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PRISTINE TRANSFINITE GRAPHS

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Abstract — A tranfinite graph is pristine if no node of any rank embraces a node of lower rank. Every transfinite graph has an equivalent pristine graph obtained by extracting from all maximal nodes every embraced node of lower rank. This leads to considerable simplifications in many parts of the theory of transfinite graphs. The present report is the first of a series of reports that will develop this simpler theory.

1   Introduction

The idea of a transfinite graph arose from an outstanding problem in the theory of infinite electrical networks; namely, what is the input resistance of an infinite ladder of resistances? If the ladder is not uniform in its element values, it can happen that this question has no unique answer unless a resistor R connected to the infinite extremity of the ladder is specified [4, Section 1.4]. In order to connect R to the ladder at infinity, a new kind of node is needed — a "1-node"; such a node is a set of one-way infinite paths that are equivalent in the sense that they are pairwise identical except for finitely many branches and nodes. Any one those paths specifies the 1-node. The needed connection of R is accomplished by inserting the conventional nodes of R (now called "0-nodes") as members of the two 1-nodes specified by two disjoint one-way infinite paths in the ladder.

This yields a "transfinite graph" (called a "graph of rank 1" or simply a "1-graph") having no counterpart in conventional graph theory. In addition to branches and 0-nodes, we now have 1-nodes. This procedure for making connections at infinite extremities can be extended to a 1- graph containing infinitely many conventional graphs connected together by 1-nodes. This leads to "2-nodes" that make connections at the infinite extremities of such a 1-graph to obtain a "graph of rank 2" or simply a "2-graph." Continuing in this way, we obtain "3-nodes," ... , "ν-nodes," and the corresponding "μ-graphs" ($\mu \leq \nu$),

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where the rank ν can range through both finite and transfinite ordinals. Moreover, a ν-node can contain a node of lower rank, which in turn can contain a node of still lower rank, and so forth. These node containments lead to considerable complications in the theory of transfinite graphs.

It turns out that a node of lower rank can be extracted from a node of higher rank without altering the connectivity of the transfinite graph. For example, the 0-nodes of R can be connected to the 1-nodes of the ladder through one-way infinite paths of electrical shorts instead of inserting the 0-nodes as members of the 1-nodes. The result is a transfinite graph in which no node of higher rank contains any node of lower rank; this we call a "pristine transfinite graph." A further result is that the theory of transfinite graphs becomes substantially simpler when all the transfinite graphs are replaced by their pristine equivalents.

The aim of this report is to rework the theory of transfinite graphs by restricting the discussions to pristine graphs and making simplifications wherever possible. This will be just the first of several reports, which together will rework all the theory of of transfinite graphs as it presently exists. Later on, we will do the same thing for transfinite electrical networks and random walks. The result will be a more accessible rendition of these subjects. Moreover, new results will also be presented, whose discoveries were facilitated by our restriction to pristine graphs. In fact, our initial restriction to pristine graphs will free us from a number of subsequent complicated assumptions that were needed to make the theory work in the nonpristine case.

2 Pristine Nodes

In this section we explicate how embraced nodes can be extracted from nodes of higher ranks to obtain a "pristine nodes" and thereby "pristine graphs." In order to follow the discussion, one must know what a nonpristine transfinite node of arbitrary rank is, and this entails a fairly long recursive construction of transfinite graphs. Let us not repeat what appears elsewhere [3, Chapter 5], [4, Chapter 2] and is in fact unessential to this report's discussion. Instead, we will simply address our remarks in this section to those having such

knowledge. Moreover, as occasions arise later on, we will point out how our restriction to "pristine nodes" (defined below) simplifies the general development of transfiniteness for graphs. This section and those remarks can be ignored. Everything else herein can be understood without referring to [3] and [4].

Turning to our task for this section, we start with the fact that there are several kinds of transfinite nodes. It is convenient to restrict the ranks of transfiniteness that we shall examine to the natural numbers $\mu = 0, 1, 2, \ldots$, to the arrow rank $\vec{\omega}$, and to the first transfinite-ordinal rank ω. A discussion of higher ranks requires only a repetition of what we say here.

A transfinite node n^μ whose rank is a positive natural number μ is a set consisting of at least one and in general many $(\mu - 1)$-tips and possibly a single node of lower rank n^α $(\alpha < \mu)$. When n^μ does not contain any node of lower rank, n^μ will be called *pristine*. However, when n^α does exist as a member of n^μ, n^α may in turn contain another node of still lower rank, which in turn may contain another node of still lower rank, and so on through finitely many nodes. Thus, we can have a finite sequence of nodes $\{n^{\mu_k}\}_{k=1}^K$ of natural number ranks μ_k with $\mu_1 < \mu_2 < \ldots < \mu_K = \mu$ such that n^{μ_k} is an element of $n^{\mu_{k+1}}$ $(k < K)$. Each of the n^{μ_k} contains at least one $(\mu_k - 1)$-tip. We say that n^μ *embraces* itself, all its members, and all the members of all the n^{μ_k} for $k = 1, \ldots, K - 1$. (By definition, n^μ does not embrace any other entity such as the representative paths of its embraced tips.) Thus, n^μ is pristine if and only if n^μ does not embrace any node of lower rank, that is, if and only if n^μ is simply a nonvoid set of $(\mu - 1)$-tips.

On the other hand, n^μ is called *maximal* if it is not embraced by a node of higher rank.

Consider next a node $n^{\vec{\omega}}$ of the arrow rank $\vec{\omega}$; $\vec{\omega}$ is the only rank satisfying $\mu < \vec{\omega} < \omega$ for all natural number ranks μ, where ω is the first transfinite-ordinal rank. $n^{\vec{\omega}}$ contains no tips of any ranks. Instead, there is an infinite sequence $\{n^{\mu_k}\}_{k=1}^\infty$ of nodes of natural number ranks μ_k such that $\mu_k < \mu_{k+1}$ and n^{μ_k} is a member of $n^{\mu_{k+1}}$ for every k. Thus, each $n^{\mu_{k+1}}$ embraces every node preceding it in the sequence as well as all the tips in every such node. (n^{μ_1} does not embrace a node of lower rank.) Then, $n^{\vec{\omega}}$ is defined as the set $\{n^{\mu_k}\}_{k=1}^\infty$. There is no such thing as a "pristine $\vec{\omega}$-node."

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Consider, finally, a node n^ω of rank ω. This is a set consisting of at least one $\vec{\omega}$-tip and possibly (but not necessarily) one node of rank less than ω. As before, $n^{\vec{\omega}}$ is called *pristine* if it does not contain a node of lower rank. If, on the other hand, it does do so, n^ω will embrace infinitely many nodes if it contains an $\vec{\omega}$-node but will embrace only finitely many nodes if it contains a μ-node, where μ is a natural number as before.

A node of any rank is called *maximal* if it is not embraced by a node of higher rank.

When all the nodes of a transfinite graph (or network) are pristine, that is, when no node embraces a node of lower rank, the graph (or network) will be called *pristine* too. Thus, a graph is pristine if and only if all its nodes are maximal. It turns out that the theory of pristine graphs and networks is substantially simpler than the theory for transfinite graphs and networks given in [3] and [4].

For example, in a general transfinite graph $\mathcal{G}^\nu$ of rank ν, a *ρ-section* ($\rho \leq \nu$) can be taken to be a branch-induced maximal subgraph of the ρ-graph of $\mathcal{G}^\nu$ whose branches are ρ-connected, that is, connected through an α-path where $\alpha \leq \rho$. (This is a somewhat sharper definition of a ρ-section than that given in [4, page 49].) On the other hand, a *$((\rho + 1)-)$-subsection* is a branch-induced maximal subgraph of the ρ-graph of $\mathcal{G}^\nu$ whose branches are ρ-connected by paths that do not meet γ-nodes in $\mathcal{G}^\nu$ where $\gamma \geq \rho + 1$. (Here too, this is a somewhat sharper definition of a $((\rho + 1)-)$-subsection than that given in [4, page 81].) In general, ρ-sections and $((\rho + 1)-)$-subsections are different entities, but they may be identical in particular cases. For instance, consider

Example 2.1. Refer to the 2-graph of Fig. 1(a). There is therein a one-ended 1-path $P^1 = \{n_1^1, P_1^0, n_2^1, P_2^0, n_3^1, P_3^0, \ldots\}$, where the P_k^0 are endless 0-paths. P^1 reaches the 2-node n^2 through its 1-tip, and n^2 contains a 0-node n^0. Furthermore, a finite 0-path containing the branches b_1, b_2, b_3, and b_4 passes through n^0 and thereby through n^2.

P^1 is both a 1-section and a $(2-)$-subsection. Also, each P_k^0 is a 0-section and a $(1-)$-subsection. On the other hand, the branches b_1, b_2, b_3, and b_4 induce a 0-section $\mathcal{S}^0$, which is partitioned into two $(1-)$-subsections: $\mathcal{S}_{b1}^{1-}$ induced by b_1 and b_2 and $\mathcal{S}_{b2}^{1-}$ induced by b_3 and b_4. $\mathcal{S}^0$, $\mathcal{S}_{b1}^{1-}$, and $\mathcal{S}_{b2}^{1-}$ are all different. Here, all nodes are both pristine and maximal except for n^2 and n^0; n^2 is maximal but not pristine, and n^0 is pristine (all 0-nodes are

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pristine) but not maximal. ♣

It may appear that pristine graphs are more restrictive than transfinite graphs in general. However, any transfinite graph can be converted into a pristine one by extracting embraced nodes. This will not change the connectivity relationships within the graph. The procedure is as follows.

First consider the case of a maximal μ-node n^μ, where μ is a positive natural number. Let $\{n^{\mu_k}\}_{k=1}^K$ be the sequence of nodes embraced by n^μ with $\mu_k < \mu_{k+1}$ for $k = 1, \ldots, K-1$ and $\mu_K = \mu$. Thus, n^{μ_k} is a member of $n^{\mu_{k+1}}$. Let us first remove n^{μ_1} from n^{μ_2}, thus making n^{μ_1} a pristine maximal node and converting n^{μ_2} into a pristine node (we use the same notation for the new nodes); then, let us append a one-ended $(\mu_2 - 1)$-path $P^{\mu_2 - 1}$ that terminates at n^{μ_1}; finally, let us add the $(\mu_2 - 1)$-tip of $P^{\mu_2 - 1}$ to the members of n^{μ_2}. All the nodes of $P^{\mu_2 - 1}$ are pristine and maximal. We obtain hereby a new μ-node that embraces only the nodes $n^{\mu_2}, \ldots, n^{\mu_K}$ and is connected to n^{μ_1} through $P^{\mu_2 - 1}$. We continue this way by extracting in turn $n^{\mu_2}, \ldots, n^{\mu_K}$. More specifically, consider the case where we have extracted $n^{\mu_1}, \ldots, n^{\mu_{k-1}}$ and are ready to treat n^{μ_k}. We remove n^{μ_k} from $n^{\mu_{k+1}}$ thus making n^{μ_k} a pristine maximal node, then we append a one-ended $(\mu_{k+1} - 1)$-path $P^{\mu_{k+1} - 1}$ that terminates at n^{μ_k}, and finally we add the $(\mu_{k+1} - 1)$-tip of $P^{\mu_{k+1} - 1}$ to the members of $n^{\mu_{k+1}}$. Here too, all the nodes of $P^{\mu_{k+1} - 1}$ are pristine and maximal. We call $P^{\mu_{k+1} - 1}$ the *extraction path along which n^{μ_k} is extracted from $n^{\mu_{k+1}}$*. $P^{\mu_{k+1} - 1}$ is *isolated* in the sense that it meets or reaches the rest of the graph only through its terminal nodes or tips. After treating all the n^{μ_k} $(k = 1, \ldots, K-1)$, the result will be pristine maximal nodes $\hat{n}^{\mu_k}$ $(k = 1, \ldots, K-1)$ and in addition a pristine maximal node $\hat{n}^\mu$. (We have now changed notation by adding a "hat.")

Example 2.2. Consider again the 2-graph of Fig. 1(a). The 0-node n^0 contained in the 2-node n^2 of that graph is extracted to obtain the pristine 2-graph of Fig. 1(b). This is accomplished by first removing n^0 from n^2 and then appending the one-ended 1-path $Q^1 = \{n^0, Q_1^0, m_2^1, Q_2^0, m_3^1, Q_3^0, \ldots\}$ that terminates at n^0 and reaches the pristine node $\hat{n}^2$. Q_1^0 is a one-ended 0-path, but the Q_k^0 $(k = 2, 3, \ldots)$ are endless 0-paths. Note that n^0 is trivially 2-connected to n^2 in part (a) and is 2-connected to $\hat{n}^2$ through the extraction path

Q^1 in part (b). In this way, the connectivity ranks are not changed. ♣

For the case of a maximal $\vec{\omega}$-node $n^{\vec{\omega}}$, we have an infinite sequence $\{n^{\mu_k}\}_{k=1}^{\infty}$ of nodes embraced by $n^{\vec{\omega}}$, and moreover $n^{\vec{\omega}}$ does not contain any tip. Proceeding as above through $k = 1, 2, 3, \ldots$, we obtain an infinity of pristine maximal nodes $\hat{n}_k^{\mu_k}$ ($k = 1, 2, 3, \ldots$) such that each $\hat{n}_k^{\mu_k}$ is connected to $\hat{n}_{k+1}^{\mu_{k+1}}$ through an appended one-ended extraction path $P^{\mu_{k+1}-1}$ starting at $\hat{n}_k^{\mu_k}$ and reaching $\hat{n}_{k+1}^{\mu_{k+1}}$. In this case, $n^{\vec{\omega}}$ disappears and is not replaced by any node after all extractions have been made. On the other hand, the appended one-ended extraction paths form altogether a one-ended $\vec{\omega}$-path.

Finally, consider a maximal nonpristine ω-node n^{ω}. It may contain either a node n^{μ_K} of natural-number rank μ_K or an $\vec{\omega}$-node. Thus, it may embrace either finitely many nodes $n^{\mu_1}, \ldots, n^{\mu_K}$ or infinitely many nodes $n^{\mu_1}, n^{\mu_2}, \ldots$. In the first case, the procedure removes all the n^{μ_k} ($k = 1, \ldots, K$) and then appends finitely many one-ended paths which together form an $\vec{\omega}$-path that starts at n^{μ_1} and passes through $n^{\mu_2}, \ldots, n^{\mu_K}$ in turn; the $\vec{\omega}$-tip of that $\vec{\omega}$-path is adjoined to the remaining members of n^{ω} to get a pristine maximal ω-node $\hat{n}^{\omega}$. In the second case, there are infinitely many appended extraction paths which are all of natural-number ranks but together form an $\vec{\omega}$-path $P^{\vec{\omega}}$; again, the $\vec{\omega}$-tip of $P^{\vec{\omega}}$ is adjoined to the remaining members of n^{ω} to get a pristine maximal ω-node $\hat{n}^{\omega}$.

This procedure for extracting embraced nodes can be continued in the very same way through ranks higher than ω.

If we perform such extractions for all the nonmaximal nodes in a graph, the result will be a pristine graph. Any two branches or nodes that are ρ-connected in the original network will remain ρ-connected in the resulting pristine network, except for the $\vec{\omega}$-nodes — they disappear.[1] In this way, many results concerning arbitrary transfinite graphs can be established simply by examining pristine graphs. The same is true for the electrical behavior of transfinite networks so long as the appended paths that implement the extractions of nonmaximal nodes are taken to be paths of shorts.

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[1] If n^{ρ} embraces n^{γ}, where $\rho > \gamma$, n^{ρ} and n^{γ} are taken to be ρ-connected through a trivial ρ-path.

3 0-Graphs and 1-Graphs

0-graphs are conventional graphs. However, we shall define them in an unconventional way. In our approach, a *branch* is a set with two elements, each of which is called either an *elementary tip* or a $\vec{0}$-*tip* or a (-1)-*tip*. Here, $\vec{0}$ or -1 is the *rank* of the tip. Thus, as ranks, we have $\vec{0} = -1$. In words, we refer to $\vec{0}$ as "zero-arrow." (We will meet other "arrow ranks" later on; they are the immediate predecessors of the "limit-ordinal ranks.") Each $\vec{0}$-tip belongs to exactly one branch. To conform with some terminology we will use later on, we also refer to a branch as a (-1)-*section* and as an *endless* (-1)-*path*. Also, if $t_1^{\vec{0}}$ and $t_2^{\vec{0}}$ are the two elementary tips of a branch b (i.e., if $b = \{t_1^{\vec{0}}, t_2^{\vec{0}}\}$), we say that b *traverses* $t_1^{\vec{0}}$ and $t_2^{\vec{0}}$ and that b is a *representative* of $t_1^{\vec{0}}$ and of $t_2^{\vec{0}}$. All this may seem quite arbitrary and unnecessary, but it simply reflects some terminology that will be needed for graphs of higher ranks. A branch may be assigned an *orientation*, that is, on ordering of its two $\vec{0}$-tips, in which case it is called an *oriented branch*.[2]

Let $\mathcal{B} = \{b_j\}_{j \in J}$ denote a nonvoid set of branches, where J is a set of indices for the branches. Also, let $T^{\vec{0}} = \cup_{j \in J} b_j$ be the set of all $\vec{0}$-tips for all the branches in $\mathcal{B}$. The cardinality $\overline{\overline{\mathcal{B}}} = \overline{\overline{J}}$ of $\mathcal{B}$ is unrestricted. Thus, $\overline{\overline{T^{\vec{0}}}} = 2\overline{\overline{\mathcal{B}}}$ if $\mathcal{B}$ is a finite set, and $\overline{\overline{T^{\vec{0}}}} = \overline{\overline{\mathcal{B}}}$ if $\mathcal{B}$ is an infinite set. Let $\mathcal{N}^0 = \{n_k^0\}_{k \in K}$ be a partitioning of $T^{\vec{0}}$, where K is an index set for the partitioning. Each n_k^0 is called a 0-*node* and corresponds to a conventional node. The *degree* of n_k^0 is its cardinality. If two or more $\vec{0}$-tips are members of the same 0-node n_k^0, those $\vec{0}$-tips are said to be *shorted together* by n_k^0 (or simply *shorted*). A *singleton* 0-node is one having exactly one (-1)-tip; otherwise, it is called a *nonsingleton* 0-node. The sole $\vec{0}$-tip of a singleton 0-node is said to be *open*.

A *self-loop* is a branch having both its $\vec{0}$-tips shorted. If a branch b has at least one of its $\vec{0}$-tips $t^{\vec{0}}$ in a 0-node n^0, b and n^0 are said to be *incident*, and b is said to *reach* n^0 through $t^{\vec{0}}$. Note that, according to this construction, n^0 is not a member of b; n^0 and b merely intersect as sets of $\vec{0}$-tips. If two branches are incident to the same 0-node, we say that they are *adjacent*. If two branches are incident to the same two 0-nodes, we say that

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[2] Branch orientations will be needed for electrical networks when branch currents and branch voltages are measured with respect to those orientations.

they are *in parallel* or are *parallel branches*. (Two self-loops that are incident to the same 0-node are not taken to be in parallel.)

Now, let $\mathcal{B}$ and thereby $\mathcal{T}^{\vec{0}}$ be given, and let $\mathcal{N}^0$ be a chosen partitioning of $\mathcal{T}^{\vec{0}}$. Then, the pair

$$\mathcal{G}^0 = \{\mathcal{B}, \mathcal{N}^0\} \tag{1}$$

is called a 0-*graph*. $\mathcal{G}^0$ is called *finite, infinite, countable,* and *uncountable* according to whether $\mathcal{B}$ has those properties respectively. Also, $\mathcal{G}^0$ is called *locally finite* if all its 0-nodes are of finite degree.

A "subgraph" of a 0-graph $\mathcal{G}^0 = \{\mathcal{B}, \mathcal{N}^0\}$ is defined as follows. Let $\mathcal{B}_s$ be a nonvoid subset of $\mathcal{B}$, and let $\mathcal{N}_s^0$ be the (nonvoid) subset of $\mathcal{N}^0$ consisting of those 0-nodes each of which contains at least one $\vec{0}$-tip belonging to a branch of $\mathcal{B}_s$. Then, $\mathcal{G}_s^0 = \{\mathcal{B}_s, \mathcal{N}_s^0\}$ is called a *subgraph* (or a 0-*subgraph*) of $\mathcal{G}^0$. For more specificity, we also refer to $\mathcal{G}_s^0$ as the 0-*subgraph of $\mathcal{G}^0$ induced by $\mathcal{B}_s$* (or *by the branches of $\mathcal{B}_s$*). Note that $\mathcal{G}_s^0$ is not in general a graph because there may be a $\vec{0}$-tip in one of the nodes of $\mathcal{G}_s^0$ that does not belong to any branch of $\mathcal{G}_s^0$.[3] Any member of $\mathcal{B}_s \cup \mathcal{N}_s^0$ is said to be *in $\mathcal{G}_s^0$*. Furthermore, if $\mathcal{B}_t \subset \mathcal{B}_s$, the subgraph $\mathcal{G}_t^0$ induced by $\mathcal{B}_t$ is a *subgraph of the subgraph $\mathcal{G}_s^0$*, and $\mathcal{G}_t^0$ is said to be *in $\mathcal{G}_s^0$*

The *union* (resp. *intersection*) of two subgraphs of a 0-graph is the subgraph induced by the union (resp. intersection) of the branch sets of the two subgraphs. Two subgraphs (resp. a 0-node and a subgraph) are said to *meet* or to be *incident* if they have a common 0-node (resp. the 0-node is in the subgraph).[4] Otherwise, they are called *disjoint*. Incident subgraphs need not contain a common branch, and disjoint subgraphs will not contain a common branch.

A 0-*path* P^0 is an alternating sequence of 0-nodes and branches

$$P^0 = \{\ldots, n_m^0, b_m, n_{m+1}^0, b_{m+1}, \ldots\} \tag{2}$$

in which no 0-node and therefore no branch appears more than once and moreover every branch and 0-node that are adjacent in the sequence (2) are incident in the graph. The

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[3] Were we to eliminate all such $\vec{0}$-tips from all the nodes of $\mathcal{G}_s^0$, we would obtain a 0-graph, which we might call a "reduced graph"; see [3, page 8]. We will not bother with this particularity.

[4] Later on, when dealing with transfinite graphs, "meet" and "incident" will mean different things, "meet" being a stronger concept than "incident."

indices $\ldots, m, m+1, \ldots$ are restricted to the integers (i.e., they do not extend to the transfinite ordinals). Thus, P^0 is in fact a conventional path. We can and do identify the 0-path (2) with the 0-subgraph its branches induce.

An *orientation* is assigned to P^0 by choosing one of the two possible orderings of (2) that maintain (2) as a path. P^0 is called *finite* or *one-ended* or *endless* if respectively the sequence (2) has only finitely many terms or extends infinitely in exactly one direction or extends infinitely in both directions. A finite 0-path is also called *two-ended*. If (2) terminates on either side, we require that the terminal element on that side be a 0-node. P^0 is called *nontrivial* if it has at least one branch, in which case it will have at least two 0-nodes as well. P^0 is called *trivial* if it contains exactly one term, a 0-node. A 0-*loop* is defined as is a finite nontrivial 0-path except that its two terminal elements are required to be the same 0-node. (More precisely, it is a circulant sequence; the 0-node chosen as the terminal elements when writing down the sequence is immaterial to its definition.) As with a 0-path, one of the two possible orderings of that circulant sequence may be chosen to obtain an *orientation* of the 0-loop.

Two 0-nodes are said to be 0-*connected* if there is a 0-path (perforce finite) terminating at those 0-nodes. Two branches are called 0-*connected* if a 0-node incident to one branch and a 0-node incident to the other branch are 0-connected, and similarly for a branch and a 0-node.[5] If all the branches of $\mathcal{G}^0$ are 0-connected, we say that $\mathcal{G}^0$ itself is 0-*connected*. A 0-*section* of a 0-graph is a subgraph induced by a maximal set of 0-connected branches. Thus, a 0-section is identical to (what in conventional terminology is called) a *component* of the 0-graph. Later on, when transfinite graphs are discussed, a 0-section will be in general different from a component of the graph.

Our next objective is to define the "infinite extremities" of a 0-graph $\mathcal{G}^0 = \{\mathcal{B}, \mathcal{N}^0\}$ having at least one component containing a one-ended 0-path. Two one-ended 0-paths will be called *equivalent* if they are identical except for finitely many branches and nodes. This is truly an equivalence relationship, and it partitions the set of all one-ended 0-paths in $\mathcal{G}^0$ into equivalence classes, called 0-*tips*. Each one-ended 0-path is a *representative* of the 0-tip

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[5] A single branch is 0-connected to itself through a trivial 0-path.

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in which it resides. The 0-tips are taken to be the "infinite extremities" mentioned above, and in a moment we shall use them to define a new kind of node that can connect infinite components together at their infinite extremities.

Subgraphs of $\mathcal{G}^0$ may also have 0-tips; those 0-tips are defined in exactly the same way except that the one-ended 0-paths are required to be in the subgraph. We say that a subgraph of $\mathcal{G}^0$ *traverses* each of its 0-tips and also each of the $\vec{0}$-tips of its branches. A one-ended (resp. endless) 0-path, considered as a subgraph, has exactly one 0-tip (resp. two 0-tips).

Let $\mathcal{T}^0$ be the set of 0-tips of $\mathcal{G}^0$. Arbitrarily partition $\mathcal{T}^0$ into a set $\mathcal{N}^1$ of subsets n_k^1 ($k \in K_1$), where K_1 is the index set for the subsets. Each n_k^1 is thus a set of 0-tips; it is called a 1-*node*. We may think of n_k^1 as a *shorting together* of some (or all) of the 0-tips of $\mathcal{G}^0$; n_k^1 provides a new kind of connection[6] between the infinite extremities of $\mathcal{G}^0$. The *rank* of n_k^1 is 1. All this results in a transfinite graph of the first rank, defined to be the triple:

$$\mathcal{G}^1 = \{\mathcal{B}, \mathcal{N}^0, \mathcal{N}^1\} \tag{3}$$

and called a 1-*graph*.

A one-ended or endless 0-path P^0 is said to *reach* a 1-node n^1 if a 0-tip of P^0 is a member of n^1.

4 μ-Graphs and $(\mu + 1)$-Graphs

To continue our recursive construction of transfinite graphs, let us now assume that, for some natural number $\mu - 1$ and for each rank $\gamma = 0, \ldots, \mu - 1$, γ-graphs:

$$\mathcal{G}^\gamma = \{\mathcal{B}, \mathcal{N}^0, \ldots, \mathcal{N}^\gamma\}, \tag{4}$$

γ-paths:

$$P^\gamma = \{\ldots, n_m^\gamma, P_m^{\gamma-1}, n_{m+1}^\gamma, P_{m+1}^{\gamma-1}, \ldots\} \tag{5}$$

and γ-tips, have all been defined, where $\mathcal{N}^\gamma$ is the set of γ-nodes n_k^γ. The definition of a γ-path includes the idea of a $(\gamma - 1)$-path reaching a γ-node and of two $(\gamma - 1)$-paths

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[6] In contrast to the constructions given in [3] and [4], the components of $\mathcal{G}^0$ can only be connected together at their 0-tips because all nodes are now pristine; see Section 1.2.

being disjoint. All this was done in the preceding section for $\gamma = 0$, where branches b_m took the role of the paths $P_m^{\gamma-1}$ in (5). (We view a branch as an endless path of rank -1, and it reaches any 0-node containing one of its (-1)-tips. Branches are trivially disjoint.) Moreover, the ideas of 0-path reaching a 1-node and of two 0-paths being disjoint were also defined.

Let $\mathcal{T}^{\mu-1}$ denote the set of all $(\mu - 1)$-tips of $\mathcal{G}^{\mu-1}$. If $\mathcal{T}^{\mu-1}$ is nonvoid, we can continue our recursive construction. Assuming this, we arbitrarily partition $\mathcal{T}^{\mu-1}$ into a set $\mathcal{N}^{\mu}$ of subsets n_k^{μ} ($k \in K_{\mu}$) of $\mathcal{T}^{\mu-1}$, where K_{μ} is an index set for the partition. Each n_k^{μ} is called an μ-node, and μ is its *rank*. If n_k^{μ} contains more than one $(\mu-1)$-tip, we say that n_k^{μ} *shorts* them, and we think of n_k^{μ} as a joining together of some (or all) of the infinite extremities of $\mathcal{G}^{\mu-1}$. In this case of two or more $(\mu - 1)$-tips in n_k^{μ}, we call n_k^{μ} a *nonsingleton* node. If n_k^{μ} has exactly one $(\mu - 1)$-tip, n_k^{μ} is called a *singleton* node, in which case we say that its sole $(\mu - 1)$-tip is *open*.

All this yields a *transfinite graph $\mathcal{G}^{\mu}$ of rank μ* or synonymously a *μ-graph*, which is by definition the $(\mu + 2)$-tuple:

$$\mathcal{G}^{\mu} = \{\mathcal{B}, \mathcal{N}^0, \ldots, \mathcal{N}^{\mu}\}. \tag{6}$$

For $\mu > 0$, $\mathcal{B}$ is perforce an infinite set. Moreover, for every $\gamma < \mu$, there will be an infinity of nonsingleton γ-nodes. Indeed, there will be at least one $(\gamma + 1)$-node, and therefore at least one γ-tip with a one-ended representative path P^{γ}, which perforce will have an infinity of nonsingleton γ-nodes. It follows that $\mathcal{N}^{\gamma}$ is infinite too, but there may be finitely many singleton γ-nodes — or none at all. On the other hand, $\mathcal{N}^{\mu}$ may be either a finite or infinite set.

If a $(\mu - 1)$-tip $t^{\mu-1}$ of a one-ended or endless $(\mu - 1)$-path $P^{\mu-1}$ is contained in the μ-node n^{μ}, we say that $P^{\mu-1}$ *reaches n^{μ} through $t^{\mu-1}$*. For each $\gamma = 0, \ldots, \mu - 1$, the $(\gamma + 2)$-tuple

$$\mathcal{G}^{\gamma} = \{\mathcal{B}, \mathcal{N}^0, \ldots, \mathcal{N}^{\gamma}\} \tag{7}$$

is called the *γ-graph of $\mathcal{G}^{\mu}$*.

Next, let $\mathcal{B}_s$ be a nonvoid subset of $\mathcal{B}$. For each $\gamma = 0, \ldots, \mu$, let $\mathcal{N}_s^{\gamma}$ be the set of all γ-nodes n^{γ} in $\mathcal{N}^{\gamma}$ such that n^{γ} contains at least one $(\gamma - 1)$-tip having a representative

all of whose branches are in $\mathcal{B}_s$. (The branches of a representative can be identified in principle by expanding every $(\gamma-2)$-path in it into a sequence containing $(\gamma-3)$-paths and then repeating this process through decreasing ranks for the paths until a set of branches is achieved.) $\mathcal{N}_s^0$ will not be void, but, for $\gamma \geq 1$, $\mathcal{N}_s^\gamma$ may be void. However, there will be some maximum rank δ $(0 \leq \delta \leq \mu)$ for which all the $\mathcal{N}_s^\gamma$ $(0 \leq \gamma \leq \delta)$ are nonvoid. When $\delta < \mu$, then all the $\mathcal{N}_s^\gamma$ $(\delta < \gamma \leq \mu)$ will be void. This because, if $\mathcal{N}_s^\gamma$ is void, then there are no γ-paths and therefore no representatives of γ-tips that can be constructed out of the branches of $\mathcal{B}_s$ alone; hence, $\mathcal{N}_s^{\gamma+1}$ must be void too.

For each $\gamma = 0, \ldots, \delta$, the $(\gamma+2)$-tuple

$$\mathcal{G}_s^\gamma = \{\mathcal{B}_s, \mathcal{N}_s^0, \ldots, \mathcal{N}_s^\gamma\} \tag{8}$$

is called the *γ-subgraph of $\mathcal{G}^\mu$ induced by $\mathcal{B}_s$* (or *by the branches of $\mathcal{B}_s$*). By the *subgraph of $\mathcal{G}^\mu$ induced by $\mathcal{B}_s$* we mean the δ-subgraph of $\mathcal{G}^\mu$ induced by $\mathcal{B}_s$. Thus, $\mathcal{G}_s^\gamma$ is the subgraph of the γ-graph of $\mathcal{G}^\mu$ induced by $\mathcal{B}_s$. Here too, $\mathcal{G}_s^\gamma$ is not in general a graph because some node of some $\mathcal{N}_s^\beta$ $(0 \leq \beta \leq \gamma)$ may have a $(\beta-1)$-tip with no representative consisting exclusively of branches of $\mathcal{B}_s$. Any member of $\mathcal{B}_s \cup \mathcal{N}_s^0 \cup \ldots \cup \mathcal{N}_s^\gamma$ is said to be *in $\mathcal{G}_s^\gamma$*, and any β-subgraph induced by a subset of $\mathcal{B}_s$ is said to be *in $\mathcal{G}_s^\gamma$* or alternatively a *subgraph of $\mathcal{G}_s^\gamma$*.

The *union* (resp. *intersection*) of two subgraphs of $\mathcal{G}^\mu$ is the subgraph induced by the union (resp. intersection) of the branch sets of the two subgraphs.

Example 4.1. It can happen that the union of two subgraphs can have more nodes than the nodes of the two subgraphs. In fact, the union can be of higher rank than the ranks of both subgraphs. Consider the 1-graph $\mathcal{G}^1$ of Fig. 2, wherein a one-ended 0-path reaches a 1-node n^1. Let $\mathcal{G}_a^0$ (resp. $\mathcal{G}_b^0$) be the subgraph induced by the branches a_k (resp. b_k), where $k = 1, 2, 3, \ldots$. Each of these subgraphs, being of rank 0, does not contain any 1-node, but their union is $\mathcal{G}^1$ and contains the 1-node n^1. ♣

Two subgraphs of possibly different ranks (resp. a node and a subgraph) are said to *meet* if they have a common node of any rank (resp. if the node is in the subgraph). Otherwise, they are said to be *disjoint*. Note that two different nodes are perforce disjoint whatever be their ranks (because all nodes are pristine). As before, subgraphs that meet need not have

a common branch, and disjoint subgraphs will not have a common branch.

We identify any $(\mu-1)$-path with the subgraph induced by its branches. (Those branches can be identified as was done above for representatives.) Thus, the idea of disjointness can be applied to two or more $(\mu-1)$-paths.

For subgraphs of ranks larger than 0, "meeting" and "being incident" can be different ideas. However, "reaching" is synonymous with "being incident to." Specifically, a γ-subgraph $\mathcal{G}^\gamma$ is said to *reach* or to be *incident to* a λ-node n^λ if there is a one-ended $(\lambda-1)$-path $(\lambda-1 \leq \gamma)$ all of whose branches are in $\mathcal{G}^\gamma_s$ and whose $(\lambda-1)$-tip is in n^λ. If $\lambda-1 < \gamma$, n^λ is in $\mathcal{G}^\gamma_s$, in which case we say that $\mathcal{G}^\gamma_s$ *meets* n^λ. However, if $\lambda-1 = \gamma$, n^λ will not be in $\mathcal{G}^\gamma_s$, and we do not say that $\mathcal{G}^\gamma_s$ meets n^λ. Two subgraphs *reach* (resp. *meet*) each other if they reach (resp. meet) a common node. Thus, two subgraphs may reach each other without meeting.

We are now ready to define a "μ-path," where μ is a positive natural number. P^μ is called a *nontrivial μ-path* if it is an alternating sequence of the form

$$P^\mu = \{\ldots, n^\mu_m, P^{\mu-1}_m, n^\mu_{m+1}, P^{\mu-1}_{m+1}, \ldots\} \tag{9}$$

where the indices m are restricted to the integers, the $P^{\mu-1}_m$ are $(\mu-1)$-paths, and the n^μ_m are μ-nodes satisfying certain conditions. However, when the sequence (9) terminates on a side (possibly both sides), the terminating element is a γ-node where $0 \leq \gamma \leq \mu$. The required conditions are these:

(a) There is at least one $(\mu-1)$-path and at least one μ-node.

(b) Every two elements in the sequence (9) are disjoint, except when there is a terminating γ-node n^γ of P^μ with $\gamma < \mu$, in which case n^γ is the terminating element of its adjacent $(\mu-1)$-path in (9). (There may be two such terminating nodes.)

(c) If n^μ is adjacent to $P^{\mu-1}$ in (9), then $P^{\mu-1}$ reaches n^μ.

As a consequence of this definition and the fact that all nodes are pristine,[7] we have that, when a $(\mu-1)$-path $P^{\mu-1}$ in (9) is adjacent to two μ-nodes in (9), $P^{\mu-1}$ is endless.

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[7]As a result of our restriction to pristine nodes, this is a much simpler definition of a μ-path than those given in [3, page 144] or in [4, page 39]. ([4] uses a tighter definition than [3] so as to avoid an ambiguity in the rank of a path.)

However, when $P^{\mu-1}$ is adjacent in (9) to a terminating γ-node n^γ with $\gamma < \mu$, then $P^{\mu-1}$ is one-ended and has n^γ as its terminating node. In this latter case, $P^{\mu-1}$ can be written out in the form

$$P^{\mu-1} \;=\; \{n^\gamma, Q^\gamma, n^{\gamma+1}, Q^{\gamma+1}, n^{\gamma+2}, Q^{\gamma+2}, \ldots, n^{\mu-1}, Q^{\mu-1}\} \qquad (10)$$

where, for each $k = 0, \ldots, \mu - \gamma - 1$, $Q^{\gamma+k}$ is a one-ended $(\gamma + k)$-path having $n^{\gamma+k}$ as its terminating node (thus, $n^{\gamma+k}$ is a member of $Q^{\gamma+k}$), and $Q^{\gamma+k}$ reaches $n^{\gamma+k+1}$ with its $(\gamma + k)$-tip. ($Q^{\mu-1}$ reaches the μ-node adjacent to $P^{\mu-1}$ in (9).) In any case, a nontrivial μ-path has at least one μ-node incident to a $(\mu - 1)$-path.

The nontrivial μ-path P^μ is called *two-ended* or, synonymously, a *finite* μ-path if (9) terminates on both sides at different nodes; thus, a two-ended nontrivial μ-path ($\mu > 0$) has only finitely many μ-nodes but infinitely many nodes of lower ranks. P^μ is called *one-ended* if (9) terminates on exactly one side. P^μ is called *endless* if (9) extends infinitely on both sides.

A *trivial μ-path* is a singleton $\{n^\mu\}$, where n^μ is a μ-node.

A μ-loop is defined exactly as is a two-ended nontrivial μ-path except that the two terminating elements in (9) are the same μ-node. (Actually, (9) for a μ-loop is a circulant sequence, and which of its μ-nodes is chosen as the terminating elements when writing (9) is of no importance.)

We identify a nontrivial μ-path or a μ-loop with the μ-subgraph of $\mathcal{G}^\mu$ that its branches induce. Here too, we can assign an *orientation* to a nontrivial μ-path or a μ-loop by choosing one of the two orderings of (9) that maintain (9) as a μ-path or μ-loop.

The next step in our recursive construction of transfinite graphs is to define the "μ-tips" of a μ-graph $\mathcal{G}^\mu$; these represent the infinite extremities of $\mathcal{G}^\mu$. Two one-ended μ-paths will be called *equivalent* if their sequences of the form (9) are the same except for finitely many of the μ-nodes and $(\mu-1)$-paths. This too is a proper equivalence relationship, and it partitions the set of all one-ended μ-paths in $\mathcal{G}^\mu$ into equivalence classes, called *μ-tips*. Each one-ended μ-path in a μ-tip is a *representative* of that μ-tip. $\mathcal{G}^\mu$ may have no one-ended path and therefore no μ-tip, but, if it does have a μ-tip, we can continue our recursive construction on to $(\mu + 1)$-graphs. So, assuming the latter case, let $\mathcal{T}^\mu$ be the set of all

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μ-tips for $\mathcal{G}^\mu$. Partition $\mathcal{T}^\mu$ into a set $\mathcal{N}^{\mu+1}$ of subsets $n_k^{\mu+1}$ ($k \in K_{\mu+1}$) of $\mathcal{T}^\mu$, where $K_{\mu+1}$ denotes the set of indices for the partitioning. Each $n_k^{\mu+1}$ is thus a set of μ-tips and is called a $(\mu+1)$-node; its *rank* is $\mu + 1$.

The $(\mu+1)$-*graph* $\mathcal{G}^{\mu+1}$ is the $(\mu+3)$-tuple:

$$\mathcal{G}^{\mu+1} = \{\mathcal{B}, \mathcal{N}^0, \ldots, \mathcal{N}^{\mu+1}\}. \tag{11}$$

We can now define the "γ-graph of $\mathcal{G}^{\mu+1}$" ($0 \leq \gamma \leq \mu$) and "branch-induced γ-subgraphs of $\mathcal{G}^{\mu+1}$" ($0 \leq \gamma \leq \mu + 1$) exactly as was done for $\mathcal{G}^\mu$. Also, the "meeting," "reaching," "incidence," "disjointness," "union," and "intersection" of subgraphs of $\mathcal{G}^{\mu+1}$ are defined as before. We also say that a γ-subgraph (and therefore any γ-path or γ-loop) *traverses* each of the β-tips ($0 \leq \beta \leq \gamma$) having a representative in that γ-subgraph.

5 $\vec{\omega}$-Graphs and ω-Graphs

Since μ is any natural number in the preceding section, our recursive construction of transfinite graphs has hereby been accomplished for all natural-number ranks. We can now assume that there is an entity consisting of an infinite set $\mathcal{B}$ of branches and in which nodes of all natural-number ranks have been constructed by repeating the constructions in the preceding section without ever stopping. Thus, for every natural number μ, we have a nonvoid set $\mathcal{N}^\mu$ of μ-nodes. This entity is called an $\vec{\omega}$-graph and is specified by the infinite set of sets:

$$\mathcal{G}^{\vec{\omega}} = \{\mathcal{B}, \mathcal{N}^0, \mathcal{N}^1, \ldots\} \tag{12}$$

In this case, every set $\mathcal{N}^\mu$ has infinitely many nonsingleton μ-nodes and possibly finitely or infinitely many singleton μ-nodes.

For each natural number γ, the $(\gamma+2)$-tuple:

$$\mathcal{G}^\gamma = \{\mathcal{B}, \mathcal{N}^0, \ldots, \mathcal{N}^\gamma\}$$

is called the γ-*graph of* $\mathcal{G}^{\vec{\omega}}$. Also, for any nonvoid subset $\mathcal{B}_s$ of $\mathcal{B}$, the *subgraph* $\mathcal{G}_s$ *of* $\mathcal{G}^{\vec{\omega}}$ *induced by* $\mathcal{B}_s$ (or *by the branches of* $\mathcal{B}_s$) is either the finite set

$$\mathcal{G}_s^\delta = \{\mathcal{B}_s, \mathcal{N}_s^0, \ldots, \mathcal{N}_s^\delta\} \tag{13}$$

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or the infinite set

$$\mathcal{G}_s^{\vec{\omega}} = \{\mathcal{B}_s, \mathcal{N}_s^0, \mathcal{N}_s^1, \ldots\} \tag{14}$$

where each $\mathcal{N}_s^\gamma$ is defined exactly as in the preceding section. The finite set (13) arises when $\mathcal{N}_s^\delta$ is nonvoid but there are no one-ended δ-paths in $\mathcal{G}_s^\delta$, and the infinite set (14) occurs when no such stopping of the recursive construction occurs. Similarly, for the natural number γ, we define the γ-subgraph of $\mathcal{G}^{\vec{\omega}}$ induced by $\mathcal{B}_s$ to be the set

$$\mathcal{G}_s^\gamma = \{\mathcal{B}_s, \mathcal{N}_s^0, \ldots, \mathcal{N}_s^\gamma\}$$

so long as $\mathcal{N}_s^\gamma$ is nonvoid. This is the subgraph of the γ-graph of $\mathcal{G}^{\vec{\omega}}$ induced by $\mathcal{B}_s$.

For these subgraphs, "meeting," "reaching" or synonymously "incident to," "disjoint-ness," "union," "intersection," and the "traversing of tips" are defined exactly as in the preceding section.

However, an "$\vec{\omega}$-path" is defined[8] rather differently than is a μ-path, μ being a natural number here and below. A one-ended $\vec{\omega}$-path $P^{\vec{\omega}}$ is a one-way infinite, alternating sequence:

$$P^{\vec{\omega}} = \{n_0^\mu, P_0^\mu, n_1^{\mu+1}, P_1^{\mu+1}, n_2^{\mu+2}, P_2^{\mu+2}, \ldots\} \tag{15}$$

where, for each $k = 0, 1, 2, \ldots$ (k is restricted to the natural numbers), $n_k^{\mu+k}$ is a $(\mu+k)$-node and $P_k^{\mu+k}$ is a one-ended $(\mu + k)$-path such that

(a) $P_k^{\mu+k}$ has $n_k^{\mu+k}$ as its terminating element (thus, $n_k^{\mu+k}$ is a member of $P_k^{\mu+k}$),

(b) other than the terminating condition (a), every two elements in (15) are disjoint (that is, for each k, $P_k^{\mu+k}$ contains $n_k^{\mu+k}$ but is disjoint from all other elements of (15)), and

(c) $P_k^{\mu+k}$ reaches $n_{k+1}^{\mu+k+1}$.

An endless $\vec{\omega}$-path is the union of two one-ended $\vec{\omega}$-paths that meet only at a common initial node. Thus, it has the form

$$P^{\vec{\omega}} = \{\ldots, m_2^{\mu+2}, Q_1^{\mu+1}, m_1^{\mu+1}, Q_0^\mu, m_0^\mu = n_0^\mu, P_0^\mu, n_1^{\mu+1}, P_1^{\mu+1}, n_2^{\mu+2}, \ldots\} \tag{16}$$

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[8]This definition too is much simpler than those given in [3, page 147] and [4, pages 40-41].

where the leftward sequence of $(\mu + k)$-nodes $m_k^{\mu+k}$ and $(\mu + k)$-paths $Q_k^{\mu+k}$ $(k = 0, 1, \ldots)$ fulfills the same conditions as does (15), and moreover every element to the left of $m_0^\mu = n_0^\mu$ is disjoint from every element to the right of $m_0^\mu = n_0^\mu$ except that Q_0^μ and P_0^μ share $m_0^\mu = n_0^\mu$.

There is no such thing as a two-ended $\vec{\omega}$-path or a trivial $\vec{\omega}$-path or an $\vec{\omega}$-loop. (Indeed, terminating (15) on the right at a δ-node would yield a δ-path for some natural number δ.)

As always, we identify an $\vec{\omega}$-path with the subgraph induced by its branches, those branches being identified through successive expansions of paths.

We now prepare for the definition of an ω-graph, ω being the first transfinite ordinal. Assume that the $\vec{\omega}$-graph $\mathcal{G}^{\vec{\omega}}$ given by (12) has a nonvoid set of one-ended $\vec{\omega}$-paths. Partition that set into equivalence classes, where two such paths are taken to be equivalent if they are identical except for finitely many terms in their sequences (15). Each equivalence class is called an $\vec{\omega}$-*tip* and represents an "infinite extremity" of $\mathcal{G}^{\vec{\omega}}$. Let $\mathcal{T}^{\vec{\omega}}$ denote the set of all $\vec{\omega}$-tips for $\mathcal{G}^{\vec{\omega}}$. Next, partition $\mathcal{T}^{\vec{\omega}}$ in any arbitrary fashion into subsets n_k^ω ($k \in K_\omega$, where K_ω is the index set for the partition.) Each n_k^ω is called an ω-*node*. Let $\mathcal{N}^\omega$ denotes the set of them. Then the *graph* $\mathcal{G}^\omega$ *of rank* ω or synonymously the ω-*graph* $\mathcal{G}^\omega$ is defined to be the infinite set of sets:

$$\mathcal{G}^\omega = \{\mathcal{B}, \mathcal{N}^0, \mathcal{N}^1, \ldots, \mathcal{N}^\omega\} \tag{17}$$

where the ellipsis ... represents the same sequence of node sets as that in (12). Since we are confining ourselves to pristine nodes, there is no such thing as an $\vec{\omega}$-node [4, page 37], and thus $\mathcal{N}^{\vec{\omega}}$ does not appear in (17). "Singleton" and "nonsingleton" ω-nodes are defined as were their counterpart μ-nodes in Section 2.2.

Again, for γ now being any natural number or $\vec{\omega}$, we define the "γ-graph of $\mathcal{G}^\omega$" in the same way as was done for a μ-graph in Section 2.2 and for an $\vec{\omega}$-graph in this section; that is, for μ a natural number, the μ-*graph of* $\mathcal{G}^\omega$ is $\{\mathcal{B}, \mathcal{N}^0, \ldots, \mathcal{N}^\mu\}$, and the $\vec{\omega}$-*graph of* $\mathcal{G}^\omega$ is $\{\mathcal{B}, \mathcal{N}^0, \mathcal{N}^1, \ldots\}$. In addition, given a subset $\mathcal{B}_s$ of $\mathcal{B}$, we define the "γ-subgraph of $\mathcal{G}^\omega$ induced by $\mathcal{B}_s$," (γ being either a natural number or $\vec{\omega}$ or ω) in the same way as before, with the "subgraph of $\mathcal{G}^\omega$ induced by $\mathcal{B}_s$" being the δ-subgraph as before. Similarly, "subgraphs of subgraphs," "meeting," "reaching" or "incident to," "disjointness," "union," "intersection," and the "traversing of tips" are defined as in Section 2.2.

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A *nontrivial ω-path P^ω* is an alternating sequence of the form

$$P^\omega \;=\; \{\ldots, n_m^\omega, P_m^{\vec{\omega}}, n_{m+1}^\omega, P_{m+1}^{\vec{\omega}}, \ldots\} \tag{18}$$

where the indices $\ldots, m, m+1, \ldots$ are restricted to the integers, the $P_m^{\vec{\omega}}$ are $\vec{\omega}$-paths, and the n_m^ω are ω-nodes satisfying the conditions given below. If the sequence (18) terminates on either side, the terminating element is a γ-node where now γ denotes either a natural number or ω. Here are the required conditions:

(a) There is at least one $\vec{\omega}$-path and at least one ω-node.

(b) Every two elements in the sequence (18) are disjoint, except when the sequence terminates on a side at a μ-node n^μ where μ is a natural number, in which case n^μ is the terminating element of its adjacent $\vec{\omega}$-path in (18).

(c) If n^ω and $P^{\vec{\omega}}$ are adjacent in (18), then $P^{\vec{\omega}}$ reaches n^ω.

It follows that, when $P^{\vec{\omega}}$ is adjacent to two ω-nodes in (18), $P^{\vec{\omega}}$ is endless. However, when $P^{\vec{\omega}}$ is adjacent to a terminating μ-node n^μ (μ a natural number), then $P^{\vec{\omega}}$ is one-ended and has n^μ as its terminating node (in which case, n^μ is a member of $P^{\vec{\omega}}$).

The adjectives "two-ended," "one-ended," and "endless" are applied to the nontrivial ω-path P^ω in exactly the same way as they were for P^μ in Section 2.2. A *trivial ω-path* is a singleton $\{n^\omega\}$ containing an ω-node n^ω. An ω-loop is defined as is a nontrivial two-ended ω-path except that the two terminating elements in the sequence are the same ω-node. (That is, an ω-loop is a circulant sequence with the choice of the terminating ω-node when writing (18) being immaterial.)

6 Transfinite Graphs of Higher Ranks

With ω-paths in hand, we can continue our recursions on toward ranks higher than ω. Indeed, endless $\vec{\omega}$-paths have taken the role that branches played in Section 2.1, and thus $\vec{\omega}$-tips correspond to $\vec{0}$-tips, ω-nodes correspond to 0-nodes, and ω-paths correspond to 0-paths. Then, by repeating the constructions given in Section 2.1 with these replacements,

we can define ω-tips, $(\omega + 1)$-nodes, and finally $(\omega + 1)$-graphs. The latter have the form

$$\mathcal{G}^{\omega+1} = \{\mathcal{B}, \mathcal{N}^0, \ldots, \mathcal{N}^\omega, \mathcal{N}^{\omega+1}\},$$

where the ellipsis ... represents node sets of all positive natural-number ranks μ. Then, by repeating the recursive arguments of Section 2.2 with μ replaced by $\omega + \mu$, we obtain $(\omega + \mu)$-graphs for every natural number μ:

$$\mathcal{G}^{\omega+\mu} = \{\mathcal{B}, \mathcal{N}^0, \ldots, \mathcal{N}^\omega, \mathcal{N}^{\omega+1}, \ldots, \mathcal{N}^{\omega+\mu}\}$$

Then, the arguments of Section 2.3 with $\vec{\omega}$ replaced by $\omega + \vec{\omega}$ and ω replaced by $\omega + \omega = \omega \cdot 2$ produce $(\omega + \vec{\omega})$-graphs

$$\mathcal{G}^{\omega+\vec{\omega}} = \{\mathcal{B}, \mathcal{N}^0, \ldots, \mathcal{N}^\omega, \mathcal{N}^{\omega+1}, \mathcal{N}^{\omega+2}, \ldots\},$$

where the last ellipsis denotes node sets of all ranks of the form $\omega + \mu$ above $\omega + 2$, and finally $(\omega \cdot 2)$-graphs

$$\mathcal{G}^{\omega\cdot2} = \{\mathcal{B}, \mathcal{N}^0, \mathcal{N}^1, \ldots, \mathcal{N}^\omega, \mathcal{N}^{\omega+1}, \ldots, \mathcal{N}^{\omega\cdot2}\}.$$

At this point we have repeated a cycle of recursions twice, once from ranks 0 to ω and secondly from ranks ω to $\omega \cdot 2$. We can continue with more cycles of recursions to obtain transfinite graphs of still higher ranks. It is tempting to speculate that this process can be continued through all the countable-ordinal ranks. Indeed, we might suppose that transfinite graphs have been constructed for all ranks up to some arbitrarily chosen, countable, limit-ordinal rank ν. Then, another cycle of recursions can produce graphs of the ranks $\nu + 1, \ldots, \nu + \vec{\omega}, \nu + \omega$. But, can such a supposition be justified? Very large countable ordinals have strange properties [2, pages 64-73], and what about the still larger countable ordinals that have not been named and explored? Perhaps our cycle of recursions may collapse for sufficiently large ranks. What we can say is that our cycles of recursions can be carried far beyond ω.

Henceforth, we will present detailed arguments for the ranks up to and including ω. Repetitions of those arguments will yield results for many higher ranks. From now on, whenever we discuss a ν-graph $\mathcal{G}^\nu$ it will be understood that the rank ν satsifies $0 \leq \nu \leq \omega$ (possibly $\nu = \vec{\omega}$).

7 Nondisconnectable Tips and Connectedness

Let us repeat some definitions mentioned in Section 1.2. for the ν-graph $\mathcal{G}^\nu$ ($0 \leq \nu \leq \omega$). Let ρ denote any rank such that $0 \leq \rho \leq \nu$. Two nodes n_1^γ and n_2^δ are said to be ρ-*connected* if there is a two-ended α-path with $\alpha \leq \rho$ that terminates at n_1^γ and n_2^δ. (If $\rho = \vec{\omega}$, we must have $\alpha < \vec{\omega}$ because there is no two-ended path of rank $\vec{\omega}$. Moreover, because all nodes are pristine, we must also have $\gamma \leq \alpha$ and $\delta \leq \alpha$.) Two branches are said to be ρ-*connected* if they are incident to 0-nodes that are ρ-connected. Furthermore, a ρ-*section* $\mathcal{S}^\rho$ in a ν-graph $\mathcal{G}^\nu$ is a subgraph of the ρ-graph of $\mathcal{G}^\nu$ that is induced by a maximal set of ρ-connected branches and which contains at least one ρ-node if $\rho \neq \vec{\omega}$ and contains nodes of all natural-number ranks if $\rho = \vec{\omega}$.[9] Unlike subgraphs in general, a ρ-section $\mathcal{S}^\rho$ is a graph because every tip of every node in $\mathcal{S}^\rho$ will have a representative all of whose branches are in $\mathcal{S}^\rho$. This is a consequence of the fact that the branch set that defines $\mathcal{S}^\rho$ is a maximal set of branches that are ρ-connected.

There can be a node that is incident to a ρ-section $\mathcal{S}^\rho$ but is not in $\mathcal{S}^\rho$. Such a node must be of rank $\rho + 1$. Indeed, by the definition of incidence that node will have a tip with a representative in $\mathcal{S}^\rho$. Since $\mathcal{S}^\rho$ is a ρ-graph, the rank of that tip cannot be less than ρ, for otherwise the node, being pristine, would be of rank no greater than ρ and therefore would be in $\mathcal{S}^\rho$; also, that tip's rank cannot be greater than ρ because any path of rank greater than ρ cannot be in $\mathcal{S}^\rho$. Hence, the tip's rank is ρ, and therefore that node, being pristine, is of rank $\rho + 1$.

We can classify the $(\rho + 1)$-nodes incident to a ρ-section $\mathcal{S}^\rho$ as follows. A *bordering node* $m^{\rho+1}$ of $\mathcal{S}^\rho$ is a node of rank $\rho + 1$ that is incident to $\mathcal{S}^\rho$; in other words, $m^{\rho+1}$ contains a ρ-tip traversed by $\mathcal{S}^\rho$. A *boundary node* $n^{\rho+1}$ of $\mathcal{S}^\rho$ is a node of rank $\rho + 1$ that is incident to $\mathcal{S}^\rho$ and also to another ρ-section; in other words, $n^{\rho+1}$ contains a ρ-tip traversed by $\mathcal{S}^\rho$ and another ρ-tip not traversed by $\mathcal{S}^\rho$. Thus, a boundary node is a special case of a bordering node. All boundary nodes are nonsingletons, but a bordering node may be a singleton.[10]

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[9]This is a somewhat sharper definition of a ρ-section than that given in [4, page 49] because there a ρ-section was merely required to be a subgraph of $\mathcal{G}^\nu$ whereas now a ρ-section is required to be a subgraph of the ρ-graph of $\mathcal{G}^\nu$. In this regard see also www.ee.sunysb.edu/~zeman for the Errata for [4, page 49].

[10]These definitions conform as special cases to those given in [4, pages 49 and 81].

Let also note that, if $\mathcal{G}^\nu$ is ν-connected, we can simplify the definition of a ρ-section $\mathcal{S}^\rho$ by dropping the requirement about the ranks of nodes in $\mathcal{S}^\rho$. In particular, if $\mathcal{G}^\nu$ is ν-connected, a ρ-section $\mathcal{S}^\rho$ is simply a subgraph of the ρ-graph of $\mathcal{G}^\nu$ that is induced by a maximal set of ρ-connected branches. Indeed, for $\rho \neq \vec{\omega}$, every branch will then be ρ-connected to some ρ-node, and therefore a maximal set of ρ-connected branches will induce a subgraph having at least one ρ-node. Similarly, for $\rho = \vec{\omega}$, nodes of all ranks less than $\vec{\omega}$ will automatically exist.

By a "component" we will mean something that is in general different from a section. Let $\mathcal{G}$ be a graph; for example, $\mathcal{G}$ may be $\mathcal{G}^\nu$ or $\mathcal{S}^\rho$. Let $\mathcal{G}_s$ be a subgraph of $\mathcal{G}$ induced by some subset $\mathcal{B}_s$ of the branch set of $\mathcal{G}$. A *component* of $\mathcal{G}_s$ is a subgraph induced by a maximal set of branches in $\mathcal{B}_s$ that are connected through $\mathcal{G}_s$ with any rank of connectedness. The essential difference between a section and a component is the following: A section is defined in $\mathcal{G}$ for some rank ρ, and different ρ-sections may be γ-connected through $\mathcal{G}$ for $\gamma > \rho$. However, for different components of the subgraph $\mathcal{G}_s$ there will not be any paths of any ranks in $\mathcal{G}_s$ connecting them.

Lemma 7.1. *Let the ν-graph $\mathcal{G}^\nu$ be ν-connected and such that, for each $\rho < \nu$, every ρ-section has only finitely many incident nonsingleton $(\rho + 1)$-nodes. Then, for each such ρ, there exist infinitely many $(\rho - 1)$-sections in each ρ-section.*

Proof. For each $\rho < \nu$, each ρ-section $\mathcal{S}^\rho$ has at least one incident $(\rho + 1)$-node $n^{\rho+1}$ by virtue of the ν-connectedness of $\mathcal{G}^\nu$. Therefore, $\mathcal{S}^\rho$ contains a representative P^ρ of a ρ-tip in $n^{\rho+1}$. P^ρ is a one-ended ρ-path and therefore must pass through infinitely many nonsingleton ρ-nodes. Since every $(\rho - 1)$-section has only finitely many incident nonsingleton ρ-nodes, there must be infinitely many $(\rho - 1)$-sections in $\mathcal{S}^\rho$. ♣

Conventional connectedness is 0-connectedness. It is an equivalence relationship among 0-nodes and hence among branches as well. Indeed, if n_a^0, n_b^0, and n_c^0 are 0-nodes with n_a^0 and n_b^0 being 0-connected and with n_b^0 and n_c^0 being 0-connected, then n_a^0 and n_c^0 are also 0-connected. In fact, if P_1^0 is a 0-path connecting n_a^0 and n_b^0 and if P_2^0 is a 0-path connecting n_b^0 and n_c^0, then the union $P_1^0 \cup P_2^0$ will contain a 0-path connecting n_a^0 and n_c^0. Hence, 0-connectedness is a transitive binary relationship between 0-nodes and is also

reflexive and symmetric obviously.

Transitivity does not extend to higher ranks of connectedness in general. In particular, if the nodes n_a^γ and n_b^δ are ρ-connected ($\rho \geq 1$) through a two-ended α-path P_1^α ($\alpha \leq \rho$) and if the nodes n_b^δ and n_c^ζ are ρ-connected through a two-ended β-path P_2^β ($\beta \leq \rho$), there need not be a two-ended θ-path ($\theta \leq \rho$) terminating at n_a^γ and n_c^ζ. As a result, ρ-sections may overlap. (See Examples 3.1-5 and 3.1-6 in [4] for illustrations of this difficulty.)

Transitivity for ρ-connectedness can be assured by imposing a rather simple requirement on the tips within $\mathcal{G}^\nu$. To state it, we need some more definitions. Two tips are said to be *disconnectable* if one can find two representatives, one for each tip, that are disjoint. Thus, any two elementary tips are trivially disconnectable since their representatives are branches. As the negation of "disconnectable," we say that two tips are *nondisconnectable* if every representative of one of them meets every representative of the other tip infinitely often. We can state this somewhat differently as follows. Two tips t_1^γ and t_2^δ are called nondisconnectable if P_1^γ and P_2^δ meet at least once whenever P_1^γ is a representative of t_1^γ and P_2^δ is a representative of t_2^δ. Finally, by an *isolated path* P in the network $\mathcal{G}^\nu$ we will mean a path that does not meet $\mathcal{G}^\nu \backslash P$ except possibly at the terminal nodes of P; this too can be stated differently by asserting that all the tips in all the nodes of P are traversed by P except possibly for some tips in the terminal nodes of P. If P is endless, it can only reach nodes of $\mathcal{G}^\nu \backslash P$; if P is one-ended or two-ended, it may meet $\mathcal{G}^\nu \backslash P$ at one or both of its terminal nodes.

We say that two tips in different nodes are *connected through a path Q* if those two nodes are connected through Q. Let us now restrict all the tips in our graph $\mathcal{G}^\nu$ as follows.

Condition 7.2. *If the ρ_1-tip $t_1^{\rho_1}$ and the ρ_2-tip $t_2^{\rho_2}$ belong to different nonsingleton nodes, then they are either disconnectable or are connected through an isolated two-ended γ-path where $\gamma = \max\{\rho_1 + 1, \rho_2 + 1\}$.*

The isolated path can arise as an extraction path (defined in Section 1.2) along which, say, the node $n_1^{\rho_1+1}$ containing $t_1^{\rho_1}$ is extracted from the node $n_2^{\rho_2+1}$ containing $t_2^{\rho_2}$, assuming $\rho_1 < \rho_2$. Thus, there may be many such isolated paths if the pristine graph at hand has

been obtained by extracting nodes from nodes of higher ranks in a nonpristine graph.[11]

There is another way of stating Condition 7.2.

Lemma 7.3. *Condition 7.2 holds if and only if the following is true. If the ρ_1-tip $t_1^{\rho_1}$ and the ρ_2-tip $t_2^{\rho_2}$ are nondisconnectable and not connected through an isolated two-ended γ-path, where $\gamma = \max\{\rho_1 + 1, \rho_2 + 1\}$, then either they are shorted together (and therefore of the same rank) or at least one of them is open.*

Example 7.4. Fig. 3 illustrates two tips of different ranks that are nondisconnectable. A one-ended 1-path is shown horizontally therein as a sequence of dots, dashes, and small circles; it reaches a 2-node n^2 through a 1-tip t^1. The arcs represent branches comprising a one-ended 0-path that reaches a 1-node n^1 through a 0-tip t^0. Those two tips are nondisconnectable. For Condition 7.2 to be fulfilled, the following would be needed. Either t^0 or t^1 is open (so that n^1 or n^2 is a singleton), or both t^0 and t^1 are open, or there is there is an isolated two-sided 2-path connecting n^1 and n^2 (thus connecting t^0 and t^1). The case where t^0 and t^1 are shorted together is not allowed because they are of different ranks; such a shorting would yield a nonpristine node, namely, n^2 with n^1 as its embraced node.

There are of course infinitely many other 1-tips in the graph of Fig. 3 belonging to one-ended 1-paths, each of which switches back and forth infinitely often between the horizontal and arc parts of the graph; these 1-tips too are nondisconnectable from each other and from t^0 and t^1. If in addition these other 1-tips are all open, Condition 7.2 is completely fulfilled. Alternatively, if they are all shorted to t^1 and if t^0 is either open or connected to t^1 through an isolated two-ended 2-path, then Condition 7.2 is again completely fulfilled. Still another way to fulfill Condition 7.2 is to leave some of the 1-tips open, connect all pairs of the nonopen 1-tips through shorts or isolated paths, and have t^0 either open or connected to all the nonopen 1-tips through isolated paths. ♣

The proof of our main result regarding the transitivity of ρ-connectedness (given by Theorem 7.6 below) requires another idea, namely, "path cuts." Let P^ρ be a ρ-path with an orientation. Let $\mathcal{X}$ be the set of all branches and all nodes of all ranks in P^ρ. The orientation of P^ρ totally orders $\mathcal{X}$. With x_1 and x_2 being two members of $\mathcal{X}$, we say that

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[11]At the end of this section we summarize the simpler case where the isolated paths of Condition 7.2 are absent.

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x_1 is *before* x_2 and that x_2 is *after* x_1 if in a tracing of P^ρ in the direction of its orientation x_1 is met before x_2 is met. A *path cut* $\{\mathcal{B}_1, \mathcal{B}_2\}$ for P^ρ is a partitioning of the set of branches of P^ρ into two nonvoid subsets $\mathcal{B}_1$ and $\mathcal{B}_2$ such that every branch of $\mathcal{B}_1$ is before every branch of $\mathcal{B}_2$. Another way of stating this is as follows. The partition $\{\mathcal{B}_1, \mathcal{B}_2\}$ of the branch set of P^ρ comprises a path cut for P^ρ if and only if, for each branch $b \in \mathcal{B}_1$, every branch of P^ρ before b is also a member of $\mathcal{B}_1$.

Lemma 7.5. *For each path cut* $\{\mathcal{B}_1, \mathcal{B}_2\}$ *for* P^ρ *there is a unique node* n^γ $(\gamma \leq \rho)$ *such that every branch* $b_1 \in \mathcal{B}_1$ *is before* n^γ *and every branch* $b_2 \in \mathcal{B}_2$ *is after* n^γ.

Note. We shall say that the path cut *occurs at* n^γ. It follows that all the nodes of P^ρ other than n^γ are also partitioned into two sets, the nodes of one set being before n^γ and the nodes of the other set being after n^γ.

Proof. This is obvious if the rank ρ of P^ρ is 0. For higher ranks we argue inductively. If ρ is a positive natural number μ, we assume this lemma is true for every rank $\gamma < \mu$. Let P^μ be oriented in the direction of increasing indices m in the expression (9) for P^μ. If the path cut occurs within a $(\mu - 1)$-path $P_m^{\mu-1}$ of P^μ, then the branches of $P_m^{\mu-1}$ are appropriately partitioned at some γ-node n^γ of $P_m^{\mu-1}$, where $\gamma \leq \mu - 1$, and this in turn partitions all the $(\mu - 1)$-paths and thereby all the branches in P^μ according to $\{\mathcal{B}_1, \mathcal{B}_2\}$. The only other possibility is that the branch set of P^μ is partitioned by $\{\mathcal{B}_1, \mathcal{B}_2\}$ at some μ-node of P^μ.

For $\rho = \vec{\omega}$, our inductive argument applied to (15) or (16) yields a unique γ-node of natural-number rank γ at which the path cut occurs. The argument also works for an ω-path (18), but now the rank γ of the node at which the path cut occurs can be either a natural number or ω. ♣

Theorem 7.6. *Let* $\mathcal{G}^\nu$ $(0 \leq \nu \leq \omega)$ *be a ν-graph for which Condition 7.2 is satisfied. Let* n_a, n_b, *and* n_c *be disjoint nonsingleton nodes (possibly of differing ranks) in* $\mathcal{G}^\nu$ *such that* n_a *and* n_b *are ρ-connected and* n_b *and* n_c *are ρ-connected. Then,* n_a *and* n_b *are ρ-connected.*

Proof. Let P_{ab}^α $(\alpha \leq \rho)$ be a two-ended α-path that terminates at n_a and n_b and is oriented from n_a to n_b, and let P_{bc}^β $(\beta \leq \rho)$ be a two-ended β-path that terminates at n_b and n_c and is oriented from n_b to n_c. Let P_{ba}^α be P_{ab}^α but with the reverse orientation. P_{ba}^α

cannot have infinitely many α-nodes because it is two-ended.

Let $\{n_i\}_{i \in I}$ be the set of nodes at which P_{ba}^α and P_{bc}^β meet, and let $\mathcal{N}_1$ be that set of nodes with the order induced by the orientation of P_{ba}^α. If $\mathcal{N}_1$ has a last node n_x, then a tracing along P_{ab}^α from n_a to n_x followed by a tracing along P_{bc}^β from n_x to n_c yields a path of rank no larger than ρ that connects n_a and n_c. Thus, n_a and n_c are ρ-connected in this case. This will certainly be so when $\{n_i\}_{i \in I}$ is a finite set.

So, assume $\mathcal{N}_1$ is an infinite ordered set (ordered as stated) without a last node. Let Q_1 be the path induced by those branches of P_{ba}^α that lie between nodes of $\mathcal{N}_1$ (i.e., as P_{ba}^α is traced from n_b onward, such a branch is traced after some node of $\mathcal{N}_1$ and before another node of $\mathcal{N}_1$). Let $\mathcal{B}_1$ be the set of those branches. Since P_{ba}^α extends beyond the nodes of $\mathcal{N}_1$, we also have a nonvoid set $\mathcal{B}_2$ consisting of those branches in P_{ba}^α that are not in $\mathcal{B}_1$. $\{\mathcal{B}_1, \mathcal{B}_2\}$ is a path cut for P_{ba}^α. Therefore, by Lemma 7.5, there is a unique $(\rho_1 + 1)$-node $n_d^{\rho_1+1}$, where $\rho_1 + 1 \leq \alpha \leq \rho$, at which that path cut occurs. Thus, Q_1 terminates at $n_d^{\rho_1+1}$. Let $t_1^{\rho_1}$ be the ρ_1-tip through which Q_1 reaches $n_d^{\rho_1+1}$ (thus, $n_d^{\rho_1+1}$ contains $t_1^{\rho_1}$). Every representative of $t_1^{\rho_1}$ contains infinitely many nodes of $\mathcal{N}_1$.

Similarly, let $\mathcal{N}_2$ be $\{n_i\}_{i \in I}$ with the order induced by the orientation of P_{bc}^β. If $\mathcal{N}_2$ has a last node n_y, we can in much the same way as before conclude that n_a and n_c are ρ-connected (this time trace from n_c to n_y to n_a). So, assume that $\mathcal{N}_2$ also does not have a last node. Let Q_2 be the path induced by those branches of P_{bc}^β that lie between nodes of $\mathcal{N}_2$. Since P_{bc}^β extends beyond the nodes of $\mathcal{N}_2$, we have by the same argument as for Q_1 that Q_2 terminates at some $(\rho_2 + 1)$-node $n_e^{\rho_2+1}$, where $\rho_2 + 1 \leq \beta \leq \rho$. Let $t_2^{\rho_2}$ be the ρ_2-tip through which Q_2 reaches $n_e^{\rho_2+1}$ (thus, $n_e^{\rho_2+1}$ contains $t_2^{\rho_2}$). Every representative of $t_2^{\rho_2}$ contains infinitely many nodes of $\mathcal{N}_2$.

Thus, the tips $t_1^{\rho_1}$ and $t_2^{\rho_2}$ are nondisconnectable. Moreover, neither of them can be open (i.e., be in a singleton node); indeed, those tips are traversed by P_{ba}^α and P_{bc}^β respectively, and the nodes of those paths are all nonsingletons. Furthermore, those tips are not shorted, for, if they were, there would be a last node for $\mathcal{N}_1$ and also for $\mathcal{N}_2$, a case we have already treated and then assumed away. So, by Condition 7.2, there is an isolated two-ended path P_{de}^γ connecting the nonsingleton node $n_d^{\rho_1+1}$ that contains $t_1^{\rho_1}$ and the nonsingleton node

$n_e^{\rho_2+1}$ that contains $t_2^{\rho_2}$, where the rank γ satisfies $\gamma = \max\{\rho_1 + 1, \rho_2 + 1\} \leq \rho$. Thus, the two-ended path obtained by tracing the part of P_{ab}^α from n_a to $n_d^{\rho_1+1}$, then tracing P_{de}^γ, and finally tracing the part of P_{bc}^β from $n_e^{\rho_2+1}$ to n_c is of rank no larger than ρ and connects n_a and n_c.

All possible cases have been treated. ♣

The last proof has established the following result.

Corollary 7.7. *Let $\mathcal{G}^\nu$, n_a, n_b, and n_c be as in Theorem 7.6. Let P_{ab}^α be a two-ended α-path connecting n_a and n_b, and let P_{bc}^β be a two-ended β-path connecting n_b and n_c. Then, there is a two-ended γ-path P^γ ($\gamma \leq \max\{\alpha,\beta\}$) connecting n_a and n_c that lies in $P_{ab}^\alpha \cup P_{bc}^\beta$ except possibly for one isolated subpath of P^γ.*

Theorem 7.6 asserts that ρ-connectedness is transitive and therefore an equivalence relationship between branches since ρ-connectedness is obviously reflexive and symmetric. Thus, the branch set of $\mathcal{G}^\nu$ is partitioned by ρ-connectedness. It follows from the definition of a ρ-section that the branch sets of the ρ-sections in $\mathcal{G}^\nu$ comprise a partition of the branch set of $\mathcal{G}^\nu$. This is what we will mean when we say that the *ρ-sections of $\mathcal{G}^\nu$ partition $\mathcal{G}^\nu$*. A similar terminology refers to the partitioning of the branch set of a ρ-section $\mathcal{S}^\rho$ by the branch sets of the λ-sections ($\lambda < \rho$) in $\mathcal{S}^\rho$. In short, we have the following.

Corollary 7.8. *Again let $\mathcal{G}^\nu$ be as in Theorem 7.6. Then, the ρ-sections of $\mathcal{G}^\nu$ partition $\mathcal{G}^\nu$, and similarly, if $\lambda < \rho$, every ρ-section $\mathcal{S}^\rho$ is partitioned by the λ-sections within $\mathcal{S}^\rho$.*

Here too, our discussion of connectedness in the context of pristine nodes is much simpler than that given in [4, Sections 3.2 to 3.5] for the general case.

Finally, let us take note of still another simplification, this one arising from a strengthening of Condition 7.2 as follows.

Condition 7.9. *If two tips belong to different nonsingleton nodes, they are disconnectable.*

Lemma 7.3 is now replaced by

Lemma 7.10. *Condition 7.9 holds if and only if the following is true. If two tips are nondisconnectable, then either they are shorted together or at least one of them is open.*

Of course, Theorem 7.6 and Corollaries 7.7 and 7.8 continue to hold when Condition 7.2

is replaced by the stronger Condition 7.9. In fact, the proof of Theorem 7.6 can be reworked (by asserting that the tips $t_1^{\rho 1}$ and $t_2^{\rho 2}$ are shorted) to strengthen the conclusion of Corollary 7.7 by deleting the phrase "except possibly for an isolated path." For easy reference, let us restate this result as

Corollary 7.11. *Under Condition 7.9, let P_{ab}^α be a two-ended α-path connecting nodes n_a and n_b, and let P_{bc}^β be a two-ended β-path connecting nodes n_b and n_c. Then, there is a two-ended γ-path ($\gamma \leq \max\{\alpha, \beta\}$) connecting n_a and n_c that lies in $P_{ab}^\alpha \cup P_{bc}^\beta$.*

8   Transfinite Versions of König's Lemma

Another result related to connectedness concerns extensions of König's lemma [1, page 81] to transfinite graphs. That lemma can be stated as follows.

Lemma 8.1 (König's Lemma). *If a 0-graph is 0-connected, has infinitely many 0-nodes, and is locally finite (i.e., each 0-node has only finitely many incident branches), then, given any 0-node, there is at least one one-ended 0-path starting from that 0-node.*

We will now derive transfinite versions of this result. For this purpose, the open tips — and thus the singleton nodes — can be ignored because no one-ended path can pass through such a tip or node. Henceforth in this section, the only nodes we shall be referring to are the nonsingleton ones, except occasionally in some passing remarks. This will yield extensions of König's lemma, which in one way are more general than those obtained in [4, Section 4.2] because that prior development assumed that all nondisconnectable tips (not just the nonopen ones) were shorted together. On the other hand, our present development in the context of pristine nodes only will be more restricted in this other way, but this will lead to some simplifications of the prior development.

Two ρ-nodes will be called ρ-*adjacent* if they are incident to the same $(\rho - 1)$-section.

Conditions 8.2. *Let $\mathcal{G}^\nu$ be a ν-graph with $0 \leq \nu \leq \omega$ and with the following conditions satisfied.*

(a) $\mathcal{G}^\nu$ *is ν-connected.*

(b) $\mathcal{G}^\nu$ *has infinitely many nonsingleton ν-nodes.*

(c) *For each rank $\rho = 0, \ldots, \nu$, every nonsingleton ρ-node is ρ-adjacent to only finitely many nonsingleton ρ-nodes.*[12]

Condition 8.2(c) implies that every $(\rho - 1)$-section has only finitely many incident nonsingleton ρ-nodes. Moreover, it represents one way of extending the idea of local finiteness to ν-graphs. Indeed, since (-1)-sections are branches, Condition 8.2(c) implies local finiteness for 0-nodes if end branches (i.e., branches incident to 0-nodes of degree 1) are ignored. In the next section, we shall extend local-finiteness to ν-graphs in another way (Condition 9.1), which will disallow the possibility of infinitely many end branches incident to a 0-node.

Lemma 8.3. *Under Condition 8.2(c), if two tips are nondisconnectable, they are of the same rank.*

Proof. Suppose the tips t^γ and t^ρ are nondisconnectable and $\gamma < \rho$. Consider two representatives P^γ and P^ρ of t^γ and t^ρ respectively. Then, P^γ and P^ρ meet at infinitely many nodes of ranks no larger than γ; let $\mathcal{M}$ be that set of nodes. Moreover, there will be an infinite set $\mathcal{N}^\rho$ of ρ-nodes in P^ρ such that between every two nodes of $\mathcal{N}^\rho$ there will be a node of $\mathcal{M}$. (Fig. 3 illustrates a particular case where $\rho = 1$, where the 1-nodes other than n^1 comprise $\mathcal{N}^1$ and the 0-nodes of the 0-path of arcs comprise $\mathcal{M}$.) Because of P^γ, the nodes of $\mathcal{M}$ lie within a single $(\rho - 1)$-section $\mathcal{S}^{\rho-1}$, and the nodes of $\mathcal{N}^\rho$ are all incident to $\mathcal{S}^{\rho-1}$. This contradicts Condition 8.2(c). ♣

Lemma 8.4. *Assume Conditions 7.2 and 8.2(c) both hold. Let n_1^ρ and n_2^ρ be two ρ-adjacent nonsingleton ρ-nodes $(1 \leq \rho \leq \omega)$, both incident to the $(\rho - 1)$-section $\mathcal{S}^{\rho-1}$ and not connected by an isolated ρ-path. Then, there is an endless $(\rho - 1)$-path $P^{\rho-1}$ in $\mathcal{S}^{\rho-1}$ that reaches n_1^ρ and n_2^ρ. In addition, if n_3^ζ $(\zeta < \rho)$ is a ζ-node in $\mathcal{S}^{\rho-1}$, then there is a one-ended $(\rho - 1)$-path in $\mathcal{S}^{\rho-1}$ that starts at n_3^ζ and reaches n_1^ρ.*

Proof. Fig. 4 illustrates some of our arguments. Consider the first case concerning n_1^ρ and n_2^ρ. Both n_1^ρ and n_2^ρ contain $(\rho - 1)$-tips $t_1^{\rho-1}$ and $t_2^{\rho-1}$ respectively with representatives that lie in $\mathcal{S}^{\rho-1}$. Since n_1^ρ and n_2^ρ are nonsingletons and different nodes, Condition 7.2 asserts that $t_1^{\rho-1}$ and $t_2^{\rho-1}$ are disconnectable. Therefore, we can choose the representative paths $P_1^{\rho-1}$ and $P_2^{\rho-1}$ for $t_1^{\rho-1}$ and $t_2^{\rho-1}$ respectively such that they do not meet. Assume $P_1^{\rho-1}$

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[12]Remember that there are no pristine $\vec{\omega}$-nodes.

terminates at n_b and $P_2^{\rho-1}$ terminates at n_c, with n_b and n_c being different nonsingleton nodes in $\mathcal{S}^{\rho-1}$. Then, there is a two-ended β-path P_{bc}^β ($\beta \leq \rho-1$) in $\mathcal{S}^{\rho-1}$ that terminates at n_b and n_c. Moreover, there will be a nonsingleton node n_a in $P_1^{\rho-1}$ such that the one-ended path $P_{1a}^{\rho-1}$ in $P_1^{\rho-1}$ between n_1^ρ and n_a does not meet P_{bc}^β. (Were this not so, the two-ended β-path P_{bc}^β would traverse a tip that is nondisconnectable from a $(\rho-1)$-tip of n_1^ρ; therefore, by Lemma 8.3, the rank of the two-ended path P_{bc}^β would be no less than ρ, a contradiction.) Let P_{ab}^λ ($\lambda \leq \rho-1$) be the two-ended path in $P_1^{\rho-1}$ from n_a to n_b. We have that n_a is $(\rho-1)$-connected to n_b, which in turn is $(\rho-1)$-connected to n_c. So, by Corollary 7.7, there is a two-ended δ-path P_{ac}^δ ($\delta \leq \rho-1$) lying in $P_{ab}^\lambda \cup P_{bc}^\beta$ except possibly for one isolated subpath of P_{ac}^δ and such that P_{ac}^δ connects n_a and n_c. P_{ac}^δ lies in $\mathcal{S}^{\rho-1}$. P_{ac}^δ does not meet $P_{1a}^{\rho-1}$ except terminally. Similarly, there is a nonsingleton node n_d in $P_2^{\rho-1}$ such that the one-ended path $P_{2d}^{\rho-1}$ in $P_2^{\rho-1}$ between n_2^ρ and n_d does not meet P_{ac}^δ. Let P_{cd}^ξ ($\xi \leq \rho-1$) be the two-ended path in P_2^ρ from n_c to n_d. Again by Corollary 7.7, there is a θ-path P_{ad}^θ ($\theta \leq \rho-1$) lying in $P_{ac}^\delta \cup P_{cd}^\xi$ except possibly for one isolated subpath of P_{ad}^θ and such that P_{ad}^θ connects n_a and n_d. P_{ad}^θ lies in $\mathcal{S}^{\rho-1}$. P_{ad}^θ does not meet $P_{2d}^{\rho-1}$ except terminally. Then, $P_{1a}^{\rho-1} \cup P_{ad}^\theta \cup P_{2d}^{\rho-1}$ contains the endless $(\rho-1)$-path we seek. (That endless path is the subgraph of the $(\rho-1)$-graph of $\mathcal{G}^\nu$ induced by the branches of $P_{1a}^{\rho-1} \cup P_{ad}^\theta \cup P_{2d}^{\rho-1}$.)

A simple modification of this argument establishes the second conclusion. ♣

Note that in Lemma 8.4 we could allow n_1^ρ and n_2^ρ to be singleton ρ-nodes so long as their $(\rho-1)$-tips are disconnectable. For a similar reason, we did not require that n_3^ζ be a nonsingleton.

Theorem 8.5. *Let $\mathcal{G}^\nu$ be a ν-graph with $1 \leq \nu \leq \omega$, $\nu \neq \vec{\omega}$. Assume $\mathcal{G}^\nu$ satisfies Conditions 8.2. Then, given any nonsingleton ν-node n_0^ν, there is at least one one-ended ν-path starting at n_0^ν.*

Proof. Corresponding to $\mathcal{G}^\nu$ we set up a "surrogate" 0-graph $\mathcal{G}^0$ by setting up one and only one 0-node m_a^0 in $\mathcal{G}^0$ for each nonsingleton ν-node n_a^ν in $\mathcal{G}^\nu$ and inserting branches as follows: Insert a branch between two 0-nodes m_a^0 and m_b^0 of $\mathcal{G}^0$ when and only when the corresponding nonsingleton ν-nodes n_a^ν and n_b^ν in $\mathcal{G}^\nu$ are ν-adjacent. (We will identify corresponding nodes n_a^ν and m_a^0 by using the same subscripts.) By Conditions 8.2, $\mathcal{G}^0$ is a

0-connected, locally finite 0-graph with infinitely many 0-nodes. Therefore, we can invoke König's lemma (Lemma 8.1) to conclude that there is a one-ended 0-path P^0 in $\mathcal{G}^0$ starting at the 0-node m_0^0 corresponding to n_0^ν. Orient P^0 from m_0^0 onward.

Let $\mathcal{M}_0^0$ be the singleton set $\{m_0^0\}$. Also, let $\mathcal{M}_1^0$ be the finite set of all 0-nodes in $\mathcal{G}^0$ that are 0-adjacent to m_0^0. Let m_1^0 be the last node in $\mathcal{M}_1^0$ that P^0 meets. No node of P^0 beyond m_1^0 will be in $\mathcal{M}_0^0 \cup \mathcal{M}_1^0$. Let $\mathcal{M}_2^0$ be the finite set of all 0-nodes in $\mathcal{G}^0$ that are 0-adjacent to m_1^0. Let m_2^0 be the last node in $\mathcal{M}_2^0$ that P^0 meets. No node of P^0 beyond m_2^0 will be in $\mathcal{M}_0^0 \cup \mathcal{M}_1^0 \cup \mathcal{M}_2^0$. We can continue recursively this way to get an infinite alternating sequence

$$\{\mathcal{M}_0^0, m_0^0, \mathcal{M}_1^0, m_1^0, \mathcal{M}_2^0, m_2^0, \ldots\}$$

where, for each $k \geq 1$, $\mathcal{M}_k^0$ is the finite set of 0-nodes in $\mathcal{G}^\nu$ that are 0-adjacent to m_{k-1}^0 and where m_k^0 is the last node in $\mathcal{M}_k^0$ that P^0 meets. Again, no node of P^0 beyond m_k^0 will be in $\bigcup_{i=0}^k \mathcal{M}_i^0$.

Now let $\{n_0^\nu, n_1^\nu, n_2^\nu, \ldots\}$ be the sequence of nonsingleton ν-nodes in $\mathcal{G}^\nu$ corresponding bijectively to the sequence $\{m_0^0, m_1^0, m_2^0, \ldots\}$ of 0-nodes in $\mathcal{G}^0$ as stated above. As a result of how the m_k^0 were chosen and how 0-adjacency in $\mathcal{G}^0$ corresponds to ν-adjacency in $\mathcal{G}^\nu$, we have that each n_k^ν $(k \geq 1)$ is ν-adjacent to n_{k-1}^ν but not ν-adjacent to any n_i^ν for $0 \leq i \leq k-1$. Thus, for each $k \geq 1$, there is a $(\nu-1)$-section $\mathcal{S}_k^{\nu-1}$ to which n_{k-1}^ν and n_k^ν are both incident and to which no other n_i^ν $(i \neq k-1, i \neq k)$ is incident. By Condition 7.2 and Lemma 8.4, there is an endless $(\nu-1)$-path $P_k^{\nu-1}$ that reaches $n_k^{\nu-1}$ and n_k^ν and either is isolated or lies in $\mathcal{S}_k^{\nu-1}$. In either case, $P_k^{\nu-1}$ is disjoint from all other $P_i^{\nu-1}$, the latter paths being determined similarly for $i < k$. Since there are infinitely many n_k^ν, we can conclude that the branches of all the $P_k^{\nu-1}$ $(k = 1, 2, 3, \ldots)$ induce a one-ended ν-path, as asserted. ♣

Corollary 8.6. *Under the hypothesis of Theorem 8.5, given any nonsingleton node of any rank in $\mathcal{G}^\nu$, there is at least one one-ended ν-path starting at that node.*

Proof. If $\zeta < \nu$ and if n^ζ is a ζ-node, we can choose a one-ended path P_1 in the $(\nu-1)$-section $\mathcal{S}^{\nu-1}$ containing n^ζ such that P_1 starts at n^ζ and reaches a boundary ν-node n^ν of $\mathcal{S}^{\nu-1}$ (Lemma 8.4 again, second conclusion). Let P_2^ν be a one-ended ν-path starting at n^ν;

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the existence of P_2^ν is assured by Theorem 8.5. Since $\mathcal{S}^{\nu-1}$ has only finitely many boundary ν-nodes (a consequence of Condition 8.2(c)), P_2^ν will eventually be disjoint from $\mathcal{S}^{\nu-1}$. It therefore follows from Corollary 7.7 that there is a one-ended ν-path P^ν that starts at n^ς and lies in $P_1 \cup P_2^\nu$ except possibly for one isolated subpath of P^ν. ♣

Here too, a minor modification of this proof allows the starting node n^ς to be a singleton.

Let us now consider the case where $\nu = \vec{\omega}$.

Theorem 8.7. *Assume that the $\vec{\omega}$-graph $\mathcal{G}^{\vec{\omega}}$ satisfies Conditions 7.2 and 8.2 with $\nu = \vec{\omega}$. Given any μ-node n^μ ($\mu < \vec{\omega}$), there is at least one $\vec{\omega}$-path in $\mathcal{G}^{\vec{\omega}}$ starting at n^μ.*

Note. Here too, n^μ need not be a nonsingleton.

Proof. Fig. 5 illustrates some of the ideas in this proof. The rank of every node in $\mathcal{G}^{\vec{\omega}}$ is a natural number. Moreover, there is no natural number that uniformly bounds all the ranks of all the nodes of $\mathcal{G}^{\vec{\omega}}$. We can choose a μ-section $\mathcal{S}^\mu$ such that n^μ is a node of $\mathcal{S}^\mu$. Proceeding recursively, for every positive natural number $k = 1, 2, 3, \ldots$, we can choose a $(\mu + k)$-section $\mathcal{S}^{\mu+k}$ such that the boundary $(\mu + k)$-nodes of $\mathcal{S}^{\mu+k-1}$ are nodes of $\mathcal{S}^{\mu+k}$. $\mathcal{S}^{\mu+k-1}$ is a $(\mu + k - 1)$-section of $\mathcal{S}^{\mu+k}$.

Now, consider $\mathcal{G}^{\vec{\omega}} \backslash \mathcal{S}^\mu$, the subgraph of $\mathcal{G}^{\vec{\omega}}$ induced by all the branches that are not in $\mathcal{S}^\mu$. This will consist of no more than finitely many components because $\mathcal{S}^\mu$ has only finitely many boundary $(\mu + 1)$-nodes by virtue of Condition 8.2(c). At least one component C_1 of $\mathcal{G}^{\vec{\omega}} \backslash \mathcal{S}^\mu$ will be an $\vec{\omega}$-graph. C_1 will also contain at least one boundary $(\mu + 1)$-node $n_1^{\mu+1}$ of $\mathcal{S}^\mu$. Let P_0^μ be a one-ended μ-path in $\mathcal{S}^\mu$ starting at n^μ and reaching $n_1^{\mu+1}$ (see the second conclusion of Lemma 8.4); P_0^μ will not reach any other $(\mu + 1)$-node.

Next, consider $\mathcal{G}^{\vec{\omega}} \backslash \mathcal{S}^{\mu+1}$. This too will have only finitely many components. At least one of them C_2 will be an $\vec{\omega}$-subgraph of C_1 and will have a boundary $(\mu + 2)$-node $n_2^{\mu+2}$ of $\mathcal{S}^{\mu+1}$. Moreover, $C_1 \cap \mathcal{S}^{\mu+1}$ will be a $(\mu + 1)$-section of $\mathcal{G}^{\vec{\omega}} \backslash \mathcal{S}^\mu$ along with some boundary $(\mu + 2)$-nodes of $\mathcal{S}^{\mu+1}$ including $n_2^{\mu+2}$. Therefore, we can choose in $C_1 \cap \mathcal{S}^{\mu+1}$ a one-ended $(\mu + 1)$-path $P_1^{\mu+1}$ starting at $n_1^{\mu+1}$, reaching $n_2^{\mu+2}$, but not reaching any other $(\mu + 2)$-node (Lemma 8.4). In fact, $P_1^{\mu+1}$ will lie in $\mathcal{S}^{\mu+1} \backslash \mathcal{S}^\mu$. Thus, P_0^μ reaches $n_1^{\mu+1}$, $P_1^{\mu+1}$ meets $n_1^{\mu+1}$, and P_1^μ and $P_1^{\mu+1}$ are disjoint.

This process can be continued recursively for all k. Replace 1 by k and 2 by $k + 1$ in the

preceding paragraph, and consider $\mathcal{G}^{\vec{\omega}} \backslash \mathcal{S}^{\mu+k}$. This yields a component $\mathcal{C}_{k+1}$ of $\mathcal{G}^{\vec{\omega}} \backslash \mathcal{S}^{\mu+k}$, which is a $\vec{\omega}$-subgraph of $\mathcal{C}_k$ and has a boundary $(\mu+k+1)$-node $n_{k+1}^{\mu+k+1}$ of $\mathcal{S}^{\mu+k}$. Moreover, $\mathcal{C}_k \cap \mathcal{S}^{\mu+k}$ will be a $(\mu+k)$-section of $\mathcal{G}^{\vec{\omega}} \backslash \mathcal{S}^{\mu+k-1}$ along with some boundary $(\mu+k+1)$-nodes of $\mathcal{S}^{\mu+k}$ including $n_{k+1}^{\mu+k+1}$. Furthermore, $\mathcal{C}_k \cap \mathcal{S}^{\mu+k}$ will contain $n_k^{\mu+k}$. Therefore, by Lemma 8.4 again, there will be a one-ended $(\mu+k)$-path $P_k^{\mu+k}$ in $\mathcal{C}_k \cap \mathcal{S}^{\mu+k}$ starting at $n_k^{\mu+k}$ and reaching $n_{k+1}^{\mu+k+1}$. In fact, $P_k^{\mu+k}$ will lie in $\mathcal{S}^{\mu+k} \backslash \mathcal{S}^{\mu+k-1}$.

With k increasing indefinitely, we will generate in this way a one-ended path

$$\{n^\mu, P_0^\mu, n_1^{\mu+1}, P_1^{\mu+1}, \ldots\}$$

that sequentially meets infinitely many nodes whose natural-number ranks increase beyond every natural number. It will in fact be an $\vec{\omega}$-path in $\mathcal{G}^{\vec{\omega}}$ starting at n^μ. ♣

Finally, let us note that all the results of this section can be applied to any section in place of $\mathcal{G}^\nu$ or $\mathcal{G}^{\vec{\omega}}$ because a section is a graph by itself.

9 Countable Graphs

A ν-graph will be called *countable* if its branch set $\mathcal{B}$ is countable. Countability of $\mathcal{G}^\nu$ follows from Condition 7.2 along with another extension of local finiteness to ν-graphs defined by Condition 9.1 below. Two ρ-sections $\mathcal{S}_1^\rho$ and $\mathcal{S}_2^\rho$ will be called $(\rho+1)$-*adjacent* if they share a common boundary node $n^{\rho+1}$ (i.e., if the $(\rho+1)$-node $n^{\rho+1}$ contains a ρ-tip of $\mathcal{S}_1^\rho$ and a ρ-tip of $\mathcal{S}_2^\rho$). Furthermore, the $(\rho+1)$-*adjacency* of a ρ-section is the set of all other ρ-section that are $(\rho+1)$-adjacent to $\mathcal{S}^\rho$. When $\rho = -1$, $\mathcal{S}^\rho$ is a branch b, and its $(\rho+1)$-adjacency is the set of all other branches that are incident to a 0-node incident to b.

Condition 9.1. *For each rank $\rho = -1, 0, \ldots, \nu$, every ρ-section has a finite $(\rho+1)$-adjacency.*[13]

Condition 8.2(c) does not imply Condition 9.1. Indeed, a ρ-section $\mathcal{S}^\rho$ with a boundary node $n^{\rho+1}$ can have infinitely many $(\rho+1)$-adjacent ρ-sections each having only $n^{\rho+1}$ as its one and only bordering node. (See Section 2.5 for the definitions of boundary and bordering nodes.) This can satisfy Condition 8.2(c) but not Condition 9.1. Conversely, Condition 9.1

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[13]When $\rho = -1$, this says that every 0-node is of finite degree.

does not imply Condition 8.2(c) because two $(\rho + 1)$-adjacent ρ-sections can each have infinitely many bordering $(\rho+1)$-nodes, thereby satisfying Condition 9.1 but not Condition 8.2(c).

Theorem 9.2. *Let the ν-graph $\mathcal{G}^\nu$ $(0 \le \nu \le \omega)$ satisfy Conditions 7.2 and 9.1. Then, $\mathcal{G}^\nu$ is countable.*

Proof. Consider any 0-section $\mathcal{S}^0$ and choose any branch b_0 in it. Set $\mathcal{H}_0 = \{b_0\}$. Let $\mathcal{H}_1$ be the set of all branches in $\mathcal{S}^0$ that are 0-adjacent to b_0. $\mathcal{H}_1$ is a finite set by Condition 9.1. Proceeding inductively, let us assume that $\mathcal{H}_0, \mathcal{H}_1, \ldots, \mathcal{H}_{k-1}$ have been chosen as finite sets of branches in $\mathcal{S}^0$. Let $\mathcal{H}_k$ be the set of branches in $\mathcal{S}^0$ that are 0-adjacent to branches of $\mathcal{H}_{k-1}$ and are not in $\bigcup_{l=0}^{k-1} \mathcal{H}_l$. By Condition 9.1 again, $\mathcal{H}_k$ is a finite set too. Moreover, every branch in $\mathcal{S}^0$ will lie in some $\mathcal{H}_k$ because it is 0-connected to b_0 through a two-ended 0-path. Consequently, $\mathcal{S}^0$ is countable.

Next, let us assume that, for some natural number μ, every $(\mu - 1)$-section is countable. Consider any μ-section $\mathcal{S}^\mu$. It is partitioned by a set of $(\mu - 1)$-sections according to Corollary 7.8. Observe that, by Condition 9.1, for each $(\mu - 1)$-section $\mathcal{S}^{\mu-1}$ in $\mathcal{S}^\mu$, there are at most finitely many $(\mu - 1)$-sections in $\mathcal{S}^\mu$ that are μ-adjacent to $\mathcal{S}^{\mu-1}$. Now, let $\mathcal{H}_0$ be any $(\mu - 1)$-section in $\mathcal{S}^\mu$. Let $\mathcal{H}_1$ be the union of all $(\mu - 1)$-sections in $\mathcal{S}^\mu$ that are μ-adjacent to $\mathcal{H}_0$. Recursively, having chosen $\mathcal{H}_0, \mathcal{H}_1, \ldots, \mathcal{H}_{k-1}$, we let $\mathcal{H}_k$ be the union of all the $(\mu - 1)$-sections in $\mathcal{S}^\mu$ that are μ-adjacent to $(\mu - 1)$-sections in $\mathcal{H}_{k-1}$ but are not in $\bigcup_{l=0}^{k-1} \mathcal{H}_l$. Since all branches of $\mathcal{S}^\mu$ are pairwise μ-connected by two-ended μ-paths in $\mathcal{S}^\mu$, $\bigcup_{k=0}^{\infty} \mathcal{H}_k$ will be $\mathcal{S}^\mu$. (When $\mu = \nu$, the $\mathcal{H}_k$ may be void for all sufficiently large k.) By our above observation, there are only finitely many $(\mu - 1)$-sections in each $\mathcal{H}_k$. Hence, there are only countably many $(\mu - 1)$-sections in each $\mathcal{S}^\mu$, and each $(\mu - 1)$-section is countable by our inductive assumption. Consequently, $\mathcal{S}^\mu$ is countable too. Now, if $\mathcal{G}^\nu$ is a μ-graph (i.e., $\nu = \mu$), $\mathcal{G}^\nu$ is a μ-section by itself and therefore is countable.

Consider next an $(\vec{\omega})$-section $\mathcal{S}^{\vec{\omega}}$. Any two branches in $\mathcal{S}^{\vec{\omega}}$ are connected by a two-ended μ-path in $\mathcal{S}^{\vec{\omega}}$, where μ is some natural number. Consequently, upon choosing any 0-section

S^0 in $S^{\vec{\omega}}$ and letting S^{μ} be the unique μ-section in which S^0 lies (Corollary 7.8), we obtain

$$S^{\vec{\omega}} = \bigcup_{0 \leq \mu < \vec{\omega}} S^{\mu}.$$

Since each S^{μ} is countable, so to is $S^{\vec{\omega}}$. So, if $\nu = \vec{\omega}$, $\mathcal{G}^{\nu}$ is again countable.

Finally, $\mathcal{G}^{\omega}$ can have only countably many $\vec{\omega}$-sections because of Corollary 7.8 and Condition 9.1. Thus, when $\nu = \omega$, we again have that $\mathcal{G}^{\nu}$ is countable. ♣

References

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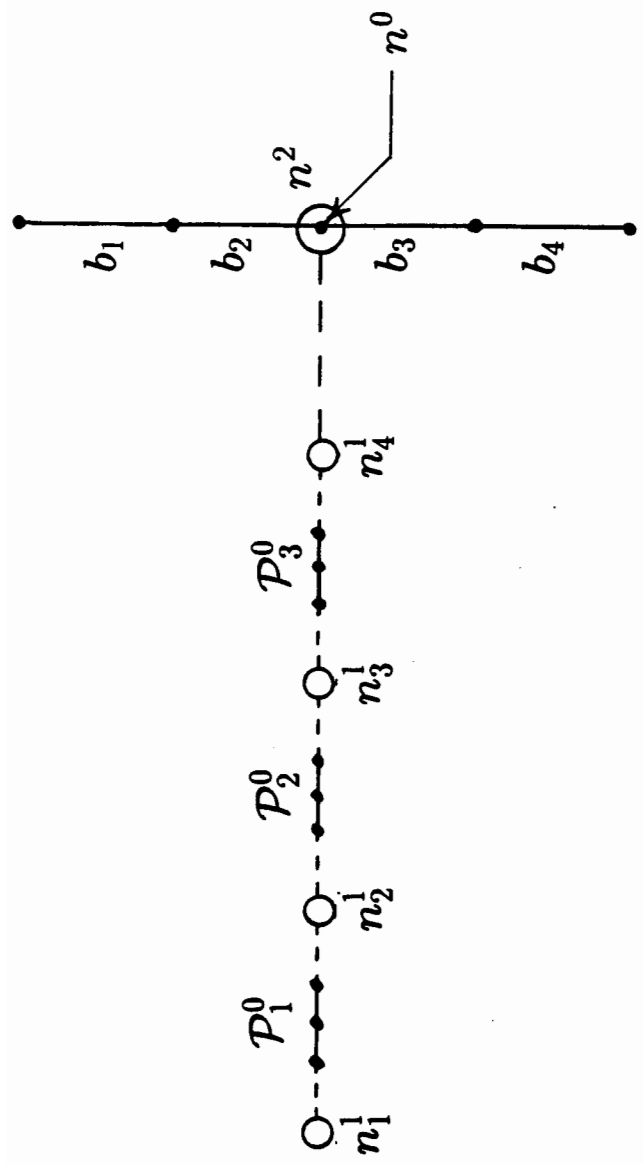
Figure Captions

Fig. 1. (a) A 2-graph containing a nonpristine 2-node n^2. The heavy dots denote 0-nodes, the smaller circles denote 1-nodes, and the larger circle denotes a 2-node.

(b) The pristine 2-graph obtained by extracting the 0-node n^0 from the 2-node n^2. The one-ended 1-path between n^0 and $\hat{n}^2$ is the extraction path along which n^0 is extracted from the nonpristine node n^2.
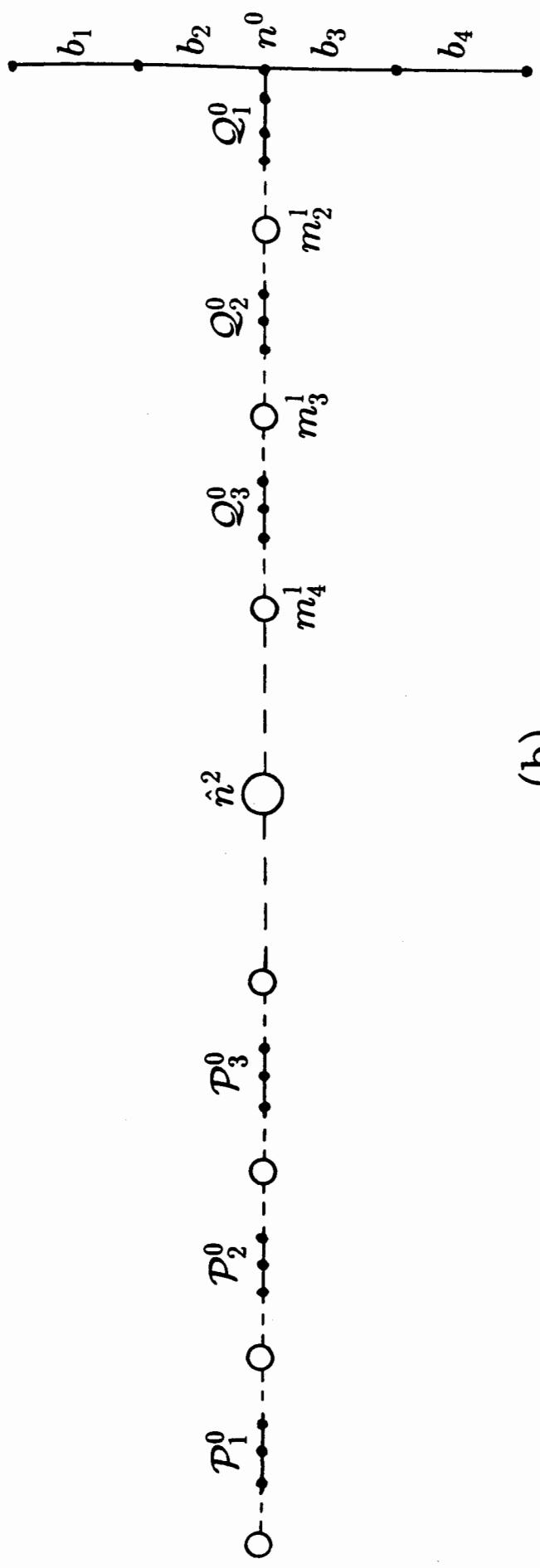
Fig. 2. The 1-graph $\mathcal{G}^1$ of Example 4.1.

Fig. 3. Illustration for Example 7.4. The dots represent 0-nodes, the small circles represent 1-nodes, and the large circle represents a 2-node.

Fig. 4. Illustration for the proof of Lemma 8.4.

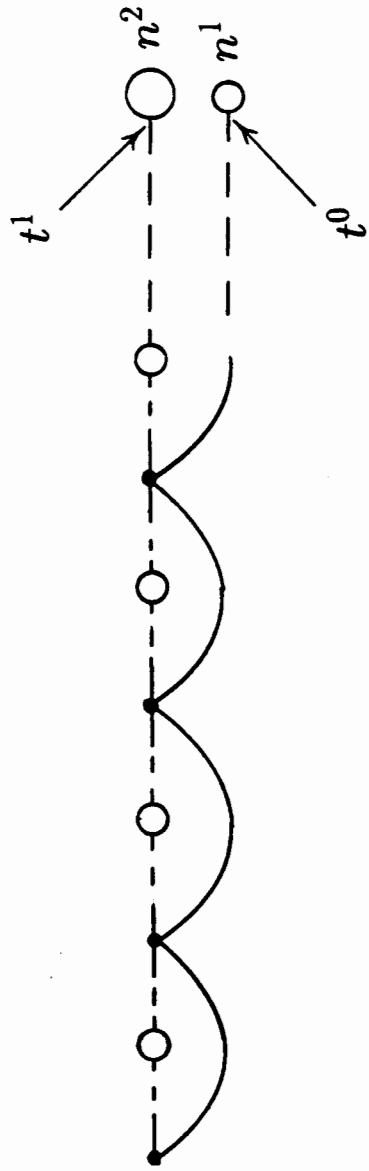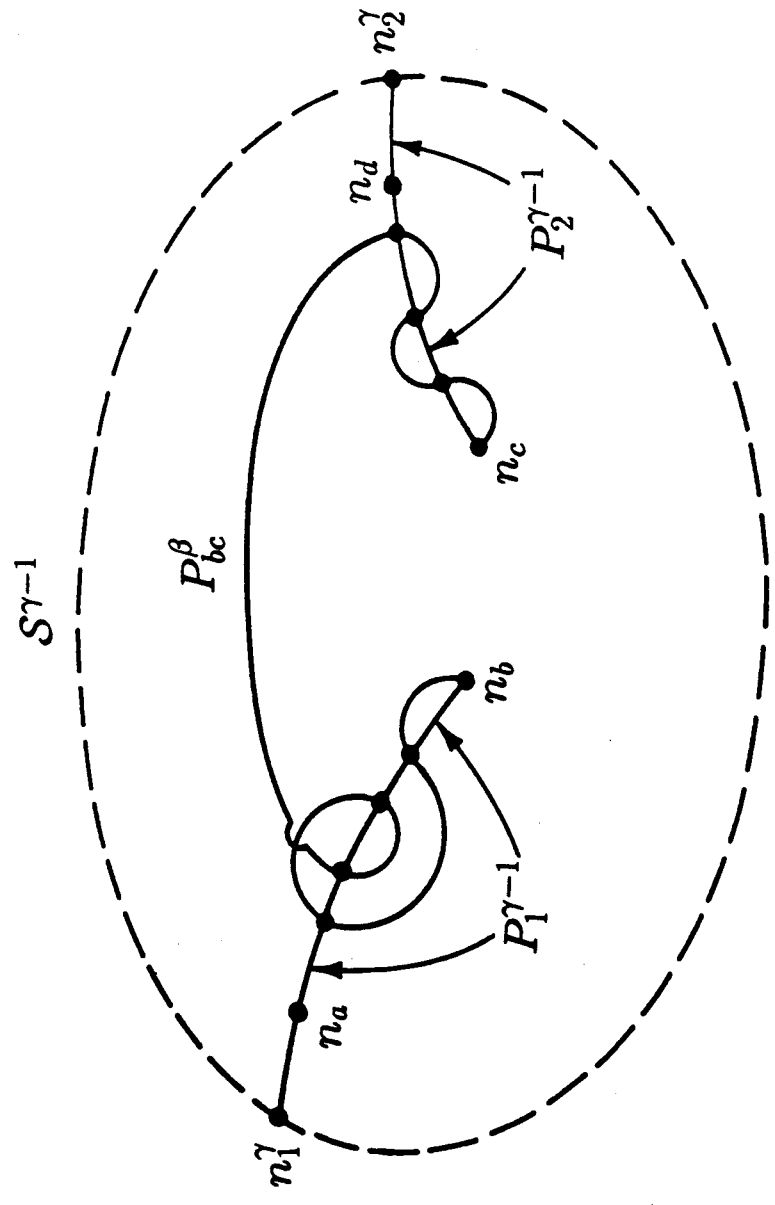**Fig. 5.** Illustration for the proof of Theorem 8.7.

Fig. 1

(a)

(b)

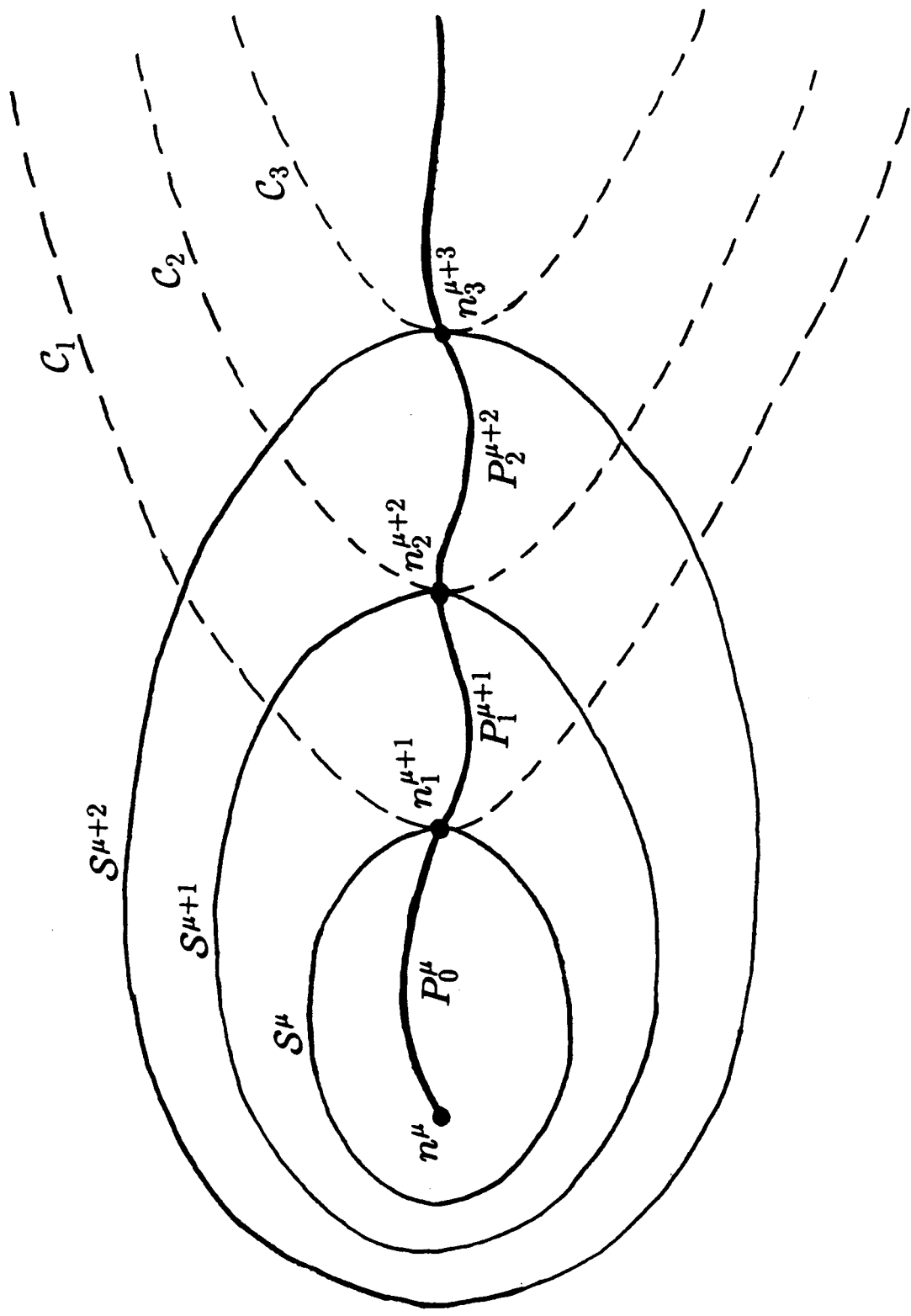a_1　b_1　a_2　b_2　a_3　b_3　n^1

Fig. 2

Fig. 3

Fig. 4

Fig. 5