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NONSTANDARD TRANSFINITE ELECTRICAL NETWORKS

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Abstract—Transfinite resistive electrical networks may (or may not) have operating points, and, even when voltages and currents do exist within them, Kirchhoff's laws may not be satisfied everywhere. Moreover, rather severe restrictions have to be placed on such networks in order to obtain such results because of the inability of standard calculus to encompass certain interchanges of limiting processes. However, the comparatively recent theory of nonstandard analysis has this ability. The idea of a nonstandard electrical network, introduced in a prior work, is expanded herein to cover all transfinite networks that are "restorable" in the sense that the network is taken to be the end result of an expanding sequence of finite networks. Kirchhoff's laws will always be satisfied in restorable transfinite networks. Several transfinite networks are then examined under nonstandard analysis, and explicit hyperreal currents and voltages are established for them.

Key Words: Transfinite networks, nonstandard networks, nonstandard graphs, hyperreal voltages and currents, restorable networks.

1 Introduction

Nonstandard electrical networks were introduced in [24] to rectify the failure of Kirchhoff's laws in various infinite resistive networks. See [22, Sec. 1.6] or [23. Examples 5.1-6 and 5.1-7] for several examples of this anomaly. The basic problem is that standard calculus does not always allow the order of applying limiting processes to be reversed. Moreover, some severe restrictions have to be placed on the infinite networks in order to get convergent expressions for the voltages and currents. Such, for instance, is the requirement of finite total power generated or dissipated, but other restrictions are also used for the same purpose. See, for

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example, [1] - [7], [8], [11], [19] - [23], [26].

However, nonstandard analysis can overcome these problems. For example, a finite current source applied to an infinite parallel circuit of 1 Ω resistors results in a violation of Kirchhoff's current law under standard calculus, but the use of infinitesimal currents in the parallel circuit restores Kirchhoff's current law. Moreover, nonstandard analysis allows us to reestablish Kirchhoff's laws even when the currents and voltages are infinite in size. It does these things by replacing the real-number system by a hyperreal-number system, which is an enlargement of the former through the addition of infinitesimals and other such entities of finite and infinite size.[1] Let us emphasize that these results are simply unattainable via standard analysis. There is no way Kirchhoff's laws can hold in the examples of transfinite networks given below if only real numbers are used. On the other hand, not only infinitesimal but also infinite voltages, currents, and powers become distinguishable and explicit with hyperreal numbers—an advantage that is unavailable with real numbers alone.

By a "nonstandard network" we will mean a conventionally infinite or transfinite resistive electrical network having countably many branches whose voltages and currents are hyperreals; in addition, the network is viewed as the end result of an expanding sequence of finite networks (i.e., networks with finitely many branches). As was determined in [24], it is not enough to specify just the graph and element values in order to uniquely designate the nonstandard network. How the network is built up by connecting together branches sequentially must also be stipulated. Indeed, the hyperreal branch currents and voltages will depend in general upon the sequence through which the branches are appended and inserted. Different sequences can lead to different hyperreal currents and voltages. even when the resulting nonstandard networks have the same graph and element values.

The purpose of this work is to define this new class of nonstandard networks, to prove they exist (i.e., they make sense), and to present some examples of them. The paper encompasses and generalizes the earlier work [24], which was overly complicated and too restrictive: in contrast, the present work only requires that the transfinite network be "restorable." as

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[1]It is conventional to say *real* instead of real number and *hyperreal* instead of hyperreal number.

defined in Sec. 5. We examine herein linear resistive networks, but our results extend imme-
diately to nonlinear resistive networks of the monotone type and to AC steady-state analysis
of linear RLC networks. Moreover, more complicated nonstandard analyses encompass the
transient behavior of transfinite RLC networks [27].

Abraham Robinson introduced nonstandard analysis in a work [14] that makes essential
use of mathematical logic and is rather inaccessible to those unfamiliar with that subject.
Other expositions are based more upon mathematical analysis. See, for example, [9] and
[10]. Shorter introductions can be found in [12] and [13]. A brief summary of some basic
ideas of nonstandard analysis is presented in [24, Sec. II] and is more than enough for an
understanding of this paper. In the next section we will state those elements of nonstandard
analysis needed for a comprehension of this paper.

Most of the earlier applications on nonstandard analysis were aimed at showing that
it provided a simpler and more elegant means of obtaining results that are nonetheless
obtainable by standard analysis. However, nonstandard analysis provides much more; it
opens up areas of mathematical endeavor in which standard analysis is useless. This paper
is a demonstration of that fact. Standard analysis certainly works very well indeed for
finite networks. It also yields results, albeit highly qualified and restricted results, for
transfinite networks [23], [26], but nonstandard analysis enables far more general results
for transfinite networks. Indeed, nonstandard networks comprise an entirely new class of
electrical networks.

2 Some Elements of Nonstandard Analysis and Other Pre-liminaries

Every hyperreal is an equivalence class of sequences of reals, as is indicated below. We will
always deal with any hyperreal by choosing a representative sequence of reals for it, that
is. by choosing one of the sequences in that equivalence class. An analysis then involves
manipulations of such sequences.[2] We are led to the use of representative sequences for

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[2] An analog to this would be the use of a particular Cauchy sequence of rational numbers for a given
real number. (For example, a decimal representation of a real number is in fact such a Cauchy sequence.)
Then real numbers would be added, multiplied, etc. by adding, multiplying, etc. corresponding terms
in the Cauchy sequences. However, in nonstandard analysis, equivalent Cauchy sequences will in general

hyperreals because an infinite network is viewed herein as the end result of an expanding sequence of finite networks. Any hyperreal voltage, current, or power in the infinite network can then be represented by the corresponding sequence of voltages, currents, or powers in those finite networks.

Furthermore, in nonstandard analysis, all sequences of real numbers are partitioned into equivalence classes with respect to a chosen and fixed "nonprincipal[3] ultrafilter" $\mathcal{F}$ on the set N, where N is the set of natural numbers: $\{0, 1, 2, \ldots\}$. In particular, $\mathcal{F}$ is a set of subsets of N satisfying certain axioms [9, pages 18-19, 24]. Then, two sequences of real numbers are considered to be *equivalent modulo* $\mathcal{F}$ or are said to *agree for almost all* n, where $n \in N$, if they agree on a subset of N that is a member of $\mathcal{F}$. More particularly, the real number sequences $\{r_n\}_{n=1}^{\infty}$ and $\{s_n\}_{n=1}^{\infty}$ are said to be *equivalent modulo* $\mathcal{F}$ if

$$\{n \in N : r_n = s_n\} \in \mathcal{F}.$$

Each such equivalence class is a *hyperreal*, and $^*\!R$ denotes the set of all hyperreals. For our purposes, the critical property is that every cofinite subset of N (i.e., every subset containing all but at most finitely many natural numbers) is a member of every nonprincipal ultrafilter [9, page 19]. Thus, every hyperreal can be specified by specifying a sequence of real numbers indexed by the positive integers. Just which hyperreal that is depends upon the choice of $\mathcal{F}$, but this need not be stipulated—as we explain below.

Correspondingly, two sequences of finite networks will be called *equivalent modulo* $\mathcal{F}$ if there is a set $F \in \mathcal{F}$ such that for each $n \in F$ the corresponding finite networks have the same graph and the same element values. In this case, we also say that the two sequences of finite networks *agree for almost all* n.[4] Such an equivalence class of sequences of finite networks will be called a *nonstandard network*. $\langle N_n^0 \rangle$ denotes that equivalence class, where $\{N_n^0\}_{n=0}^{\infty}$ is any one of the sequences of finite networks in the class. Here too, we shall choose a particular sequence from the equivalence class of sequences of finite networks and will analyze the members of that sequence in order to analyze the nonstandard

correspond to different, but infinitesimally close, limited hyperreals.

[3]Also, called a "free ultrafilter."

[4]By this "agreement" we mean that for each $n \in F$ we have a graph-isomorphism that preserves element values.

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network. Note that any finite network can be viewed as a nonstandard network by using a sequence, whose every member is that finite network, as a representative of an equivalence class of networks. However, our attention will generally be on some expanding sequence of finite networks that fill out a given countably infinite network[5] in order to analyze the corresponding nonstandard network.

Now, upon choosing such a sequence of networks, we determine a sequence for each branch voltage or current in the infinite network. Just which hyperreal that sequence represents depends upon the choice of $\mathcal{F}$. In fact, since every cofinite subset of N is a member of every nonprincipal ultrafilter $\mathcal{F}$, the choice of an expanding sequence of finite networks will specify a nonstandard image of the infinite network for each choice of $\mathcal{F}$, in general, a different image for a different choice of $\mathcal{F}$. To repeat, the hyperreal voltages and currents in a nonstandard network are not uniquely determined by choosing that expanding sequence of finite networks; those hyperreals still depend upon the choice of $\mathcal{F}$. What can we make of this nonuniqueness? Simply this:

Nonstandard analysis provides many different nonstandard images of a given countably infinite network, each image determined by the choice of $\mathcal{F}$. Within each such image, Kirchhoff's laws will be satisfied with the use of hyperreal voltages and currents. Thus, any nonstandard image will serve to insure the satisfaction of Kirchhoff's laws. There is no need to explicate which nonstandard image is being used. All of them become available upon choosing that expanding sequence of finite networks. In short, we may simply use the corresponding sequences of voltages and currents as representatives of hyperreals determined by some unspecified choice of $\mathcal{F}$.

The terminology of certain nonstandard entities is not uniform throughout the literature. Here are some definitions we use. We shall use lower case letters for reals and upper-case letters for hyperreals. n will always denote the index for a representative sequence, say, $\{i_n\}_{n=1}^{\infty}$ of a hyperreal. In this case, the corresponding hyperreal is denoted as $I = \langle i_n \rangle$, which in fact is the equivalence class of sequences (modulo $\mathcal{F}$) having the representative sequence $\{i_n\}_{n=1}^{\infty}$.

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[5]A network is called *countably infinite* or simply *countable* if its set of branches is infinite and countable.

If a condition depending upon n holds for all n in some set $F \in \mathcal{F}$, we will simply say that it holds "almost everywhere" or simply "a.e.". For example, the hyperreals $X = \langle x_n \rangle$ and $Y = \langle y_n \rangle$ are defined to be *equal* (i.e., $X = Y$) if $\{n \in N : x_n = y_n\} = F \in \mathcal{F}$, and we say in this case that $x_n = y_n$ a.e. Furthermore, addition, multiplication, inequality, and absolute value are defined componentwise on the representatives of hyperreals. Also, $X < Y$ means $x_n < y_n$ a.e., and $X \leq Y$ is defined similarly. Furthermore, $|X| = \langle |x_n| \rangle$.

The hyperreal $\langle x_n \rangle$ is called *infinitesimal* if, for every positive real ϵ, we have $\{n \in N : |x_n| < \epsilon\} \in \mathcal{F}$, that is, if $|x_n| < \epsilon$ a.e. Also, $\langle x_n \rangle$ is called *unlimited* if $|x_n| > \epsilon$ a.e. for every positive real ϵ. Thus, the reciprocal $\langle x_n^{-1} \rangle$ of an infinitesimal $\langle x_n \rangle$ is unlimited, and conversely. A *limited* hyperreal is one that is not unlimited. Thus, $X = \langle x_n \rangle$ is limited if and only if there is a $\gamma \in R_+$ such that $|x_n| < \gamma$ a.e. A hyperreal that is neither infinitesimal nor unlimited is called *appreciable*. Around each real $X = \langle x, x, x, \ldots \rangle$ in $^*R$, there is a set of hyperreals $Y = \langle y_1, y_2, y_3, \ldots \rangle$ that are infinitesimally close to X (i.e., $|X - Y|$ is infinitesimal for each such Y). The set of such hyperreals is called the *halo* of X, and X is called the *shadow* or *standard part* of every Y in that halo.

We assume throughout that each branch consists of a positive resistor and possibly a source, a voltage source in series with the resistor (the Thevenin form) as shown in Fig.1(a) or a current source in parallel with the resistor (the Norton form) as shown in Fig. 1(b). We allow self-loops.[6]

Some basic ideas from the theory of transfinite networks are used in this paper. We explain such concepts as they arise. For more detailed explanations, one may refer to any of three books [22], [23], [26] or to the tutorial/survey paper [25].

3 Restorations of Networks

Our objective in this and the next section is to find some procedure for approximating an infinite network[7] with a sequence of finite networks by shorting some branches and opening others in such a fashion that the network obtained by restoring all branches in sequence is

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[6] A self-loop is a branch incident to a single node, and thus its two elementary tips are shorted together.

[7] By an *infinite network* N^ν we mean either a conventionally infinite network or a transfinite network. The former has no transfinite nodes and is thus a 0-network, and the latter does have them and is thus a ν-network with $\nu \geq 1$.

identical to the original infinite network. This implies among other things that the original network must be countable (i.e., have only countably many branches) if the restoration of all the branches one at a time is to be feasible.

Let us be more specific by what we mean by "opening" or "shorting" a branch. We assume that every branch has a positive resistance and possibly a source. Any branch can be represented either by its Norton circuit, shown in Fig. 1(a), or equivalently by its Thevenin circuit, shown in Fig. 1(b). To *open* the branch will mean that, with respect to its Norton representation, the branch conductance g, current source h, and current i are all set equal to 0: $g = h = i = 0$. The branch voltage v cannot be determined from Ohm's law, $i + h = gv$, and its value will not be needed until the branch is restored.[8] To *short* the branch will mean that with respect to its Thevenin representation, the branch resistance r, voltage source e, and voltage v are all set equal to 0: $r = e = v = 0$. The branch current i cannot be determined from Ohm's law, $v + e = ri$, and its value will not be needed until the branch is restored.[9]

A sequence of finite networks can be generated from a given countably infinite network N^ν as follows. First number all the branches with the natural numbers: $k = 0, 1, 2, \ldots$. Then, open and/or short all but finitely many branches. Then, "restore" branches finitely many at a time in accordance with the branch numbering. A branch is *restored* simply by restoring its original electrical-parameter values.

At each step of the restoration, we will have a finite network N_n^0 $(n = 0, 1, 2, \ldots)$ obtained by removing all opened and shorted branches that have not yet been restored and by coalescing the two nodes of each shorted branch into a single node. Each subgraph induced by a maximal set of shorted branches that are connected through branches of that set coalesces into a single 0-node.[10] Thus, at each step of the restoration process, all transfinite nodes will have disappeared, and only finitely many 0-nodes will remain. We will refer to each finite network N_n^0 obtained in this way as an *embedded finite network* of

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[8] In fact. when specifying a sequence for the hyperreal voltage V for that branch, we may arbitrarily set $v = 0$ until the branch is restored.

[9] Here. too. when specifying a sequence for the hyperreal current I, we can set $i = 0$ before the branch is restored.

[10] Restored self loops may arise at this point from restored opened branches whose two nodes lie in a path of shorted branches.

the original infinite network $\mathbf{N}^{\nu}$.

It can happen that two different maximal nodes in $\mathbf{N}^{\nu}$ remain connected through a path of shorted branches at each step of the restoration, as some later examples will show. If this never happens, the sequence $\{\mathbf{N}_n^0\}_{n=1}^{\infty}$ of embedded finite networks defines a *nonstandard network* $\mathbf{N}_{\mathrm{nst}}^{\nu} = \langle \mathbf{N}_n^0 \rangle$. The operating point of each finite embedded network determines a voltage and current for each branch that has already been restored. Before a branch has been restored, that branch's voltage and current are taken to be 0; this occurs only finitely many times. Thus, the process of restoration yields a sequence of voltages and a sequence of currents for each branch. These are representatives of the hyperreal voltage and hyperreal current for that branch. $\mathbf{N}_{\mathrm{nst}}^{\nu}$ will be called a *nonstandard image* of $\mathbf{N}^{\nu}$. In any nonstandard network $\mathbf{N}_{nst}^{\nu}$ Kirchhoff's voltage law is always satisfied around every finite or transfinite loop by the hyperreal branch voltages, and Kirchhoff's current law is always satisfied at every finite or infinite node by the hyperreal branch currents. In this way, *standard analyses of finite resistive networks can be lifted to nonstandard analyses of transfinite resistive networks without any restrictions imposed on the transfinite network other than restorability.* Restorability is discussed more fully in Sec. 6.

4 Restorations of Two Ladder Networks

To establish some ideas with respect to the sought-for procedure, let us now consider the 1-network consisting of a one-way infinite ladder network connected at its infinite extremity to a resistor r_{ω} through two 1-nodes n_1^1 and n_2^1. See Fig. 2. Here, n_1^1 (resp. n_2^1) contains the 0-tip of the upper (resp. lower) path of horizontal branches and one of the 0-nodes of r_{ω}. After numbering all the branches, we short some of them and open the others and then restore them in sequence according to the numbering. Were we to open all the branches and short none of them, r_{ω} would be disconnected from the rest of the network at each stage of the restoration, and thus would remain so in the final restored network, as shown in Fig. 3. In order to recover the connections to r_{ω}, we may use shorts instead of opens for some of the ladder's branches leading to n_1^1 and n_2^1. More specifically, we may choose two representative 0-paths for the ladder's 0-tips in n_1^1 and n_2^1 and short all their branches. The

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other branches are opened. Then, after all branches are restored, we will have recovered the original network of Fig. 2.

For the general case of any countably infinite network, we will always short the branches of a representative path for each nonopen[11] nonelementary tip in order to maintain connections through transfinite nodes. This is not done for the open tips. All the other branches are opened before the restoration begins.

Let us now consider the 2-network obtained by replacing every branch in Fig. 2 by an endless path. Every 0-node becomes a 1-node and the two 1-nodes become two 2-nodes n_1^2 and n_2^2 that embrace the 0-nodes of r_ω. This is shown in Fig. 4. The resulting network is still countable. Number its branches in any fashion using the natural numbers. To obtain an expanding sequence of finite networks that fill out this 2-network, we choose two representative one-ended 1-paths[12] for the two 1-tips in n_1^2 and n_2^2 and short their branches. We then choose representative one-ended 0-paths for all the 0-tips of all the 1-nodes and short their branches too. Some of those branches will already be shorted in the first step. At this point, we can and do choose the representative paths of the two 0-tips of each vertical endless path to be disjoint. Next, we open all the other branches. Finally, we restore the branches one-by-one in accordance with the branch numbering. At each stage, we have a finite network. Moreover, after all the branches are restored, we will have recovered the original 2-network because connections through the 1-nodes and 2-nodes will have been maintained.

The reason for choosing disjoint representative paths in each of the vertical endless paths is to insure that n_1^2 and n_2^2 do not coalesce into a single 2-node. Indeed, if an infinity of those vertical endless paths had been shorted, then at every stage of the restoration the two 2-nodes would have been coalesced into a single 0-node. As a result, they would be replaced by a single 2-node in the final restored network. Thus, it appears that some care must be taken in the way the transfinite network is restored from a sequence of finite networks.

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[11] A tip of any rank is said to be *nonopen* if it is embraced by a nonsingleton node and thus is shorted to at least one other tip; otherwise, it is said to be *open*.

[12] These can be any one-ended 1-subpaths of the upper and lower horizontal 1-paths.

5 Restorable Networks

In fact, there is a more substantial problem. Not all transfinite networks can be restored through a sequence of finite networks. For example, consider the 1-network of Fig. 5 having two nondisconnectable[13] 0-tips in two different 1-nodes n_a^1 and n_b^1. Such a network can also occur as part of a larger transfinite network. We wish to maintain the connection provided by the 1-node n_a^1 between the 0-tip of the 0-path of the a_k branches and the elementary tip of branch c_1. In order to ensure this when building up the 1-network from a sequence of finite networks, we have to short (the branches of) a representative path for that 0-tip. For a similar reason, we have to short a representative path for the 0-tip in n_b^1. We open all the other branches. Those two shorted representative paths meet infinitely often, and thus the network obtained through any sequence of branch restorations will yield the different 1-network of Fig. 6(a), in which n_a^1 and n_b^1 are coalesced into a single 1-node n_c^1. That is, at each step of the restoration sequence, n_a^1 and n_b^1 coalesce into a single 0-node through the shorting of some representative paths for their 0-tips. As a result, n_a^1 and n_b^1 have the same hyperreal voltage after the restoration process is completed. Electrically, n_a^1 and n_b^1 have been shorted, and we draw the restored network accordingly with n_a^1 and n_b^1 replaced by the single 1-node n_c^1.

On the other hand, were we to open all branches in Fig. 5, we would lose the connections through n_a^1 and n_b^1 and would end up with the disconnected 1-network of Fig. 6(b). We can conclude that the network of Fig. 5 cannot be restored through a sequence of finite networks.

Networks that can be so rebuilt this way will be called "restorable." Let us be more specific. Given a countably infinite network N^ν, consider the following procedure.

Procedure 5.1.

1. Number all the branches of N^ν using the natural numbers: $k = 0, 1, 2, \ldots$.

2. For each nonopen nonelementary tip, short one of its representative paths (i.e., short every branch in that path). Open all but finitely many of the branches that have not

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[13] Two tips of any ranks are called *nondisconnectable* if every representative path of one of them meets every representative path of the other [23, page 58], [26, Sec. 3.1].

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been shorted.

3. Restore branches sequentially finitely many at a time, in accordance with the branch numbering.

Note that there are many ways of following Procedure 5.1 because of the many ways of numbering the branches, choosing shorted representative paths, and restoring branches finitely many at a time.

At the nth step $(n = 0, 1, 2, \ldots)$ of the restoration process (item 3 in Procedure 5.1), let M_n^ν be the infinite network having the same graph as N^ν but with the electrical parameters existing at that point of restoration; thus, finitely many of the branches of M_n^ν will have the originally electrical-parameter values and the other branches of M_n^ν will be either opened or shorted. For a given M_n^ν, a maximal set of maximal nodes in M_n^ν that are pairwise connected through shorted paths will be called a *proximity*. Also, a maximal node that is not connected to any other maximal node through a shorted path will be called a *proximity*, albeit a singleton one. Every maximal node in M_n^ν will belong to some proximity in M_n^ν and the proximities in M_n^ν partition the set of maximal nodes. We also say that the embraced tips of a maximal node lie in its proximity. A proximity of a given node may shrink as n increases, but it will not expand. As $n \to \infty$, we have for each maximal node a sequence of proximities containing that node. Given two maximal nodes, we will say that their sequences of proximities are *eventually disjoint* if, for some natural number $\tilde{n}$, the corresponding two proximities are disjoint for every $n \geq \tilde{n}$.

Also, note that each M_n^ν defines a finite embedded network N_n^0 obtained by removing the opened and shorted branches in the way specified in Sec.3.

It may happen that two distinct maximal nodes of N^ν remain in the same proximity in M_n^ν for every n. This means that the node voltages at those nodes are the same for every n. In effect, those nodes—and their embraced tips as well—are shorted together throughout the restoration process. Thus, the restoration process imposes more shortings between tips than exist in N^ν. When this happens, we say that N^ν is *not restorable*. This will always happen if the following condition is violated in N^ν. It is a necessary condition for N^ν to be "restorable."

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Condition 5.2. *If two tips (of any ranks) are nondisconnectable in* N^ν, *then either they are shorted together (i.e., are embraced by the same node) or at least one of them is open.*

Indeed, if this condition is not satisfied by some pair of tips, then those tips will be nonopen, will be embraced by different maximal nodes, and will be in the same proximity in M_n^ν for every n, whatever be the choices of the shorted representative paths in Step 2 of Procedure 5.1.

In general, even when Condition 5.2 is satisfied, not every Procedure 5.1 will restore N^ν but some may. We saw this when discussing the 2-network of Fig. 4.

If, however, there is a choice of shortings of representative paths in Step 2 of Procedure 5.1 for which no additional shortings of tips persist throughout the entire restoration process, we will call N^ν *restorable* and will also say that N^ν can be *restored* with respect to an appropriately chosen sequence $\{N_n^0\}_{n=0}^\infty$ of finite embedded networks N_n^0. In this case, N^ν has a nonstandard image $N_{nst}^\nu = \langle N_n^0 \rangle$ as defined in Sec. 3, with the N_n^0 determined by the M_n^ν ($n = 0, 1, 2, \ldots$) and the graph of N_{nst}^ν taken to be that of N^ν.

Here is a necessary and sufficient condition for N^ν to be restorable. Its proof follows readily from our definitions.

Theorem 5.3. N^ν is restorable if and only if it is possible to choose a shorted representative path for every nonopen nonelementary tip in N^ν such that, for every pair of maximal nodes, the corresponding sequences of proximities in which those two nodes reside are eventually disjoint.

Proof. *Only if:* Assume it is impossible to choose shorted representative paths as stated. Then, for some pair of distinct maximal nodes in N^ν, their embraced tips will all be shorted together in M_n^ν for every n. Thus, N^ν is not restorable.

If: Assume shorted representatives can be chosen as stated. If two tips belong to the same maximal node in N^ν, they will remain in the same proximity for each M_n^ν and therefore will be shorted together for all n. If two tips belong to different maximal nodes in N^ν, they will eventually belong to disjoint proximities. Thus, they will eventually not be shorted together. So, no additional shortings of tips persist throughout the restoration process. N^ν

is restorable. □

Let us note that every countable, conventionally infinite network is restorable whatever be the choice of branch numbering. In this case, there are no transfinite nodes, and therefore no representative paths need to be shorted. All branches are first opened and then restored sequentially. We might say that such networks are "open at infinity."

At the other extreme, we have countable, conventionally infinite, locally finite networks that are "shorted at infinity." That is, the 0-tips of such a network are all shorted together to get a restorable 1-network.

In the rest of this paper we examine several restorable networks and calculate some hyperreal voltages and currents.

6 An Infinite Binary Tree Connected at Infinity

As an example of a restorable network, consider the infinite binary tree with connections at infinity, as shown in Fig. 7. That tree has uncountably many 0-tips. Were all such 0-tips connected bijectively to uncountably many branches through 1-nodes, it would be impossible to number all the branches with the natural numbers. However, we can choose countably many of those 0-tips and connect them one-to-one to countably many branches— through 1-nodes again. For instance, we can identify each one-ended 0-path starting at the apex 0-node n_0 by the number $\sum_{k=1}^{\infty} a_k/2^k$, where $a_k = 0$ (resp. $a_k = 1$) if the path proceeds from the kth node after n_0 toward the left (resp. toward the right). This labels all the 0-tips. Then, the subset of all 0-tips having only finitely many nonzero a_k is a countable set. We can short each such 0-tip through a 1-node to the elementary tip of a branch, whose other elementary tip is incident to n_0, this being done in a one-to-one fashion. All the other 0-tips are open and are not indicated in Fig. 7. Note that every two 0-tips are disconnectable; that is, they have (sufficiently small) representative 0-paths that do not meet. Thus, the necessary Condition 5.2 is satisfied.

We can now apply Procedure 5.1, numbering all the branches and then choosing a representative 0-path for each 0-tip in the chosen countable set. (Any representative 0-path will do in this case.) We can short those paths and open all the other branches. Then, upon

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restoring branches sequentially, we have at each step a finite embedded network. Moreover, since every two 0-tips are disconnectable, the sequences of proximities of any two 1-nodes are eventually disjoint. Thus, this tree with connections at infinity is restorable. So, with positive resistances and possibly sources assigned to the branches, we obtain a nonstandard network after the shorted representative paths and the order of restoring branches are specified (a fixed $\mathcal{F}$ being understood).

In particular, assume that the short on the right is replaced by a current source feeding 1 A to the apex node of the binary tree and extracting 1 A from the node of infinite degree at the bottom connected to the countably many branches there (i.e., at "infinity"). Let us short all branches and then restore them as follows. First, restore the two uppermost branches along with the two branches at infinity connected to the two paths that follow the left-most paths starting at the two nodes just below the apex node. Next, restore the next four branches and the two extra branches at infinity that are connected to the two additional left-most paths starting at two of the four nodes two rows below the apex node. Continuing this way, we see that each of the countably many branches at infinity carries the hyperreal current $\langle 2^{-n} \rangle$. Moreover, there are $\langle 2^n \rangle$ branches at infinity. Since $\langle 2^{-n} \rangle \langle 2^n \rangle = \langle 1 \rangle$, Kirchhoff's current law is satisfied at the node of infinite degree at the bottom.

7   The Fibonacci Numbers

In the rest of this work, we shall examine various nonstandard ladder networks, in which all resistances are 1 Ω and will then determine explicitly the hyperreal currents and voltages in them. Because of the 1 Ω resistance values and the ladder structure, the Fibonacci numbers occur throughout our analyses. Let us therefore review some facts about those numbers.

The Fibonacci numbers $F(k)$ comprise a sequence defined recursively by setting $F(0) = F(1) = 1$ and

$$F(k) = F(k-1) + F(k-2) \tag{1}$$

for $k = 2, 3, 4, \ldots$. See, for example, [18, page 144]. Thus, the next several values are $F(2) = 2$, $F(3) = 3$, $F(4) = 5$, $F(5) = 8$, $F(6) = 13, \ldots$. A formula can be derived for any

$F(k)$ by solving the linear difference equation (1) with constant coefficients in the standard way [16, pages 167-168]. This gives

$$F(k) = \frac{\lambda_1^{k+1} - \lambda_2^{k+1}}{\sqrt{5}}, \tag{2}$$

where $\lambda_1 = (1 + \sqrt{5})/2 = 1.618\cdots$ and $\lambda_2 = (1 - \sqrt{5})/2 = -0.618\cdots$. To accomodate a subsequent need, we also set $F(-1) = 0$.

8   A One-Way Infinite Ladder with a Source at Infinity

Let us now consider a transfinite network for which a standard analysis provides only a trivial voltage-current regime. The network we examine is the purely resistive one-way infinite ladder excited at its infinite extremity by a Thevenin branch with a 1 V voltage source. See Fig. 8(a). This network satisfies the conditions that allow a standard analysis to be applied [22, Theorem 3.3-5], [23, Theorem 5.2-8]. Every loop passing through the one and only voltage source is transfinite with an infinite sum for its resistors. Hence, the solution space $\mathcal{K}$ has no such nonzero loop current. Thus, the only solution the standard analysis gives is the one where every branch current is 0, and in this case Kirchhoff's voltage law is violated around every transfinite loop.

Far more interesting are the results provided by nonstandard analyses. They restore Kirchhoff's voltage law, albeit with hyperreal values. Now, however, there are many different solutions depending upon how the network is restored from finite ones. One way is to truncate the ladder after the mth resistor, m odd, by shorting (resp. opening) all subsequent series (resp. shunt) resistors. This is indicated in Fig. 8(b). The Thevenin branch at infinity remains unchanged. A straightforward recursive analysis[14] shows that the currents in this finite network, as shown in Fig. 8(b), have the values

$$i_{k,m} = \frac{F(k-1)}{F(m+1)}, \quad k = 1,\ldots,m, \tag{3}$$

and $i_s = F(m)/F(m+1)$. In that figure, with m fixed we have set $i_k = i_{k,m}$. If instead of truncating after a shunt resistor we were to truncate after the series resistor of index $m+1$,

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[14]Compute the sequence of driving-point resistances of the ladder to the left of each resistor, working from the left to the right. Then, compute the branch currents working from right to left.

the current value would be

$$i_{k,m+1} = \frac{F(k-1)}{F(m+2)}, \quad k = 1,\ldots,m+1, \tag{4}$$

and $i_s = F(m)/F(m+2)$.

Now, we can obtain a nonstandard network by alternately restoring the shunt and series resistors one at a time. Thus, the current in the kth resistor is the hyperreal $I_k = \langle i_{k,n} \rangle$, where k is fixed, n is the index for the representative sequence, and $i_{k,n}$ is given by (3) for $n = m$ odd and by (4) for $n = m+1$ even. I_k is an infinitesimal. On the other hand, the hyperreal source current is $I_s = \langle i_n \rangle$, where $i_n = F(n)/F(n+1)$ for n odd and $i_n = F(n-1)/F(n+1)$ for n even. I_s is limited but not infinitesimal, that is, it is appreciable. In terms of these hyperreals, Kirchhoff's laws are satisfied everywhere (including the voltage law around transfinite loops) since they are satisfied in each of the finite networks.

Let us consider another way of restoring the transfinite network of Fig. 8(a). After opening all the shunt resistors and shorting all the series resistors, we restore them starting at the left by restoring the first shunt resistor, then the first and second series resistors, then the second shunt resistor, then the third and fourth series resistors, and so forth alternately restoring one shunt and then two series resistors. In general, just after restoring a shunt resistor, we will have the network of Fig. 9. Also, just after restoring two series resistors, we will have the same network except for $(m+3)/2$ series resistors in place of the indicated $(m-1)/2$ series resistors. A recursive analysis once again yields the following results. Just after restoring a shunt resistor, we have

$$i_{k,m} = \frac{2F(k-1)}{2F(m-1)+(m+1)F(m)}, \quad k = 1,\ldots,m, \tag{5}$$

$$i_s = \frac{2F(m)}{2F(m-1)+(m+1)F(m)}.$$

Just after restoring two series resistors, we have

$$i_{k,m} = \frac{2F(k-1)}{2F(m-1)+(m+5)F(m)}, \quad k = 1,\ldots,m, \tag{6}$$

$$i_s = \frac{2F(m)}{2F(m-1)+(m+5)F(m)}.$$

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Now, the nonstandard network corresponding to this sequence of restorations has the following hyperreal currents. For the kth resistor and with n being the index for the representative sequence as before, $I_k = \langle i_{k,n} \rangle$, where $i_{k,n}$ is given by (5) for $n = m$ odd and by (6) for $n = m + 1$ even (i.e., replace m by $n - 1$ in (6)). The hyperreal current in the series circuit of $(m - 1)/2$ resistors or $(m + 3)/2$ resistors is the same as the source current I_s, which is given by $I_s = \langle i_n \rangle$, where

$$ i_n = \frac{2F(n)}{2F(n-1) + (n+1)F(n)}, \quad n \text{ odd}, $$

$$ i_n = \frac{2F(n-1)}{2F(n-2) + (n+4)F(n-1)}, \quad n \text{ even}. $$

In the present case, I_k is a smaller infinitesimal than it was for the first method of restoring resistors because of the larger denominators. Also, I_s is now an infinitesimal, in contrast to the appreciable source current obtained previously. Of course, Kirchhoff's laws are satisfied here as well in a nonstandard way.

9 Another One-Way Infinite Ladder

Consider now the ladder of Fig. 10(a). It is excited at its input by a pure current source of real value h A. All resistors are 1 Ω including the resistor r_ω connected to the ladder at its infinite extremity. Under a standard analysis, the real-valued branch currents converge to 0 as infinity is approached. As a result, we have to conclude that the real current i_ω in r_ω is 0. Under a nonstandard analysis, we can determine a nonzero hyperreal current I_ω in r_ω due to a nonzero hyperreal input current H, and can do so whether H is infinitesimal, appreciable, or unlimited.

We first have to specify how the nonstandard ladder is restored from finite ones. Let us assume that it is restored by inserting el-sections, each consisting of a shunt resistor followed by a series resistor. That is, each of the finite networks have the form shown in Fig. 10(b), which starts with a shunt resistor and ends with a series resistor before the opened and shorted branches begin. Here, k and m are odd positive integers with $k = 1, 3, \ldots, m$. The sequence of finite truncations is obtained by increasing m according to $m = 1, 3, 5, \ldots$. In

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this case, we have

$$i_k = h\frac{F(m+2-k)}{F(m+2)}, \quad i_{k+1} = h\frac{F(m+1-k)}{F(m+2)}, \quad i_\omega = i_{m+1} = h\frac{1}{F(m+2)}.$$

In order to have n as the index for each step of the expanding sequence of finite ladders ($n = 1, 2, 3, \ldots$), we set $m = 2n - 1$. Thus, for the corresponding nonstandard ladder we have the following hyperreal currents, where H is the hyperreal input source current.

$$I_k = H\left\langle\frac{F(2n+1-k)}{F(2n+1)}\right\rangle, \quad I_{k+1} = H\left\langle\frac{F(2n-k)}{F(2n+1)}\right\rangle, \quad I_\omega = H\left\langle\frac{1}{F(2n+1)}\right\rangle.$$

Using (2) again, we see that as $n \to \infty$

$$\frac{F(2n+1-k)}{F(2n+1)} \sim \frac{1}{\lambda_1^k}, \quad \frac{F(2n-k)}{F(2n+1)} \sim \frac{1}{\lambda_1^{k+1}}, \quad \frac{1}{F(2n+1)} \sim \frac{\sqrt{5}}{\lambda_1^{2n+2}} = \frac{A}{\lambda_1^{2n}}$$

where $A = \sqrt{5}/\lambda_1^2$. Thus, I_k and I_{k+1} are infinitesimal (resp. appreciable, resp. unlimited) whenever H is infinitesimal (resp. appreciable, resp. unlimited). On the other hand, If H has a representative that is $o(\lambda_1^{2n})$ as $n \to \infty$, then I_ω is infinitesimal. If H has a representative that is asymptotic to $B\lambda_1^{2n}$, where B is a nonzero constant, then I_ω is appreciable. Finally, if H has a representative $\{h_n\}_{n=1}^\infty$ such that $h_n\lambda_1^{-2n} \to \infty$, then I_ω is unlimited.

10 Two One-Way Infinite Ladders in Cascade

As a last example, let us consider a network that is more substantially transfinite than the networks we have considered so far. In particular, let that network be a cascade connection of two ladders identical to that of Fig. 10(a) except that the second ladder replaces r_ω. That is, the infinite extremity of the first ladder is connected through a 1-node to the input of the second ladder, which in turn has at its infinite extremity a resistor $r_{\omega \cdot 2}$ connected through another 1-node. We shall now denote the currents with double subscripts, the first subscript being 1 for the first ladder and 2 for the second ladder. We shall also truncate both ladders in the same way with the same number of el-sections. Thus, the last el-section in each ladder has the branches with second-subscript indices m and $m + 1$, where m is an odd positive integer. Also, we use the odd positive integers $k = 1, 3, \ldots, m$ and $p = 1, 3, \ldots, m$ to index

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shunt branches in the first and second ladders respectively, and $k+1$ and $p+1$ for the series branches. The resulting finite network can be analyzed exactly as before except that now we have twice as many el-sections for each m. Next, we restore el-sections simultaneously; that is, for a transition from m to $m+2$, we restore two el-sections, one at the end of the first finite ladder and the other at the end of the second finite ladder. Finally, to get the current expressions as sequences indexed by $n = 1, 2, 3, \ldots$, we set $m = 2n - 1$. Altogether then, the following expressions are obtained:

$$I_{1,k} = H\left\langle \frac{F(4n+1-k)}{F(4n+1)} \right\rangle, \quad I_{1,k+1} = H\left\langle \frac{F(4n-k)}{F(4n+1)} \right\rangle, \quad I_\omega = H\left\langle \frac{F(2n+1)}{F(4n+1)} \right\rangle,$$

$$I_{2,p} = H\left\langle \frac{F(2n+1-p)}{F(4n+1)} \right\rangle, \quad I_{2,p+1} = H\left\langle \frac{F(2n-p)}{F(4n+1)} \right\rangle, \quad I_{\omega \cdot 2} = H\left\langle \frac{F(1)}{F(4n+1)} \right\rangle.$$

As before, we can use the asymptotic behavior of the Fibonacci numbers, $F(n) \sim \lambda_1^{n+1}/\sqrt{5}$ as $n \to \infty$, to determine the character of each current (whether it is infinitesimal, appreciable, or unlimited) given the character of the hyperreal input-source current H.

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Figure Captions

Fig. 1.

    (a) The Norton circuit representing a branch with a positive conductance g in parallel with a current source h. The branch current i and the branch voltage v are related by Ohm's law: $i + h = gv$.

    (b) The Thevenin circuit representing a branch with a positive resistance r in series with a voltage source e. Now, Ohm's law has the form $v + e = ri$. When $r = 1/g$ and $e = -hr$, these two circuits are equivalent.

Fig. 2. A one-way infinite ladder connected at its infinite extremity to a resistor r_ω through two 1-nodes n_1^1 and n_2^1. The 0-nodes of r_ω are embraced by the 1-nodes. Any of the branches may have sources; these are not shown in the figure.

Fig. 3. The ladder of Fig. 2 except that the resistor r_ω is now disconnected from the ladder. The 1-nodes (shown by the small circles) no longer embrace the 0-nodes of r_ω.

Fig. 4. A one-way infinite ladder consisting of endless 0-paths. The ladder's connections are now made through 1-nodes (the small circles). The ladder is connected at its infinite extremity to a resistor r_ω through two 2-nodes (the double circles). Those 2-nodes embrace the 0-nodes of r_ω.

Fig. 5. A 1-network. The pairs of parallel branches extend infinitely to the right. The 1-node n_a^1 (resp. n_b^1) consists of the 0-tip of the one-ended 0-path along the upper branches a_k (resp. lower branches b_k) and an elementary tip of the branch c_1 (resp. c_2). The two 0-tips are nondisconnectable. Every branch has a positive resistor and possibly a source.

Fig. 6.

    (a) The network resulting from any sequence of restorations of the branches in Fig. 5, wherein the shorting between the upper 0-tip and a 0-node of branch c_1 and
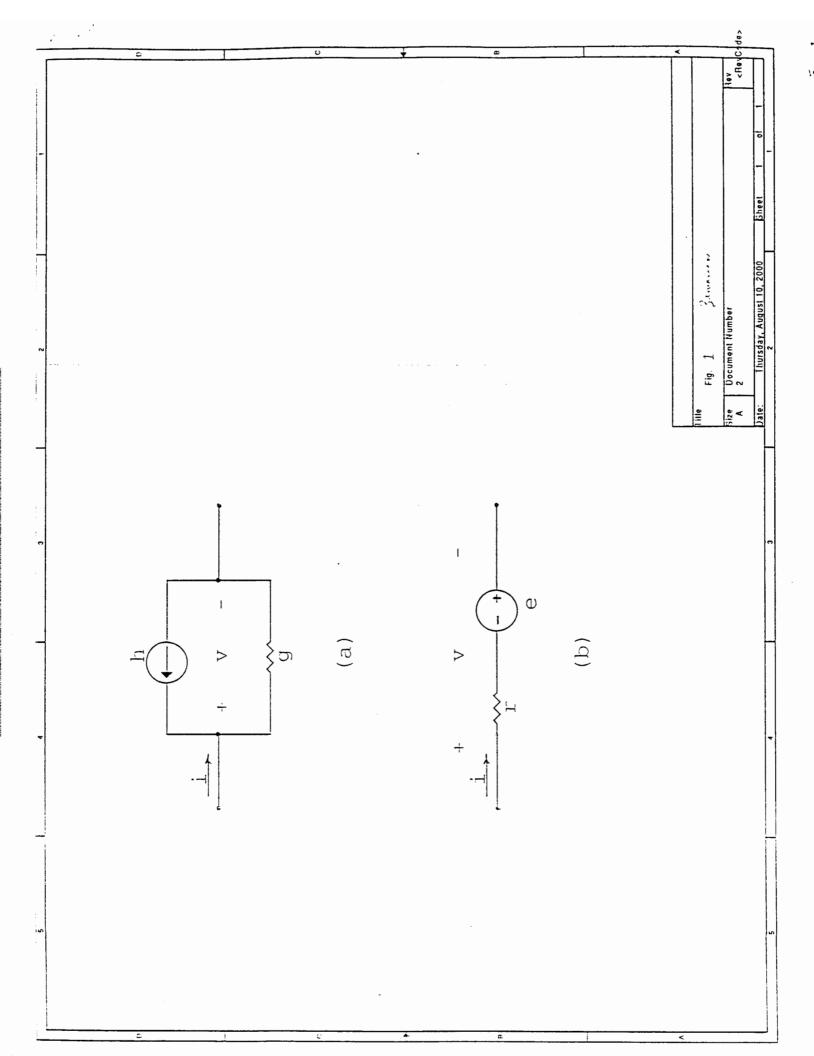
the shorting between the lower 0-tip and a 0-node of branch c_2 are maintained by initially shorting representative paths of those 0-tips.

(b) The network obtained after all the branches are opened and then restored sequentially.

Fig. 7. An infinite binary tree connected at infinity to countably many branches, which in turn are incident to the apex 0-node n_0. The small circles denote 1-nodes through which countably many of the 0-tips of the tree are connected bijectively to the branches at infinity.
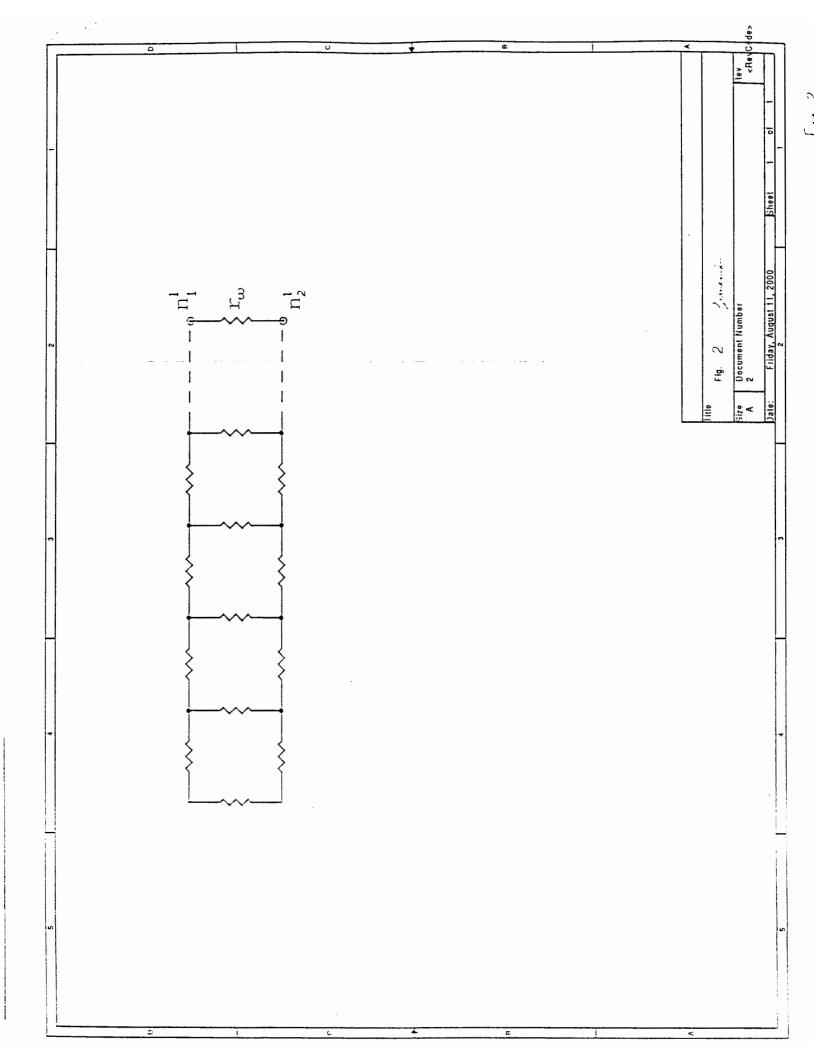
Fig. 8.

(a) A one-way infinite, purely resistive ladder excited at infinity by a 1 V voltage source in series with a resistor. All resistors are 1 Ω. Here, m is an odd positive integer.

(b) The truncated ladder. All series (resp. shunt) resistors beyond the mth resistor are shorts (resp. opens).

Fig. 9. The network obtained just after restoring a shunt resistor during the second way of building up the transfinite network of Fig. 8(a). Here again, m is an odd positive integer.
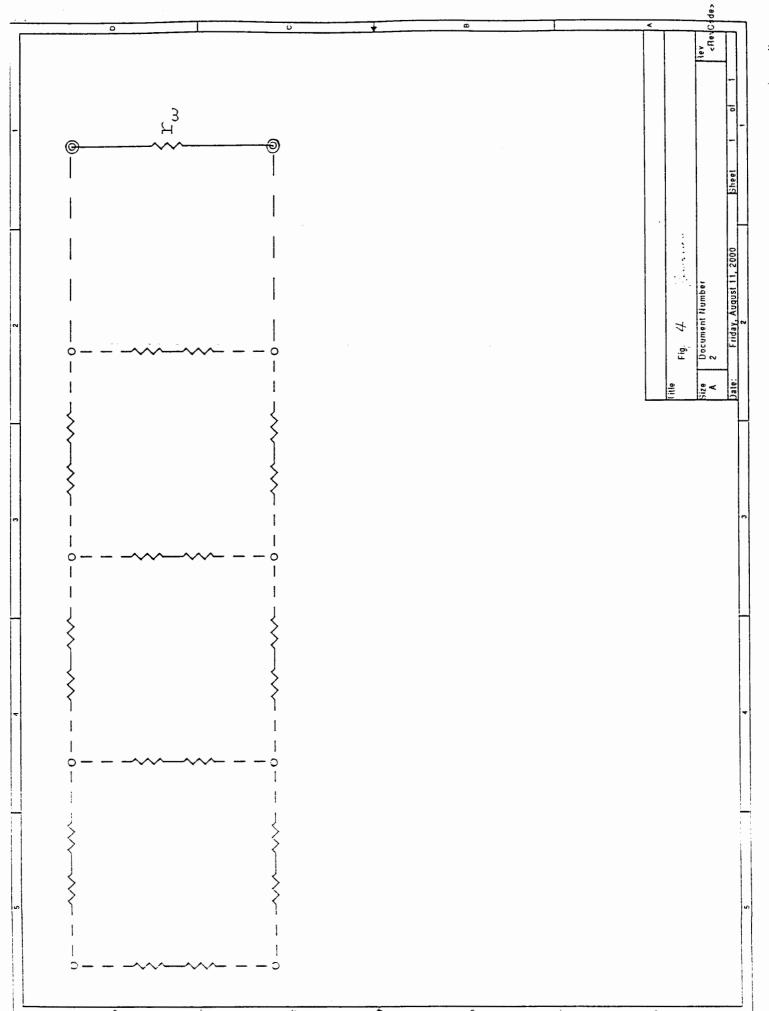
Fig. 10.

(a) A one-way infinite ladder connected at infinity to a resistor r_ω and excited by a pure current source h at its input. All resistors, including r_ω, are 1 Ω.

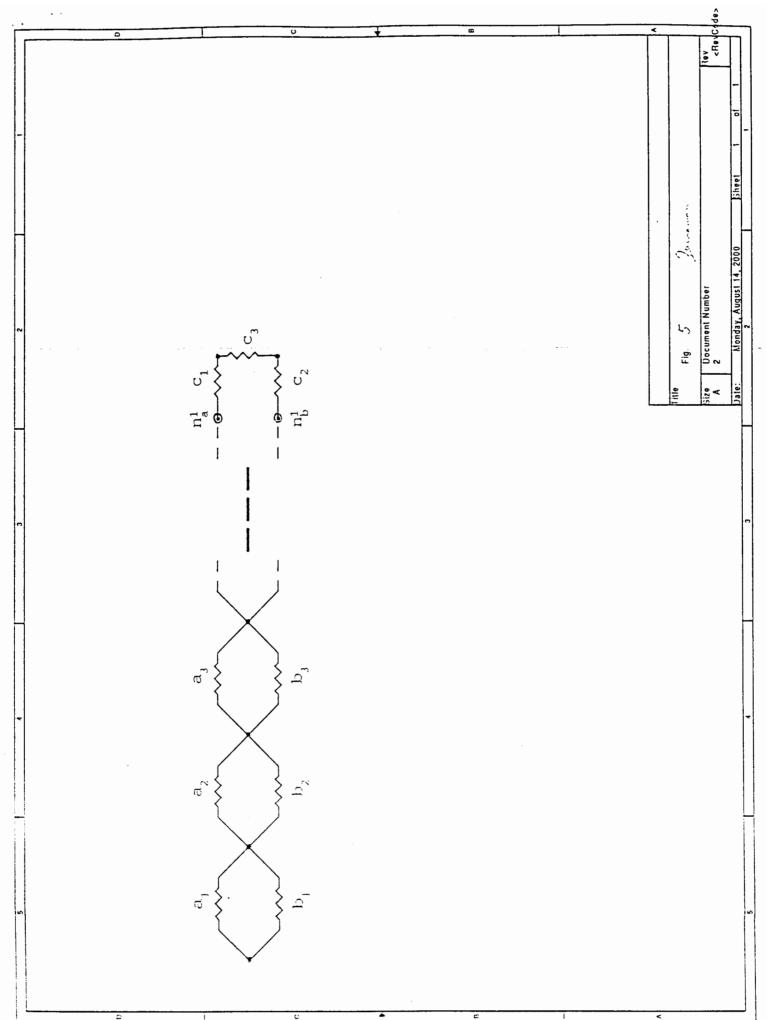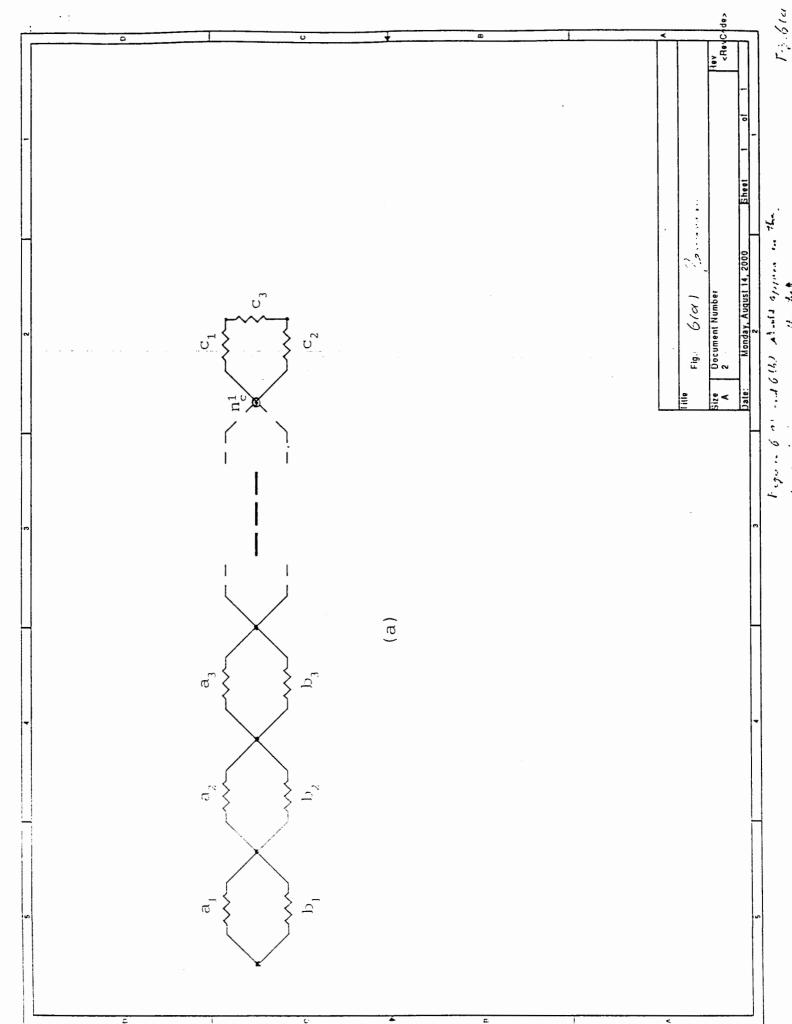(b) A finite truncation of the ladder. The ladder is built up with el-sections restored one at a time.
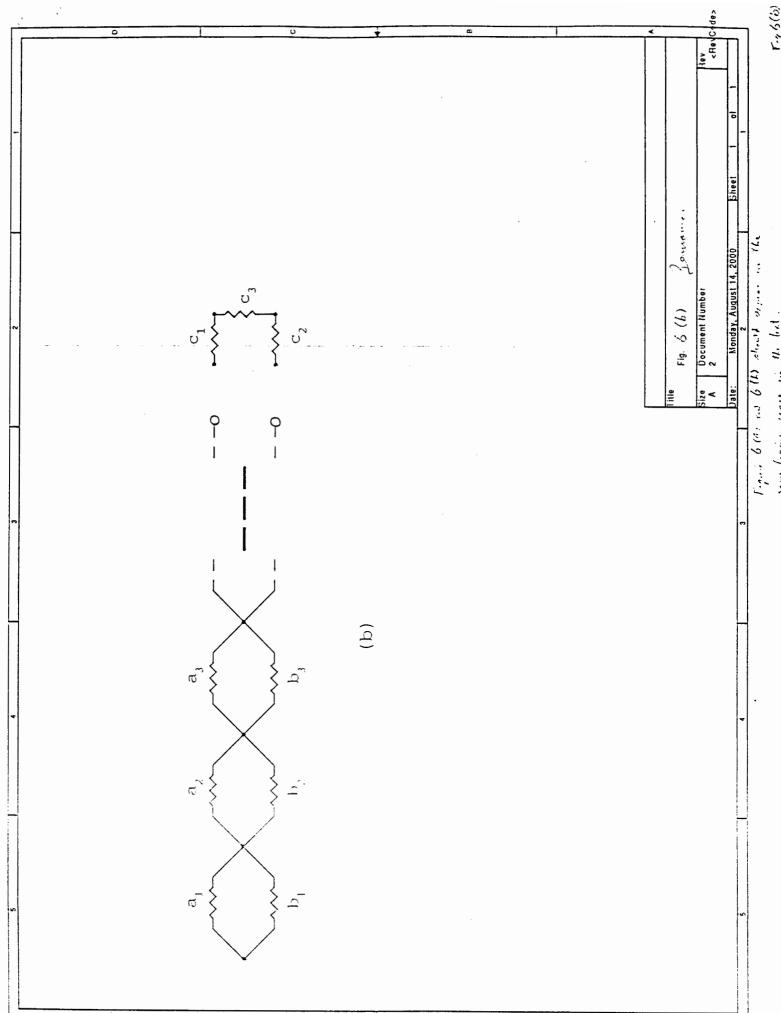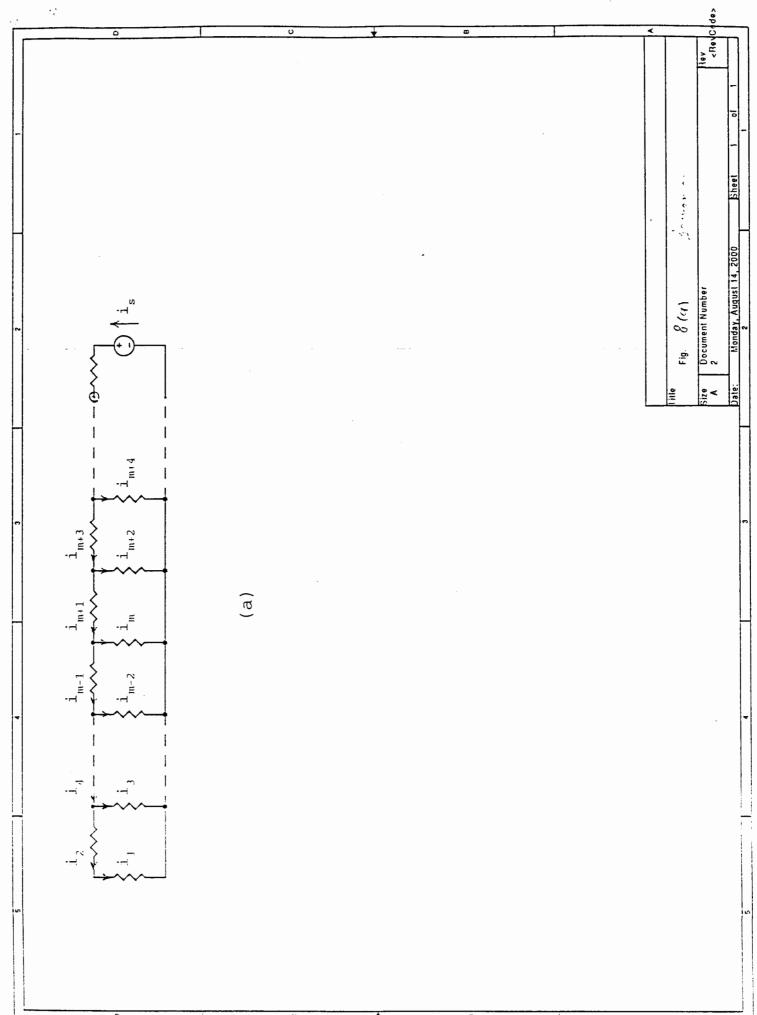
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(a)

(b)

Fig. 2

n_1^l

r_ω

n_2^l

Fig. 3 Zusatz

r_w

Fig. 4

C_3

C_1

C_2

n_a^1

n_b^1

a_3

b_3

a_2

b_2

a_1

b_1

Fig. 5

(a)

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Fig. 6(a)

(b)

Fig. 6 (b)

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Fig 6 (b)

Fig. 7

n_0

n_0

(a)

Fig. 8 (a)

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(b)

i_s

i_{m-1} i_m i_{m-2}

i_4 i_3

i_2 i_1

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Fig. 8 (b)

Fig. 9

(a)

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input lines to the ...

Fig. 10(a)

$i_\omega = i_{m+1}$

r_ω

i_{m+1}

i_m

i_{k+1}

i_k

i_4

i_3

i_2

i_1

I_1

(b)

Figure 10(a) & 10(b) should be placed on
the same figure space in the text.