

NONSTANDARD GRAPHS BASED ON TIPS AS THE INDIVIDUAL ELEMENTS

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Abstract — From any given sequence of finite or infinite graphs, a nonstandard graph is constructed. The procedure is similar to an ultrapower construction of an internal set from a sequence of subsets of the real line, but now the individual entities are the tips of the branches instead of real numbers. The transfer principle is then invoked to extend several graph-theoretic results to the nonstandard case. After incidences and adjacencies between nonstandard nodes and branches are defined, several formulas regarding numbers of nodes and branches, and nonstandard versions of Eulerian graphs, Hamiltonian graphs, and a coloring theorem are established for these nonstandard graphs. This work is a revision and expansion of CEAS Technical Report 793, University at Stony Brook, January 2002.

Key Words: Nonstandard graphs, transfer principle, ultrapower constructions.

1 Introduction

In the book [5] (see Sec. 19.1), R. Goldblatt constructed a nonstandard graph by applying the transfer principle to a set V of vertices with the set E of edges defined by a symmetric irreflexive binary relationship on V . Thus, the conventional graph $\langle V, E \rangle$ is transferred to the nonstandard graph ${}^*G = \langle {}^*V, {}^*E \rangle$. This result was used to establish in a nonstandard way the standard theorem that, if every finite subgraph of a conventional infinite graph G has a coloring with finitely many colors, then G itself has such a finite coloring.

On the other hand, in several recent works [8], [9], [10], [12], [13], the idea of nonstandard transfinite graphs and networks was introduced and investigated. The basic idea in those works was to start with a given transfinite graph, to reduce it to a finite graph by shorting and opening branches, and then to obtain an expanding sequence of finite graphs

by restoring branches sequentially. If all this is done in an appropriate fashion, it may happen that the sequence of finite graphs fills out and restores the original transfinite graph once the restoration process is completed. If in addition there is an assignment of electrical parameters to the branches, the final result may be sequences of branch currents and branch voltages, from which hyperreal currents and voltages can be derived. The latter hyperreal quantities will then automatically satisfy Kirchhoff's laws, even though Kirchhoff's laws may on occasion be violated in the original transfinite network when standard real numbers are used.

However, this is only a partial construction of a nonstandard graph in the sense that the completion of the restoration process—if successful—results in the original standard transfinite graph. The sequence of restorations only provides a means of constructing hyperreal currents and voltages satisfying Kirchhoff's laws. Another approach might start with an arbitrary sequence of conventional graphs and construct from that a nonstandard graph in much the same way as an internal set in the hyperreal line ${}^*\mathbf{R}$ is constructed from a given sequence of subsets of the real line \mathbf{R} , that is, by means of an ultrapower construction [5, page 36]. In this case, the resulting nonstandard graph has nonstandard branches and nonstandard nodes. Thus, that nonstandard graph is much different from those of [8] - [13].

Our objective in this work is to develop this latter approach to nonstandard graphs. To conform with all our prior works, we use electrical terminology by writing ‘branches’ instead of ‘edges’ and ‘nodes’ instead of ‘vertices.’ Moreover, \overline{A} or card A denotes the cardinality of a set A . \mathbf{N} denotes the set $\{0, 1, 2, \dots\}$ of all natural numbers. A sequence of elements a_n will be denoted by $\langle a_n : n \in \mathbf{N} \rangle$ or more simply by $\langle a_n \rangle$, where the index n is understood to traverse \mathbf{N} .

Actually, there are two essentially equivalent ways of accomplishing our objective. One way is to adopt the approach that leads to the construction of transfinite graphs [7]. Thus, given a sequence $\langle G_n : n \in \mathbf{N} \rangle$ of finite or conventionally infinite graphs G_n , each branch of G_n is viewed as a set of two tips,¹ with the intersection between any two branches being empty. The union of all the branches is a set \mathcal{T}_n of tips. Thus, $\overline{\overline{\mathcal{T}_n}}$ is even if G_n is a finite

¹Called ‘elementary tips’ in [7].

graph, but $\overline{\overline{\mathcal{T}_n}}$ may be any infinite cardinal number since G_n is allowed to be any, possibly infinite, conventional graph.² Then, the set \mathcal{T}_n is partitioned arbitrarily to get the nodes of G_n , each node being a set of the partition. With B_n (resp. X_n) being the set of branches (resp. nodes) of G_n , we write $G_n = \{B_n, X_n\}$ but remember that it is the tips that play the role of individuals when constructing our nonstandard graph. An individual is an entity that is not a set—it has no members—unlike a branch or a node. This is the approach we adopt in this work. Another consequence of this approach is that there are no isolated nodes in G_n . However, there may be parallel branches (i.e., two or more branches incident to the same two nodes) and also self-loops (i.e., a branch incident to just one node).

The other way of constructing the nonstandard graphs considered herein is to use the nodes of each G_n as the individual entities and then define the branches of G_n as pairs of nodes. This is a conventional way of defining a graph. The former way is readily generalized to get tips of higher ranks representing infinite extremities of graphs and yields thereby transfinite graphs. However, there is a problem (which may yet be overcome) in identifying nonstandard tips of higher transfinite ranks as the infinite extremities of nonstandard graphs. That problem arises from the fact that the infinite extremities of a graph are defined as certain equivalence classes of one-way infinite paths, and that in turn requires infinitely many conditions and thereby infinitely many sets in the chosen ultrafilter for the ultrapower construction employed herein. Those infinitely many sets may have an empty intersection, thereby preventing an ultrapower construction of a nonstandard one-way infinite path. An equivalent way of viewing this problem is to note that a symbolic sentence representing a one-way infinite path would have to be infinitely long and thereby not a legitimate sentence in symbolic logic.

The latter more conventional way also encounters the same difficulty. In any case, the two approaches are equivalent, if we disallow isolated nodes, self-loops, and parallel branches.

Because of this problem regarding the extension of our nonstandard graphs to nonstandard transfinite graphs, we will be dealing only with the lowest rank of transfiniteness. So,

²The prior works on nonstandard graphs cited above required that each G_n have at most a finite branch set; thus in this regard we have greater generality now.

we drop any reference to transfinite ranks; that is, (-1) -tips, 0-nodes, and 0-graphs are henceforth called simply tips, nodes, and graphs.

2 Nonstandard Graphs

A standard graph is a conventional (finite or infinite) graph. However, we use an unconventional definition, as given in [7, Sec. 1.3], just to conform with our prior works on transfinite graphs. In particular, each branch is taken to be a set of two tips (also called “elementary tips”) with the branches being pairwise disjoint. The set \mathcal{T} of all tips is the union of the branches. Alternatively we can start with a set \mathcal{T} consisting of either a finite even number of tips or infinitely many tips and then partition \mathcal{T} into two-element subsets, each subset being defined to be a branch. B denotes the set of branches. Next, we partition \mathcal{T} arbitrarily and define each subset of the partition to be a node. X denotes the set of nodes. Thus, a *conventional graph* or synonymously a *graph* G is a pair $G = \{B, X\}$ derived from an initially given set \mathcal{T} of tips. Under this definition, there are no isolated nodes in G .

Next, let $\langle G_n : n \in \mathbf{N} \rangle$ be a sequence of graphs.³ Thus, $G_n = \{B_n, X_n\}$ for each n , and each set \mathcal{T}_n of tips in G_n can be obtained by taking the union of all the branches in B_n or alternatively all the nodes in X_n . We allow $B_n \cap B_m \neq \emptyset$ when $n \neq m$; as a result, nodes in different G_n and G_m may also have nonempty intersections.

Furthermore, let \mathcal{F} be a nonprinciple ultrafilter in \mathbf{N} [5, pages 18-19]. A *nonstandard tip* *t is an equivalence class $[\langle t_n \rangle]$ of sequences $\langle t_n \rangle$ of tips $t_n \in \mathcal{T}_n$ where two such sequences $\langle t_n \rangle$ and $\langle s_n \rangle$ are taken to be equivalent if $\{n \in \mathbf{N} : t_n = s_n\} \in \mathcal{F}$, in which case we write “ $t_n = s_n$ a.e.” or “ $t_n = s_n$ for almost all n .” For the sake of a simpler notation, we also write “ ${}^*t = [t_n]$ ” where it is now understood that the t_n are the members of one (i.e., any one) representative sequence $\langle t_n \rangle$ in the equivalence class. That this truly defines an equivalence class can be shown as follows. Reflexivity and symmetry being obvious, consider transitivity. Given that $\langle t_n \rangle = \langle s_n \rangle$ a.e. and that $\langle s_n \rangle = \langle r_n \rangle$ a.e., let $\mathbf{N}_{ts} = \{n : t_n = s_n\} \in \mathcal{F}$, and let $\mathbf{N}_{sr} = \{n : s_n = r_n\} \in \mathcal{F}$. By the properties of an ultrafilter filter, $\mathbf{N}_{ts} \cap \mathbf{N}_{sr} \in \mathcal{F}$. Moreover, $\mathbf{N}_{tr} = \{n : t_n = r_n\} \supset (\mathbf{N}_{ts} \cap \mathbf{N}_{sr})$. Therefore, $\mathbf{N}_{tr} \in \mathcal{F}$. So, $\langle t_n \rangle = \langle r_n \rangle$ a.e.;

³More generally, we could let the index n traverse an arbitrary infinite set with hardly any alteration in our development.

transitivity holds. We let $*T$ denote the set of nonstandard tips.

Next, we define the nonstandard branches. Let $*t = [t_n]$ and $*s = [s_n]$ be two nonstandard tips. This time, let $N_{ts} = \{n : \{t_n, s_n\} \in B_n\}$ and $N_{ts}^c = \{n : \{t_n, s_n\} \notin B_n\}$. Since \mathcal{F} is an ultrafilter, exactly one of N_{ts} and N_{ts}^c is a member of \mathcal{F} . If it is N_{ts} , $*b_{ts} = [\{t_n, s_n\}]$ is defined to be a nonstandard branch; that is, $*b_{ts}$ is an equivalence class $[b_n]$ of sequences of branches with $\langle b_n \rangle$ being one of those sequences and $b_n = \{t_n, s_n\}$, $n = 0, 1, 2, \dots$. If $N_{ts}^c \in \mathcal{F}$, then $[\{t_n, s_n\}]$ is not a nonstandard branch.

We should now show that this definition is independent of the representatives chosen for the branches. Let $[\{t_n, s_n\}]$ and $[\{r_n, q_n\}]$ be nonstandard branches. We want to show that, if $s_n = r_n$ a.e., then $t_n = q_n$ a.e. Suppose $t_n \neq q_n$ a.e., but $s_n = r_n$ a.e. Then, there exists at least one n for which $t_n, s_n = r_n$, and q_n all belong to the same branch in G_n . Hence, the branch set in G_n is not a partition of the set of tips in G_n into two-element subsets—a contradiction.

Next, we show that we truly have an equivalence relationship among all sequences of standard branches. Reflexivity and symmetry again being obvious, consider transitivity. Let $*b = [\{t_n, s_n\}]$, $*\tilde{b} = [\{\tilde{t}_n, \tilde{s}_n\}]$, $*\acute{b} = [\{\acute{t}_n, \acute{s}_n\}]$, and assume that $*b = *\tilde{b}$ and $*\tilde{b} = *\acute{b}$. We want to show that $*b = *\acute{b}$. Set

$$N_b = \{n : \{t_n, s_n\} \in B_n\} \in \mathcal{F},$$

$$N_{\tilde{b}} = \{n : \{\tilde{t}_n, \tilde{s}_n\} \in B_n\} \in \mathcal{F},$$

$$N_{\acute{b}} = \{n : \{\acute{t}_n, \acute{s}_n\} \in B_n\} \in \mathcal{F}.$$

By our assumption, $N_b \cap N_{\tilde{b}} \in \mathcal{F}$ and $N_{\tilde{b}} \cap N_{\acute{b}} \in \mathcal{F}$. Because, \mathcal{F} is an ultrafilter, $N_b \cap N_{\tilde{b}} \cap N_{\acute{b}} \in \mathcal{F}$. But, $(N_b \cap N_{\acute{b}}) \supset (N_b \cap N_{\tilde{b}} \cap N_{\acute{b}})$. Hence, $N_b \cap N_{\acute{b}} \in \mathcal{F}$. Consequently, $*b = *\acute{b}$, as desired.

Because of all this, we also write $*b = \{*t, *s\}$, where $*t = [t_n]$ and $*s = [s_n]$, and we let $*B$ denote the set of all nonstandard branches $*b$.

We have yet to define the nonstandard nodes. For each n , we have a partition of \mathcal{T}_n into the nodes of G_n . If t_n and s_n are members of the same node in G_n , we say that t_n and s_n are *shorted together* and write $t_n \doteq s_n$. Similarly, let $*t = [t_n]$ and $*s = [s_n]$, and now

let $N_{ts} = \{n : t_n \doteq s_n\}$ and $N_{ts}^c = \{n : t_n \not\doteq s_n\}$. As before, exactly one of N_{ts} and N_{ts}^c is in \mathcal{F} . If it is N_{ts} (resp. N_{ts}^c), we say that $*t$ is shorted to $*s$ (resp. $*t$ is not shorted to $*s$), and we write $*t \doteq *s$ (resp. $*t \not\doteq *s$). This shorting is an equivalence relationship for the set of all nonstandard tips, as can be shown much as before; indeed, for transitivity, assume $*t \doteq *s$ and $*s \doteq *v$; since $\{n : t_n \doteq s_n\} \cap \{n : s_n \doteq v_n\} \subset \{n : t_n \doteq v_n\}$, we have $*t \doteq *v$. The resulting equivalence classes of the corresponding partition are the nonstandard nodes. Also, much as before, this definition can be shown to be independent of the representative sequences of the nonstandard tips. To be specific, let $*t = [t_n] = [\tilde{t}_n]$ and $*s = [s_n] = [\tilde{s}_n]$. Set $N_t = \{n : t_n = \tilde{t}_n\} \in \mathcal{F}$ and $N_s = \{n : s_n = \tilde{s}_n\} \in \mathcal{F}$. Assume $[t_n] \doteq [s_n]$. Thus, $N_{ts} = \{n : t_n \doteq s_n\} \in \mathcal{F}$. We want to show that $\tilde{N}_{ts} = \{n : \tilde{t}_n \doteq \tilde{s}_n\} \in \mathcal{F}$ so that $[\tilde{t}_n] \doteq [\tilde{s}_n]$. We have $(N_t \cap N_s \cap N_{ts}) \subset \tilde{N}_{ts}$, whence our conclusion.

Altogether then, we define a nonstandard node $*x$ to be any set in the partition of the nonstandard tips under the shorting together of such nonstandard tips. $*X$ will denote the set of nonstandard nodes. Finally we define a nonstandard graph $*G$ to be the pair $*G = \{*B, *X\}$.

Let us take note of a special case that arises when all the G_n are the same standard graph G , that is, each G_n is constructed out of the same set \mathcal{T} of tips in exactly the same way. In this case, we call $*G$ an *enlargement* of G , in conformity with an enlargement $*A$ of a subset A of \mathbf{R} [5, pages 28-29]. If G is a finite graph, then its tips comprise a finite set of even cardinality. For this case, each nonstandard tip of $*G$ can be identified with a standard tip of G (modulo the given ultrafilter \mathcal{F}). In fact, $*\mathcal{T} = \mathcal{T}$ because the enlargement of a finite set equals the set itself. Consequently, the nonstandard branches and nodes can be identified with the standard branches and nodes, and we have $*G = G$. On the other hand, if G is a conventionally infinite graph, \mathcal{T} is an infinite set and its enlargement $*\mathcal{T}$ has more elements, namely, the nonstandard tips that are not equal to standard tips (i.e., $*\mathcal{T} \setminus \mathcal{T}$ is not empty). When this happens, $*G$ has nonstandard branches and nodes that are different from the standard branches and nodes of G .

Example 2.1. Let G be a one-ended path P (i.e., a one-way infinite path) in a standard

graph:

$$P = \{x_0, b_0, x_1, b_1, \dots, x_m, b_m, \dots\}$$

where $m = 0, 1, 2, \dots$. Also, let $G_n = G$ for all $n \in \mathbf{N}$, and let $*G = \{*B, *X\}$ be the resulting enlargement of G . Then, in terms of tips, let us set $b_m = \{t_m, \tilde{t}_m\}$ and thus $x_m = \{\tilde{t}_{m-1}, t_m\}$ for each $m = 1, 2, 3, \dots$; however, it will be understood henceforth that $x_0 = \{t_0\}$ for $m = 0$. These are the only branches and nodes appearing in G .

Now, let $\langle k_n : n \in \mathbf{N} \rangle$ be any sequence of natural numbers. Set $*t_{*k} = [t_{k_n}]$ and $*\tilde{t}_{*k} = [\tilde{t}_{k_n}]$ where $*k = [k_n]$. These are two nonstandard tips belonging to the nonstandard branch $*b_{*k} = \{ *t_{*k}, *\tilde{t}_{*k} \} = [\{t_{k_n}, \tilde{t}_{k_n}\}]$. Then, $*x_{*k} = \{ *\tilde{t}_{*k-1}, *t_{*k} \} = [\{\tilde{t}_{k_n-1}, t_{k_n}\}]$ and $*x_{*k+1} = \{ *\tilde{t}_{*k}, *t_{*k+1} \} = [\{\tilde{t}_{k_n}, t_{k_n+1}\}]$ are two nonstandard nodes “incident to” $*b_{*k}$. We also say that $*x_{*k}$ and $*x_{*k+1}$ are “adjacent to” each other. On the other hand, if $\langle p_n : n \in \mathbf{N} \rangle$ is another sequence of natural numbers with $p_n \neq k_n$ a.e., then $[\{t_{k_n}, \tilde{t}_{p_n}\}]$ is not a nonstandard branch in $*G$ because $\{t_{k_n}, \tilde{t}_{p_n}\}$ is not a standard branch in G_n for almost all n . Similarly, $[\{\tilde{t}_{k_n-1}, t_{p_n}\}]$ is not a nonstandard node in $*G$. Note also, that, when k_n is a fixed natural number r for all n , then $*b_{*k}$ can be identified with the standard branch $b_r = \{t_r, \tilde{t}_r\}$, and $*x_{*k}$ can be identified with the standard node $x_r = \{\tilde{t}_{r-1}, t_r\}$. Altogether then, the enlargement $*G$ of G is the nonstandard path:

$$\begin{aligned} *P &= \{ *x_0, *b_0, *x_1, *b_1, \dots, *x_{*k}, *b_{*k}, *x_{*k+1}, *b_{*k+1}, \dots \} \\ &= \{ x_0, b_0, x_1, b_1, \dots, x_{*k}, *b_{*k}, *x_{*k+1}, *b_{*k+1}, \dots \} \end{aligned}$$

This generation of a nonstandard path by using an ultrapower construction is rather complicated. Fortunately, the transfer principle allows us to write $*P$ directly as a generalization of P . \square

Another special case arises when almost all the G_n are (possibly different) finite graphs. Again in conformity with the terminology used for hyperfinite internal subsets of $*R$ [5, page 149], we will refer to the resulting nonstandard graph $*G$ as a hyperfinite graph.⁴ As a result, we can lift many theorems concerning finite graphs to hyperfinite graphs. It is just a matter of writing the standard theorem as an appropriate sentence using symbolic logic and then applying the transfer principle. We let $*G_f$ denote the set of hyperfinite graphs.

⁴This should not be confused with a hypergraph—an entirely different object [2].

3 Incidences and Adjacencies for Nodes and Branches

Let us now define these ideas for nonstandard graphs both in terms of an ultrapower construction and by transfer of appropriate symbolic sentences. In the subsequent sections, we will usually confine ourselves to the transfer principle. We henceforth drop the asterisks when denoting nonstandard nodes and branches. These are specified as members of $*X$ and $*B$ respectively.

Incidence between a node and an branch: Given a sequence $\langle G_n : n \in \mathbf{N} \rangle$ of standard graphs $G_n = \{X_n, B_n\}$, a nonstandard node $x = [x_n] \in *X$ and a nonstandard branch $b = [b_n] \in *B$ are said to be *incident* if x_n and b_n share at least one tip for almost all n , that is, $\{n : x_n \cap b_n \neq \emptyset\} \in \mathcal{F}$, where as always the nonprincipal ultrafilter \mathcal{F} is understood to be chosen and fixed. Also, the image $*\emptyset$ of the empty set \emptyset under transfer is denoted simply by \emptyset .

On the other hand, we can define incidence between a standard node $x \in X$ and a standard branch $b \in B$ for the graph $G = \{X, B\}$ through the following symbolic sentence

$$(\exists x \in X)(\exists b \in B)(x \cap b \neq \emptyset)$$

By transfer, we have that $x \in *X$ and $b \in *B$ are incident when the following sentence is true.

$$(\exists x \in *X)(\exists b \in *B)(x \cap b \neq \emptyset)$$

These are equivalent definitions. We will denote the incidence between x and b by $x \triangleleft b$ or $b \triangleright x$.

Adjacency between nodes: For a standard graph $G = \{X, B\}$, two nodes $x, y \in X$ are called *adjacent* and we write $x \diamond y$ if the following sentence on the right-hand side of \leftrightarrow is true.

$$x \diamond y \leftrightarrow (\exists x, y \in X)(\exists b \in B)(x \cap b \neq \emptyset \wedge y \cap b \neq \emptyset)$$

By transfer, this becomes for a nonstandard graph $*G = \{*X, *B\}$

$$x \diamond y \leftrightarrow (\exists x, y \in *X)(\exists b \in *B)(x \cap b \neq \emptyset \wedge y \cap b \neq \emptyset)$$

Alternatively, under an ultrapower construction, we have for $*G = [G_n] = [\{X_n, B_n\}]$ that $x = [x_n] \in *X$ and $y = [y_n] \in *X$ are adjacent (i.e., $x \diamond y$) if there exists a $b = [b_n] \in *B$ such that $\{n: x_n \cap b_n \neq \emptyset \text{ and } y_n \cap b_n \neq \emptyset\} \in \mathcal{F}$.

Adjacency between branches: For a standard graph, two branches $b, c \in B$ are called *adjacent* and we write $b \bowtie c$ when the following sentence on the right-hand side of \leftrightarrow is true.

$$b \bowtie c \leftrightarrow (\exists b, c \in B) (\exists x \in X) (b \cap x \neq \emptyset \wedge c \cap x \neq \emptyset)$$

By transfer, we have for nonstandard branches b and c

$$b \bowtie c \leftrightarrow (\exists b, c \in *B) (\exists x \in *X) (b \cap x \neq \emptyset \wedge c \cap x \neq \emptyset)$$

Under an ultrapower approach, we would have $b = [b_n]$ and $c = [c_n]$ are adjacent nonstandard branches when there exists a nonstandard node $x = [x_n]$ such that

$$\{n: b_n \cap x_n \neq \emptyset \text{ and } c_n \cap x_n \neq \emptyset\} \in \mathcal{F}.$$

4 Nonstandard Hyperfinite Paths and Loops

Again, we start with a standard graph $G = \{X, B\}$. A *finite path* P in G is defined by the sentence

$$\begin{aligned} & (\exists k \in \mathbf{N} \setminus \{0\}) (\exists x_0, x_1, \dots, x_k \in X) (\exists b_0, b_1, \dots, b_{k-1} \in B) \\ & (x_0 \cap b_0 \neq \emptyset \wedge b_0 \cap x_1 \neq \emptyset \wedge x_1 \cap b_1 \neq \emptyset \wedge b_1 \cap x_2 \neq \emptyset \wedge \dots \\ & \wedge x_{k-1} \cap b_{k-1} \neq \emptyset \wedge b_{k-1} \cap x_k \neq \emptyset) \end{aligned} \quad (1)$$

That all the nodes in P are distinct is implied by the fact that the those k elements in the set X are perforce distinct. Similarly, the branches in P , being $k - 1$ elements of the set B , are perforce all distinct; they are also regular (i.e., not self-loops) because each branch is incident to two nodes. The *length* $|P|$ of P is the number of branches in P ; thus, $|P| = k$.

We may apply transfer to (1) to get the following definition of a *nonstandard path* $*P$.

$$\begin{aligned} & (\exists k \in *N \setminus \{0\}) (\exists x_0, x_1, \dots, x_k \in *X) (\exists b_0, b_1, \dots, b_{k-1} \in *B) \\ & (x_0 \cap b_0 \neq \emptyset \wedge b_0 \cap x_1 \neq \emptyset \wedge x_1 \cap b_1 \neq \emptyset \wedge b_1 \cap x_2 \neq \emptyset \wedge \dots \end{aligned}$$

$$\wedge x_{k-1} \cap b_{k-1} \neq \emptyset \wedge b_{k-1} \cap x_k \neq \emptyset) \quad (2)$$

In this case, the *length* $|*P|$ equals $k \in *N \setminus \{0\}$; in general, k is now a positive hypernatural number. We therefore call $*P$ a *hyperfinite path*. To view this fact in terms of an ultrapower construction of $*G = [G_n]$, note that the G_n may be finite graphs growing in size or indeed be conventionally infinite graphs. Thus, $*P$ may have an unlimited length, that is, its length may be a member of $*N \setminus N$.

A standard loop is defined as is a standard path except that the first and last nodes are required to be the same. Upon applying transfer, we get the following definition of a *nonstandard loop*.

$$\begin{aligned} & (\exists k \in *N \setminus \{0\}) (\exists x_0, x_1, \dots, x_{k-1} \in *X) (\exists b_0, b_1, \dots, b_{k-1} \in *B) \\ & (x_0 \cap b_0 \neq \emptyset \wedge b_0 \cap x_1 \neq \emptyset \wedge x_1 \cap b_1 \neq \emptyset \wedge b_1 \cap x_2 \neq \emptyset \wedge \dots \\ & \wedge x_{k-1} \cap b_{k-1} \neq \emptyset \wedge b_{k-1} \cap x_k \neq \emptyset) \wedge b_{k-1} \cap x_0 \neq \emptyset) \end{aligned} \quad (3)$$

5 Connected Nonstandard Graphs

A standard graph $G = \{X, B\}$ is called *connected* if, for every two nodes x and y in G , there is a finite path (1) terminating at those nodes, that is, $x_0 = x$ and $x_k = y$. Let \mathcal{C} denote the set of connected standard graphs, and let $\mathcal{P}(G)$ be the set of all finite paths in a given standard graph G . Also, for any $P \in \mathcal{P}(G)$, let $x_0(P)$ and $x_k(P)$ denote the first and last nodes of P in accordance with (1); k depends upon P . Then, the connectedness of G is defined symbolically by the truth of the following sentence to the right of \leftrightarrow .

$$G \in \mathcal{C} \leftrightarrow (\forall x, y \in X) (\exists P \in \mathcal{P}(G)) ((x_0(P) = x) \wedge (x_k(P) = y)) \quad (4)$$

By transfer, we obtain the definition of the set $*\mathcal{C}$ of all *connected* nonstandard graphs: For $*G = \{*X, *B\}$, for $*\mathcal{P}(*G)$ being the set of nonstandard paths $*P \in *G$, and for $x_0(*P)$ and $x_k(*P)$ being the first and last nonstandard nodes of $*P$, we have

$$*G \in *\mathcal{C} \leftrightarrow (\forall x, y \in *X) (\exists *P \in *\mathcal{P}(*G)) ((x_0(*P) = x) \wedge (x_k(*P) = y)) \quad (5)$$

Here, $k \in *N \setminus \{0\}$ as in (2).

Let us explicate this still further in terms of an ultrapower construction of $*G = [G_n]$ from an equivalence class of sequences of (possibly infinite) graphs, $\langle G_n \rangle$ being one of those sequences. With $*P = [P_n]$ denoting a nonstandard path obtained similarly from a representative sequence $\langle P_n \rangle$ of finite paths, P_n being in G_n , we let $x_0(P_n)$ and $x_k(P_n)$ denote the first and last nodes of P_n . (Thus, k also depends on n of course.) Then, $x_0(*P) = [x_0(P_n)]$ and $x_k(*P) = [x_k(P_n)]$ are the first and last nonstandard nodes of $*P$. (In (5), $*k$ is denoted by $k \in *N \setminus \{0\}$.) Then, $*G$ is called *connected* if and only if, given any nonstandard nodes $*x = [x_n]$ and $*y = [y_n]$ in $*G$, we have that, for almost all n , there exists a finite path P_n terminating at x_n and y_n . This can be restated by saying that there exists a hyperfinite path $*P$ in $*G$ terminating at $*x$ and $*y$.

Later on, we will need a special case of \mathcal{C} : Let \mathcal{C}_f denote the subset of \mathcal{C} consisting of all finite connected standard graphs $G = \{X, B\}$, where $|X|, |B| \in N \setminus \{0\}$. Then, $*\mathcal{C}_f$ is the subset of \mathcal{C} obtained by lifting \mathcal{C}_f through transfer to a subset of \mathcal{C} . In this case,

$$*G \in *\mathcal{C}_f \leftrightarrow (*G = \{X, B\} \in \mathcal{C}) \wedge (|X|, |B| \in N \setminus \{0\}). \quad (6)$$

We call such a $*G$ a nonstandard *hyperfinite* connected graph.

6 Nonstandard Subgraphs

If A and C are sets of nodes with $A \subseteq C$, we get upon transfer the following definition of a subset $*A$ of a set $*C$ of nonstandard nodes.

$$*A \subseteq *C \leftrightarrow (\forall x \in *A) (x \in *C)$$

Our purpose in this section is to define a nonstandard subgraph $*G_s$ of a given nonstandard graph $*G$. We let \mathcal{G} denote the set of all standard graphs. By definition, $G_s = \{X_s, B_s\}$ is a (node induced) subgraph of $G = \{X, B\} \in \mathcal{G}$ if $X_s \subseteq X$ and B_s is the set of those branches in B that are each incident to one or two nodes in X_s . Let us denote the set of all such subgraphs of G by $\mathcal{G}_s(G)$. The following symbolic definition of $G_s \in \mathcal{G}_s(G)$ uses tips and is thereby somewhat complicated because we wish to allow both self-loops and parallel branches.

$$G_s \in \mathcal{G}_s(G) \leftrightarrow$$

$$(\exists G_s = \{X_s, B_s\} \in \mathcal{G}) (\exists G = \{X, B\} \in \mathcal{G})$$

$$((X_s \subseteq X) \wedge (\forall b = \{t_b, \tilde{t}_b\} \in B_s) ((\exists x \in X_s) (t_b, \tilde{t}_b \in x)) \vee ((\exists x, y \in X_s) (t_b \in x \wedge \tilde{t}_b \in y)))$$

By transfer, we get the definition of a *nonstandard subgraph* $*G_s$ of a given nonstandard graph $*G$. In this case, $*\mathcal{G}$ denotes the set of all nonstandard graphs, and $*\mathcal{G}_s(*G)$ denotes the set of all nonstandard (nose induced) subgraphs of a given $*G \in *\mathcal{G}$.

$$*G_s \in *\mathcal{G}_s(*G) \leftrightarrow$$

$$(\exists *G_s = \{*X_s, *B_s\} \in *\mathcal{G}) (\exists *G = \{*X, *B\} \in *\mathcal{G})$$

$$((*X_s \subseteq *X) \wedge (\forall b = \{t_b, \tilde{t}_b\} \in *B_s) ((\exists x \in *X_s) (t_b, \tilde{t}_b \in x)) \vee ((\exists x, y \in *X_s) (t_b \in x \wedge \tilde{t}_b \in y)))$$

7 Nonstandard Trees

The symbols \mathcal{G} , $\mathcal{G}_s(G)$, \mathcal{C} , \mathcal{C}_f , and their nonstandard counterparts have been defined above. Now, let $\mathcal{L}(G)$ be the set of all loops in the standard graph G , and let \mathcal{T} be the set of all standard trees. Then, a tree T can be defined symbolically by

$$T \in \mathcal{T} \leftrightarrow (\exists T \in \mathcal{C}) (\neg(\exists L \in \mathcal{L}(T))) \quad (7)$$

To transfer this, we let $*\mathcal{L}(*G)$ be the set of all nonstandard loops in a given nonstandard graph $*G$, as defined by (3), and we let $*\mathcal{T}$ denote the set of *nonstandard trees* $*T$, defined as follows:

$$*T \in *\mathcal{T} \leftrightarrow (\exists *T \in *\mathcal{C}) (\neg(\exists *L \in *\mathcal{L}(*T))) \quad (8)$$

Next, let $\mathcal{T}_{sp}(G)$ be the set of all spanning trees in a given finite connected standard graph $G = \{X, B\}$. That T is such a spanning tree can be expressed symbolically as follows.

$$T \in \mathcal{T}_{sp}(G) \leftrightarrow$$

$$(\exists G = \{X, B\} \in \mathcal{C}_f) (\exists T = \{X_T, B_T\} \in \mathcal{T}) ((T \in \mathcal{G}_s(G)) \wedge (X = X_T)) \quad (9)$$

By transfer, we have the set $*\mathcal{T}_{sp}(*G)$ of all *spanning trees* of a given hyperfinite connected nonstandard graph $*G$, defined as follows:

$$*T \in *\mathcal{T}_{sp}(*G) \leftrightarrow$$

$$(\exists *G = \{*X, *B\} \in *\mathcal{C}_f) (\exists *T = \{*X_T, *B_T\} \in *\mathcal{T}) ((*T \in *\mathcal{G}_s(*G)) \wedge (*X = *X_T)) \quad (10)$$

8 Some Numerical Formulas

With these symbolic definition in hand, we can now lift some standard formulas regarding numbers of nodes and branches into a nonstandard setting. For example, if p , q , and r are the number of nodes, the number of branches, and the cyclomatic number respectively of a given connected finite graph, then $r = q - p + 1$. Symbolically, this can be stated as follows. Again, we use the notation $G = \{X, B\}$ for a graph and $T = \{X_T, B_T\}$ for a tree.

$$(\forall p, q, r \in \mathbf{N}) (\forall G \in \mathcal{C}_f) (\forall T \in \mathcal{T}_{sp}(G))$$

$$((p = |X| \wedge q = |B| \wedge r = |B| - |B_T|) \rightarrow (r = q - p + 1))$$

By transfer, we obtain the following formula in hypernatural numbers.

$$(\forall p, q, r \in {}^*\mathbf{N}) (\forall {}^*G \in {}^*\mathcal{C}_f) (\forall {}^*T \in {}^*\mathcal{T}_{sp}({}^*G))$$

$$((p = |{}^*X| \wedge q = |{}^*B| \wedge r = |{}^*B| - |{}^*B_T|) \rightarrow (r = q - p + 1))$$

Another example of such a lifting concerns the radius R and diameter D of a finite connected graph G . It is a fact that $R \leq D \leq 2R$ [3, page 37], [4, pages 20-21]. Again, we need to express ideas symbolically.

Let $A \subset \mathbf{N}$ be such that $|A| \in \mathbf{N}$ (i.e., A is a finite subset of \mathbf{N}). We use the symbols $\bar{a} = \max A$ and $\underline{a} = \min A$ as abbreviations for the following sentences.

$$\bar{a} = \max A \leftrightarrow (\forall c \in A) (\exists \bar{a} \in A) (\bar{a} \geq c)$$

$$\underline{a} = \min A \leftrightarrow (\forall c \in A) (\exists \underline{a} \in A) (\underline{a} \leq c)$$

This transfers to

$$\bar{a} = \max {}^*A \leftrightarrow (\forall c \in {}^*A) (\exists \bar{a} \in {}^*A) (\bar{a} \geq c)$$

$$\underline{a} = \min {}^*A \leftrightarrow (\forall c \in {}^*A) (\exists \underline{a} \in {}^*A) (\underline{a} \leq c),$$

where now *A is a hyperfinite subset of ${}^*\mathbf{N}$ and \bar{a} and \underline{a} are hypernatural numbers. *A does have a maximum element and a minimum element so that these definitions are valid [5, pages 149-150].

Now, for a given finite connected graph $G \in \mathcal{C}_f$, let \mathcal{P}_x denote the set of all paths P_x starting at x . The length $|P_x|$ of any $P_x \in \mathcal{P}_x$ is the number of branches in P_x . Also, let $\mathcal{E}(G)$ be the set of eccentricities of the nodes in $G = \{X, B\}$. Symbolically, we have

$$e_x \in \mathcal{E}(G) \leftrightarrow (\exists x \in X) (\forall P_x \in \mathcal{P}_x) (e_x = \max\{|P_x| : P_x \in \mathcal{P}_x\}).$$

So, for a hyperfinite connected graph $*G = \{*X, *B\} \in *\mathcal{C}_f$, we have by transfer

$$e_x \in \mathcal{E}(*G) \leftrightarrow (\exists x \in *X) (\forall *P_x \in *\mathcal{P}_x) (e_x = \max\{|*P_x| : *P_x \in *\mathcal{P}_x\}),$$

where now $\mathcal{E}(*G)$ is the set of hypernatural eccentricities in $*G$ and $*P_x$ is any nonstandard hyperfinite path in $*G$ starting at the nonstandard node x .

Then, for any $G = \{X, B\} \in \mathcal{C}_f$, the radius $R(G)$ is defined by

$$(\forall e_x \in \mathcal{E}(G)) (\exists R(G) \in \mathbf{N}) (R(G) = \min\{e_x : x \in X\}),$$

which by transfer gives the following definition of the hypernatural radius $R(*G) \in *\mathbf{N}$ of any $*G = \{*X, *B\} \in *\mathcal{C}_f$:

$$(\forall e_x \in *\mathcal{E}(G)) (\exists R(*G) \in *\mathbf{N}) (R(*G) = \min\{e_x : x \in *X\}),$$

Similarly, the diameter $D(G)$ of G is defined by

$$(\forall e_x \in \mathcal{E}(G)) (\exists D(G) \in \mathbf{N}) (D(G) = \max\{e_x : x \in X\}),$$

which by transfer gives the hypernatural diameter $D(*G) \in *\mathbf{N}$ of $*G$.

$$(\forall e_x \in \mathcal{E}(*G)) (\exists D(*G) \in *\mathbf{N}) (D(*G) = \max\{e_x : x \in *X\}),$$

So, we have the following sentence for the standard result:

$$(\forall G \in \mathcal{C}_f) (R(G) \leq D(G) \leq 2R(G)),$$

which by transfer yields the nonstandard result

$$(\forall G \in *\mathcal{C}_f) (R(*G) \leq D(*G) \leq 2R(*G)).$$

9 Eulerian Graphs

A finite trail is defined much as a finite path is defined except that the condition that all the nodes be distinct is relaxed; however, branches are still required to be distinct. Thus, the truth of the following sentence defines a *trail* T in a finite graph $G = \{X, B\}$, with T having two or more branches. This time, we use the notation $b_m = \{t_m, \tilde{t}_m\}$ to designate the two tips of the m -th branch. (Remember that \doteq denotes a shorting between tips.)

$$(\exists k \in \mathbf{N} \setminus \{0\}) (\exists b_0, b_1, \dots, b_k \in B) (\forall m \in \{0, \dots, k-1\}) (\tilde{t}_m \doteq t_{m+1})$$

That B is a set insures that the branches b_0, b_1, \dots, b_k are all distinct. On the other hand, this sentence allows nodes to repeat in a trail.

For a *closed trail*, we have the truth of the following sentence as its definition.

$$(\exists k \in \mathbf{N} \setminus \{0, 1\}) (\exists b_0, b_1, \dots, b_k \in B) (\forall m \in \{0, \dots, k-1\}) (\tilde{t}_m \doteq t_{m+1}) \wedge (\tilde{t}_k \doteq t_0)$$

With Q denoting a trail, we denote the set of branches in Q by $B(Q)$. Also, we let $\mathcal{Q}(G)$ denote the set of closed trails in a given graph $G = \{X, B\}$.

By attaching asterisks as usual, we obtain by transfer the corresponding sentence for trails in a given nonstandard graph $*G = \{*X, *B\}$. Thus, a *nonstandard trail* $*Q$ is defined by the truth of the following sentence, where now $b_m = \{t_m, \tilde{t}_m\}$ is a nonstandard branch with the nonstandard tips t_m and \tilde{t}_m !

$$(\exists k \in *\mathbf{N} \setminus \{0\}) (\exists b_0, b_1, \dots, b_k \in *B) (\forall m \in \{0, \dots, k-1\}) (\tilde{t}_m \doteq t_{m+1})$$

A similar expression holds for a *nonstandard closed trail* (just append $\tilde{t}_k \doteq t_0$). With $*Q$ denoting a nonstandard trail, we denote the set of nonstandard branches in $*Q$ by $*B(*Q)$. Also, we let $*\mathcal{Q}(*G)$ denote the set of nonstandard closed trails in a given nonstandard graph $*G = \{*X, *B\}$.

A finite graph $G = \{X, B\} \in \mathcal{C}_f$ is called *Eulerian* if it contains a closed trail that meets every node of X . The degree d_x of $x \in X$ is the natural number $d_x = |\{b \in B : b \triangleright x\}|$. The nonstandard version of this definition is as follows: Given $*G = \{*X, *B\} \in *\mathcal{C}_f$, for any $x \in *X$, the *degree* of x is $d_x = |\{b \in *B : b \triangleright x\}|$. In this case, d_x may be an unlimited

hypernatural number when $*G$ is a hyperfinite graph. However, $*G$ might happen to be a finite graph $G \in \mathcal{C}_f$, which from the point of view of an ultrapower construction can occur if all the G_n for $*G = [G_n]$ are the same finite graph $G \in \mathcal{G}_f$; in this case, d_x will be a natural number for all $x \in *X$.

Let \mathcal{E}_u (resp. $*\mathcal{E}_u$) denote the set of all standard Eulerian graphs (resp. nonstandard Eulerian graphs). Then, Eulerian graphs can be defined by asserting the truth of the following sentence to the right of \leftrightarrow , where as usual $G = \{X, B\}$.

$$G \in \mathcal{E}_u \leftrightarrow (\exists Q \in \mathcal{Q}(G)) ((\forall b \in B) (b \in B(Q)))$$

By transfer, the truth of the following right-hand side defines *nonstandard Eulerian graphs*. Now, $*G = \{*X, *B\}$.

$$*G \in *\mathcal{E}_u \leftrightarrow (\exists *Q \in *\mathcal{Q}(G)) ((\forall b \in *B) (b \in *B(*Q)))$$

Now an ancient theorem of Euler asserts that a graph G is Eulerian if and only if the degree of every node of $G = \{X, B\}$ is an even natural number. Symbolically, this can be stated as follows.

$$G \in \mathcal{E}_u \leftrightarrow (\forall x \in X) (d_x/2 \in \mathbf{N})$$

Transferring this, we get the nonstandard version of this theorem of Euler:

$$*G \in *\mathcal{E}_u \leftrightarrow (\forall x \in *X) (d_x/2 \in *\mathbf{N})$$

10 Hamiltonian Graphs

In this section, it is assumed that each graph $G = \{X, B\}$ is connected and finite and has at least three nodes (i.e., $|X| \geq 3$). A graph G is called *Hamiltonian* if it contains a loop that meets every node in the graph. Let $\mathcal{L}(G)$ denote the set of all loops in G . Also, for any loop $L \in \mathcal{L}(G)$, let $X(L)$ denote the node set of L . Then, a Hamiltonian graph $G = \{X, B\} \in \mathcal{C}_f$ is also defined by the truth of the following sentence to the right of \leftrightarrow . The set of Hamiltonian graphs will be denoted by \mathcal{H} , where $\mathcal{H} \subset \mathcal{C}_f$.

$$G \in \mathcal{H} \leftrightarrow (\exists L \in \mathcal{L}(G)) ((\forall x \in X) (x \in X(L)))$$

By transfer of this sentence, we define a *nonstandard Hamiltonian graph* as follows, where now ${}^*\mathcal{H}$ is the set of nonstandard hyperfinite Hamiltonian graphs, $\mathcal{L}({}^*G)$ is the set of all nonstandard loops in *G , ${}^*X(L)$ is the set of nonstandard nodes in $L \in \mathcal{L}({}^*G)$, and ${}^*G = \{{}^*X, {}^*B\}$.

$${}^*G \in {}^*\mathcal{H} \leftrightarrow (\exists L \in \mathcal{L}({}^*G)) ((\forall x \in {}^*X) (x \in {}^*X(L))).$$

A simple criterion for a graph $G = \{X, B\}$ to be Hamiltonian is that the degree d_x of each of its nodes x be no less than one half of $|X|$ [1, page 134], [3, page 79]. Symbolically, this condition is expressed as follows:

$$((\forall x \in X) (d_x \geq |X|/2)) \rightarrow G \in \mathcal{H}.$$

By transfer, we get the following criterion for a nonstandard Hamiltonian graph.

$$((\forall x \in {}^*X) (d_x \geq |{}^*X|/2)) \rightarrow {}^*G \in {}^*\mathcal{H}.$$

A more general criterion due to Ore asserts that $G = \{X, B\}$ is Hamiltonian if, for every pair of nonadjacent nodes x and y , $d_x + d_y \geq |X|$ [1, page 134], [3, page 79]. Symbolically, we have

$$((\forall x, y \in X) ((\neg(x \diamond y)) \rightarrow (d_x + d_y \geq |X|))) \rightarrow G \in \mathcal{H},$$

which by transfer becomes

$$((\forall x, y \in {}^*X) ((\neg(x \diamond y)) \rightarrow (d_x + d_y \geq |{}^*X|))) \rightarrow {}^*G \in {}^*\mathcal{H}.$$

Still more general is Posa's theorem [1, page 132], [3, page 79]: If, for every $j \in \mathbf{N}$ satisfying $1 \leq j < |X|/2$, the number of nodes of degree no larger than j is less than j , then the graph $G = \{X, B\}$ is Hamiltonian. The following symbolic sentence states this criterion.

$$((\forall j \in \mathbf{N}) (\forall x \in X) ((1 \leq j < |X|/2) \rightarrow (|\{x \in X : d_x \leq j\}| < j))) \rightarrow G \in \mathcal{H}$$

By transfer the following criterion holds for nonstandard graphs ${}^*G = \{{}^*X, {}^*B\}$.

$$((\forall j \in {}^*\mathbf{N}) (\forall x \in {}^*X) ((1 \leq j < |{}^*X|/2) \rightarrow (|\{x \in {}^*X : d_x \leq j\}| < j))) \rightarrow {}^*G \in {}^*\mathcal{H}$$

11 A Coloring Theorem

A simple graph-coloring theorem that is not restricted to planar graphs asserts that, if the largest of the degrees for the nodes of a graph $G = \{X, B\}$ is k , then the graph is $(k + 1)$ -colorable [6, page 82].⁵ To express this symbolically, first let $M(X, \mathbf{N}_{k+1})$ denote the set of all functions that map a set X into the set \mathbf{N}_{k+1} of those natural numbers j satisfying $1 \leq j \leq k + 1$. Then, the following restates this theorem for the given graph G .

$$(\exists k \in \mathbf{N}) (\forall x, y \in X)$$

$$((d_x \leq k) \rightarrow ((\exists f \in M(X, \mathbf{N}_{k+1})) ((x \diamond y) \rightarrow (f(x) \neq f(y)))))$$

To transfer this, we first let $*M(*X, \mathbf{N}_{k+1})$ be the set of all internal functions mapping the enlargement $*X$ into \mathbf{N}_{k+1} . Then, this theorem is transferred to nonstandard graphs simply by appending asterisks, as usual:

$$(\exists k \in \mathbf{N}) (\forall x, y \in *X)$$

$$((d_x \leq k) \rightarrow ((\exists *f \in *M(*X, \mathbf{N}_{k+1})) ((x \diamond y) \rightarrow (*f(x) \neq *f(y)))))$$

Note here that the assumption of a natural-number bound k on the degrees of all the nonstandard nodes has been maintained. This conforms to the fact that the enlargement of the finite set \mathbf{N}_{k+1} is \mathbf{N}_{k+1} . As a consequence, the conclusion remains strong after transfer.

On the other hand, we could generalize this transferred theorem as follows: In terms of an ultrapower construction, we could replace \mathbf{N}_{k+1} by an internal set $*\mathbf{N}_{k+1}$ obtained from a sequence $\langle \mathbf{N}_{k_n+1} : n \in \mathbf{N} \rangle$ of finite sets \mathbf{N}_{k_n+1} , one set for each G_n with regard to $*G = [G_n]$. But then, our conclusion would be weakened to a coloring with a hypernatural number $*k = [k_n]$ of colors.

12 A Final Comment

Undoubtedly, other standard results for graphs can be lifted in this way to nonstandard settings.

⁵There exists a function f that assigns to each node one of $k + 1$ colors such that no two adjacent nodes have the same color.

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