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END-GENERATED TRANSFINITE MONOTONE NETWORKS

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Abstract — A prior work established the existence of an operating point for a transfinite nonlinear electrical network having the first rank of transfiniteness. This work inductively extends that result to all natural-number ranks of transfiniteness and then to the first transfinite-ordinal rank ω. It can also be extended to still higher ranks in the same way. As before, it is assumed that every branch characteristic is maximal monotone, and thus this paper presents a further extension of a classical result of Minty. Moreover, the idea of transfinite ends is developed and used to define transfinite nodes as finite combinations of transfinite ends. The network is so constructed that such ends are limits under certain metrics on the nodes, and as a result all ends of all ranks are "permissive" in the sense that currents are able to flow through them. All this yields a transfinite network more amenable to an electrical network theory and empowers the theory of monotone networks presented herein. Furthermore, "no-gain" bounds are established for the operating point. Finally, the uniqueness of the operating point is established when branch characteristics are restricted to continuous, strictly monotonic curves.

1 Introduction

This paper extends some of the results of [2] to higher ranks of transfiniteness. That prior work established the existence and uniqueness of operating points for nonlinear electrical networks possessing the simplest kind of transfiniteness, that is, in the terminology of [14] and [15], for 1-networks. It also established bounds on the branch currents and branch

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voltages much like those established by Wolaver [13] and sharpened by Calvert [1]. By an "operating point" we mean a set of branch currents and branch voltages that satisfy Kirchhoff's laws and conform with the branch characteristics, which we take to be maximally monotone curves in the current-voltage plane. The basic idea for obtaining an operating point for a ν-network, where the transfiniteness rank ν is a natural number no less than 2, is to let the role played by branches in [2] be taken over by γ-sections, where $0 \leq \gamma \leq \nu - 2$. (Branches can be taken to be (-1)-sections.) γ-sections are maximal subnetworks of rank γ whose branches and nodes are connected by transfinite paths of ranks no larger than γ. This extension necessitates a more complicated analysis than that of [2], but the general strategy, when dealing with a single rank of transfiniteness, is like that of [2]. We will present an inductive argument to obtain results for all the natural-number ranks. Then, changes in our arguments enable a construction of an operating point for an ω-network, where the rank ω is the first transfinite ordinal. Operating points for networks of still higher ordinal ranks can be obtained by repeating this two-step approach, the first for successor-ordinal ranks and the second for limit-ordinal ranks. How far repetitions of this two-step procedure will carry through the countable ordinal ranks is not answered in this paper.

Not all the results of [2] are extended herein. Only end-generated transfinite networks are considered. The idea of an "end" for a conventionally infinite graph has been quite useful; see, for example, [3], [4], [5], [6], [9], [11], [12]. In this paper, we extend the idea of an end transfinitely. Later on, we impose an additional restriction; namely, the ends of a conventionally infinite graph or of a transfinite graph are taken to be isolated from each other in the sense that the deletion of a sufficiently large finite set of branches in the conventional case or of sections in the transfinite case will leave different ends in separate components.

This paper is based upon the theory of transfinite graphs and networks presented in [14] and [15] and uses the terminology explicated in those books. The unconventional terms regarding transfiniteness used herein are provided with references to those sources. (For brevity, we often say "in" in place of "embraced by" [14], [15]; the context in which 'in" is used should dispell any confusion.) Furthermore, the more specialized transfinite structure developed in [16] is also used, and we summarize the needed ideas in Section 2. Thus, this paper can be read independently from [16]. In Section 3 we specify the assumptions (Conditions 3.2) imposed upon our network. Section 4 presents the first step of our inductive argument; it shows how a finite 0-network $\mathbf{N}^0$ satisfies Conditions 3.2 and then invokes a classical theorem of Minty [7] to establish an operating point for $\mathbf{N}^0$ satisfying certain bounds. The inductive argument from ν-networks to $(\nu+1)$-networks, ν being a natural number, is presented in Sections 5 through 7. Section 5 examines the flow of current toward the extremities (i.e., "μ-ends") of a μ-section $(0 \leq \mu < \nu)$ and explains what Kirchhoff's current law means for a $(\mu+1)$-node. Section 6 derives the existence of a potential function on all the nodes of a $(\nu+1)$-network; it does this by invoking the Arzela-Ascoli theorem as applied to a sequence of potential functions on the nodes of a sequence of ν-networks that fill out a given $(\nu+1)$-network. The obtained limiting potential function will be continuous at all nodes of ranks greater than 0 and therefore will generate branch voltages that satisfy Kirchhoff's voltage law around all γ-loops, where $\gamma = 1, \ldots, \mu+1$. (All 0-nodes are isolated, and therefore the continuity of the potential function is not an issue for them.) These results are then combined in Section 7 to obtain an operating point for the $(\nu+1)$-network, and bounds on the branch currents and branch voltages are also established. The uniqueness of that operating point is established in Section 8 under an additional restriction on the monotone branch characteristics, namely, that they have neither horizontal nor vertical segments. ω-networks are then considered. Current flows and Kirchhoff's current law are examined again in Section 9. Potentials and finally the existence and uniqueness of an operating point are discussed in Section 10.

PART I: PRELIMINARIES

2 Permissively Structured Transfinite Networks

We use the terminology and notation defined in either [14] or [15]. $\mathbf{N}^\nu$ will denote a transfinite network of rank ν, and $(\mathcal{B}, \mathcal{N}^0, \mathcal{N}^1, \ldots, \mathcal{N}^\nu)$ will denote its transfinite graph, where $\mathcal{B}$ is its set of branches, and $\mathcal{N}^\gamma$ is its set of nodes of rank γ $(0 \leq \gamma \leq \nu)$. In this paper, ν will be either a natural number or ω, and μ will be a rank such that $-1 \leq \mu < \nu$.

When $\nu = \omega$, $\nu - 1$ denotes the arrow rank $\vec{\omega}$ [15, pages 4 and 36], and, when $\mu = \vec{\omega}$, $\mu + 1$ denotes ω. When $0 \leq \mu < \nu$, we take a μ-section to be a subgraph of the μ-graph of $\mathbf{N}^\nu$, in accordance with [15, pages 32-33] (but we refer to all graphs and subgraphs as networks and subnetworks). Thus, a μ-section embraces nodes of ranks μ or less but does not embrace its incident $(\mu+1)$-nodes. Furthermore, for $\mu = -1$, a (-1)-section is taken to be a branch, and its two (-1)-tips are then its two "(-1)-terminals" — analogous to the "μ-terminals" defined below for $\mu \geq 0$. Also, $\mathbf{N}^\nu$ is itself a ν-section.

$\boldsymbol{R}^n$ denotes n-dimensional Euclidean space with $\boldsymbol{R}^1$ replaced by $\boldsymbol{R}$. $\boldsymbol{R}_+$ is the set of positive numbers in $\boldsymbol{R}$. The adjective "nonlinear" is used inclusively, with "linear" being a special case of it. The notation $x \dashv Q$ means that the entity x is embraced by the entity Q. Every branch b of our transfinite network $\mathbf{N}^\nu$ will relate current and voltage nonlinearly. Nonetheless, in Section 3 we shall also assign to b a linear positive resistance R_b derived from that nonlinearity in order to treat the nodes of $\mathbf{N}^\nu$ as the elements of a metric space. The *resistive length* $|P|$ of a path P of any rank is the sum of all the R_b values of its embraced branches; in symbols, $|P| = \sum_{b \dashv P} R_b$. We say that P is *permissive* if $|P| < \infty$; otherwise, *nonpermissive*. Similarly, a tip of any rank [14, page 140], [15, page 30] is *permissive* (resp. *nonpermissive*) if it has a permissive representative path (resp. all its representative paths are nonpermissive). Every nonpermissive tip will be assigned to a singleton node, and that node will also be called *nonpermissive*. The remaining nodes will be *permissive* in the sense that all their tips are permissive.

We henceforth assume that our transfinite network $\mathbf{N}^\nu$ is permissively structured [16]. Thus, it has the following properties: No node embraces a node of lower rank. Therefore, sections [15, page 49] and subsections [15, page 81] coincide. Also, there are no $\vec{\omega}$-nodes when $\nu = \omega$ [15, page 37]. Now, let $\mathbf{S}^\mu$ be any μ-section of $\mathbf{N}^\nu$, where now $0 \leq \mu \leq \nu - 1$. (As noted above, $\nu - 1 = \vec{\omega}$ when $\nu = \omega$.) Let $\mathcal{N}_{\mathbf{S}^\mu}$ denote the set of all permissive nodes of ranks no larger than μ embraced by $\mathbf{S}^\mu$. With $m, n \in \mathcal{N}_{\mathbf{S}^\mu}$, let $\mathcal{P}(\mathbf{S}^\mu, m, n)$ denote the set of all two-ended permissive paths of all ranks no larger than μ that terminate at m and n. (The trivial path consisting of only a single node is included.) There will be at least one such path for every m and n [16, Proposition 5.3]. Define the mapping $d^\mu \colon \mathcal{N}_{\mathbf{S}^\mu} \times \mathcal{N}_{\mathbf{S}^\mu} \mapsto \boldsymbol{R}$

by

$$d^\mu(m,n) \;=\; \inf\{|P|: P \in \mathcal{P}(\mathbf{S}^\mu, m, n)\}. \tag{1}$$

Then, d^μ is a metric on $\mathcal{N}_{\mathbf{S}^\mu}$ [16, Proposition 5.6 and Section 6]. This same definition works when $\mu = \omega$, in which case d^ω is a metric on the set $\mathcal{N}_{\mathbf{N}^\omega}$ of all nodes in $\mathbf{N}^\omega$.

Returning to the case where $0 \leq \mu \leq \nu - 1$, take the completion $\widehat{\mathcal{N}}_{\mathbf{S}^\mu}$ of $\mathcal{N}_{\mathbf{S}^\mu}$ under d^μ. The obtained limit points having the property that they are not the limit points of any ρ-section $\mathbf{S}^\rho$ embraced by $\mathbf{S}^\mu$ ($\rho < \mu$) are called the *μ-terminals* of $\mathbf{S}^\mu$. It is a fact that the nodes of every representative of any permissive μ-tip of $\mathbf{S}^\mu$ converge to some μ-terminal [16, Propositions 5.10 and 6.4]. Thus, each μ-terminal of $\mathbf{S}^\mu$ can be identified with a set of permissive μ-tips of $\mathbf{S}^\mu$, and thereby the μ-terminals of $\mathbf{S}^\mu$ partition the set of permissive μ-tips of $\mathbf{S}^\mu$. It is an assumed property of a permissively structured network $\mathbf{N}^\nu$ that every μ-section of $\mathbf{N}^\nu$ has only finitely many μ-terminals. Moreover, each permissive $(\mu+1)$-node of $\mathbf{N}^\nu$ is also assumed to consist of all the permissive μ-tips in only finitely many of the μ-terminals of the various μ-sections of $\mathbf{N}^\nu$. Thus, each $(\mu+1)$-node is incident to only finitely many μ-sections of $\mathbf{N}^\nu$. All these properties are also assumed to hold for branches as well; indeed, branches can be viewed as sections of rank -1. Thus, every 0-node is of finite degree. On the other hand, every nonpermissive μ-tip ($\mu \geq 0$) is open; this means that it is the sole member of a nonpermissive $(\mu+1)$-node. Consequently, no path or loop can pass through a nonpermissive node; it can only terminate there.

We can summarize all this by saying that a permissively structured ν-network is defined by the following conditions, where as always μ denotes an arbitrary rank less than ν, that is, $-1 \leq \mu < \nu$.

Conditions 2.1.

(a) *No node of* $\mathbf{N}^\nu$ *embraces a node of lower rank.*

(b) *Every section of every rank* μ *has no more than finitely many* μ *terminals.*

(c) *Every node is either permissive or nonpermissive. If* μ *is a natural number (resp. if* $\mu = \vec{\omega}$*), then every nonpermissive* $(\mu+1)$*-node (resp. nonpermissive* ω*-node) is a singleton consisting of exactly one nonpermissive* μ*-tip (resp. nonpermissive* $\vec{\omega}$*-tip).*

If μ is -1 or a natural number (resp. if $\mu = \vec{\omega}$), every permissive $(\mu+1)$-node (resp. ω-node) consists of all the μ-tips (resp. $\vec{\omega}$-tips), perforce permissive, in only finitely many μ-terminals (resp. $\vec{\omega}$-terminals).

As an immediate consequence of Conditions 2.1(b) and (c), we have

Lemma 2.2. *Every μ-section has only finitely many incident permissive $(\mu+1)$-nodes, and every permissive $(\mu+1)$-node is incident to only finitely many μ-sections.* (When $\mu = \vec{\omega}$, $\mu + 1$ denotes ω.)

With a branch b being taken as a (-1)-section with exactly two (-1)-terminals, which we identify with the two (-1)-tips of b, Condition 2.1(c) asserts that every 0-node is permissive and of finite degree. There are no nonpermissive 0-nodes.

Henceforth, we discuss only permissive nodes and ignore the nonpermissive nodes. So, we often drop the adjective "permissive." Thus, it will always be tacitly understood that any node we refer to is permissive.

Because every nonpermissive tip is open (Condition 2.1(c)), we have

Lemma 2.3. *Every two-ended path and every loop is permissive.*

3 Conditions Assumed on the Transfinite Network

Each branch $b \in \mathcal{B}$ is assigned an electrical parameter characterized by a nonlinear curve in the current-voltage plane as follows. The current i_b and voltage v_b for branch b will always be measured with respect to an orientation assigned to b, with the flow of i_b and the potential drop of v_b being in the direction of that orientation. We will freely switch the orientation of b depending on whether we want to measure current flow and potential drop toward or away from a node; i_b and v_b are replaced by their negative values when the orientation is switched. A potential p is a function defined on the node set $\mathcal{N}^0 \cup \ldots \cup \mathcal{N}^\nu$ with values in $\boldsymbol{R}$ such that, for each b with incident nodes n_1^0 and n_2^0, $v_b = p(n_1^0) - p(n_2^0)$, where b is oriented from n_1^0 to n_2^0. We shall seek a potential that is continuous at permissive nodes of ranks higher than 0. As a consequence, the values of p on $\mathcal{N}^0$ determine its values on the permissive nodes in $\mathcal{N}^1 \cup \ldots, \mathcal{N}^\nu$ through continuity.

For each $b \in \mathcal{B}$, M_b will denote a characteristic curve in the current-voltage plane. We let x denote current and y denote voltage. M_b is assumed to be a maximal monotone characteristic mapping a subset of (possibly all of) $\boldsymbol{R}$ into $2^{\boldsymbol{R}}$ [7]. In particular, M_b is a set-valued function of x that is *monotone* in the sense that

$$(M_b(x)^* - M_b(x')^*)(x - x') \geq 0$$

where $M_b(x)^*$ denotes a point in the set $M_b(x)$. We also view M_b as being a subset of $\boldsymbol{R}^2$ and treat $(x, y) \in M_b$ and $y \in M_b(x)$ as being equivalent expressions. M_b^{-1} denotes the inverse of M_b: $(y, x) \in M_b^{-1}$ if and only if $(x, y) \in M_b$. We also write $x \in M_b^{-1}(y)$. Furthermore, M_b is *maximal monotone* in the sense that the two conditions $(x, y) \in \boldsymbol{R}^2$ and $(x - x')(y - y') \geq 0$ for all $(x', y') \in M_b$ imply that $(x, y) \in M_b$. Note that M_b serves as a resistance function in that it maps each current into a set of voltage values, but it may implicitly contain a source; the latter occurs when M_b does not contain the origin of $\boldsymbol{R}^2$. We assume that there is at least one such M_b not containing that origin. (Otherwise, a "solution," as defined below, would be the trivial one where all branch voltages and currents are 0.

We assume still more conditions on every M_b, namely:

Conditions 3.1.

(a) *There is at least one y with $(0, y) \in M_b$ (i.e., the origin of the current axis is in the domain of M_b). Thus, we can set*

$$\delta''(0, 0, M_b) = \min\{|y| : (0, y) \in M_b\}.$$

(b) *There is at least one x with $(x, 0) \in M_b$ (i.e., the origin of the voltage axis is in the range of M_b). Thus, we can set*

$$\delta'(0, 0, M_b) = \min\{|x| : (x, 0) \in M_b\}.$$

(c) *Finally,*

$$\boldsymbol{I} \equiv \sum_{b \in \mathcal{B}} \delta'(0, 0, M_b) < \infty$$

and

$$\boldsymbol{V} \equiv \sum_{b \in \mathcal{B}} \delta''(0, 0, M_b) < \infty.$$

Now, to each $b \in \mathcal{B}$ we assign a real positive number R_b as follows: If M_b is a Lipschitz continuous function on $[-\boldsymbol{I}, \boldsymbol{I}]$, R_b is the Lipschitz constant of M_b on $[-\boldsymbol{I}, \boldsymbol{I}]$ so long as $R_b > 0$. If that Lipschitz constant is 0, we assign a positive value R_b to b in such a way that $\sum_{b \in \mathcal{B}_s} R_b < \infty$, where $\mathcal{B}_s$ is the subset of $\mathcal{B}$ consisting of all branches for which those Lipschitz constants equal 0. On the other hand, if M_b is not Lipschitz continuous on $[-\boldsymbol{I}, \boldsymbol{I}]$, we set $R_b = 1$. R_b plays the role that the linear resistances r_b played in [16].

We now turn to some strictly graph-theoretic ideas. They are independent of the values R_b. Nonetheless, we continue to speak in terms of networks. By a *subnetwork* of $\mathbf{N}^\nu$, we mean a branch-induced subgraph [15, page 32] of $\mathbf{N}^\nu$ with the same assignment M_b to each branch $b \in \mathcal{B}$. $\mathbf{N}(\mathcal{B}_1)$ denotes the subnetwork induced by the subset $\mathcal{B}_1$ of $\mathcal{B}$. Let $\mathbf{N}(\mathcal{B}_1)$ and $\mathbf{N}(\mathcal{B}_2)$ be subnetworks of $\mathbf{N}^\nu$. We say that $\mathbf{N}(\mathcal{B}_1)$ is a subnetwork of $\mathbf{N}(\mathcal{B}_2)$ if $\mathcal{B}_1 \subset \mathcal{B}_2$.

Let us now define the "transfinite ends" of any μ-section $\mathbf{S}^\mu$ $(0 \leq \mu \leq \nu - 1)$ for the case where ν is a natural number. (We consider the case where $\nu = \omega$ later on.) First of all, when $\mu = 0$, we have the customary definition for an end of a 0-section $\mathbf{S}^0$, which we now refer to as a "0-end." With $\mathcal{E}$ denoting any finite subset of the branch set $\mathcal{B}^{-1}(\mathbf{S}^0)$ of $\mathbf{S}^0$, we let $\mathbf{S}^0 \backslash \mathcal{E}$ denote the 0-subnetwork of $\mathbf{S}^0$ induced by all the branches of $\mathbf{S}^0$ that are not in $\mathcal{E}$. Then, a *0-end* e^0 of $\mathbf{S}^0$ is a mapping that takes every finite subset $\mathcal{E}$ of $\mathcal{B}^{-1}(\mathbf{S}^0)$ into an infinite component $e^0(\mathcal{E})$ of $\mathbf{S}^0 \backslash \mathcal{E}$ in accordance with the following rule: If $\mathcal{E}$ and $\mathcal{F}$ are finite subsets of $\mathcal{B}^{-1}(\mathbf{S}^0)$ with $\mathcal{E} \subset \mathcal{F}$, then $e^0(\mathcal{F})$ is a 0-subnetwork of $e^0(\mathcal{E})$.

Proceeding transfinitely, consider the case where μ is a positive natural number. Since no node embraces a node of lower rank, a μ-section $\mathbf{S}^\mu$ must have infinitely many $(\mu - 1)$-sections. We now let $\mathcal{E}^{\mu-1}$ denote any finite subset of the set $\mathcal{B}^{\mu-1}(\mathbf{S}^\mu)$ of $(\mu - 1)$-sections in $\mathbf{S}^\mu$, and let $\mathbf{S}^\mu \backslash \mathcal{E}^{\mu-1}$ denote the μ-subnetwork of $\mathbf{S}^\mu$ induced by all the branches of all the $(\mu - 1)$-sections of $\mathbf{S}^\mu$ that are not in $\mathcal{E}^{\mu-1}$. (See [15, page 32] for "branch-induced subgraphs"; here, we refer to them as "subnetworks.") A *μ-end* e^μ is a mapping that takes every finite subset $\mathcal{E}^{\mu-1}$ of $\mathcal{B}^{\mu-1}(\mathbf{S}^\mu)$ into a component of $\mathbf{S}^\mu \backslash \mathcal{E}^{\mu-1}$ having infinitely many $(\mu - 1)$-sections, in accordance with the following rule: If $\mathcal{E}^{\mu-1}$ and $\mathcal{F}^{\mu-1}$ are finite subsets of $\mathcal{B}^{\mu-1}(\mathbf{S}^\mu)$ with $\mathcal{E}^{\mu-1} \subset \mathcal{F}^{\mu-1}$, then $e^\mu(\mathcal{F}^{\mu-1})$ is a μ-subnetwork of $e^\mu(\mathcal{E}^{\mu-1})$. Later on, we will identify μ-ends with μ-terminals bijectively and will construct the $(\mu + 1)$-nodes out of the μ-ends.

Finally, let us prepare for a definition of an "$\vec{\omega}$-end" of an $\vec{\omega}$-section $S^{\vec{\omega}}$. The construction in this case is quite a bit different from that for the μ-ends given above. First, note that, given any 0-node n^0 in $S^{\vec{\omega}}$, there is a unique sequence $\{S^{\mu}_{n^0}\}_{\mu=0}^{\infty}$ of μ-sections $S^{\mu}_{n^0}$, each containing n^0. In fact, $S^{\mu}_{n^0} \subset S^{\mu+1}_{n^0}$ for every μ. We call $\{S^{\mu}_{n^0}\}_{\mu=0}^{\infty}$ a *nested sequence of sections induced by* n^0.

Lemma 3.2. *$\{S^{\mu}_{n^0}\}_{\mu=0}^{\infty}$ fills out $S^{\vec{\omega}}$ in the sense that, given any node m^{γ} of any rank γ embraced by $S^{\vec{\omega}}$, there will be some rank μ_1 such that $S^{\mu_1}_{n^0}$ embraces m^{γ}, and similarly for any branch embraced by $S^{\vec{\omega}}$.*

Proof. Since $S^{\vec{\omega}}$ is $\vec{\omega}$-connected, there is a two-ended μ_1-path P^{μ_1} $(\gamma \leq \mu_1 < \vec{\omega})$ embraced by $S^{\vec{\omega}}$ that terminates at n^0 and m^{γ}. P^{μ_1}, and therefore m^{γ} too, will be embraced by $S^{\mu_1}_{n^0}$, one of the sections in $\{S^{\mu}_{n^0}\}_{\mu=0}^{\infty}$. The same argument works for any branch of $S^{\vec{\omega}}$. □

Lemma 3.3. *Two nested sequences $\{S^{\mu}_{n^0}\}_{\mu=0}^{\infty}$ and $\{S^{\mu}_{m^0}\}_{\mu=0}^{\infty}$ induced by two different 0-nodes n^0 and m^0 in $S^{\vec{\omega}}$ are eventually identical.*

Proof. By Lemma 3.2, n^0 and m^0 are both in $S^{\mu_1}_{n^0}$ and also in $S^{\mu_1}_{m^0}$ for some μ_1. Therefore, $S^{\mu_1}_{n^0} = S^{\mu_1}_{m^0}$. Hence, $S^{\mu}_{n^0} = S^{\mu}_{m^0}$ for all $\mu \geq \mu_1$. □

Lemma 3.4. *Given any γ-section S^{γ} in $S^{\vec{\omega}}$, where γ is a natural number, and given a nested sequence of sections induced by any 0-node $\{S^{\mu}_{n^0}\}_{\mu=0}^{\infty}$, S^{γ} is eventually embraced by $\{S^{\mu}_{n^0}\}_{\mu=0}^{\infty}$ (i.e., there is a rank μ_1 such that $S^{\gamma} \subset S^{\mu}_{n^0}$ for all $\mu \geq \mu_1$.)*

Proof. Choose any 0-node m^0 of S^{γ} and then μ_1 such that $S^{\mu_1}_{n^0}$ embraces m^0, as in Lemma 3.2. □

We are now ready to define an $\vec{\omega}$-end of $S^{\vec{\omega}}$. With S^{μ} being any μ-section in $S^{\vec{\omega}}$, we now let $S^{\vec{\omega}} \backslash S^{\mu}$ denote the $\vec{\omega}$-subnetwork of $S^{\vec{\omega}}$ induced by all the branches of $S^{\vec{\omega}}$ that are not in S^{μ}. An *$\vec{\omega}$-end* is a mapping that takes every μ-section S^{μ} of $S^{\vec{\omega}}$ (of every natural number rank μ) into a component of $S^{\vec{\omega}} \backslash S^{\mu}$ in accordance with the following two rules:

- That component is an $\vec{\omega}$-network (i.e., has nodes of all natural number ranks).
- If S^{μ} and S^{γ} are a μ-section and a γ-section respectively in $S^{\vec{\omega}}$, where $0 \leq \mu < \gamma$ and where S^{μ} is a μ-subnetwork of S^{γ}, then $S^{\vec{\omega}} \backslash S^{\gamma}$ is an $\vec{\omega}$-subnetwork of $S^{\vec{\omega}} \backslash S^{\mu}$.

Here too, we will identify $\vec{\omega}$-ends with the $\vec{\omega}$-terminals bijectively and will construct ω-nodes out of the $\vec{\omega}$-ends.

Here are the assumptions we shall impose throughout this work upon any transfinite network $\mathbf{N}^\nu$. As before, we restrict the rank ν to being either a natural number or ω; also, when $\nu = \omega$, $\nu - 1$ denotes $\vec{\omega}$.

Conditions 3.5.

(a) $\mathbf{N}^\nu$ *is ν-connected and has only finitely many (permissive) ν-nodes.*

(b) $\mathbf{N}^\nu$ *is permissively structured (i.e., Conditions 2.1 are fulfilled).*

(c) *For each μ-section $\mathbf{S}^\mu$, where $0 \le \mu \le \nu - 1$, the set $\mathcal{N}_{\mathbf{S}^\mu}$ of all nodes of all ranks in $\mathbf{S}^\mu$ (and thus of ranks no larger than μ) is a totally bounded set under the metric d^μ.*

(d) *Every μ-end of every μ-section (where again $0 \le \mu \le \nu - 1$) is isolated and coincides with a unique μ-terminal, and there is a bijection in this way between the μ-ends and μ-terminals.*

(e) *Every branch of $\mathbf{N}^\nu$ is characterized electrically by a maximal monotone curve in the current-voltage plane satisfying Conditions 3.1.*

Let us be more explicit about Condition 3.5(d). First, consider the case where μ is a natural number. That a μ-end e^μ of a μ-section $\mathbf{S}^\mu$ is *isolated* means that there is a finite set $\mathcal{E}^{\mu-1}$ of $(\mu-1)$-sections (e.g., of branches if $\mu = 0$) in $\mathbf{S}^\mu$ such that $e^\mu(\mathcal{E}^{\mu-1})$ has exactly one μ-end; that is, for no other finite set $\mathcal{F}^{\mu-1}$ of $(\mu-1)$-sections in $\mathbf{S}^\mu$ with $\mathcal{F}^{\mu-1} \supset \mathcal{E}^{\mu-1}$ will $e^\mu(\mathcal{E}^{\mu-1})\backslash\mathcal{F}^{\mu-1}$ have two or more components with infinitely many $(\mu-1)$-sections. That end is said to *coincide* with a μ-terminal T^μ of $\mathbf{S}^\mu$ if, for all $\mathcal{E}^{\mu-1}$ sufficiently large, $e^\mu(\mathcal{E}^{\mu-1})$ has exactly one μ-terminal, namely, T^μ; that is, the set of all μ-nodes of $e^\mu(\mathcal{E}^{\mu-1})$ has only T^μ as its unique limit point under d^μ whenever $\mathcal{E}^{\mu-1}$ is chosen large enough. In this way we can identify the terminal T^μ with the end e^μ.

Now, consider an $\vec{\omega}$-end $e^{\vec{\omega}}$ of an $\vec{\omega}$-section $\mathbf{S}^{\vec{\omega}}$. That it is *isolated* means that there is a μ-section in $\mathbf{S}^{\vec{\omega}}$, where μ is a natural number, such that $e^{\vec{\omega}}(\mathbf{S}^\mu)$ has exactly one $\vec{\omega}$-end; that is, for no other γ-section $\mathbf{S}^\gamma$ in $\mathbf{S}^{\vec{\omega}}$ with $\mathbf{S}^\gamma \supset \mathbf{S}^\mu$ will $e^{\vec{\omega}}(\mathbf{S}^\mu)\backslash\mathbf{S}^\gamma$ have two or more components that are each $\vec{\omega}$-subnetworks of $\mathbf{S}^{\vec{\omega}}$. That $\vec{\omega}$-end is said to *coincide* with an $\vec{\omega}$-terminal $T^{\vec{\omega}}$ of $\mathbf{S}^{\vec{\omega}}$ if, for all μ-sections with sufficiently large natural-number ranks μ, $e^{\vec{\omega}}(\mathbf{S}^{\mu})$ has exactly one $\vec{\omega}$-terminal, namely, $T^{\vec{\omega}}$.

Lemma 3.6. *$\mathbf{N}^{\nu}$ has only finitely many $(\nu-1)$-sections.*

Proof. This follows directly from Conditions 2.1(c) and 3.5(a) and the fact that each $(\nu-1)$-terminal belongs to exactly one $(\nu-1)$-section. □

Finally, let us state what we mean by an *operating point* of $\mathbf{N}^{\nu}$. This is synonymous with a *solution* for $\mathbf{N}^{\nu}$.

Definition 3.7. An *operating point* for a transfinite network $\mathbf{N}^{\nu}$ is a pair (i, v) consisting of a set $i = \{i_b\}_{b\in\mathcal{B}}$ of branch currents satisfying Kirchhoff's current law at every node of every rank in $\mathbf{N}^{\nu}$ and a set $v = \{v_b\}_{b\in\mathcal{B}}$ of branch voltages derived from a potential p on all the nodes of $\mathbf{N}^{\nu}$ with p being continuous at every node of rank greater than 0 such that $(i_b, v_b) \in M_b$ for every $b \in \mathcal{B}$.

We have yet to define what we mean by the satisfaction of Kirchhoff's current law by i at every node. Since every 0-node is of finite degree, this has the usual meaning at 0-nodes. This was also defined for 1-nodes in [2, Section 3]. We shall define it for all μ-nodes for all natural-number ranks μ in Section 5 and for ω-nodes in Section 9. On the other hand, each branch voltage v_b will be obtained from the potential p in the usual way as the oriented difference in the potentials at the two 0-nodes incident to b. The fact that p is continuous at all nodes of positive ranks insures that $v = \{v_b\}_{b\in\mathcal{B}}$ satisfies Kirchhoff's voltage law around every loop of every rank, as we shall note again in Sections 7 and 10.

PART II: NETWORKS WITH NATURAL-NUMBER RANKS

4 The First Inductive Step and a Summary of Subsequent Steps

We shall argue by induction on the ranks of transfinite networks. The first step, the existence of an operating point for a finite 0-network, is a known result, which we now restate in a form more suitable for generalization.

Let $\mathbf{N}^0$ be a finite 0-connected network whose branch characteristics are maximal monotone and satisfy Conditions 3.1. It follows that $\mathbf{N}^0$ satisfies all of Conditions 3.5 so long as "(-1)-sections" and "(-1)-terminals" are interpreted appropriately. Indeed, part (a) of Conditions 3.5 is obviously satisfied. For 3.5(b), turn to Conditions 2.1. Condition 2.1(a) is fulfilled because there are no nodes of ranks less than 0. Furthermore, every branch b can be taken to be a section of rank -1, and its two elementary tips (i.e., its two (-1)-tips) can be taken to be its two permissive (-1)-terminals and also its two isolated (-1)-ends, in which case Conditions 2.1(b), 2.1(c), and 3.5(d) are fulfilled. Furthermore, b as a (-1)-section has no sections of lower rank and no nodes, and so Condition 3.5(c) is fulfilled trivially. Conditions 3.5(e) has already been assumed. Conversely, we can also start with Conditions 3.5 and reverse these arguments. In summary, we have

Lemma 4.1. *Under the stated interpretation of any branch as a (-1)-section with the branch's two (-1)-tips taken to be two permissive (-1)-terminals and also two isolated (-1)-ends, a 0-network $\mathbf{N}^0$ is finite, 0-connected, and has maximal-monotone branch characteristics satisfying Conditions 3.1 if and only if $\mathbf{N}^0$ satisfies Conditions 3.5 (that is, with $\nu = 0$ and $\mu = -1$).*

By virtue of this lemma, we can invoke [7, Theorem 8.1] (see also [1] for a simpler derivation and an extension to conventionally infinite networks) to conclude that $\mathbf{N}^0$ has an operating point satisfying certain bounds. In particular, we have

Theorem 4.2. *Let $\mathbf{N}^0$ be a 0-network satisfying Conditions 3.5. Then, $\mathbf{N}^0$ has an operating point (i, v), and, for each branch b, $|i_b| \leq \boldsymbol{I}$ and $|v_b| \leq \boldsymbol{V}$.*

The aim of our inductive argument is to show that, if an extension of Theorem 4.2 to any arbitrary natural-number rank ν (resp. to all natural-number ranks ν) holds, then it also holds for the rank $\nu + 1$ (resp. for the rank ω). We start by positing the following.

Assertion 4.3. *Let $\mathbf{N}^\nu$ be a ν-network satisfying Conditions 3.5, where ν is any arbitrary natural number. Then, $\mathbf{N}^\nu$ has an operating point (i, v), and, for each branch b, $|i_b| \leq \boldsymbol{I}$ and $|v_b| \leq \boldsymbol{V}$.*

In the next three sections, we shall prove that, if a $(\nu + 1)$-network $\mathbf{N}^{\nu+1}$, where ν is a natural number, satisfies Conditions 3.5 with ν replaced by $\nu + 1$, then it too has an operating point satisfying the stated bounds. This will establish the truth of Assertion 4.3 for every natural-number rank ν. The case where $\nu + 1$ is replaced by ω is taken up in Sections 9 and 10.

5 Current Flows through Transfinite Nodes

Eventually, we will assert Kirchhoff's current law at permissive transfinite nodes (that is, at permissive nodes of ranks higher than 0), and for this purpose we need to identify what is the total current incident at such a node. How we will do the latter recursively in this section can be summarized as follows. Under Conditions 3.5 every $(\mu + 1)$-node (μ being a natural number now) can be separated from all other $(\mu + 1)$-nodes by selecting a finite number of $(\mu - 1)$-sections, as we shall see. Moreover, if Kirchhoff's current law is satisfied at every α-node for all ranks α with $0 \leq \alpha \leq \mu$, the current flow through each $(\mu - 1)$-section can be determined, and the sum of those currents through the selected $(\mu - 1)$-sections can be identified as the total current incident at the $(\mu + 1)$-node because that total current will be shown to be independent of the choice of the separating set of $(\mu - 1)$-sections. This allows us to assert Kirchhoff's current law at the $(\mu + 1)$-node. That this law is truly satisfied when currents correspond to an operating point will be shown in Section 7.

The first step of our recursive development concerns the satisfaction of Kirchhoff's current law at 0-nodes, the conventional case, and this we have from Theorem 4.2. The second step concerns Kirchhoff's current law at 1-nodes, that is, when $\mu + 1 = 1$; in this case, $(\mu - 1)$-sections are branches, and the currents in them are immediately identified. This second step has already been considered in [2]. Our present discussion of Kirchhoff's current law at $(\mu + 1)$-nodes, where $\mu + 1 \geq 1$, subsumes the second step as a special case, but some of our present terminology is different from that of [2]. For instance, a (-1)-section is simply a branch, and the branch's two (-1)-tips comprise its two (-1)-terminals. Quite a bit more is involved when $\mu + 1 > 1$. So, let us proceed directly to this general case, since what we say for it will indeed hold when $\mu + 1 = 1$.

By Condition 2.1(c), every $(\mu + 1)$-node consists of only finitely many μ-terminals. In order to analyze the currents incident at that $(\mu+1)$-node, we have to consider the μ-sections to which its μ-terminals belong. So, consider any μ-section $\mathbf{S}^{\mu}$, and let T_k^{μ} $(k = 1, \ldots, K)$ be its incident μ-terminals — also finite in number according to Condition 2.1(b). $\mathbf{S}^{\mu}$ consists of infinitely many $(\mu - 1)$-sections.

Let $\mathbf{D}^{\mu}(T_k^{\mu}, \epsilon_k)$ denote the μ-subnetwork of $\mathbf{N}^{\nu}$ (ν a natural number and $0 \leq \mu \leq \nu$) induced by all the branches of all the $(\mu - 1)$-sections whose incident μ-nodes are all no further than $\epsilon_k > 0$ from T_k^{μ}, as measured by d^{μ} restricted to $\mathbf{S}^{\mu}$. (Thus, the incident μ-nodes of such $(\mu - 1)$-sections are in $\mathbf{D}^{\mu}(T_k^{\mu}, \epsilon_k)$.) That the μ-terminals and μ-ends coincide (Condition 3.5(d)) implies that the ϵ_k can be chosen so small that there does not exist any $(\mu - 1)$-section having one incident μ-node in $\mathbf{D}^{\mu}(T_k^{\mu}, \epsilon_k)$ and the same or another incident μ-node in $\mathbf{D}^{\mu}(T_l^{\mu}, \epsilon_l)$, where $k \neq l$. Indeed, if there are any such $(\mu - 1)$-sections, they will be finite in number, for otherwise the removal of only finitely many $(\mu - 1)$-sections would not separate all the μ-terminals, in violation of the coincidence between μ-terminals and μ-ends. Moreover, each $(\mu - 1)$-section has only finitely many incident μ-nodes according to Lemma 2.2. So, we can decrease the ϵ_k sufficiently to eliminate all such $(\mu - 1)$-sections. When this is so, we shall refer to each $\mathbf{D}^{\mu}(T_k^{\mu}, \epsilon_k)$ as an *ϵ_k-vicinity* of T_k^{μ}.

Lemma 5.1. *There are only finitely many μ-nodes lying outside all the $\mathbf{D}^{\mu}(T_k^{\mu}, \epsilon_k)$ for all the μ-terminals T_k^{μ} $(k = 1, \ldots, K)$ of $\mathbf{S}^{\mu}$.*

Proof. Suppose this is not so. Then, the set of μ-nodes lying outside of all the ϵ_k-vicinities is an infinite set $\mathcal{M}$ in the complete metric space consisting of all the μ-nodes of $\mathbf{S}^{\mu}$ along with the μ-terminals of $\mathbf{S}^{\mu}$. $\mathcal{M}$ has no limit point because the only limit points in that metric space are the μ-terminals and these reside only in the $\mathbf{D}^{\mu}(T_k^{\mu}, \epsilon_k)$. Therefore, $\mathcal{M}$ is not relatively compact and thus not totally bounded, in contradiction to Condition 3.5(c). □

Now, let us drop the subscript k and consider an arbitrary ϵ-vicinity $\mathbf{D}^{\mu}(T^{\mu}, \epsilon)$ for a given μ-terminal T^{μ} of a μ-section $\mathbf{S}^{\mu}$. Any $(\mu - 1)$-section in $\mathbf{S}^{\mu}$ that is incident to a μ-node in $\mathbf{D}^{\mu}(T^{\mu}, \epsilon)$ and also incident to a μ-node not in $\mathbf{D}^{\mu}(T^{\mu}, \epsilon)$ will be called a *surface* $(\mu - 1)$-*section* of $\mathbf{D}^{\mu}(T^{\mu}, \epsilon)$. Such surface $(\mu - 1)$-sections are not in $\mathbf{D}^{\mu}(T^{\mu}, \epsilon)$. A μ-node that is incident to a surface $(\mu - 1)$-section and also to a $(\mu - 1)$-section of $\mathbf{D}^{\mu}(T^{\mu}, \epsilon)$ will be called a *surface μ-node* of $\mathbf{D}^{\mu}(T^{\mu}, \epsilon)$. (Surface μ-nodes do belong to $\mathbf{D}^{\mu}(T^{\mu}, \epsilon)$.) Furthermore, we partition the $(\mu-1)$-terminals of a surface μ-node into two sets, those $(\mu-1)$-terminals belonging to surface $(\mu-1)$-sections and those not doing so (that is, those belonging to $(\mu-1)$-sections of $\mathbf{D}^\mu(T^\mu,\epsilon)$). The former $(\mu-1)$-terminals are called *exterior* and the latter *interior surface $(\mu-1)$-terminals* of $\mathbf{D}^\mu(T^\mu,\epsilon)$. By Conditions 2.1(a) and (b) and Lemmas 2.2 and 5.1, $\mathbf{D}^\mu(T^\mu,\epsilon)$ has only finitely many surface $(\mu-1)$-sections, surface μ-nodes, and exterior and interior surface $(\mu-1)$-terminals.

These definitions are illustrated in Fig. 1, which shows some of the infinitely many $(\mu-1)$-sections of a μ-section $\mathbf{S}^\mu$. T^μ, shown on the right, is a μ-terminal for $\mathbf{S}^\mu$, and that diagram illustrates two ϵ_k-vicinities $\mathbf{D}^\mu(T^\mu,\epsilon_1)$ and $\mathbf{D}^\mu(T^\mu,\epsilon_2)$, where $0<\epsilon_2<\epsilon_1$. Of all the indicated $(\mu-1)$-sections only $\mathbf{S}_1^{\mu-1}$, $\mathbf{S}_2^{\mu-1}$, and $\mathbf{S}_4^{\mu-1}$ are surface $(\mu-1)$-sections for $\mathbf{D}^\mu(T^\mu,\epsilon_1)$, and only $\mathbf{S}_3^{\mu-1}$ and $\mathbf{S}_4^{\mu-1}$ are surface $(\mu-1)$-sections for $\mathbf{D}^\mu(T^\mu,\epsilon_2)$. The μ-nodes are denoted by heavy dots and connect adjacent $(\mu-1)$-sections. Some of the μ-nodes incident to the shown surface $(\mu-1)$-sections are labeled 1 and/or 2 to indicate that they are respectively surface μ-nodes for $\mathbf{D}^\mu(T^\mu,\epsilon_1)$ and $\mathbf{D}^\mu(T^\mu,\epsilon_2)$. $\mathbf{S}_4^{\mu-1}$ is a surface $(\mu-1)$-section for both $\mathbf{D}^\mu(T^\mu,\epsilon_1)$ and $\mathbf{D}^\mu(T^\mu,\epsilon_2)$, and therefore some of its incident μ-nodes have both labels accordingly. Note that, among all the nodes of all ranks within or incident to a given $(\mu-1)$-section, the nodes closest to T^μ must be μ-nodes because the only way a path can enter a $(\mu-1)$-section is through an incident μ-node. Also, a $(\mu-1)$-section of $\mathbf{D}^\mu(T^\mu,\epsilon_1)$ (which therefore is not a surface $(\mu-1)$-section of $\mathbf{D}^\mu(T^\mu,\epsilon_1)$) may have some internal nodes (of ranks less than μ) that are further away from T^μ than ϵ_1; one such $(\mu-1)$-section is shown in Fig. 1.

Our next objective is to define recursively the idea of current flowing through a μ-terminal and thereby through a $(\mu+1)$-node. This will allow us assert Kirchhoff's current law at a $(\mu+1)$-node. To this end, let us take as true the following recursive assumptions for all ranks $\rho=-1,0,1,\ldots,\mu-1$.

Properties 5.2.

(a) *For each ρ-terminal T^ρ there is a current i_{T^ρ} leaving T^ρ through the ρ-section $\mathbf{S}^\rho$ to which T^ρ belongs.* (Thus, upon changing the orientation, we get $-i_{T^\rho}$ as the current entering T^ρ through $\mathbf{S}^\rho$.)

(b) *Let T_k^ρ $(k = 1, \ldots, K)$ be the finitely many ρ-terminals belonging to a ρ-section $\mathbf{S}^\rho$ (Condition 2.1(b)), and let i_k be the current leaving T_k^ρ through $\mathbf{S}^\rho$ as in (a). Then, $\sum_{k=1}^K i_k = 0$.*

(c) *Kirchhoff's current law holds at every $(\rho+1)$-node in the following way: Let T_m^ρ $(m = 1, \ldots, M)$ be the finitely many ρ-terminals in a $(\rho+1)$-node $n^{\rho+1}$, and let i_m be the current leaving T_m^ρ as in (a). Then, $\sum_{m=1}^M i_m = 0$.*

Fig. 2 illustrates Properties 5.2(a) and (b) for $\rho = \mu - 1$. Fig. 3 illustrates Properties 5.2(a) and (c) for $\rho = \mu - 1$.

Under Property 5.2(b) we can partition the ρ-terminals of $\mathbf{S}^\rho$ arbitrarily into two sets with index sets K_1 and K_2; then, $\sum_{k \in K_1} i_k = \sum_{k \in K_2}(-i_k)$. We can view $\sum_{k \in K_1} i_k$ as the current passing through $\mathbf{S}^\rho$ from all the T_k^ρ $(k \in K_1)$ to all the T_k^ρ $(k \in K_2)$. (Of course, any i_k may be positive, negative, or 0.)

Properties 5.2(a) and (b) clearly hold when $\rho = -1$, for then $\mathbf{S}^{-1}$ is a branch with two (-1)-terminals; moreover, Property 5.2(c) then asserts Kirchhoff's current law at a 0-node.

Our next step is to show that Property 5.2(a) can be defined in a consistent way for a μ-terminal T^μ. Having chosen two vicinities $\mathbf{D}^\mu(T^\mu, \epsilon_1)$ and $\mathbf{D}^\mu(T^\mu, \epsilon_2)$ of T^μ, where $0 < \epsilon_2 < \epsilon_1$, let $\mathcal{N}_*^\mu$ denote the set of all μ-nodes of $\mathbf{D}^\mu(T^\mu, \epsilon_1)$ that are not in $\mathbf{D}^\mu(T^\mu, \epsilon_2)$ and also all surface μ-nodes of $\mathbf{D}^\mu(T^\mu, \epsilon_2)$. (Such μ-nodes are labeled by an asterisk * in Fig. 1.) Then, let I_* be the sum of all the $(\mu - 1)$-terminal currents leaving the μ-nodes of $\mathcal{N}_*^\mu$. By Property 5.2(c), $I_* = 0$. Furthermore, by Property 5.2(b), the $(\mu - 1)$-terminal currents entering a $(\mu - 1)$-section, all of whose incident μ-nodes are in $\mathcal{N}_*^\mu$, cancel out of I_*. What is left is the sum of the exterior surface $(\mu - 1)$-terminal currents leaving $\mathbf{D}^\mu(T^\mu, \epsilon_1)$ and the interior surface $(\mu - 1)$-terminal currents entering $\mathbf{D}^\mu(T^\mu, \epsilon_2)$, that sum being equal to 0. By Kirchhoff's current law applied to the surface μ-nodes of $\mathbf{D}^\mu(T^\mu, \epsilon_2)$, we can conclude that the total current leaving $\mathbf{D}^\mu(T^\mu, \epsilon_1)$ through its exterior $(\mu - 1)$-terminals (equivalently, through its surface μ-nodes) is equal to the total current leaving $\mathbf{D}^\mu(T^\mu, \epsilon_2)$ through its exterior surface $(\mu - 1)$-terminals (equivalently, through its surface μ-nodes). All this remains true no matter what the values of ϵ_1 and ϵ_2 are so long as $0 < \epsilon_2 < \epsilon_1$ and $\mathbf{D}^\mu(T^\mu, \epsilon_1)$ and $\mathbf{D}^\mu(T^\mu, \epsilon_2)$ are ϵ-vicinities. Because of this, we are justified in treating the current leaving $\mathbf{D}^\mu(T^\mu, \epsilon_1)$ through its exterior surface $(\mu-1)$-terminals (or, equivalently, through its surface μ-nodes) as the current i_{T^μ} leaving T^μ through $\mathbf{S}^\mu$. In this way, we obtain Property 5.2(a) with ρ replaced by μ.

The extension of Property 5.2(b) to the rank μ is much like that for Property 5.2(a). Given a μ-section $\mathbf{S}^\mu$ with the μ-terminals $T_1^\mu, \ldots, T_K^\mu$, choose an ϵ_k-vicinity $\mathbf{D}^\mu(T_k^\mu, \epsilon_k)$ for each T_k^μ. Consider the μ-subnetwork $\mathbf{M}^\mu = \mathbf{S}^\mu \backslash \cup_{k=1}^K \mathbf{D}^\mu(T_k^\mu, \epsilon_k)$ induced by all the branches of $\mathbf{S}^\mu$ not in every $\mathbf{D}^\mu(T_k^\mu, \epsilon_k)$. $\mathbf{M}^\mu$ has only finitely many μ-nodes. Apply Kirchhoff's current law (Property 5.2(c)) at all the μ-nodes of $\mathbf{M}^\mu$ and sum all those equations. That sum equals 0. Then, apply Property 5.2(b) to all the $(\mu-1)$-sections in $\mathbf{M}^\mu$ to cancel terms. It follows that the sum of all $(\mu-1)$-terminal currents leaving all the ϵ_k-vicinities through their interior surface $(\mu-1)$-terminals (or equivalently through their exterior surface $(\mu-1)$-terminals) equals 0. The latter is the sum of the currents leaving all the μ-terminals T_k^μ of $\mathbf{S}^\mu$ according to our extension of Property 5.2(a) obtained above. Thus, we have Property 5.2(b) extended to the rank μ.

On the other hand, Kirchhoff's current law, as expressed by Property 5.2(c), cannot as yet be established for a $(\mu+1)$-node. This will be accomplished when we derive the operating point for our transfinite network $\mathbf{N}^{\mu+1}$. When doing so, we shall employ a different but entirely equivalent form of Kirchoff's current law than the one indicated in Property 5.2(c) (with ρ replaced by μ). Let us derive it here. We shall show that the current i_{T^μ} leaving a μ-terminal through the exterior surface $(\mu-1)$-terminals of $\mathbf{D}^\mu(T^\mu, \epsilon)$ is equal to the sum of the currents in a finite set of branches contained in the surface $(\mu-1)$-sections for $\mathbf{D}^\mu(T^\mu, \epsilon)$. That finite set of branches will be called a *branch-cut* for T^μ. It will separate T^μ from all the other μ-terminals of the μ-section to which T^μ belongs.

We have that i_{T^μ} is by definition the current leaving an ϵ-vicinity $\mathbf{D}^\mu(T^\mu, \epsilon)$ through its exterior surface $(\mu-1)$-terminals. These terminals are finite in number. We can choose an ϵ-vicinity (in general, a different ϵ) of rank $\mu-1$ for each of them. Consider the exterior surface $(\mu-2)$-terminals for all the latter ϵ-vicinities. These too are finite in number. By virtue of Properties 5.2, that is, by virtue of how currents leaving terminals have been defined recursively, we have that i_{T^μ} is equal to the sum of the currents leaving those exterior surface $(\mu - 2)$-terminals.

Continuing in this fashion through decreasing ranks, we eventually arrive at currents leaving finitely many (-1)-terminals; these currents also sum to i_{T^μ}. But, the currents leaving those (-1)-terminals are simply branch currents oriented away from T^μ. In this way, we obtain a finite branch-cut for T^μ, whose branch currents oriented away T^μ sum to i_{T^μ}. The branch-cut resides within the surface $(\mu - 1)$-sections of $\mathbf{D}^\mu(T^\mu, \epsilon)$.

Now, consider any $(\mu + 1)$-node $n^{\mu+1}$. It has finitely many μ-terminals, each of which has a finite branch-cut. Upon taking the union of those branch-cuts, we obtain a finite *branch-cut* $\mathcal{C}_{n^{\mu+1}}$ for $n^{\mu+1}$, which separates $n^{\mu+1}$ from all other $(\mu + 1)$-nodes of $\mathbf{N}^{\nu+1}$ in the sense that any path connecting $n^{\mu+1}$ to any other $(\mu + 1)$-node must pass through a branch of $\mathcal{C}_{n^{\mu+1}}$.

Thus, Kirchhoff's current law for a $(\mu + 1)$-node $n^{\mu+1}$ can be stated as follows: The sum of the currents in the branches of $\mathcal{C}_{n^{\mu+1}}$ all oriented the same way (either away from or toward $n^{\mu+1}$) is equal to 0. This version of Kirchhoff's current law will be established in Section 7 for the nodal rank $\mu + 1$ and thereby inductively for all natural-number nodal ranks.

6 Potentials on the Nodes of $\mathbf{N}^{\nu+1}$

Our objective now is to show that, if Assertion 4.3 holds for the natural-number rank ν, then a $(\nu + 1)$-network $\mathbf{N}^{\nu+1}$ satisfying Conditions 3.5 (with ν replaced by $\nu + 1$) has a potential p that is continuous at all nodes of all ranks from 1 to $\nu + 1$. That potential p will be obtained as the limit of potentials for a sequence of ν-networks obtained by shorting ϵ-vicinities of the $(\nu + 1)$-nodes of $\mathbf{N}^{\nu+1}$ and then contracting those ϵ-vicinites. Our argument will proceed through a series of five steps, which we display below. We assume henceforth that $\mathbf{N}^{\nu+1}$ satisfies Conditions 3.5 with ν replaced by $\nu + 1$. We will need

Lemma 6.1. *For every two nodes m and n of $\mathbf{N}^{\nu+1}$ there is a path terminating at them, and every such path is permissive.*

Proof. There is a two-sided path terminating at m and n by virtue of the connectedness of $\mathbf{N}^\nu$ (Condition 3.5(a)). Therefore, this lemma follows from Lemma 2.3. □

6a. *The set of all permissive nodes in* $\mathbf{N}^{\nu+1}$ *is a compact metric space under* $d^{\nu+1}$*:*

Let $\mathcal{N}_{\mathbf{N}^{\nu+1}}$ denote the set of all permissive nodes of all ranks in $\mathbf{N}^{\nu+1}$. We first show that $\mathcal{N}_{\mathbf{N}^{\nu+1}}$ is a complete metric space under $d^{\nu+1}$. Let $\{n_k\}_{k=1}^{\infty}$ be a Cauchy sequence in $\mathcal{N}_{\mathbf{N}^{\nu+1}}$ under $d^{\nu+1}$. If $\{n_k\}_{k=1}^{\infty}$ lies within some γ-section $\mathbf{S}^{\gamma}$, where $0 \leq \gamma \leq \nu$, then every n_k is of rank no larger than γ, and by [16, Proposition 5.7], $\{n_k\}_{k=1}^{\infty}$ is a Cauchy sequence under d^{γ}. Therefore, $\{n_k\}_{k=1}^{\infty}$ converges to some λ-node n^{λ} in or incident to $\mathbf{S}^{\gamma}$, where $1 \leq \lambda \leq \gamma + 1$, because the union of $\mathcal{N}_{\mathbf{S}^{\gamma}}$ and the set of terminals of $\mathbf{S}^{\gamma}$ is complete under d^{γ} according to our recursive procedure of generating $\mathbf{N}^{\nu}$. (If $\{n_k\}_{k=1}^{\infty}$ converges under d^{γ} to a γ-terminal of $\mathbf{S}^{\gamma}$, it converges under d^{γ} to the $(\gamma+1)$-node containing the γ-terminal.) On the other hand, if $\{n_k\}_{k=1}^{\infty}$ contains nodes in two or more ν-sections of $\mathbf{N}^{\nu+1}$, it can be partitioned into two or more — but finitely many (Lemma 3.6) — subsequences where each subsequence is contained in a ν-section of $\mathbf{N}^{\nu+1}$. These subsequences will be Cauchy under d^{ν} as well, again according to [16, Proposition 5.7]. They must converge to ν-terminals, all of which are in a single $(\nu+1)$-node, for otherwise $\{n_k\}_{k=1}^{\infty}$ would not be Cauchy under $d^{\nu+1}$. Thus, every Cauchy sequence under $d^{\nu+1}$ converges, and $\mathcal{N}_{\mathbf{N}^{\nu+1}}$ is complete.

Now, Condition 3.5(c), as applied to $\mathbf{N}^{\nu+1}$, asserts that, for each ν-section $\mathbf{S}^{\nu}$, $\mathcal{N}_{\mathbf{S}^{\nu}}$ is a totally bounded set under d^{ν}. Since $d^{\nu+1}(m,n) \leq d^{\nu}(m,n)$ for all $m, n \in \mathcal{N}_{\mathbf{S}^{\nu}}$, $\mathcal{N}_{\mathbf{S}^{\nu}}$ is also totally bounded under $d^{\nu+1}$. (Indeed, any finite ϵ-net for $\mathcal{N}_{\mathbf{S}^{\nu}}$ under d^{ν} is a finite ϵ-net for $\mathcal{N}_{\mathbf{S}^{\nu}}$ under $d^{\nu+1}$.) Moreover, since $\mathbf{N}^{\nu+1}$ has only finitely many ν-sections (Lemma 3.6), $\mathcal{N}_{\mathbf{N}^{\nu+1}}$ is totally bounded under $d^{\nu+1}$. By completeness and total-boundedness, we have that $\mathcal{N}_{\mathbf{N}^{\nu+1}}$ is compact under $d^{\nu+1}$.

6b. *Approximating* $\mathbf{N}^{\nu+1}$ *by a sequence of* ν*-networks:*

Let $n_q^{\nu+1}$ $(q = 1, \ldots, Q)$ be the finitely many permissive $(\nu+1)$-nodes of $\mathbf{N}^{\nu+1}$. $\mathbf{D}^{\nu+1}(n_q^{\nu+1}, \epsilon)$ will denote the subnetwork of $\mathbf{N}^{\nu+1}$ induced by all the branches of all the ϵ-vicinities of all the ν-terminals in $n_q^{\nu+1}$. We will call this an *ϵ-vicinity* of $n_q^{\nu+1}$. $\mathbf{D}^{\nu+1}(n_q^{\nu+1}, \epsilon)$ has exactly one $(\nu+1)$-node, namely, $n_q^{\nu+1}$, and infinitely many nodes for each nodal rank less than $\nu+1$. Exactly as for vicinities of terminals, we can define *surface $(\nu-1)$-sections, surface ν-nodes*, and *exterior* and *interior surface $(\nu-1)$-terminals* for $\mathbf{D}^{\nu+1}(n_q^{\nu+1}, \epsilon)$. There are only finitely many of these because $n_q^{\nu+1}$ has only finitely many ν-terminals, which in turn have only finitely many of these entities. Let us choose a null sequence $\{\epsilon_k\}_{k=1}^{\infty}$ with $\epsilon_k > \epsilon_{k+1} > 0$ for all k and $\epsilon_k \to 0$, starting with a sufficiently small ϵ_1 such that $\mathbf{D}^{\nu+1}(n_q^{\nu+1}, \epsilon_k)$ is an ϵ_k-vicinity for every $n_q^{\nu+1}$ and every k. For each k, let us short every $\mathbf{D}^{\nu+1}(n_q^{\nu+1}, \epsilon_k)$ to obtain a ν-network $\mathbf{N}_k^{\nu}$; with k fixed, this "shorting" is accomplished by replacing every $\mathbf{D}^{\nu+1}(n_q^{\nu+1}, \epsilon_k)$ by a ν-node $n_{q,k}^{\nu}$ consisting of all the external $(\nu-1)$-terminals of all the surface nodes of $\mathbf{D}^{\nu+1}(n_q^{\nu+1}, \epsilon_k)$. Every branch and every γ-node $(0 \leq \gamma \leq \nu)$ of $\mathbf{N}^{\nu+1}$ will eventually appear in the $\mathbf{N}_k^{\nu}$ (i.e., eventually lie outside all the ϵ_k-vicinities as $k \to \infty$). In this way, the $\mathbf{N}_k^{\nu}$ form an approximating sequence that fills out $\mathbf{N}^{\nu+1}$; the $(\nu+1)$-nodes never appear in any $\mathbf{N}_k^{\nu}$.

Each $\mathbf{N}_k^{\nu}$ satisfies Conditions 3.5. Indeed, Lemma 5.1 insures the fulfillment of the second part of Condition 3.5(a). Condition 3.5(b) (i.e., Conditions 2.1) are also fulfilled because each $\mathbf{D}^{\nu+1}(n_q^{\nu+1}, \epsilon_k)$ has only finitely many exterior surface $(\nu-1)$-terminals. All the other requirements are obviously fulfilled since $\mathbf{N}^{\nu+1}$ fulfills Conditions 3.5 when ν is replaced by $\nu+1$. So, we can invoke Assertion 4.3, which is our inductive assumption, to affirm that, for each k, $\mathbf{N}_k^{\nu}$ has an operating point (i_k, v_k) satisfying $|i_{k,b}| \leq \boldsymbol{I}_k$ and $|v_{k,b}| \leq \boldsymbol{V}_k$ for every branch b in $\mathbf{N}_k^{\nu}$, where $\boldsymbol{I}_k$ and $\boldsymbol{V}_k$ are the quantities of Condition 3.1(c) for $\mathbf{N}_k^{\nu}$. We will argue in Section 7 that these operating points converge in a certain way to an operating point for $\mathbf{N}^{\nu+1}$ as $k \to \infty$. When doing so, we can replace $\boldsymbol{I}_k$ and $\boldsymbol{V}_k$ by the quantities $\boldsymbol{I}$ and $\boldsymbol{V}$ for $\mathbf{N}^{\nu+1}$ because $\boldsymbol{I}_k \leq \boldsymbol{I}$ and $\boldsymbol{V}_k \leq \boldsymbol{V}$ for all k.

Next, choose and fix a ground node n_g that lies outside every $\mathbf{D}^{\nu+1}(n_q^{\nu+1}, \epsilon_1)$. Every potential discussed below will be measured with respect to n_g as ground; that is, it will take the value 0 at n_g.

By the definition of an operating point, there is, for each k, a potential $\tilde{p}_k$ that assigns a real number to each node of any rank in $\mathbf{N}_k^{\nu}$. The branch voltages in $\mathbf{N}_k^{\nu}$ are obtained from $\tilde{p}_k$, and $\tilde{p}_k$ is continuous on $\mathcal{N}_{\mathbf{N}_k^{\nu}}$ (that is, at all permissive nodes of $\mathbf{N}_k^{\nu}$ with positive ranks). We extend $\tilde{p}_k$ onto the nodes of all ranks within each $\mathbf{D}^{\nu+1}(n_q^{\nu+1}, \epsilon_k)$ by assigning to each such node the value $\tilde{p}_k(n_q^{\nu})$, where n_q^{ν} is the ν-node in $\mathbf{N}_k^{\nu}$ that replaces $\mathbf{D}^{\nu+1}(n_q^{\nu+1}, \epsilon_k)$ when shorting the latter vicinity. Let p_k be the resulting potential; p_k is defined on the set $\mathcal{N}_{\mathbf{N}^{\nu+1}}$ of all the permissive nodes of all ranks in $\mathbf{N}^{\nu+1}$, including its $(\nu+1)$-nodes, and p_k

is constant on the nodes of each $\mathbf{D}^{\nu+1}(n_q^{\nu+1}, \epsilon_k)$. Hence, it is continuous on $\mathcal{N}_{\mathbf{N}^{\nu+1}}$ since it is continuous on $\mathcal{N}_{\mathbf{N}_k^\nu}$.

We will invoke the Arzela-Ascoli theorem in order to show that a subsequence of $\{p_k\}_{k=1}^\infty$ converges uniformly to a continuous potential on $\mathcal{N}_{\mathbf{N}^{\nu+1}}$. Items 6c and 6d below constitute the hypothesis of that theorem.

6c. *$\{p_k\}_{k=1}^\infty$ is pointwise (i.e., nodewise) relatively compact:*

Consider any p_k (k fixed). For any branch b embraced by a vicinity $\mathbf{D}^{\nu+1}(n_q^{\nu+1}, \epsilon_k)$, its branch voltage is $v_{k,b} = 0$ because p_k is constant on that vicinity. Next, assume that b lies outside every $\mathbf{D}^{\nu+1}(n_q^{\nu+1}, \epsilon_k)$, and apply Assertion 4.3 to $\mathbf{N}_k^\nu$. (Recall the linear branch resistances R_b assigned just after Conditions 3.1.) If M_b has a finite Lipschitz constant on $[-\boldsymbol{I}, \boldsymbol{I}]$, then $|v_{k,b}| \leq R_b(|i_{k,b}| + \delta'(0,0,M_b)) \leq 2\boldsymbol{I}R_b$. On the other hand, if M_b does not have a finite Lipschitz constant on $[-\boldsymbol{I}, \boldsymbol{I}]$, we can nonetheless write $|v_{k,b}| \leq \boldsymbol{V}$. Then, for any node $n \in \mathcal{N}_{\mathbf{N}^{\nu+1}}$, we may choose a permissive path P from n to n_g according to Lemma 6.1. (It was with the R_b that permissive paths were defined.) Let $\mathcal{B}_1$ (resp. $\mathcal{B}_2$) be the set of branches in P having (resp. not having) finite Lipschitz constants on $[-\boldsymbol{I}, \boldsymbol{I}]$. Recall that, for $b \in \mathcal{B}_2$, $R_b = 1$. We may now bound branch voltages along P to write

$$|p_k(n)| \leq \sum_{b \dashv P} |v_{k,b}| \leq 2\boldsymbol{I} \sum_{b \in \mathcal{B}_1} R_b + \sum_{b \in \mathcal{B}_2} \boldsymbol{V} \leq \max(2\boldsymbol{I}, \boldsymbol{V}) \sum_{b \dashv P} R_b.$$

Since the right-hand side is finite and independent of k, we can conclude that $\{p_k(n)\}_{k=1}^\infty$ is a bounded set in $\boldsymbol{R}$ and therefore that $\{p_k\}_{k=1}^\infty$ is a pointwise relatively compact set of potentials on $\mathcal{N}_{\mathbf{N}^{\nu+1}}$.

6d. *$\{p_k\}_{k=1}^\infty$ is equicontinuous at each node of positive rank:*

Consider any node n of positive rank in $\mathbf{N}^{\nu+1}$. Choose any $\epsilon > 0$, and set $\delta = \epsilon / \max(2\boldsymbol{I}, \boldsymbol{V})$. Now, for any node m in $\mathbf{N}^{\nu+1}$ such that $d^{\nu+1}(m,n) < \delta$, take a path P terminating at m and n such that $\sum_{b \dashv P} R_b < \delta$. Then, as above we have for all k

$$|p_k(m) - p_k(n)| \leq \max(2\boldsymbol{I}, \boldsymbol{V}) \sum_{b \dashv P} R_b < \epsilon. \tag{2}$$

This proves the assertion 6d.

6e. *The existence of a continuous potential p on $\mathcal{N}_{\mathbf{N}^{\nu+1}}$:*

Let $\mathcal{C}$ be the complete metric space of all continuous functions mapping $\mathcal{N}_{\mathbf{N}^{\nu+1}}$ into $\boldsymbol{R}$. The metric for $\mathcal{N}_{\mathbf{N}^{\nu+1}}$ is $d^{\nu+1}$ as always, and the metric $d_{\mathcal{C}}$ for $\mathcal{C}$ is taken to be

$$d_{\mathcal{C}}(p_1, p_2) \;=\; \sup\{|p_1(n) - p_2(n)| : n \in \mathcal{N}_{\mathbf{N}^{\nu+1}}\}.$$

By virtue of 6a, 6c and 6d above and the Arzela-Ascoli theorem [8, page 149], we can conclude that $\{p_k\}_{k=1}^{\infty}$ is a relatively compact subset of $\mathcal{C}$. Thus, it contains a subsequence that converges in $\mathcal{C}$ under the metric $d_{\mathcal{C}}$. Let p be its limit in $\mathcal{C}$. Thus, p is a continuous potential on $\mathcal{N}_{\mathbf{N}^{\nu+1}}$.

7 The Existence of a Solution for $\mathbf{N}^{\nu+1}$

So far, we have found a subsequence $\{p_{k_j}\}_{j=1}^{\infty}$ of $\{p_k\}_{k=1}^{\infty}$ that converges uniformly to a continuous potential p on $\mathcal{N}_{\mathbf{N}^{\nu+1}}$. Hence, for each branch b of $\mathbf{N}^{\nu+1}$, the corresponding subsequence of branch voltages $\{v_{k_j,b}\}_{j=1}^{\infty}$ converges to a voltage vector v for $\mathbf{N}^{\nu+1}$. By Assertion 4.3, $|v_{k_j,b}| \le \boldsymbol{V}_k \le \boldsymbol{V}$ for all k_j, and therefore $|v_b| \le \boldsymbol{V}$. Again by Assertion 4.3, there is for each k_j a current vector i_{k_j} satisfying $|i_{k_j,b}| \le \boldsymbol{I}_k \le \boldsymbol{I}$. Thus, the corresponding current vectors satisfy $i_{k_j} \in [-\boldsymbol{I}, \boldsymbol{I}]^{\mathcal{B}}$. By Tychonoff's theorem [10, page 119], $[-\boldsymbol{I}, \boldsymbol{I}]^{\mathcal{B}}$ is compact under the product topology. Therefore, we can choose a subsequence $\{p_l^*\}_{l=1}^{\infty}$ of $\{p_{k_j}\}_{j=1}^{\infty}$ such that the current vectors corresponding to $\{p_l^*\}_{l=1}^{\infty}$ converge in the product topology to a current vector $i \in [-\boldsymbol{I}, \boldsymbol{I}]^{\mathcal{B}}$ and therefore converge branchwise as well.

At the end of Section 5 we noted that, for each $(\mu+1)$-node $n^{\mu+1}$ $(1 \le \mu+1 \le \nu+1)$, there is a finite branch cut $C_{n^{\mu+1}}$, which isolates $n^{\mu+1}$ from all other $(\mu+1)$-nodes; moreover, the branch currents for $C_{n^{\mu+1}}$ measured, say, away from $n^{\mu+1}$ sum to the current incident away from $n^{\mu+1}$. By Assertion 4.3 again, the current vector i_l^* corresponding to p_l^* satisfies Kirchhoff's current law on $C_{n^{\mu+1}}$ for all l sufficiently large. By the branchwise convergence noted above, i too satisfies Kirchhoff's current law on $C_{n^{\mu+1}}$ and thus at every $(\mu+1)$-node. Similarly, i satisfies Kirchhoff's current law at every 0-node as well.

Finally, by Assertion 4.3 once again, for each branch b the branch current and branch voltage corresponding to p_l^* comprise a point on M_b again for all l sufficiently large. We have shown that such points converge in $\boldsymbol{R}^2$ as $l \to \infty$. Since M_b is a closed set, the limit point (i_b, v_b) is also a member of M_b.

Altogether then, we have proven that $\mathbf{N}^{\nu+1}$ has a solution (i.e., an operating point) as defined at the end of Section 3 and is bounded as stated above. By induction, this is true for all natural-number ranks of our transfinite networks. We restate this as the following one of two main theorems of this paper, where now $\nu+1$ is replaced by ν. (The second main theorem is given in Section 10.

Theorem 7.1. *Let* $\mathbf{N}^{\nu}$ *be a transfinite electrical network of arbitrary natural-number rank* ν, *and let* $\mathbf{N}^{\nu}$ *satisfy Conditions 3.5. Then,* $\mathbf{N}^{\nu}$ *has an operating point* (i,v), *and, for every branch* b, $|i_b| \leq \boldsymbol{I}$ *and* $|v_b| \leq \boldsymbol{V}$.

Finally, let us explicitly note the satisfaction of Kirchhoff's voltage law. Let m and n be any two nodes of any ranks in $\mathbf{N}^{\nu}$, and let P be a path terminating at m and n; P is permissive (Lemma 6.1). Orient P from m to n. By Theorem 7.1 and the definition of an operating point, we have a potential $p: \mathcal{N}_{\mathbf{N}^{\nu}} \rightsquigarrow \boldsymbol{R}$ with p being continuous at every node of rank larger than 0. Then, as before, we have

$$p(m) - p(n) \;=\; \sum_{b \dashv P} v_b$$

where every branch $b \dashv P$ is oriented conformably with P.

By Lemma 2.3, every loop L of any rank in $\mathbf{N}^{\nu}$ is *permissive*. Orient L. Since L is the union of two oppositely oriented paths with common terminal nodes, it follows that, if Theorem 7.1 holds, then Kirchhoff's voltage law:

$$\sum_{b \dashv L} v_b \;=\; 0 \qquad (3)$$

is satisfied around every loop L. (In (3), L and every $b \dashv L$ are oriented the same way.)

8 Uniqueness of an Operating Point for $\mathbf{N}^{\nu}$

We have established the existence of an operating point for $\mathbf{N}^{\nu}$ according to Theorem 7.1, but not its uniqueness. Indeed, that operating point need not be unique. For example, suppose there is a 0-loop L^0 in $\mathbf{N}^{\nu}$ such that, for every $b \dashv L^0$, M_b has a horizontal segment with (i_b, v_b) being inside that segment, where (i_b, v_b) is a branch voltage-current pair corresponding to an operating point for $\mathbf{N}^{\nu}$. Then, for each $b \dashv L^0$ we can shift (i_b, v_b)

on that horizontal segment without altering any v_b and thereby the potential for that operating point. If the shifts are all the same for every $b \dashv L^0$, we have in fact introduced a loop current around L^0. The new branch currents will again satisfy Kirchhoff's current law. Moreover, the shifts may be small enough to satisfy the bounds $|i_b| \leq \boldsymbol{I}$. In this way, we may find another operating point for $\mathbf{N}^\nu$.

Similarly, if there is a finite branch-cut $\mathcal{C}^0$ that separates $\mathbf{N}^\nu$ into two components and if, for every $b \in \mathcal{C}^0$, M_b has a vertical segment with (i_b, v_b) being inside that segment, where again (i_b, v_b) corresponds to an operating point for $\mathbf{N}^\nu$, we can vertically shift (i_b, v_b) for each $b \dashv \mathcal{C}^0$ by the same amount on each segment to find a new operating point for $\mathbf{N}^\nu$, and the bound $|v_b| \leq \boldsymbol{V}$ might still be satisfied. In this case, the potential for $\mathbf{N}^\nu$ will change, but by the same amount across $\mathcal{C}^0$. The new potential will still be continuous at all transfinite nodes.

A sufficient condition for the uniqueness of the operating point can be stated. Specifically, the requirement is that every branch characteristic M_b have neither a horizontal segment nor a vertical segment. This is equivalent to the hypothesis on M_b in the next theorem.

A branch-cut $\mathcal{C}_{n^\mu}$ for a μ-node was defined at the end of Section 5. $\mathcal{C}_{n^\mu}$ is a finite set of branches that separates n^μ from all other λ-nodes ($\mu \leq \lambda \leq \nu$) of $\mathbf{N}^\nu$. We will need the ideas of the "interior" and "exterior" of $\mathcal{C}_{n^\mu}$ and of the corresponding "0-node-cut" for n^μ. The *interior* of $\mathcal{C}_{n^\mu}$ is the set of all nodes of all ranks (perforce no larger than μ) for which there exist paths connecting those nodes to n^μ that do not pass through any branch of $\mathcal{C}_{n^\mu}$. The *exterior* of $\mathcal{C}_{n^\mu}$ is the set of all nodes of all ranks not in the interior of $\mathcal{C}_{n^\mu}$. Furthermore, the *0-node-cut* $\mathcal{C}^0_{n^\mu}$ corresponding to $\mathcal{C}_{n^\mu}$ will be the set of all 0-nodes in the interior of $\mathcal{C}_{n^\mu}$ that are incident to the branches of $\mathcal{C}_{n^\mu}$. $\mathcal{C}^0_{n^\mu}$ is also a finite set. When saying that Kirchhoff's current law is applied to $\mathcal{C}^0_{n^\mu}$, we will mean that it is applied to $\mathcal{C}_{n^\mu}$ and that the sum of branch currents for $\mathcal{C}_{n^\mu}$ oriented toward n^μ (i.e., oriented toward the 0-node-cut) is equal to 0. Note also that by the continuity of the potential p we can choose $\mathcal{C}^0_{n^\mu}$ close enough to n^μ to make $\max\{|p(n^0) - p(n^\mu)| : n^0 \in \mathcal{C}^0_{n^\mu}\}$ as small as we wish.

Theorem 8.1. *Under the hypothesis of Theorem 7.1, assume furthermore that, for* *every branch b, M_b is a continuous, strictly monotonically increasing function of i_b. Then, the operating point (i, v) for $\mathbf{N}^\nu$ is unique.*

Note. The domain and range of M_b as a function of i_b may be proper subsets of $\boldsymbol{R}$.

Proof. Suppose that (i, v) and $(i + \Delta i, v + \Delta v)$ are two different operating points for $\mathbf{N}^\nu$. Orient every branch b such that $\Delta i \geq 0$. Then, $\Delta v \geq 0$ too. In fact, for every branch b, either $\Delta i_b = \Delta v_b = 0$ or $\Delta i_b > 0$ and $\Delta v_b > 0$. Furthermore, Δi satisfies Kirchhoff's current law, and Δv is generated by a continuous potential Δp on $\mathcal{N}_{\mathbf{N}^\nu}$.

Suppose furthermore that there is a branch a with $\Delta i_a > 0$ and $\Delta v_b > 0$. Let n_1^0 and n_2^0 be the 0-nodes incident to a with $\Delta p(n_1^0) < \Delta p(n_2^0)$. Let W^ν denote the open interval $(\Delta p(n_1^0), \Delta p(n_2^0))$. By virtue of the continuity of Δp and the fact that there are only finitely many ν-nodes in $\mathbf{N}^\nu$, we can choose a 0-node-cut $\mathcal{C}_{n^\nu}^0$ around each ν-node n^ν of $\mathbf{N}^\nu$ close enough to n^ν to satisfy the following condition: There is a nonvoid open subinterval $W^{\nu-1}$ of W^ν such that, for each 0-node-cut $\mathcal{C}_{n^\nu}^0$, either $\Delta p(n^0) < \inf W^{\nu-1}$ for all $n^0 \in \mathcal{C}_{n^\nu}^0$ or $\Delta p(n^0) > \sup W^{\nu-1}$ for all $n^0 \in \mathcal{C}_{n^\nu}^0$. Δi satisfies Kirchhoff's current law on every $\mathcal{C}_{n^\nu}^0$.

Now, consider the intersection $\mathcal{X}^{\nu-1}$ of all the exteriors of all the branch-cuts $\mathcal{C}_{n^\nu}$ corresponding to $\mathcal{C}_{n^\nu}^0$. $\mathcal{X}^{\nu-1}$ will contain nodes of rank $\nu - 1$ (otherwise, $\mathcal{C}_{n^\nu}$ would not separate n^ν from all other ν-nodes), and those $(\nu-1)$ nodes will be finite in number (this follows from Condition 3.5(c) and an argument just like the proof of Lemma 5.1). Choose a 0-node-cut $\mathcal{C}_{n^{\nu-1}}^0$ around each $(\nu - 1)$-node in $\mathcal{X}^{\nu-1}$ close enough to $n^{\nu-1}$ to satisfy the following condition: There is a nonvoid open subinterval $W^{\nu-2}$ of $W^{\nu-1}$ such that, for each 0-node-cut $\mathcal{C}_{n^{\nu-1}}^0$, either $\Delta p(n^0) < \inf W^{\nu-2}$ for all $n^0 \in \mathcal{C}_{n^{\nu-1}}^0$ or $\Delta p(n^0) > \sup W^{\nu-2}$ for all $n^0 \in \mathcal{C}_{n^{\nu-1}}^0$. Again, Δi satisfies Kirchhoff's current law on every $\mathcal{C}_{n^{\nu-1}}^0$.

Next, we argue in the same way for the intersection $\mathcal{X}^{\nu-2}$ of all the exteriors of all the $\mathcal{C}_{n^\nu}$ and $\mathcal{C}_{n^{\nu-1}}$, where $\mathcal{C}_{n^{\nu-1}}$ is the branch-cut corresponding to $\mathcal{C}_{n^{\nu-1}}^0$. Again, $\mathcal{X}^{\nu-2}$ will contain at least one and no more than finitely many $(\nu - 2)$-nodes.

Continuing in this way, we can proceed through decreasing ranks to reach the rank 0. The resulting intersection $\mathcal{X}^0$ of all exteriors will contain only finitely many 0-nodes. Furthermore, there will be a nonvoid open subinterval W^0 with $W^0 \subset W^1 \subset \cdots \subset W^{\nu-1} \subset W^\nu$ such that, for every rank $\gamma = 1, \ldots, \nu$ and for every 0-node-cut $\mathcal{C}_{n^\gamma}^0$, either $\Delta p(n^0) <$

$\inf W^0$ for all $n^0 \in \mathcal{C}^0_{n^\gamma}$ or $\Delta p(n^0) > \sup W^0$ for all $n^0 \in \mathcal{C}^0_{n^\gamma}$. Also, Kirchhoff's current law will be satisfied at every $\mathcal{C}^0_{n^\gamma}$. Note, moreover, that there are only finitely many 0-node-cuts $\mathcal{C}^0_{n^\gamma}$ of all ranks $\gamma = 1, \ldots, \nu$.

Next, let $\mathcal{Y}$ denote the union of the set of all 0-nodes n^0 of $\mathcal{X}^0$ satisfying $\Delta p(n^0) < \inf W^0$ and all 0-node-cuts $\mathcal{C}^0_{n^\gamma}$ of all ranks γ satisfying $\max\{\Delta p(n^0) \colon n^0 \in \mathcal{C}^0_{n^\gamma}\} < \inf W^0$. Now, sum the equations obtained by applying Kirchhoff's current law to all members of $\mathcal{Y}$. That sum $I_{\mathcal{Y}}$ of branch currents will equal 0.

On the other hand, every branch having both of its 0-nodes in $\mathcal{Y}$ will contribute two current terms to $I_{\mathcal{Y}}$, one the negative of the other. Therefore, $I_{\mathcal{Y}}$ is also equal to the sum of branch currents for all branches having one node in $\mathcal{Y}$ and the other node with a potential value greater than $\sup W^0$. The branch a in one of them. Since every branch has been oriented from its node of higher potential value to its node of lower potential value, we have that $I_{\mathcal{Y}} \geq \Delta i_a > 0$. This is a contradiction. The theorem is proven. □

PART III: ω-NETWORKS

We now extend our results to an ω-network $\mathbf{N}^\omega$ satisfying Conditions 3.5 with $\nu = \omega$. Much of the needed arguments are the same as those of Part II with merely changes of notations and wording, but there are significant differences. So, in this Part III we shall explicate the arguments that are substantially different from the preceding and will merely summarize the arguments that are essentially the same.

9 Current Flows through ω-Networks

Consider an $\vec{\omega}$-section $\mathbf{S}^{\vec{\omega}}$ in $\mathbf{N}^\omega$, and let $\mathcal{T}^{\vec{\omega}} = \{T_k^{\vec{\omega}}\}_{k=1}^K$ be the finite set of ω-terminals $T_k^{\vec{\omega}}$ of $\mathbf{S}^{\vec{\omega}}$. Also, choose any μ-node n^μ embraced by $\mathbf{S}^{\vec{\omega}}$, where μ is a natural number, and set

$$d^{\vec{\omega}}(n^\mu, \mathcal{T}^{\vec{\omega}}) \;=\; \min_{1 \leq k \leq K} d^{\vec{\omega}}(n^\mu, T_k^{\vec{\omega}}),$$

where as always $d^{\vec{\omega}}$ is the metric (1) (with $\mu = \vec{\omega}$) on the nodes and $\vec{\omega}$-terminals of $\mathbf{S}^{\vec{\omega}}$. $d^{\vec{\omega}}(n^\mu, \mathcal{T}^{\vec{\omega}})$ is a positive number. Finally, set

$$x_\mu \;=\; \sup\{d^{\vec{\omega}}(n^\mu, \mathcal{T}^{\vec{\omega}}) \colon n^\mu \dashv \mathbf{S}^{\vec{\omega}}\}.$$

Lemma 9.1. *$x_\mu \to 0$ as $\mu \to \infty$ (i.e., as μ increases through all the natural numbers).*

Note: This lemma says that all the μ-nodes of $\mathbf{S}^{\vec{\omega}}$ are contained in smaller and smaller $d^{\vec{\omega}}$-spheres around the $T_k^{\vec{\omega}}$ as $\mu \to \infty$.

Proof. Suppose this lemma is not true. Then there exists an $\epsilon > 0$ and a subsequence $\{\mu_j\}_{j=1}^\infty$ of the sequence $\{\mu\}_{\mu=1}^\infty$ such that $x_{\mu_j} > \epsilon$ for all j. Therefore, for each j, there is a μ_j-node n^{μ_j} for which $d^{\vec{\omega}}(n^{\mu_j}, \mathcal{T}^{\vec{\omega}}) > \epsilon$. Now, $\{n^{\mu_j}\}_{j=1}^\infty$ is an infinite subset of the complete metric space consisting of all nodes of all ranks in $\mathbf{S}^{\vec{\omega}}$ along with the $\vec{\omega}$-terminals of $\mathbf{S}^{\vec{\omega}}$. By Condition 3.5(c) (with μ replaced by $\vec{\omega}$), $\{n^{\mu_j}\}_{j=1}^\infty$ has a limit point in that metric space. That limit point must be an $\vec{\omega}$-terminal $T_L^{\vec{\omega}}$ of $\mathbf{S}^{\vec{\omega}}$ because the ranks of the n^{μ_j} increase indefinitely and no node of natural-number rank can be the limit of such a sequence under Condition 2.1(a). (If $\{n^{\mu_j}\}_{j=1}^\infty$ were to converge to a node n^λ of natural-number rank λ, $\{n^{\mu_j}\}_{j=1}^\infty$ would have to be contained eventually in a $(\lambda+1)$-section; the nodes incident to $\mathbf{S}^{\lambda+1}$ are of ranks no larger than $\lambda+2$, and this violates the fact that $\mu_j \to \infty$.) Thus, $T_L^{\vec{\omega}}$ must be different from all the $\vec{\omega}$-terminals of $\mathbf{S}^{\vec{\omega}}$, a contradiction. □

Let us now define "vicinities" for the $\vec{\omega}$-terminals of $\mathbf{S}^{\vec{\omega}}$. The construction here is substantially different from that of an ϵ-vicinity of a terminal with a natural-number rank. Instead of depending upon a choice of $\epsilon > 0$ as before, the present vicinity will depend upon a choice of a section of sufficiently large rank in a chosen nested sequence of sections (recall the construction given in Section 3).

So, choose any 0-node n^0, and let $\mathcal{S}_{n^0} = \{\mathbf{S}_{n^0}^\mu\}_{\mu=0}^\infty$ be the resulting nested sequence of sections induced by n^0. Next, let μ be so large that $\mathbf{S}_{n^0}^\mu$ isolates all $\vec{\omega}$-terminals of $\mathbf{S}^{\vec{\omega}}$. (This can always be done; see Condition 3.5(d) and Lemma 3.4.) Also, let $\mathbf{S}^{\vec{\omega}} \backslash \mathbf{S}_{n^0}^\mu$ denote the $\vec{\omega}$-subnetwork of $\mathbf{S}^{\vec{\omega}}$ induced by all branches of $\mathbf{S}^{\vec{\omega}}$ not in $\mathbf{S}_{n^0}^\mu$. Thus, every $\vec{\omega}$-component of $\mathbf{S}^{\vec{\omega}} \backslash \mathbf{S}_{n^0}^\mu$ has exactly one $\vec{\omega}$-terminal and therefore exactly one $\vec{\omega}$-end (Condition 3.5(d)). For each $\vec{\omega}$-terminal $T^{\vec{\omega}}$ of $\mathbf{S}^{\vec{\omega}}$, the $\vec{\omega}$-component of $\mathbf{S}^{\vec{\omega}} \backslash \mathbf{S}_{n^0}^\mu$ having $T^{\vec{\omega}}$ as its $\vec{\omega}$-terminal will be called an (n^0, μ)-*vicinity* of $T^{\vec{\omega}}$ and will be denoted by $\mathbf{D}^{\vec{\omega}}(T^{\vec{\omega}}, n^0, \mu)$. It will always be understood that the isolation of all the $\vec{\omega}$-terminals of $\mathbf{S}^{\vec{\omega}}$ occurs whenever we refer to vicinities of the $\vec{\omega}$-terminals of $\mathbf{S}^{\vec{\omega}}$.

Note also that, if $e^{\vec{\omega}}$ is the $\vec{\omega}$-end of $\mathbf{S}^{\vec{\omega}}$ corresponding to $T^{\vec{\omega}}$, then $e^{\vec{\omega}}(\mathbf{S}_{n^0}^\mu) = \mathbf{D}^{\vec{\omega}}(T^{\vec{\omega}}, n^0, \mu)$.

By virtue of Lemma 3.3, given any two 0-nodes n_1^0 and n_2^0, $\mathbf{D}^{\vec{\omega}}(T^{\vec{\omega}}, n_1^0, \mu)$ and $\mathbf{D}^{\vec{\omega}}(T^{\vec{\omega}}, n_2^0, \mu)$ eventually coincide for all μ sufficiently large, this being true for every $\vec{\omega}$-terminal $T^{\vec{\omega}}$ of $\mathbf{S}^{\vec{\omega}}$. However, there will not be any μ for which the (n^0, μ)-vicinities of $T^{\vec{\omega}}$ coincide for all the 0-nodes n^0 in $\mathbf{S}^{\vec{\omega}}$.

These vicinites for $\vec{\omega}$-terminals induce vicinities for ω-nodes. Given any ω-node n^ω, let $T_m^{\vec{\omega}}$ $(m = 1, \ldots, M)$ be its finitely many $\vec{\omega}$-terminals. Choose a 0-node n_m^0 in each $\vec{\omega}$-section incident to n^ω. Thus, $n_{m_1}^0 = n_{m_2}^0$ if $T_{m_1}^{\vec{\omega}}$ and $T_{m_2}^{\vec{\omega}}$ belong to the same $\vec{\omega}$-section. Then, choose μ sufficiently large to isolate each $T_m^{\vec{\omega}}$ within the $\vec{\omega}$-section to which it belongs. We now define a *vicinity* $\mathbf{D}^{\vec{\omega}}(n^\omega, \mu)$ of n^ω as the ω-subnetwork of $\mathbf{N}^\omega$ induced by all the branches in all the $\mathbf{D}^{\vec{\omega}}(T_m^{\vec{\omega}}, n^0, \mu)$, where $m = 1, \ldots, M$. Thus, $\mathbf{D}^\omega(n^\omega, \mu)$ embraces n^ω, in contrast to the $\mathbf{D}^{\vec{\omega}}(T_m^{\vec{\omega}}, n^0, \mu))$, which do not embrace n^ω. (Note that we have suppressed the n_m^0 in the notation for $\mathbf{D}^\omega(n^\omega, \mu)$. That $\mathbf{D}^\omega(n^\omega, \mu)$ depends upon the choice of $\{n_m^0\}_{m=1}^M$ will be understood. For two different choices, the $\mathbf{D}^\omega(n^\omega, \mu)$ will be the same for all μ sufficiently large, by virtue of Lemma 3.3.)

Having chosen the μ-section $\mathbf{S}_{n^0}^\mu$ that determines the (n^0, μ)-vicinities of all the $\vec{\omega}$-terminals of $\mathbf{S}^{\vec{\omega}}$, consider the $(\mu + 1)$-nodes incident to $\mathbf{S}_{n^0}^\mu$. These are finite in number. Given an $\vec{\omega}$-terminal $T^{\vec{\omega}}$ of $\mathbf{S}^{\vec{\omega}}$, the $(\mu + 1)$-nodes incident to both $\mathbf{S}_{n^0}^\mu$ and $\mathbf{D}^{\vec{\omega}}(T^{\vec{\omega}}, n^0, \mu)$ will be called the *surface nodes* of $\mathbf{D}^{\vec{\omega}}(T^{\vec{\omega}}, n^0, \mu)$. Those surface nodes belong to (i.e., are embraced by) $\mathbf{D}^{\vec{\omega}}(T^{\vec{\omega}}, n^0, \mu)$ and are incident to both $\mathbf{S}_{n^0}^\mu$ and to one or more μ-sections in $\mathbf{D}^{\vec{\omega}}(T^{\vec{\omega}}, n^0, \mu)$. Consider any surface node $n^{\mu+1}$ of $\mathbf{D}^{\vec{\omega}}(T^{\vec{\omega}}, n^0, \mu)$ and its μ-terminals; such a μ-terminal belonging to (resp. not belonging to) a μ-section in $\mathbf{D}^{\vec{\omega}}(T^{\vec{\omega}}, n^0, \mu)$ will be called an *interior* (resp. *exterior*) *surface μ-terminal* of $n^{\mu+1}$ and also of $\mathbf{D}^{\vec{\omega}}(T^{\vec{\omega}}, n^0, \mu)$. There are only finitely many interior and exterior surface μ-terminals of $\mathbf{D}^{\vec{\omega}}(T^{\vec{\omega}}, n^0, \mu)$ by virtue of Lemma 2.2 applied to $\mathbf{S}_{n^0}^\mu$ and of Condition 2.1(c).

Fig. 4 illustrates these ideas. $\mathbf{S}_{n^0}^\mu$ and $\mathbf{S}_{n^0}^{\mu+1}$ are two successive sections in a nested sequence of sections for the $\vec{\omega}$-section $\mathbf{S}^{\vec{\omega}}$. Thus, $\mathbf{S}_{n^0}^\mu \subset \mathbf{S}_{n^0}^{\mu+1} \subset \mathbf{S}^{\vec{\omega}}$. $\mathbf{D}^{\vec{\omega}}(T^{\vec{\omega}}, n^0, \mu)$ and $\mathbf{D}^{\vec{\omega}}(T^{\vec{\omega}}, n^0, \mu + 1)$ consist of all the sections to the right of $\mathbf{S}_{n^0}^\mu$ and of $\mathbf{S}_{n^0}^{\mu+1}$ respectively. The indicated $\mathbf{S}_k^\mu$ $(k = 1, \ldots, 5)$ are some of the μ-sections in $\mathbf{D}^{\vec{\omega}}(T^{\vec{\omega}}, n^0, \mu)$. $n_1^{\mu+1}$ and $n_2^{\mu+1}$ are the surface nodes of $\mathbf{D}^{\vec{\omega}}(T^{\vec{\omega}}, n^0, \mu)$. The exterior terminals of $n_1^{\mu+1}$ and $n_2^{\mu+2}$ (and also of $\mathbf{D}^{\vec{\omega}}(T^{\vec{\omega}}, n^0, \mu))$ belong to $\mathbf{S}^{\mu}_{n^0}$. Each interior terminal of $n_1^{\mu+1}$ and $n_2^{\mu+1}$ (and also of $\mathbf{D}^{\vec{\omega}}(T^{\vec{\omega}}, n^0, \mu))$ belongs to one of $\mathbf{S}^{\mu}_1$, $\mathbf{S}^{\mu}_2$, and $\mathbf{S}^{\mu}_3$. The surface nodes $n_3^{\mu+2}$ and $n_4^{\mu+2}$ for $\mathbf{D}^{\vec{\omega}}(T^{\vec{\omega}}, n^0, \mu+1)$ are also shown.

Let us now discuss the flow of current at an $\vec{\omega}$-terminal $T^{\vec{\omega}}$. It is now assumed that Kirchhoff's current law holds at every node of every natural-number rank μ. We have already defined the currents incident away from μ-terminals. We now define the *current incident away from* $T^{\vec{\omega}}$ as the sum of the currents incident away from the (finitely many) exterior surface μ-terminals of $\mathbf{D}^{\vec{\omega}}(T^{\vec{\omega}}, n^0, \mu)$. Since Kirchhoff's current law holds at every surface node of $\mathbf{D}^{\vec{\omega}}(T^{\vec{\omega}}, n^0, \mu)$, the current incident away from $T^{\vec{\omega}}$ is also equal to the sum of the currents incident toward the interior surface μ-terminals of $\mathbf{D}^{\vec{\omega}}(T^{\vec{\omega}}, n^0, \mu)$. (We can also interchange "away from" and"toward" throughout the last three sentences.)

We check that this definition is consistent for various choices of n^0 and μ as follows. First, let n^0 be fixed. Let $\mathcal{F}^{\mu}$ be any finite subset of the set of μ-sections in $\mathbf{D}^{\vec{\omega}}(T^{\vec{\omega}}, n^0, \mu) \backslash \mathbf{D}^{\vec{\omega}}(T^{\vec{\omega}}, n^0, \mu+1)$. $\mathbf{S}^{\mu}_{n^0}$ is not a member of $\mathcal{F}^{\mu}$. Consider the set $\mathcal{M}^{\mu+1}$ of $(\mu+1)$-nodes that are either incident to μ-sections of $\mathcal{F}^{\mu}$ or are surface μ-nodes of $\mathbf{D}^{\vec{\omega}}(T^{\vec{\omega}}, n^0, \mu)$, or both. Any μ-terminal of a $(\mu+1)$-node in $\mathcal{M}^{\mu+1}$ that does not belong to any μ-section in $\mathcal{F}^{\mu} \cup \{\mathbf{S}^{\mu}_{n^0}\}$ will be called an *adjoining μ-terminal* of $\mathcal{F}^{\mu} \cup \{\mathbf{S}^{\mu}_{n^0}\}$. Apply Kirchhoff's current law to all the $(\mu+1)$-nodes in $\mathcal{M}^{\mu+1}$, and then sum those equations. The result equals 0. By the cancellation of μ-terminal currents within each μ-section of $\mathcal{F}^{\mu}$ (Property 5.2(b)), we are left with the currents incident away from the exterior μ-terminals of $\mathbf{D}^{\vec{\omega}}(T^{\vec{\omega}}, n^0, \mu)$ and also the currents incident away from the adjoining μ-terminals of $\mathcal{F}^{\mu} \cup \{\mathbf{S}^{\mu}_{n^0}\}$, and these currents sum to 0 too. Since this is true for every choice of $\mathcal{F}^{\mu}$, we can take the sum of the currents incident away from the exterior surface μ-terminals of $\mathbf{D}^{\vec{\omega}}(T^{\vec{\omega}}, n^0, \mu)$ as being the sum of the currents incident away from the exterior surface $(\mu+1)$-terminals of $\mathbf{D}^{\vec{\omega}}(T^{\vec{\omega}}, n^0, \mu+1)$. By recursion, we get that sum as being the sum of the currents incident away from the exterior surface λ-terminals of $\mathbf{D}^{\vec{\omega}}(T^{\vec{\omega}}, n^0, \lambda)$ for every $\lambda > \mu$. This in turn allows us to define the current incident away from $T^{\vec{\omega}}$ as that sum again. We have consistency with respect to different choices of μ.

With regard to different choices of n^0, we get consistency directly from the foregoing result and Lemma 3.3.

We can now state Kirchhoff's current law for an ω-node n^ω in $\mathbf{N}^\omega$. Let $T_m^{\vec{\omega}}$ ($m = 1, \ldots, M$) be the finitely many $\vec{\omega}$-terminals of n^ω, and let i_m be the current flowing away from $T_m^{\vec{\omega}}$. Kirchhoff's current law asserts that $\sum_{m=1}^{M} i_m = 0$. That this law is truly satisfied at every ω-node will be established when we establish the existence of a solution for $\mathbf{N}^\omega$

We will also need to state Kirchhoff's current law in terms of a branch-cut for n^ω. For each m, consider $\mathbf{D}^{\vec{\omega}}(T_m^{\vec{\omega}}, n^0, \mu)$ and then consider its (finitely many) exterior μ-terminals $T_{m,k}^\mu$ ($k = 1, \ldots, K$). These belong to the μ-section $\mathbf{S}_{m,n^0}^\mu$ that induces $\mathbf{D}^{\vec{\omega}}(T_m^{\vec{\omega}}, n^0, \mu)$. Now, exactly as was done at the end of Section 5, we can find a finite branch-cut within $\mathbf{S}_{m,n^0}^\mu$ for every $T_{m,k}^\mu$ that separates $T_{m,k}^\mu$ from every other μ-terminal of $\mathbf{S}_{m,n^0}^\mu$. The union of those branch-cuts for all $k = 1, \ldots, K$ is a branch-cut of finitely many branches that separates $T_m^{\vec{\omega}}$ from all other $\vec{\omega}$-terminals of $\mathbf{S}^{\vec{\omega}}$. In this way, we have a finite branch-cut for each $T_m^{\vec{\omega}}$ in n^ω, and the union of those latter branch-cuts for all m is a finite branch-cut $\mathcal{C}_{n^\omega}$ that separates $n^\omega$from all the other ω-nodes of $\mathbf{N}^\omega$. We shall simply refer to $\mathcal{C}_{n^\omega}$ as a *branch-cut* for n^ω. Finally, the sum of the currents in the branches of $\mathcal{C}_{n^\omega}$ is the same as the total current incident at n^ω (with — as always — all the branches oriented the same way with respect to n^ω). Thus, an equivalent version of Kirchhoff's current law for n^ω asserts that the sum of the currents in (the branches of) $\mathcal{C}_{n^\omega}$ is equal to 0. It is this form of Kirchhoff's current law that will be used when we discuss an operating point for $\mathbf{N}^\omega$.

10 A Solution for $\mathbf{N}^\omega$

The remaining arguments are much like those in Sections 6 through 8. We start by assuming that Assertion 4.3 holds for every natural number ν. Lemma 6.1 remains true when $\mathbf{N}^{\nu+1}$ is replaced by $\mathbf{N}^\omega$. Then, as in Subsection 6a, the set $\mathcal{N}_{\mathbf{N}^\omega}$ of all nodes of all ranks in $\mathbf{N}^\omega$ can be shown to be a compact metric space under the metric d^ω, which is defined as in (1) with μ replaced by ω and $\mathcal{P}(\mathbf{S}^\mu, m, n)$ replaced by the set $\mathcal{P}(\mathbf{N}^\omega, m, n)$ of all two-ended paths (perforce permissive) in $\mathbf{N}^\omega$ terminating at the nodes m and n of $\mathcal{N}_{\mathbf{N}^\omega}$.

As for the extension of Subsection 6b, we now approximate $\mathbf{N}^\omega$ with an expanding sequence $\{\mathbf{N}^\mu\}_{\mu=\mu_0}^{\infty}$ of μ-networks constructed as follows. We start by choosing a 0-node in each of the finitely many $\vec{\omega}$-sections of $\mathbf{N}^\omega$, and then choosing μ_0 so large that we obtain vicinities $\mathbf{D}^{\vec{\omega}}(T^{\vec{\omega}}, n^0, \mu_0)$ for all the $\vec{\omega}$-terminals $T^{\vec{\omega}}$ and thereby vicinities $\mathbf{D}^\omega(n_q^\omega, \mu_0)$ for all the ω-nodes n_q^ω ($q = 1, \ldots, Q$) of $\mathbf{N}^\omega$. Upon shorting those vicinites of the ω-nodes (i.e., by replacing the vicinity $\mathbf{D}^\omega(n_q^\omega, \mu_0)$ of each ω-node n_q^ω by a μ_0-node $n_q^{\mu_0}$ consisting of all the exterior surface μ_0-terminals of all the $\mathbf{D}^{\vec{\omega}}(T^{\vec{\omega}}, n^0, \mu_0)$ in each $\mathbf{D}^\omega(n_q^\omega, \mu_0)$), we obtain a μ_0-network $\mathbf{N}^{\mu_0}$. ($\mathbf{N}^{\mu_0}$ depends of course on our initial choices of the 0-nodes in the $\vec{\omega}$-sections, but Lemma 3.3 assures us that this is of no consequence.) We perform the same construction for each $\mu > \mu_0$ to get $\mathbf{N}^\mu$. Then, as $\mu \to \infty$, $\{\mathbf{N}^\mu\}_{\mu=\mu_0}^\infty$ fills out $\mathbf{N}^\omega$ in the sense that each branch and each node of finite rank in $\mathbf{N}^\omega$ is eventually in some $\mathbf{N}^\mu$ and in all subsequent $\mathbf{N}^\lambda$ ($\lambda > \mu$) as well. (This is a consequence of Lemma 3.2.) Each such $\mathbf{N}^\mu$ satisfies Conditions 3.5 with ν replaced by μ. So, by Assertion 4.3, $\mathbf{N}^\mu$ has an operating point (i, v) with $|i_b| \leq \boldsymbol{I}_\mu \leq \boldsymbol{I}$ and $|v_b| \leq \boldsymbol{V}_\mu \leq \boldsymbol{V}$ for each branch b in $\mathbf{N}^\mu$, where $\boldsymbol{I}$ and $\boldsymbol{V}$ are the quantities of Condition 3.1(c) for $\mathcal{B}$ being the set of all branches in $\mathbf{N}^\omega$. We now choose a ground 0-node for $\mathbf{N}^\omega$ that lies outside the vicinity $\mathbf{D}^\omega(n_q^\omega, \mu_0)$ of every ω-node n_q^ω ($q = 1, \ldots, Q$) in $\mathbf{N}^\omega$. This yields a unique continuous potential $\tilde{p}_\mu$ on the nodes in $\mathbf{N}^\mu$ for each $\mu \geq \mu_0$. We then extend $\tilde{p}_\mu$ onto all nodes of all ranks in each $\mathbf{D}^\omega(n_q^\omega, \mu)$ by continuity and constancy to obtain a potential p_μ on $\mathcal{N}_{\mathbf{N}^\omega}$.

The rest of the argument requires no changes other than for notation. We can show that $\{p_\mu\}_{\mu=\mu_0}^\infty$ is pointwise relatively compact and pointwise equicontinuous as in Subsections 6c and 6d (just replace $\mathbf{D}^{\nu+1}(n_q^{\nu+1}, \epsilon_k)$ by $\mathbf{D}^\omega(n_q^\omega, \mu)$. This allows us to invoke the Arzela-Ascoli theorem to assert the existence of a continuous potential p on $\mathcal{N}_{\mathbf{N}^\omega}$ as the limit of a subsequence of $\{p_\mu\}_{\mu=\mu_0}^\infty$.

Next, we can establish the existence of a bounded solution for $\mathbf{N}^\omega$ as in Section 7. Just replace $\{p_k\}_{k=1}^\infty$ by $\{p_\mu\}_{\mu=\mu_0}^\infty$, and then select subsequences as before. In the present case, we use the finite branch-cut for each ω-node in $\mathbf{N}^\omega$, specified at the end of Section 9, when establishing Kirchhoff's current law at each ω-node. Thus, we have arrived at the following, which is the second main theorem of this paper.

Theorem 10.1. *Let the ω-network $\mathbf{N}^\omega$ satisfy Conditions 3.5 with $\nu = \omega$. Then, $\mathbf{N}^\omega$ has an operating point (i, v), and, for every branch b, $|i_b| \leq \boldsymbol{I}$ and $|v_b| \leq \boldsymbol{V}$.*

Moreover, v will satisfy Kirchhoff's voltage law as before.

Finally the argument establishing the uniqueness of the operating point when every branch characteristic M_b has neither horizontal nor vertical segments requires only one additional step as follows. We define the *interior*, *exterior*, and 0-*node-cut* of a branch-cut for each ω-node as in Section 8. We then choose a finite branch-cut and thereby a finite 0-node-cut $\mathcal{C}^0_{n^\omega}$ around each ω-node n^ω to satisfy the requirements that $\mathcal{C}^0_{n^\nu}$ fulfilled in the proof of Theorem 8.1. The important point here is the following: Since there are only finitely many ω-nodes in $\mathbf{N}^\omega$, there will be a minimum natural-number rank ν such that the intersection $\mathcal{X}^{\nu-1}$ of all the exteriors of all the chosen branch-cuts for all the ω-nodes does not contain any nodes of ranks ν or larger. In fact, $\nu = \mu_0 + 1$, where $\mu_0 + 1$ is the rank of all the surface nodes of all the vicinities of all the ω-nodes corresponding to which the branch-cuts are chosen. The argument then proceeds with $\mathcal{X}^{\nu-1}$, $\mathcal{X}^{\nu-2}, \ldots, \mathcal{X}^0$ as in the proof of Theorem 8.1. We thus obtain

Theorem 10.2. *Under the hypothesis of Theorem 10.1, assume furthermore that, for each branch b, M_b is a continuous, strictly monotonically increasing function of i_b. Then, the operating point (i, v) for $\mathbf{N}^\omega$ is unique.*

11 Final Remarks

(i) We have presented a theory for the operating points of end-generated transfinite monotone networks of all ranks up to and including ω. We can continue to do so for ranks higher than ω, needing only changes in notation, by following the arguments of Sections 5 through 8 for successor-ordinal ranks and the arguments of Sections 9 and 10 for limit-ordinal ranks. However, we cannot assert that our rank-by-rank extensions will carry through all the countable ordinals because we have not presented a transfinitely inductive argument. It is not clear how to make an inductive assumption for any arbitrary countable-ordinal rank.

Furthermore, how to define transfinite graphs and networks of ranks $\aleph_1$ and higher remains an open problem.

(ii) When $\mathbf{N}^\nu$ is a linear network, we can relate the operating point found in this paper to that given in [14, Chapter 5], [15, Chapter 5] as follows. Let M_b be a straight line with positive slope r_b for every b. Thus, r_b is the linear resistance of branch b. M_b intersects the current axis at, say, $-h_b$ and the voltage axis at $e_b = h_b r_b$. h_b is the value of the current source when b has the Norton form, and e_b is the value of the voltage source when b has the Thevenin form. Under Conditions 3.1, $\delta'(0,0,M_b) = |h_b|$, $\delta''(0,0,M_b) = |e_b|$, $\sum |h_b| < \infty$, and $\sum |e_b| < \infty$, where $\sum$ denotes a summation over all $b \in \mathcal{B}$. Consequently, $\sum e_b^2 < \infty$ and $\sum h_b^2 < \infty$, and we have

$$\sum \frac{e_b^2}{r_b} = \sum |e_b||h_b| \leq \left(\sum e_b^2 \sum h_b^2\right)^{1/2} < \infty.$$

Thus, the hypothesis of the fundamental theorem [14, Theorem 5.5-1], [15, Theorem 5.2-8] is fulfilled. We can conclude that the unique operating point given by Theorems 7.1 and 10.1 herein is the same as that dictated by the fundamental theorem given in [14] and [15].

Note that, in the linear case, Condition 3.1(c) herein is stronger than the condition $\sum e_b^2/r_b < \infty$ assumed in the prior fundamental theorem.

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Figure Captions

Fig. 1. Illustration of some $(\mu-1)$-sections related to an ϵ_1-vicinity $\mathbf{D}^\mu(T^\mu,\epsilon_1)$ and an ϵ_2-vicinity $\mathbf{D}^\mu(T^\mu,\epsilon_2)$ for a μ-terminal T^μ (T^μ is indicated at the right of the figure). Here, $0<\epsilon_2<\epsilon_1$. The singly crosshatched areas denote $(\mu-1)$-sections of $\mathbf{D}^\mu(T^\mu,\epsilon_1)$. The doubly crosshatched areas denote $(\mu-1)$-sections of $\mathbf{D}^\mu(T^\mu,\epsilon_2)$ — and therefore of $\mathbf{D}^\mu(T^\mu,\epsilon_1)$ too. The heavy dots denote μ-nodes. They are labeled as follows. The label 1 indicates a surface μ-node of $\mathbf{D}^\mu(T^\mu,\epsilon_1)$, 2 indicates a surface μ-node of $\mathbf{D}^\mu(T^\mu,\epsilon_2)$, and the asterisk * indicates either a μ-node in $\mathbf{D}^\mu(T^\mu,\epsilon_1)\backslash\mathbf{D}^\mu(T^\mu,\epsilon_2)$ or a surface μ-node of $\mathbf{D}^\mu(T^\mu,\epsilon_2)$. The areas labeled $\mathbf{S}_1^{\mu-1}$, $\mathbf{S}_2^{\mu-1}$, and $\mathbf{S}_4^{\mu-1}$ indicate surface $(\mu-1)$-sections of $\mathbf{D}^\mu(T^\mu,\epsilon_1)$ (these are not crosshatched), and $\mathbf{S}_3^{\mu-1}$ and $\mathbf{S}_4^{\mu-1}$ indicate surface $(\mu-1)$-sections of $\mathbf{D}^\mu(T^\mu,\epsilon_2)$.

Fig. 2. Illustration of the terminal currents $i_{T_{k,j}^{\mu-1}}$ $(k=1,2,3)$ entering a $(\mu-1)$-section $\mathbf{S}^{\mu-1}$ through its incident μ-nodes n_k^μ. The apex of each "horn-like" shape denotes a $(\mu-1)$-terminal (and therefore a $(\mu-1)$-end too, by Condition 3.2(d)). $i_{T_{k,j}^{\mu-1}}$ is the current leaving the jth $(\mu-1)$-terminal $T_{k,j}^{\mu-1}$ belonging to both n_k^μ and $\mathbf{S}^{\mu-1}$. Property 5.2(b) asserts that $i_{T_{1,1}^{\mu-1}}+i_{T_{2,1}^{\mu-1}}+i_{T_{2,2}^{\mu-1}}+i_{T_{3,1}^{\mu-1}}=0$.

Fig. 3. Illustration of the terminal currents $i_{T_m^{\mu-1}}$ $(m=1,2,3,4)$ leaving a μ-node n^μ through its $(\mu-1)$-terminals $T_m^{\mu-1}$. Kirchhoff's current law asserts that $\sum_{m=1}^4 i_{T_m^{\mu-1}}=0$.

Fig. 4. Illustration of two sections $\mathbf{S}_{n^0}^\mu$ and $\mathbf{S}_{n^0}^{\mu+1}$ in a nested sequence of sections within an $\vec{\omega}$-section $\mathbf{S}^{\vec{\omega}}$. $T^{\vec{\omega}}$ is an $\vec{\omega}$-terminal for $\mathbf{S}^{\vec{\omega}}$, and n^ω is an ω-node containing $T^{\vec{\omega}}$. $\mathbf{S}_{n^0}^\mu$ is to the left of the $(\mu+1)$-nodes $n_1^{\mu+1}$ and $n_2^{\mu+1}$. $\mathbf{S}_{n^0}^{\mu+1}$ contains everything shown to the left of the $(\mu+2)$-nodes $n_3^{\mu+2}$ and $n_4^{\mu+2}$.

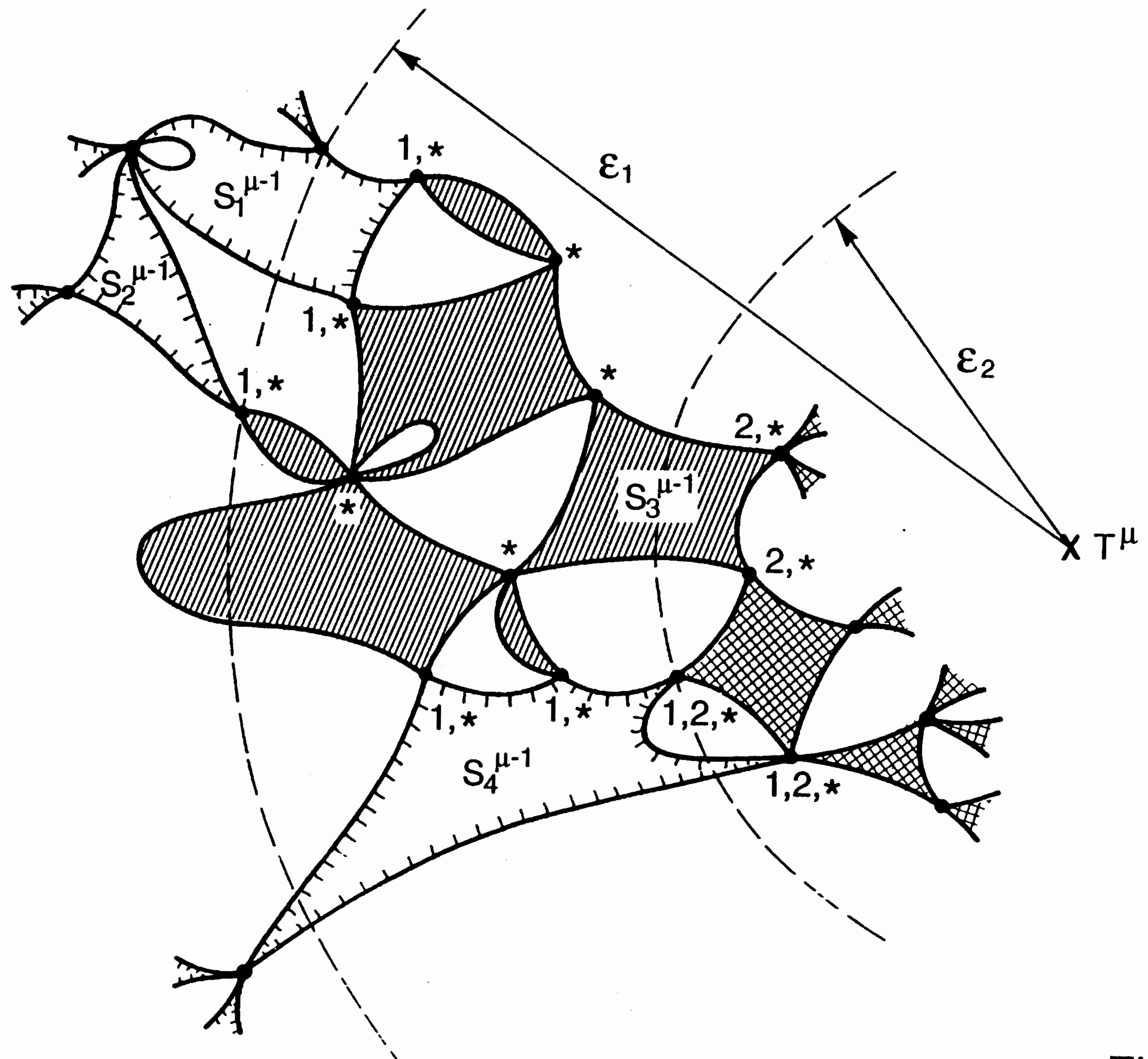
ε_1
ε_2
T^{μ}
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Fig. 1

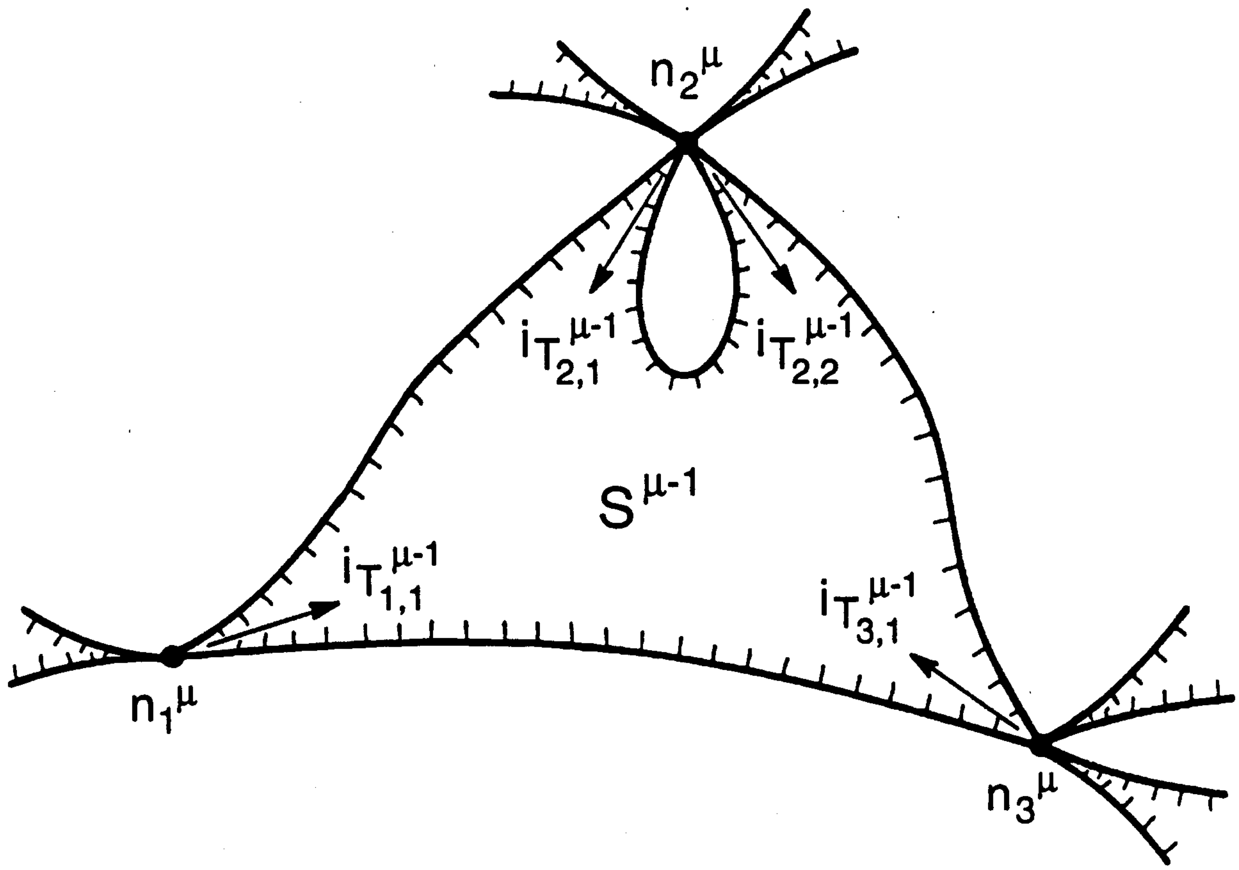


Fig. 2

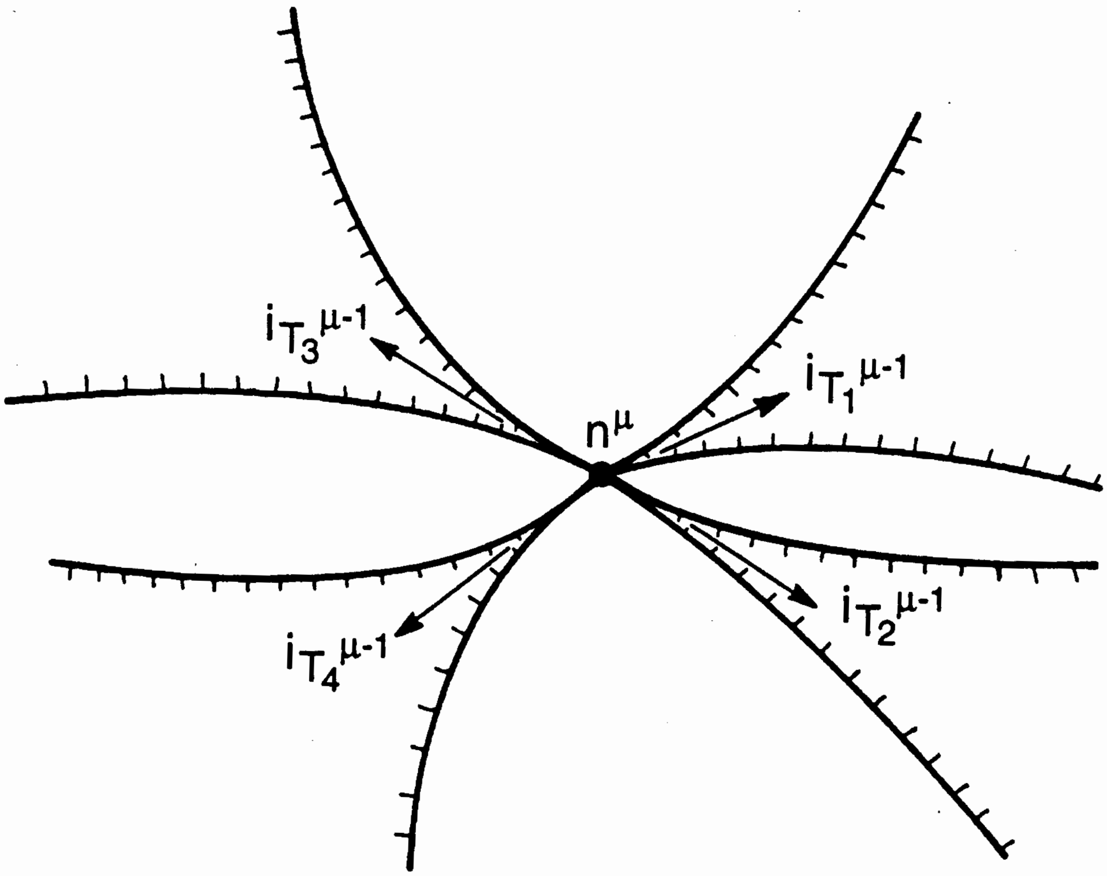
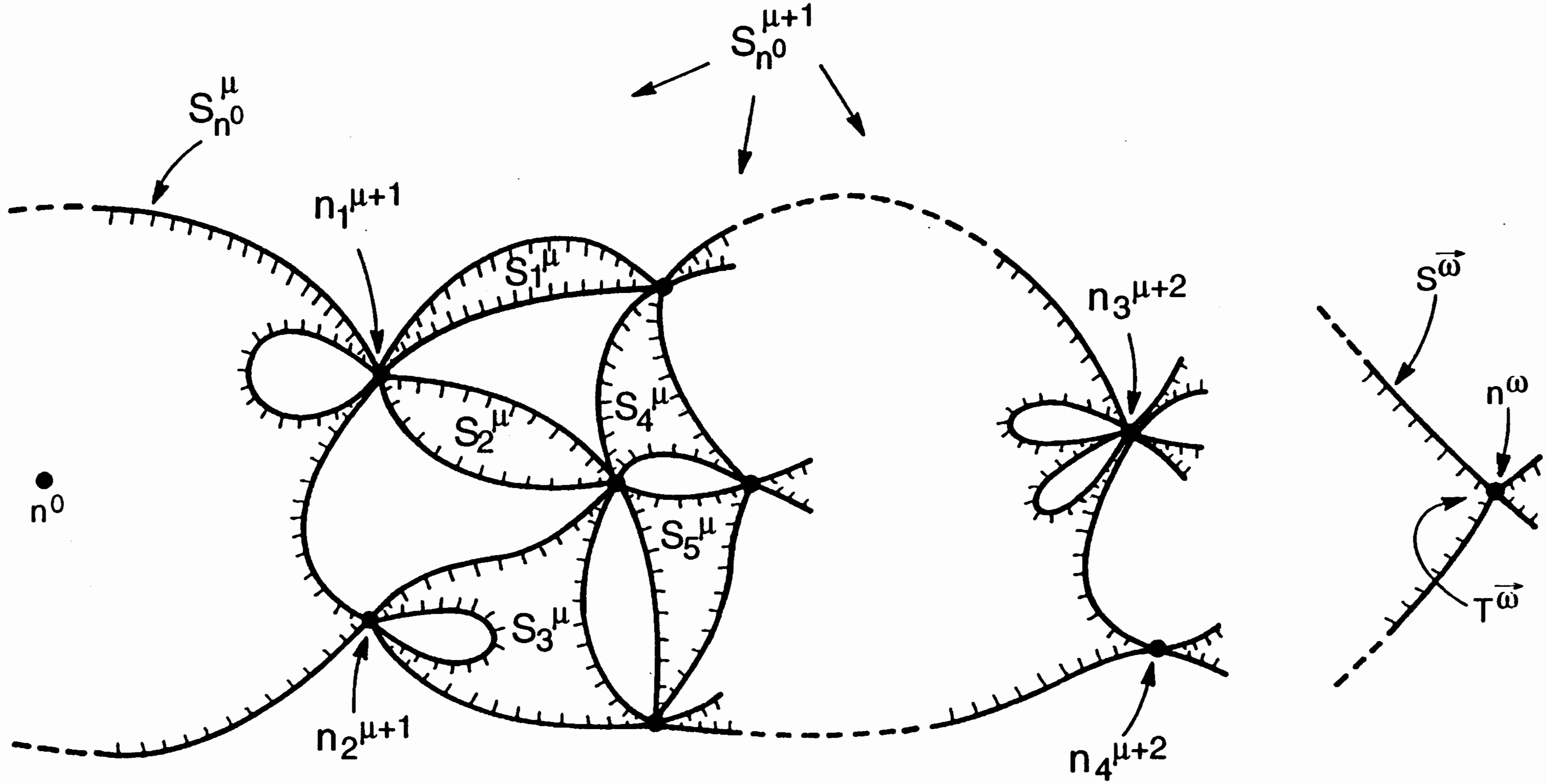


Fig. 3



$S_{n^0}^{\mu+1}$
$S_{n^0}^{\mu}$
$n_1^{\mu+1}$
S_1^{μ}
$n_3^{\mu+2}$
$S^{\vec{\omega}}$
n^{ω}
S_4^{μ}
S_2^{μ}
n^0
S_5^{μ}
S_3^{μ}
$T^{\vec{\omega}}$
$n_2^{\mu+1}$
$n_4^{\mu+2}$

Fig. 4