$ is partitioned by its $(\alpha-)$-subsections. We can arrange the latter by choosing any one of them, say, $\mathcal{S}_{b,0,1}^{\alpha-}$, then letting $\mathcal{S}_{b,1,l}^{\alpha-}$ ($l = 1, \ldots, L_1$) be the $(\alpha-)$-subsections that are α-adjacent to $\mathcal{S}_{b,0,1}^{\alpha-}$, and similarly, for each $m = 2, 3, \ldots$, letting $\mathcal{S}_{b,m,l}^{\alpha-}$ ($l = 1, \ldots, L_m$) be the $(\alpha-)$-subsections that are α-adjacent to at least one of the $\mathcal{S}_{b,m-1,l}^{\alpha-}$ but not α-adjacent to any $\mathcal{S}_{b,p,l}^{\alpha-}$, where $p < m - 1$.

Let us assume that a spanning tree $T_{m,l}$ has been constructed for each ($\alpha-$)-subsection $\mathcal{S}_{b,m,l}^{\alpha-}$ ($m = 0, 1, 2, \ldots; l = 1, \ldots, L_m; L_0 = 1$). Thus, $T_{0,1}$ is a spanning tree for $\mathcal{S}_{b,0,1}^{\alpha-}$. Then, $T_{0,1} \bigcup \left(\bigcup_{l=1}^{L_1} T_{1,l} \right)$ will have only finitely many loops. These can be broken by opening no more than finitely many branches without disconnecting the resulting graph. We get a spanning tree T_1 for $\mathcal{S}_{b,0,1}^{\alpha-} \bigcup \left(\bigcup_{l=1}^{L_1} \mathcal{S}_{b,1,l}^{\alpha-} \right)$.

We proceed inductively for $m = 2, 3, \ldots$. Let T_{m-1} be the spanning tree obtained after appending all the spanning trees $T_{p,l}$ up to and including the index $p = m - 1$ and breaking loops. Again $T_{m-1} \bigcup \left(\bigcup_{l=1}^{L_m} T_{m,l} \right)$ will have only finitely many loops, and these too can be broken by opening finitely many branches to get a spanning tree T_m for $\bigcup_{p=0}^{m} \bigcup_{l=1}^{L_p} \mathcal{S}_{b,p,l}^{\alpha-}$. Continuing this procedure through all m, we obtain a spanning tree for the α-graph of $\mathcal{S}_b^{\alpha}$, where again α is the internal rank of $\mathcal{S}_b^{\alpha}$. As before, we need break no more than finitely many loops — those created by the bordering nodes of $\mathcal{S}_b^{\alpha}$ — to get a spanning tree for $\mathcal{S}_b^{\alpha}$.

In view of Corollary 3.2, we can now conclude that, for each $\alpha = 1, \ldots, \mu$, every α-subsection of $\mathcal{G}^{\mu}$ has a spanning tree, and in particular $\mathcal{G}^{\mu}$ itself has a spanning tree. ♣

It is a fact that no $\vec{\omega}$−graph can be finitely structured [1, Lemma 4.5-1(ii)]. So, let us proceed to an ω-graph.

Theorem 4.2. *Every finitely structured ω-graph $\mathcal{G}^{\omega}$ has a spanning tree.*

Proof. Every ($\omega-$)-subsection $\mathcal{S}_b^{\omega-}$ of $\mathcal{G}^{\omega}$ has a spanning tree. We can conclude this by induction on α as in the last paragraph of the preceding proof, where now α progresses through all the natural numbers. (Remember that every loop in $\mathcal{S}_b^{\omega-}$ must have a natural number rank, and therefore every loop created in our procedure of appending trees will be broken at some natural number for α.) Next, upon considering the shortings of tips imposed by the finitely many ω-nodes of $\mathcal{G}^{\omega}$, we can break finitely many loops again to obtain a spanning tree for $\mathcal{G}^{\omega}$. ♣

We have thus constructed a spanning tree for any finitely structured graph of any rank up to and including ω. Clearly, our procedure can be continued to still higher ranks of countable ordinals for both successor and limit ordinals. (No finitely structured graph can have an arrow-rank — as is the case for $\vec{\omega}$.) All that is required for extensions to higher ranks is changes in notations. The procedure works for all finitely structured graphs that can be defined.

References

[1] A.H.Zemanian, *Transfiniteness for Graphs, Electrical Networks, and Random Walks*, Birkhauser, Boston, 1995.

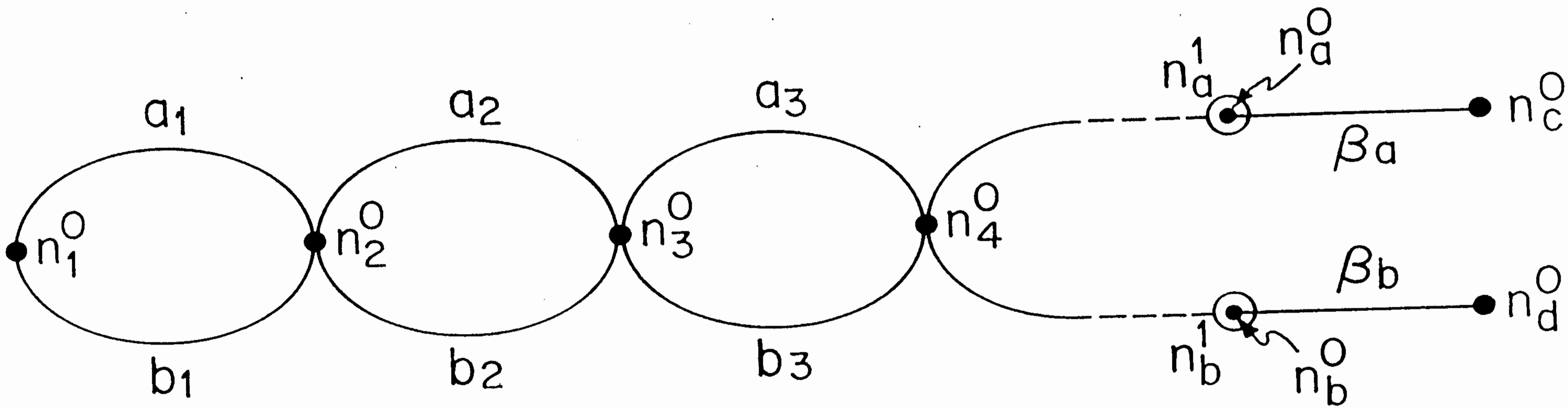

Figure 1. In this graph heavy dots denote 0-nodes and small circles denote nonsingleton 1-nodes. Not shown are the singleton 1-nodes, each containing a 0-tip with a representative passing through both a_k and b_j branches.

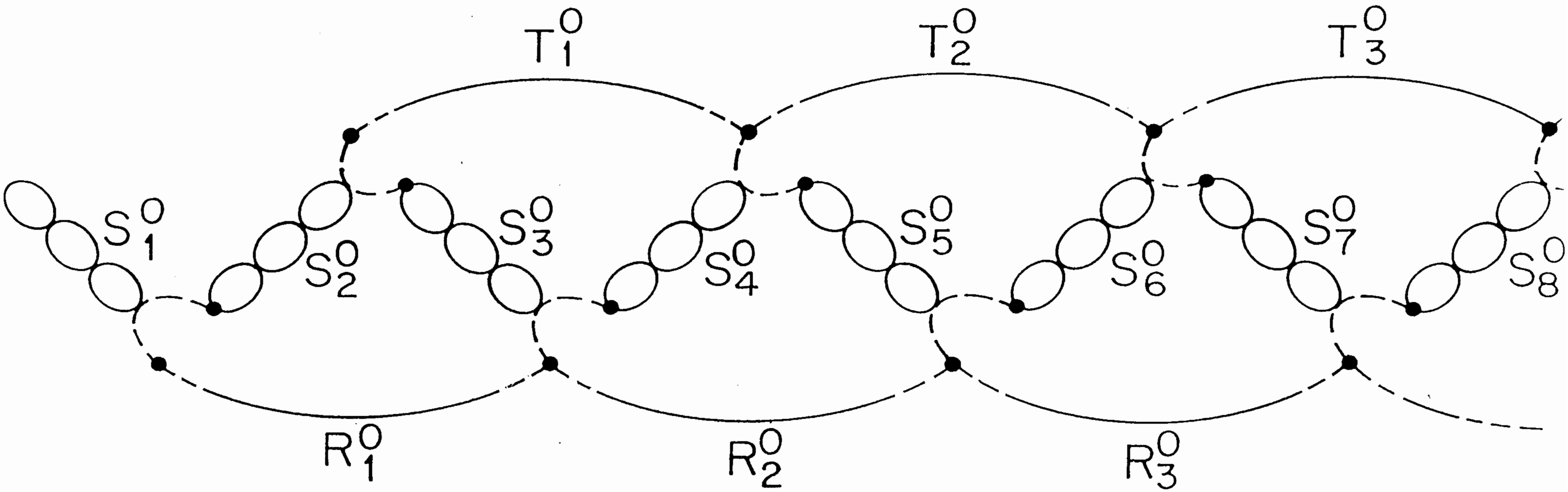

Figure 2. In this graph the S_k^0 represent the 0-graph occurring in Figure 1 to the left of n_a^1 and n_b^1. The R_k^0 and T_k^0 represent endless 0-paths. The heavy dots now denote nonsingleton 1-nodes. This graph extends infinitely toward the right, maintaining the pattern shown.