# NETS AND COMPACTNESS

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In a uniform space completeness can be defined in terms of the convergence of each member of a particular class of nets. It will be shown in this paper that compactness for an arbitrary topological space can be characterized in the same way. Covering properties associated with certain other nets will also be considered in this paper.

Terms and expressions not defined in this paper will have the same meaning as in Kelley [5]. To save space only a few of the terms more directly related to our topic will be defined here.

Let M be a nonempty set with null set N and let $\mathfrak{J}$ be a topology for M. Denote complementation with respect to M by c; thus cA = M - A for $A \subset M$.

Let D be a nonempty set and let $\geq$ direct D. Let I be a net that maps D into M. For brevity let us agree that I is eventually in $A \subset M$ iff there is an element m of D such that if $n \in D$ and $n \geq m$ then $I_n \in A$.

Definition. A topological space is compact iff every open cover has a finite subcover.

Definition. A net I is a $\mathfrak{J}$-net (or C-net) iff for each $A \in \mathfrak{J}$ and every point $x \in A$

1. there is $B \in \mathfrak{J}$ such that $x \in B \subset A$ and

2. I is eventually in B or eventually in cB.

It is obvious that I is a $\mathfrak{J}$-net iff X is a closed set and $x \in cX$ imply there is a closed set Y such that $X \subset Y$, $x \in cY$ and I is eventually in Y or eventually in cY.

Theorem 1. The topological space $(M,\mathfrak{J})$ is compact iff every $\mathfrak{J}$-net in M converges.

Proof. Let the space be compact and let I be a $\mathfrak{J}$-net in M. If possible, let I not converge. Then $x \in M$ implies there is an open set A containing x such that I is not eventually in A. Now there is an open set $B_x$ such that $x \in B_x \subset A$ and I is eventually in $B_x$ or eventually in $cB_x$. Hence I is eventually in $cB_x$. Let $\mathfrak{B} = \{B_x : x \in M\}$. Then $\mathfrak{B}$ is an open cover and so there is a finite number $B_1, \ldots, B_n$, of members of $\mathfrak{B}$, which covers M. But I is eventually in each $cB_i$, $i = 1, \ldots, n$ and so I is eventually in their intersection which is the empty set N and this is a contradiction.

To prove the converse, assume every $\mathfrak{J}$-net converges and let $\mathfrak{F}$ be a family of closed sets with the finite intersection property. Then the family $\mathfrak{A}$ of all finite intersections of members of $\mathfrak{F}$ is directed by $\subset$. Choosing a point $I_A$ from each A in $\mathfrak{A}$ we find $I = \{I_A : A \in \mathfrak{A}\}$ is a net in M. Suppose $\mathfrak{A}$ has empty intersection. Let X be a closed set and let $x \in cX$. Now there is $A \in \mathfrak{A}$ such that $x \in cA$ and so $Y = A \cup X$ is a closed set such that $x \in cY$, $X \subset Y$ and I is eventually in Y. Hence I is a $\mathfrak{J}$-net and so converges to some point y. It is easily verified that y is in each member of $\mathfrak{A}$ which is a contradiction and this proves the theorem.

Definition. A point x is a cluster point of a net I iff I is eventually in the complement of no neighborhood of x.

Theorem 2. A net I is a C-net iff I converges to each of its cluster points.

Proof. Let I be a C-net and let x be a cluster point of I. If A is an open set containing x then there is an open set B such that $x \in B \subset A$ and I is eventually in B or eventually in cB. Hence I is eventually in B and so is eventually in A.

To prove the converse assume I is a net that converges to each of its cluster points. If A is an open set and $x \in A$ then x is either a cluster point or not a cluster point of I. In the former case $x \in A \subset A$ and I is eventually in A. In the latter case there is an open set E such that $x \in E$ and I is eventually in cE. Then $B = A \cap E$ is open, $x \in B \subset A$ and I is eventually in cB.

Corollary. A topological space is compact iff every C-net has a cluster point. Hence a topological space is compact iff every C-net has a convergent subnet.

Corollary. In a uniform space every Cauchy net is a C-net and so compactness implies completeness.

One way of defining Cauchy nets and completeness for arbitrary topological spaces is shown in [7]. Here, too, every Cauchy net turns out to be a C-net. Hence we can simply say compactness implies completeness.

Every convergent net in a uniform space is a Cauchy net. Hence in a compact uniform space every C-net is a Cauchy net and so these two kinds of nets are identical. If a C-net in a uniform space is not a Cauchy net then the space is not compact.

It is easy to prove the next theorem.

Theorem 3. A net with no cluster points is a C-net. Hence if a net I is not a C-net then I has a cluster point.

Corollary. A topological space is compact iff every net has a cluster point.

The result in this corollary is well known and has been deduced by other methods.

Corollary. If a net I in a uniform space is not a C-net then I has a convergent Cauchy subnet. Hence a uniform space is compact iff it is complete and every C-net has a Cauchy subnet. Also, a uniform space is totally bounded iff every C-net has a Cauchy subnet.

A net I is a universal net iff $A \subset M$ implies I is eventually in A or eventually in cA. Obviously a universal net converges to each of its cluster points and hence a universal net is a C-net. By virtue of the axiom of choice every C-net has a universal subnet and so we can say a space is compact iff every universal net converges or every universal net has a cluster point.

If I is a C-net and J is a subnet of I then J is also a C-net.

Let I be a net which is such that A is a neighborhood of x implies there is a neighborhood B of x, $B \subset A$ and I is eventually in B or eventually in cB. Then I converges to each of its cluster points and so is a C-net. That every C-net satisfies the conditions imposed on I is obvious. Hence a net J is a C-net iff A is a neighborhood of x implies there is a neighborhood B of x such that $B \subset A$ and J is eventually in B or eventually in cB.

Lemma. If a net I in a compact space has one and only one cluster point x then I converges to x.

Proof. Let A be an open set containing x. If $y \in cA$ then there is an open neighborhood $B_y$ of y such that I is eventually in $cB_y$. Now $\mathfrak{B} = \{B_y: y \in cA\}$ is an open cover of cA and cA is compact. Hence a finite subfamily $B_1, \ldots, B_n$ of $\mathfrak{B}$ covers cA. But I is eventually in each of $cB_1, \ldots, cB_n$ and so is eventually in their intersection which is a subset of A. Hence I converges to x.

The next theorem now follows easily.

Theorem 4. If a net I in a compact space has only one cluster point then I is a C-net.

Theorem 5. A convergent net in a Hansdorff space is a C-net.

Proof. Let a net I converge to a point x in a Hansdorff space. If y is a point of the space distinct from x then there are disjoint neighborhoods A and B for x and y respectively and I is eventually in A. Hence x is the only cluster point of I. Consequently, I converges to each of its cluster points. This completes the proof.

Let us call a net I a C-subnet when I is both a C-net and a subnet of some net. The following corollaries are then immediate.

Corollary. In a Hansdorff space every net, which is not a C-net, has a convergent C-subnet.

Corollary. In a Hansdorff space every net has a C-subnet.

Corollary. A Hansdorff space is compact iff every net has a convergent C-subnet.

Let $(M,\mathfrak{J})$ be a topological space, A a subset of M and $\mathfrak{U}$ the relativization of $\mathfrak{J}$ to A. Since A is compact iff $(A,\mathfrak{U})$ is compact we see that A is compact iff every C-net in A converges in the topological space $(A,\mathfrak{U})$.

Theorem 5 can be used to give a new proof of the well-known result that a compact subset of a Hansdorff space is closed. Let A be a compact subset of a Hansdorff space and let x be a point of the closure of A. There is then a net I in A converging to x. Hence I is a C-net and therefore $x \in A$.

Let $(M,\mathfrak{J})$ be a topological space, $(D,\leq)$ a directed set and $I:D \to M$ a C-net in $(M,\mathfrak{J})$. Let $A \subset M$ and $\mathfrak{U}$ the relativization of $\mathfrak{J}$ to A. Suppose I has the property that for each $m \in D$ there is $n \in D$ such that $n \geq m$ and $I_n \in A$. Take $E = \{n: n \in D, I_n \in A\}$. Define $J: E \to M$ by $J_n = I_n$ for

$n \in E$. Then $(E,\leq)$ is a directed set and $J$ is a C-subnet of $I$ in $(M,\mathfrak{J})$. Also $J$ is a C-net in $(A,\mathfrak{U})$.

Theorem 6. Let $(M,\mathfrak{J})$ be a topological space, $A$ a closed subset of $M$ and $\mathfrak{U}$ the relativization of $\mathfrak{J}$ to $\mathfrak{U}$. If $I$ is a $\mathfrak{U}$-net then $I$ is a $\mathfrak{J}$-net.

Proof. If $X \in \mathfrak{J}$ then $Y = (X \cap cA) \in \mathfrak{J}$ and $Z = (X \cap A) \in \mathfrak{U}$. Hence $x \in Y$ implies $x \in Y \subset X$ and $I$ is eventually in $cY$. If $x \in Z$ then there is $B \in \mathfrak{U}$ such that $x \in B \subset Z$ and $I$ is eventually in $B$ or eventually in $cB \cap A$. But $B = A \cap F$ for some $F \in \mathfrak{J}$. So $x \in Z$ implies there is $G = (F \cap X) \in \mathfrak{J}$ such that $x \in G \subset X$ and $I$ is eventually in $G$ or eventually in $cG$.

Corollary. A closed subset of a compact space is compact.

Let $(M,\mathfrak{J})$ and $(P,\mathfrak{U})$ be topological spaces and let $f\colon M \to P$ be continuous. If $I$ is a C-net in $M$ then $f \circ I$ need not necessarily be a C-net in $P$ even if $f$ is an open continuous map. But if $M$ is compact, $P$ is Hausdorff and $f$ is onto then the image $f \circ I$ of a C-net in $M$ is a C-net in $P$.

Definition. Let $\mathfrak{M}$ denote the family of all subsets of $M$. A net $I$ in $M$ is a $\mathfrak{M}$-net iff $A \in \mathfrak{M}$ and $x \in A$ imply there is $B \in \mathfrak{M}$ such that $x \in B \subset A$ and $I$ is eventually in $B$ or eventually in $cB$.

It is easily verified that a net $I$ is a $\mathfrak{M}$-net iff $X \in \mathfrak{M}$ and $x \in cX$ imply there is $Y \in \mathfrak{M}$ such that $x \in cY$, $X \subset Y$ and $I$ is eventually in $Y$ or eventually in $cY$.

Definition. The space $(M,\mathfrak{J})$ is $\mathfrak{M}$-compact iff every family of subsets of $M$, the interiors of whose members cover $M$, has a finite subcover.

Theorem 7. The space $(M,\mathfrak{J})$ is $\mathfrak{M}$-compact iff every $\mathfrak{M}$-net has a cluster point.

This theorem can be proved in the same way as theorem 1.

Lemma. Every $\mathfrak{M}$-net has a cluster point iff every C-net has a cluster point.

Proof. If every $\mathfrak{M}$-net has a cluster point but a C-net has none, then it is easily seen that the C-net is a $\mathfrak{M}$-net and so has a cluster point. The converse is obvious.

Theorem 8. The space $(M,\mathfrak{J})$ is compact iff every $\mathfrak{M}$-net has a cluster point.

Corollary. Compactness and $\mathfrak{M}$-compactness are identical.

It is also obvious from their definitions that compactness and $\mathfrak{M}$-compactness are identical. This is related to the fact that the Kuratowski closure function defining a topology is idempotent. But the situation can change if the function defining the space is no longer idempotent as in Fréchet space, Čech space or some of the abstract spaces studied by Hammer [ 3].

A universal net is a $\mathfrak{M}$-net. A C-net may not be a $\mathfrak{M}$-net and a $\mathfrak{M}$-net may not be a C-net. A subnet of a $\mathfrak{M}$-net is also a $\mathfrak{M}$-net. The image of a $\mathfrak{M}$-net under an open continuous map need not be a $\mathfrak{M}$-net.

A net I is a C-net iff A is open and $x \in A$ imply there is B open such that $x \in B$ and I is eventually in $B \cap A$ or eventually in cB. Also a net I is a C-net iff A is open and $x \in A$ imply there is B open such that $x \in B$ and I is eventually in $A \cap B$ or eventually in $c(A \cap B)$. Obviously $\mathfrak{M}$-nets can also be characterized in a similar way.

Let $\mathfrak{B}$ be a base for the topology $\mathfrak{J}$. Let us call a net I a $\mathfrak{B}$-net iff $A \in \mathfrak{B}$ and $x \in A$ imply there is $B \in \mathfrak{B}$ such that $x \in B \subset A$ and I is eventually in B or eventually in cB. Then every cover of M by members of $\mathfrak{B}$ has a finite subcover iff every $\mathfrak{B}$-net converges. It is also obvious that a net is a C-net iff it is a $\mathfrak{B}$-net, and that a $\mathfrak{B}$-net can be defined in a few other equivalent ways.

Definition. Let $\mathfrak{S}$ be a subbase for the topology $\mathfrak{J}$. A net I is a $\mathfrak{S}'$-net iff $A \in \mathfrak{J}$ and $x \in A$ imply there is $S \in \mathfrak{S}$ such that $x \in S$ and I is eventually in $S \cap A$ or eventually in cS.

The following theorem can then be proved.

Theorem 9. Every cover of M by members of $\mathfrak{S}$ has a finite subcover iff every $\mathfrak{S}'$-net converges.

For convenience let us say that (1) the space M is $\mathfrak{S}$-compact iff every cover of M by members of $\mathfrak{S}$ has a finite subcover (2) x is a $\mathfrak{S}$-cluster point of a net I iff $x \in S \in \mathfrak{S}$ imply I is not eventually in cS (3) a net I $\mathfrak{S}$-converges to a point-x iff $x \in S \in \mathfrak{S}$ imply I is eventually in S and (4) a net I is a $\mathfrak{S}$-net iff $A \in \mathfrak{S}$ and $x \in A$ imply there is $B \in \mathfrak{S}$ such that $x \in B$ and I is eventually in $A \cap B$ or eventually in cB. The following results can then be proved.

Lemma. A net $\mathfrak{S}$-converges to x iff it converges to x.

Theorem 10. A space is $\mathfrak{S}$-compact iff every $\mathfrak{S}$-net $\mathfrak{S}$-converges.

Theorem 11. A net I is a $\mathfrak{S}$-net iff I $\mathfrak{S}$-converges to each of its $\mathfrak{S}$-cluster points.

Corollary. A net without any $\mathfrak{S}$-cluster points is a $\mathfrak{S}$-net. Hence if a net is not a $\mathfrak{S}$-net then it has a $\mathfrak{S}$-cluster point.

Corollary. A space is $\mathfrak{S}$-compact iff every $\mathfrak{S}$-net has a $\mathfrak{S}$-cluster point. Hence a space is $\mathfrak{S}$-compact iff every net has a $\mathfrak{S}$-cluster point.

Theorem 12. A net I is a $\mathfrak{S}$-net iff I is a $\mathfrak{S}'$-net.

A cluster point of a net is obviously a $\mathfrak{S}$-cluster point of it and so a $\mathfrak{S}$-net converges to each of its cluster points and a $\mathfrak{S}$-net is a C-net. A space is $\mathfrak{S}$-compact iff each $\mathfrak{S}$-net has a cluster point.

A universal net is a $\mathfrak{S}$-net and so $\mathfrak{S}$-compactness of a space implies compactness. That compactness implies $\mathfrak{S}$-compactness is evident; this can also be seen from the fact that a $\mathfrak{S}$-net is a C-net.

Let us say a net I is a $\mathfrak{S}_1$-net iff $A \in \mathfrak{S}$, $x \in A$ imply there is $B \in \mathfrak{S}$ such that $x \in B \subset A$ and I is eventually in B or eventually in cB. It is clear that a $\mathfrak{S}_1$-net is a $\mathfrak{S}$-net and so $\mathfrak{S}$-compactness implies every $\mathfrak{S}_1$-net converges. But a universal net is a $\mathfrak{S}_1$-net and so a space is compact iff every $\mathfrak{S}_1$-net converges or every $\mathfrak{S}_1$-net has a cluster point.

Let us call a net I a $\mathfrak{S}_2$-net iff $A \in \mathfrak{S}$ and $x \in A$ imply there is $B \in \mathfrak{S}$ such that $x \in B$ and I is eventually in $A \cap B$ or eventually in $c(A \cap B)$. It is then easy to see that a $\mathfrak{S}$-net is a $\mathfrak{S}_2$-net. If $\mathfrak{A} = \{S \cap T : S \in \mathfrak{S},\ T \in \mathfrak{J}\}$ then every cover of M by members of $\mathfrak{A}$ has a finite subcover implies every $\mathfrak{S}_2$-net converges. A $\mathfrak{S}_2$-net converges to each of its cluster points and so is a C-net. Since a universal net is a $\mathfrak{S}_2$-net we find that a space is compact iff every $\mathfrak{S}_2$-net converges.

It is also easy to see that a net I is a C-net if A is open and $x \in A$ imply there is $S \in \mathfrak{S}$ such that $x \in S$ and I is eventually in $A \cap S$ or eventually in $c(A \cap S)$. The axiom of choice enables us to say that every C-net converges if every $\mathfrak{S}$-net does so.

Next let $\mathfrak{A}$ be an open cover of the topological space $(M,\mathfrak{J})$. For the sake of brevity let us agree to say that (1) the space is $\mathfrak{A}$-compact iff every cover of M by members of $\mathfrak{A}$ has a finite subcover (2) x is a $\mathfrak{A}$-cluster point of a net I iff $x \in A \in \mathfrak{A}$ imply I is not eventually in cA and (3) a net I $\mathfrak{A}$-converges to x iff $x \in A \in \mathfrak{A}$ imply I is eventually in A.

Definition. A net I is a $\mathfrak{A}$-net iff $A \in \mathfrak{A}$ and $x \in A$ imply there is $B \in \mathfrak{A}$ such that $x \in B$ and I is eventually in $A \cap B$ or eventually in cB.

$\mathfrak{A}$ is a subbase of a certain topology $\mathfrak{J}_1$ for M and hence the following results are obvious.

Theorem 13. A net $\mathfrak{A}$-converges iff it converges in the topology $\mathfrak{J}_1$.

Theorem 14. The space is $\mathfrak{A}$-compact iff every $\mathfrak{A}$-net $\mathfrak{A}$-converges.

Theorem 15. A net is an $\mathfrak{A}$-net iff it $\mathfrak{A}$-converges to each of its $\mathfrak{A}$-cluster points.

Corollary. If a net has no $\mathfrak{A}$-cluster points then it is an $\mathfrak{A}$-net. Therefore every net which is not an $\mathfrak{A}$-net has an $\mathfrak{A}$-cluster point.

Corollary. The space is $\mathfrak{A}$-compact iff every $\mathfrak{A}$-net has an $\mathfrak{A}$-cluster point. So the space is $\mathfrak{A}$-compact iff every net has an $\mathfrak{A}$-cluster point.

A universal net is an $\mathfrak{A}$-net and so the space is $\mathfrak{A}$-compact iff every universal net $\mathfrak{A}$-converges. $\mathfrak{A}$ is a subbase for $\mathfrak{J}_1$ and every $\mathfrak{J}_1$-net $\mathfrak{A}$-converges iff every universal net $\mathfrak{A}$-converges. Thus $\mathfrak{A}$-compactness is equivalent to the compactness of $(M,\mathfrak{J}_1)$.

Definition. Let $\mathfrak{A}$ be an open cover of $(M,\mathfrak{J})$. A net I is an $\mathfrak{A}'$-net iff $X \in \mathfrak{J}$ and $x \in X$ imply there is $A \in \mathfrak{A}$ such that $x \in A$ and I is eventually in $A \cap X$ or eventually in $cA$.

Theorem 16. The space is $\mathfrak{A}$-compact iff every $\mathfrak{A}'$-net converges in $(M,\mathfrak{J})$.

Corollary. Every $\mathfrak{A}$-net $\mathfrak{A}$-converges iff every $\mathfrak{A}'$-net converges.

An $\mathfrak{A}'$-net converges to each of its cluster points as well as to each of its $\mathfrak{A}$-cluster points and so an $\mathfrak{A}'$-net is an $\mathfrak{A}$-net. An $\mathfrak{A}'$-net is a C-net and compactness implies $\mathfrak{A}$-compactness. A cluster point of a net is an $\mathfrak{A}$-cluster point of it and an $\mathfrak{A}$-net $\mathfrak{A}$-converges to each of its cluster points.

We can also define $\mathfrak{A}_1$ and $\mathfrak{A}_2$ nets in a way similar to those of $\mathfrak{S}_1$ and $\mathfrak{S}_2$ nets. It is easy to see what their properties would be. It is also obvious that a net I is a C-net if $X \in \mathfrak{J}$ and $x \in X$ implies there is $A \in \mathfrak{A}$ such that $x \in A$ and I is eventually in $A \cap X$ or eventually in $c(A \cap X)$.

Finally these considerations can be extended to an arbitrary cover of M. Let $\alpha$ be an arbitrary family of subsets of M such that each point of M is contained in a member of $\alpha$. Now $\alpha$ is the subbase for a topology $\mathfrak{U}$ for M. We can study covering properties of the topological space $(M,\mathfrak{U})$ in the same way as that of $(M,\mathfrak{J})$.

If $(M,d)$ is a pseudometric space and $\mathfrak{J}$ is the topology of the pseudometric d then we can use sequences to characterize compactness. For instance we can define a sequence I to be a C-sequence iff A is open and $x \in A$ implies there is an open set B such that $x \in B$ and I is eventually in $A \cap B$ or eventually in cB.

Theorem 17. A sequence I in a pseudometric space is a C-sequence iff I converges to each of its cluster points.

It is obvious that a sequence without any cluster points is a C-sequence. Hence if a sequence is not a C-sequence then it has a cluster point. Now a pseudometric space is campact iff every sequence has a cluster point. Thus we have proved

Theorem 18. A pseudometric space is compact iff every C-sequence converges.

We know that if a net has no cluster points then it is a $\mathfrak{M}$-net and so if a net is not a $\mathfrak{M}$-net then it has a cluster point. Defining a $\mathfrak{M}$-sequence in a way analogous to that of a $\mathfrak{M}$-net we find similarly that if a sequence is not a $\mathfrak{M}$-sequence then it has a cluster point.

Theorem 19. A pseudometric space is compact iff every $\mathfrak{M}$-sequence has a cluster point.

All of the results involving nets can also be expressed in terms of directed functions (as in McShane and Botts [ 6 ]), filters or filterbases (as in Bourbaki [ 1 ]). A nonempty family $\varphi$ of nonempty subsets of M is a filterbase iff A, B $\epsilon$ $\varphi$ imply there is C $\epsilon$ $\varphi$ such that $C \subset A \cap B$. A filter is a filterbase $\varphi$ such that $A \subset B$ and A $\epsilon$ $\varphi$ imply B $\epsilon$ $\varphi$. An ultrafilter is a maximal filter. A filterbase $\varphi$ may be defined to be eventually in a set A iff there is F $\epsilon$ $\varphi$ such that $F \subset A$. Then a C-filterbase can be defined to be a filterbase $\varphi$ which has the property that A is open and x $\epsilon$ A imply there is an open set B such that x $\epsilon$ B and $\varphi$ is eventually in $A \cap B$ or eventually in cB. Let us say a filterbase $\varphi$ converges to a point x iff $\varphi$ is eventually in each neighborhood of x. It is obvious what the other definitions and results would be. Directed functions may be handled in the same way.

We can also express the results in terms of families of sets with the finite intersection property. Let us say a family $\varphi$, of subsets of M, with the finite intersection property is a C-family iff A is open and x $\epsilon$ A imply there exist an open set B and a F $\epsilon$ $\varphi$ such that x $\epsilon$ B and $F \subset A \cap B$ or $F \subset cB$. If we define convergence of a C-family in the obvious way then we can say a topological space is compact iff every C-family converges. It is clear how to obtain other reseults of this sort.

We can also use filterbases all of whose members are closed sets. For instance a space is compact iff every C-filterbase, all of whose members are closed sets, converges. Other results of this kind are obvious.

Several other variations are also possible.

Let us say a point x is a cluster point of a filterbase $\varphi$ iff $\varphi$ is not eventually in the complement of any neighborhood of x. Let $f : M \to M'$ be a continuous onto map and let $(M, \mathfrak{J})$ and $(M', \mathfrak{J}')$ be topological spaces. If a filterbase $\varphi$ in M has a cluster point x $\epsilon$ M then f(x) is a cluster point of $f\varphi$. This fact can be used to prove that $(M', \mathfrak{J}')$ is compact if $(M, \mathfrak{J})$ is. For if $\varphi'$ is a C-filterbase in $M'$ then $f^{-1}\varphi'$ has a cluster point in M when $(M, \mathfrak{J})$ is compact and so $\varphi'$ has a cluster point in $M'$. An immediate consequence of this is the result that the coordinate spaces are compact if the product space is compact. If $\varphi'$ is a C-filterbase in $M'$ then $f^{-1}\varphi'$ need not be a C-filterbase in M.

Let us agree that a filterbase $\varphi$ is finer than a filterbase $\varphi'$ iff $F' \in \varphi'$ implies there is $F \in \varphi$ such that $F \subset F'$. It is easy to show that if $\varphi$ is a C-filterbase in the product space of a finite number of compact spaces then there is a filterbase $\varphi'$ in the product-space which has a cluster point and which is finer than $\varphi$; hence this product space is compact.

When we come to the product of an arbitrary number of compact spaces it appears C-filterbases by themselves are unable to show that the product space is compact. But this is hardly surprising since Kelley [ 4 ] has proved that the Tychonoff product theorem on compact spaces is equivalent to the axiom of choice. In this connection it may be noted that to prove the result that a space is compact iff every C-filterbase converges it is not necessary to use the axiom of choice while it was used in the proof of Theorem 1 where C-nets were employed.

The following results are immediate consequences of the Tychonoff product theorem. Let $(M,\mathfrak{J})$ be the product of the compact spaces $(M_j,\mathfrak{J}_j)$, $j \in J$ where J is an index set. Let I be a C-net in $(M,\mathfrak{J})$. Then the projection of I into each coordinate space converges. If a coordinate space $(M_j,\mathfrak{J}_j)$ is Hausdorff then the projection of I into this space is a C-net. Every C-net in the product space of compact uniform spaces must be a Cauchy net.

It is evident that a product space is compact iff each projection, of every C-net in the product space, converges.

Let us now consider a different kind of covering property.

Definition. A topological space $(M,\mathfrak{J})$ is $\alpha$-compact iff every family $\mathfrak{N}$, of subsets of M with the properties

1. the interiors of members of $\mathfrak{N}$ cover M

2. X is open implies there is $A \in \mathfrak{N}$ such that $X \subset A$ or $cX \subset A$,

has a finite subfamily which covers M.

Definition. A filterbase $\varphi$ in a topological space $(M,\mathfrak{J})$ is an $\alpha$-filterbase iff X is open implies $\varphi$ is eventually in X or eventually in cX.

Lemma. An $\alpha$-filterbase converges to each of its cluster points.

Lemma. A topological space is $\alpha$-compact iff every family $\mathfrak{N}$, of subsets of M with the finite intersection property and with the property that X is open implies there is $A \in \mathfrak{N}$ such that $A \subset X$ or $A \subset cX$, has the property that the closures of the members of $\mathfrak{N}$ have a nonempty intersection.

Theorem 20. A space is $\alpha$-compact iff every $\alpha$-filterbase converges.

Proof. Let every $\alpha$-filterbase converge and let $\mathfrak{N}$ be a family of sets with the finite intersection property and with the property that X is open implies there is $A \in \mathfrak{N}$ such that $A \subset X$ or $A \subset cX$. Then the family $\mathfrak{B}$ of all finite intersections of members of $\mathfrak{N}$ is an $\alpha$-filterbase and so has a cluster point. Hence the required result follows.

To prove the converse let us suppose that if $\mathfrak{N}$ is a family of sets with the finite intersection property and with the property that X is open implies there is $A \in \mathfrak{N}$ such that $A \subset X$ or $A \subset cX$ then the closures of members of $\mathfrak{N}$ have a nonempty intersection. Let $\varphi$ be an $\alpha$-filterbase. Then $\varphi$ has a cluster point.

Corollary. A space is $\alpha$-compact iff every $\alpha$-filterbase has a cluster point.

An ultrafilter is an $\alpha$-filterbase and so we can say a space is compact iff each $\alpha$-filterbase converges. Hence a space is compact iff it is $\alpha$-compact.

It has been shown by Thampuran [ 7 ] that given a topological space $(M,\mathfrak{J})$ there is a quasiuniform structure $\mathfrak{U}$, from which we can get back the topology $\mathfrak{J}$, such that Cauchy filterbases defined with respect to $\mathfrak{U}$ are identical with $\alpha$-filterbases. It can also be shown that starting from a given topological space we can find a syntopogenous structure $\mathfrak{S}$, from which the given topology can be derived, such that the compressed grills, of Császár [ 2 ], defined with respect to $\mathfrak{S}$ also coincide with $\alpha$-filterbases.

If we agree that the terms Cauchy filterbase and $\alpha$-filterbase are synonymous, that a space is complete iff every Cauchy filterbase converges and that a family $\mathfrak{N}$ of subsets of M contains small sets iff X is open implies there is $A \in \mathfrak{N}$ such that $A \subset X$ or $A \subset cX$ then we can say: a topological space is complete iff every family $\mathfrak{N}$ of sets which contains small sets and which has the finite intersection property is such that the closures of members of $\mathfrak{N}$ have a nonempty intersection. This is similar to the corresponding characterization of completeness for uniform spaces.

Among nets the analogue of a $\alpha$-filterbase is a net I which is such that X is open implies I is eventually in X or eventually in cX.

Let $(M,\mathfrak{J})$ be a topological space. Denote by $\tau$ the family of all subsets X of M such that X is open or cX is open.

Definition. A nonempty family $\varphi$ of subsets of M is a $\tau$-family iff $\varphi$ is a subfamily of $\tau$ and $\varphi$ has the finite intersection property.

Let us say $\varphi$ is a maximal $\tau$-family iff no $\tau$-family properly contains $\varphi$.

Lemma. $\varphi$ is a maximal $\tau$-family iff X is open implies $X \in \varphi$ or $cX \in \varphi$.

If $\varphi$ is a $\tau$-family there is an ultrafilter $\psi$ containing $\varphi$. If $\varphi'$ is the intersection of $\psi$ and $\tau$ then $\varphi'$ is a maximal $\tau$-family.

Theorem 21. Each $\tau$-family is contained in a maximal $\tau$-family.

Let us say a maximal $\tau$-family $\varphi$ is eventually in a subset A of M iff there is $F \in \varphi$ such that $F \subset A$. Define cluster points, convergence, etc., of $\tau$-families in the same way as for filterbases. Then a maximal $\tau$-family converges to each of its cluster points and all the $\alpha$-filterbases converge iff each maximal $\tau$-family converges. Therefore a space is $\alpha$-compact (and hence compact) iff each maximal $\tau$-family converges.

For brevity let us say a $\tau$-family $\varphi$ is a $C'$-family iff A is open and $x \in A$ imply there is $F \in \varphi$ such that F is open and $x \in F \subset A$ or F is closed and $x \in cF \subset A$. Then every C-filterbase converges iff every $C'$-family converges. Hence a space is compact iff every $C'$-family converges.

These considerations can be easily extended to bases, subbases etc. For instance the following results can be stated.

Definition. Let $(M,\mathfrak{J})$ be a topological space and let $\mathfrak{A}$ be a subfamily of $\mathfrak{J}$. A filterbase $\varphi$ in M is a $\beta$-filterbase iff $A \in \mathfrak{A}$ implies $\varphi$ is eventually in A or eventually in cA.

Let $(M,\mathfrak{J})$ be a topological space and let $\mathfrak{A}$ be an open cover of M. Let $\mathfrak{U}$ be the topology for M which has $\mathfrak{A}$ as subbase. Let us say a filterbase $\varphi$ is an $\mathfrak{A}$-filterbase iff $A \in \mathfrak{A}$ and $x \in A$ imply there is $B \in \mathfrak{A}$ such that $x \in B$ and $\varphi$ is eventually in $A \cap B$ or eventually in cB.

Theorem 22. In the space $(M,\mathfrak{U})$ every $\beta$-filterbase converges iff every $\mathfrak{A}$-filterbase converges.

Corollary. The space $(M,\mathfrak{U})$ is compact iff every $\beta$-filterbase converges in $(M,\mathfrak{U})$.

By the $\mathfrak{U}$-interior of a subset X of M we will mean the interior of X in the space $(M,\mathfrak{U})$ and by the interior of X we will mean the interior of X in the space $(M,\mathfrak{J})$.

Definition. Let $(M,\mathfrak{J})$ be a topological space and let $\mathfrak{A}$ be an open cover of M. Then M is $\beta$-compact iff every family $\mathfrak{R}$, of subsets of M with the properties (1) $A \in \mathfrak{A}$ implies there is $R \in \mathfrak{R}$ such that $A \subset R$ or $cA \subset R$ and (2) the $\mathfrak{U}$-interiors of members of $\mathfrak{R}$ cover M, has a finite subfamily which covers M.

Theorem 23. M is $\beta$-compact iff every $\beta$-filterbase converges in $(M,\mathfrak{U})$.

Corollary. $(M,\mathfrak{U})$ is compact iff it is $\beta$-compact.

Definition. Let $(M,\mathfrak{J})$ be a topological space and let $\mathfrak{A}$ be a family of open subsets of M. Then M is $\gamma$-compact iff every family $\mathfrak{R}$, of subsets of M with the properties (1) $A \in \mathfrak{A}$ implies there is $R \in \mathfrak{R}$ such that $A \subset R$ or $cA \subset R$ and (2) the interiors of members of $\mathfrak{R}$ cover M, has a finite subfamily which covers M.

Theorem 24. M is $\gamma$-compact iff every $\beta$-filterbase has a cluster point in $(M,\mathfrak{J})$.

Corollary. $(M,\mathfrak{J})$ is compact iff M is $\gamma$-compact.

It is obvious that $\gamma$-compactness implies $\beta$-compactness.

Let $(M,\mathfrak{J})$ be a topological space and let $\mathfrak{A}$ be an open cover of M. Denote by $\tau'$ the family of sets obtained by adjoining to $\mathfrak{A}$ the complement of each member of $\mathfrak{A}$. Let us call a nonempty subfamily $\varphi$ of $\tau'$ a $\tau'$-family iff $\varphi$ has the finite intersection property and that $A \in \mathfrak{A}$ implies $A \in \varphi$ or $cA \in \varphi$. Then M is $\beta$-compact iff every $\tau'$-family $\mathfrak{A}$-converges. If $(M,\mathfrak{J})$ is compact then every $\tau'$-family has a cluster point and if each $\tau'$-family converges then $(M,\mathfrak{J})$ is compact. If $\mathfrak{A}$ is a subbase for $\mathfrak{J}$ then $(M,\mathfrak{J})$ is compact iff every $\tau'$-family converges.

A filterbase finer than an $\alpha$-filterbase is also an $\alpha$-filterbase. The continuous image of an $\alpha$-filterbase is also an $\alpha$-filterbase. Let $f : M \to M'$ be continuous and $\varphi'$ be an $\alpha$-filterbase in $M'$; then $f^{-1}\varphi'$ need not be an $\alpha$-filterbase even if f is an open map. If M is $\alpha$-compact and f is onto then $M'$ is $\alpha$-compact. A product space is $\alpha$-compact iff each coordinate space is $\alpha$-compact. A more detailed investigation of the properties of $\alpha$-filterbases will be found in [ 7 ].

Let $(M,\mathfrak{I})$ be the product of the spaces $(M_j,\mathfrak{I}_j)$ for j in some index set J. Let $p_j$ denote projection into the j-th coordinate space and let $\mathfrak{U}_j$ be an open cover of $M_j$. Denote by $\mathfrak{U}$ the family of all sets of the form $p_j^{-1}\mathfrak{U}_j$, $j \in J$. The product space is then $\beta$-compact iff each coordinate space is. As a particular case of this we get the result that the product of compact spaces is compact. Several other results of this kind can be easily obtained.

Finally let us consider ultrafilters and their covering properties.

Definition. A filterbase $\varphi$ in M is an ultrafilterbase iff X is a subset of M implies there is $F \in \varphi$ such that $F \subset X$ or $F \subset cX$.

Definition. A topological space $(M,\mathfrak{I})$ is ultracompact iff every family $\mathfrak{R}$, of subsets of M with the properties (1) $X \subset M$ implies there is $R \in \mathfrak{R}$ such that $X \subset R$ or $cX \subset R$ and (2) the interiors of members of $\mathfrak{R}$ is a cover of M, has a finite subfamily which covers M.

It is easy to prove the following theorem.

Theorem 25. A topological space is ultracompact iff every ultrafilterbase converges.

Corollary. A space is ultracompact iff every ultrafilter converges.

Corollary. A space is compact iff it is ultracompact.

For arbitrary families of subsets of M it is possible to get several results on the same lines as those for families of open subsets of M. The following are a few of such results.

Definition. Let $\mathfrak{B}$ be a family of subsets of M. Then a filterbase $\varphi$ in M is a $\delta$-filterbase iff $B \in \mathfrak{B}$ implies $\varphi$ is eventually in B or eventually in cB.

If $\mathfrak{B}$ is a cover of M denote by $\mathfrak{U}$ the topology for M which has $\mathfrak{B}$ as subbase.

Definition. Let $\mathfrak{B}$ be a cover of M. Then M is $\delta$-compact iff every family $\mathfrak{R}$, of subsets of M with the properties (1) $B \in \mathfrak{B}$ implies there is $R \in \mathfrak{R}$ such that $B \subset R$ or $cB \subset R$ and (2) the $\mathfrak{U}$-interiors of the members of $\mathfrak{R}$ cover M, has a finite subfamily which is a cover of M.

Theorem 26. M is $\delta$-compact iff every $\delta$-filterbase converges in $(M,\mathfrak{U})$.

Corollary. $(M,\mathfrak{U})$ is compact iff it is $\delta$-compact.

Definition. Let $(M,\mathfrak{J})$ be a topological space and let $\mathfrak{B}$ be a family of subsets of M. Then M is $\lambda$-compact iff every family $\mathfrak{R}$, of subsets of M with the properties (1) $B \in \mathfrak{B}$ implies there is $R \in \mathfrak{R}$ such that $B \subset R$ or $cB \subset R$ and (2) the interiors of the members of $\mathfrak{R}$ cover M, has a finite subfamily which covers M.

Theorem 27. M is $\lambda$-compact iff every $\delta$-filterbase has a cluster point in the space $(M,\mathfrak{J})$.

Corollary. $(M,\mathfrak{J})$ is compact iff it is $\lambda$-compact.

A filterbase finer than an ultrafilterbase is also an ultrafilterbase. An ultrafilterbase converges to each of its cluster points. The continuous image of an ultrafilterbase is an ultrafilterbase. The continuous image of an ultracompact space is ultracompact. A product space is ultracompact iff each coordinate space is ultracompact; one consequence of this result is the Tychonoff product theorem. The inverse image of an ultrafilterbase under an open continuous map need not be an ultrafilterbase.

Some aspects of covering properties associated with certain nets and filterbases have been considered in the preceding pages; other aspects, such as the role nets play in compactification, will be taken up in subsequent papers.

There are certain similarities between the covering properties associated with nets or filterbases defined with respect to open sets and those defined with respect to all the subsets of the space. For instance:

1. Space is compact iff every C-net converges; space is compact iff every $\mathfrak{M}$-net has a cluster point.

2. Space is $\mathfrak{A}$-compact iff every $\mathfrak{A}$-net $\mathfrak{A}$-converges; space is $\mathfrak{A}$-compact iff every $\mathfrak{M}$-net has an $\mathfrak{A}$-cluster point.

3. Space is $\alpha$-compact iff every $\alpha$-filterbase converges; space is $\alpha$-compact iff every ultrafilterbase converges.

It is also interesting to note that a net I is an $\mathfrak{A}'$-net iff I converges to each of its $\mathfrak{A}$-cluster points. This result more or less summarizes the relationship between convergence and $\mathfrak{A}$-cluster points.

Only a few nets and filterbases have been studied in this paper. Covering properties associated with certain others are considered in a forthcoming paper.

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