Two Aspects of Sasakian Geometry

A Dissertation Presented

by

Weixin Guo

to

The Graduate School

in Partial Fulfillment of the

Requirements

for the Degree of

Doctor of Philosophy

in

Mathematics

Stony Brook University

December 2011

Stony Brook University

The Graduate School

Weixin Guo

We, the dissertation committee for the above candidate for the Doctor of Philosophy degree, hereby recommend acceptance of this dissertation.

Claude LeBrun - Advisor
Professor, Department of Mathematics

H. Blaine Lawson Jr. - Chairperson of Defense
Distinguished Professor, Department of Mathematics

Denson Hill
Professor, Department of Mathematics

Xianfeng David Gu
Professor, Department of Computer Science

This dissertation is accepted by the Graduate School.

Lawrence Martin
Dean of the Graduate School

Abstract of the Dissertation

The Aspects of Sasakian Geometry

by

Weixin Guo

Doctor of Philosophy

in

Mathematics

Stony Brook University

2011

Sasakian structures are the counterparts of Kähler structures in odd dimensions. A Sasakian manifold is a strictly pseudo-convex CR manifold with a Reeb vector field that generates CR automorphisms.

We first study Type I deformations of Sasakian structures, which amount to different choices of Reeb vector field on a fixed CR manifold. Here we show that the CR automorphism group of a Sasakian manifold is severely constrained by mild curvature assumptions.

We then study products of pairs of compact Sasakian manifolds. Such products are shown to always yield compact complex manifolds that do not admit Kähler metrics, generalizing a remarkable construction due to Calabi and Eckmann. As a consequence, any product of two compact Sasaki-Einstein manifolds yields an Einstein Hermitian metric on a compact complex manifold which does not admit Kähler metrics. This result stands in marked contrast to the situation in real dimension 4, where LeBrun showed that Einstein Hermitian metrics on compact complex surfaces are always conformally Kähler.

To my parents

Contents

Ackowledgements

Thanks are due to many people.

First of all, to my parents for their unwavering support.

I owe a deep debt of gratitude to my adviser Claude LeBrun for his time and encouragement. Without his guidance, this work could not have been completed. Most of all I thank him for being so supportive at the time that I don't believe in myself.

I would like to thank Xiaojiu Chen, Ning Hao for their help and suggestions.

Thanks also are due to many friends here in Stony Brook, especially Loy Weng, Yinghua Li, Xiaojie Wang and Wentao Jiang for all the joys through these years.

Chapter 1

Type I deformation of Sasakian manifolds

1.1 Sasakian manifolds

A CR-manifold (of hypersurface type) is a $(2n+1)$ dimensional manifold with a fixed real rank-$2n$ subbundle D of the tangent bundle, where D is equipped with a (almost) complex structure J.

Let η be a 1-form on M (locally if necessary) so that $D = \ker\eta$. A CR structure is said to be strictly pseudo convex, if the Levi form $d\eta$ is positive/negative definite on D. Note that this does not depend on the choice of the 1-form η. In this case $d\eta(\cdot, J\cdot)|_D$ defines a metric on D, which is compactible with J.

For any sections $X, Y \in D$, we have $\eta([X, JY]) = X\eta(JY) - JY\eta(X) - 2d\eta(X, JY) = -\eta([JX, Y])$. Hence, $[X, JY] + [JX, Y]$ is a also section of D. So then the Nijenhause 'tensor' on D can be defined, i.e. $N_J(X, Y) = ([X, Y] - [JX, JY]) - J([X, JY] + [JX, Y])$ for any sections $X, Y \in D$. The CR structure is said to be integrable if this Nijenhause 'tensor' vanishes.

A CR-manifold is sometimes defined in complex terms: A $(2n+1)$ dimensional CR manifold is a smooth manifold with a fixed n-dimensional complex subbundle $H \subset T_{\mathbb{C}}M$, such that $H \cap \bar{H} = 0$. The real subbundle $D = \mathrm{Re} H \oplus \bar{H}$, and the integrability condition is equivalent to $[H, H] \subset H$.

In the following, a CR structure is always assume to be of hypersurface type, integrable and strictly pseudo convex.

We will also further assume that the sub-bundle D is coorientable, equivalently, we assume the manifold to be orientable (since D is natrually oriented by its complex structure). In such a case, we can globally fix a choice of a real 1-from η. With such a choice, (M, D, J, η) is then called a pseudo hermitian structure. There are several tensor fields that can be associated to a pseudo-hermitian structure.

(1) A vector field, called the Reeb vector field is uniquely determined by

requiring $\eta(T) = 1$ and $d\eta(T, \cdot) = 0$.

(2) A Riemannian metric g is determined by requiring that $g|_D = d\eta(\cdot, J\cdot)$, $g(T,T) = 1$ and $g(T,D) = 0$.

(3) The complex structure J defined on D can be also extended uniquely to a (1,1)-tensor field, which for simplicity we will still denote by J, by requiring $J(T) = 0$. (in the literature, this (1,1) tensor field is usually denoted by Φ)

With respect to the metric g, we might take an orthonormal basis of real covectors $(\eta, \eta^\alpha, J\eta^\alpha)$. Let $\theta^\alpha = \eta^\alpha + \sqrt{-1}J\eta^\alpha$, then by definition, $d\eta = -\sqrt{-1}\sum \theta^\alpha \wedge \theta^{\bar{\alpha}}$.

Pseudo-hermitian structure is defined and studied by Webster [**?**], where he defined the Webster connection.

Proposition 1.1.1 [**?**] Let (M, η) be an integral, strictly pseudo-convex, pseudo-hermitian manifold, there exist connection forms ω_α^β, and torsion forms $\tau_\beta = A_{\beta\alpha}\theta^\alpha$, where $A_{\alpha\beta} = A_{\beta\alpha}$, such that

$$d\theta^\beta = \theta^\alpha \wedge \omega_\alpha^\beta + \eta \wedge \tau^\beta \tag{1.1}$$

$$\omega_{\alpha\bar{\beta}} + \omega_{\bar{\beta}\alpha} = 0 \tag{1.2}$$

Let $(T, Z_\alpha, Z_{\bar{\alpha}})$ be a dual basis to the chosen $(\eta, \theta^\alpha, \theta^{\bar{\alpha}})$. We can set

$$\dot{\nabla} Z_\alpha = \omega_\alpha^\beta Z_\beta \quad \text{and} \quad \dot{\nabla} Z_{\bar{\alpha}} = \omega_{\bar{\alpha}}^{\bar{\beta}} Z_{\bar{\beta}}$$

The $\dot{\nabla}$ so defined is a connection for the vector bundle D over M. By definition, this connection is complex, namely, $\dot{\nabla} JX = J\dot{\nabla} X$ for any $X \in D$ and by (**??**), it is a metric connection, $\dot{\nabla} g(X,Y) = g(\dot{\nabla} X, Y) + g(X, \dot{\nabla} Y)$ for any $X, Y \in D$

Since $D \subset TM$, we might consider the 'torsion tensor' $C(X,Y) = \dot{\nabla}_X Y - \dot{\nabla}_Y X - [X,Y]$ for $X, Y \in D$. By applying $Z_i \wedge Z_j$ and $Z_i \wedge \bar{Z}_j$ to (**??**), we see $C(X,Y)$ has no component in D, In fact,

$$C(X,Y) = \eta(\dot{\nabla}_X Y - \dot{\nabla}_Y X - [X,Y])T = 2d\eta(X,Y)T \tag{1.3}$$

Likewise, the Webster torsion τ can be captured by the tensor

$$\tau(X) \doteq \dot{\nabla}_T X - [T, X] \tag{1.4}$$

Proposition 1.1.2 [**?**] The Webster torsion vanishes iff the Reeb vector field preserves the CR structure.

Proof. By applying $T \wedge Z_i$ to (**??**), we see

$$\dot{\nabla}_T Z_\alpha - [T, Z_\alpha] = A_\alpha^{\bar{\beta}} Z_{\bar{\beta}}$$

Note that the right hand side is anti-holomorphic, hence, the tensor $\tau(X) = \dot{\nabla}_T X - [T, X]$ is J anti-invariant. Note also $\dot{\nabla}_T JX = J\dot{\nabla}_T X$, hence, $\tau(X)$ is J invariant iff $[T, JX] = J[T, X]$, i.e. $L_T JX = JL_T X$ for any $X \in D$. Thus we conclude that $\tau = 0$ if and only if T preserves the CR structure.

Thus, when the Reeb vector field of the pseudo-hermitian manifold is a CR vector field, the Webster connection is a complex connection which is also metric and with minimal torsion.

Definition 1.1.3 Let M be strictly pseudo-convex, integrable, pseudo-hermitian manifold. M is called Sasakian if its Reeb vector field is a CR vector field.

The Webster connection $\dot{\nabla}$ defined for the bundle $D \subset TM$ can now be extended to the Levi-Civita connection on M, which we will call the Sasakian connection.

Proposition 1.1.4 The Sasakian connection is given by

$$\nabla_X Y = \dot{\nabla}_X Y - g(JX, Y)T, \quad X, Y \in D \tag{1.5}$$

$$\nabla_T X = \dot{\nabla}_T X + JX, \qquad X \in D \tag{1.6}$$

$$\nabla_X T = JX, \nabla_T T = 0 \qquad X \in D \tag{1.7}$$

Proof. We only need to check ∇ is torsion free and metric.

By (**??**), we see $\nabla_X Y - \nabla_Y X = [X, Y]$ for $X, Y \in D$. Since the Webster torsion vanishes, from (**??**) we have $[T, X] = \dot{\nabla}_T X$. It is then easy to see $\nabla_X T - \nabla_T X = [X, T]$ by the definitions. Hence, ∇ is torsion free.

Using the fact that the Webster connection is metric, it is a case by case check that ∇ is also a metric connection.

Remark. On a manifold that is not necessarily Sasakian, we may extend the Webster connection by requiring the Reeb vector field to be parallel, i.e $\nabla T = 0$. The linear connection obtain in this way is called the Webster-Tanaka connection [**?**] or Webster-Stanton connection [**?**]. It is metric, but has torsion.

On the other hand, if we begin with the Sasakian connection, then,

Proposition 1.1.5 the Webster connection can be reconstructed by setting:

$$\dot{\nabla}_X Y = (\nabla_X Y)^D \tag{1.8}$$

$$\dot{\nabla}_T Y = [T, Y]^D \tag{1.9}$$

where the supercript ^{D}, means projecting (with respect to the metric g) to the subbundle $D \subset TM$.

Proof. (**??**) is clear in view of (**??**). (**??**) is just the fact that the Webster torsion is 0.

Since $L_T\eta = d(\eta(T)) - d\eta\neg T = 0$ and $L_T J = 0$, by definition of $g(\cdot,\cdot) = d\eta(\cdot, J\cdot) + \eta \otimes \eta$, we deduce that $L_T g = 0$. In other words, the Reeb vector field T is a killing field. The foliation defined by T is then a Riemanian foliation and the metric g is bundle-like [**?**]. We define the transverse metric $g^D(V, W) = g(V^D, W^D)$ on M for any $V, W \in TM$, where supersript ^{D} means projection to D. The transverse metric can be considered as the metric on the leaf space M/T. Especially, when the leaves are compact, the leaf space is a Riemanian orbifold [**?**], and we have a Riemanian submersion $\pi : M \to M/T$.

The induced transverse (Levi-Civita) connection ∇^D for g^D is defined exactly by (**??**) and (**??**). In other words,

Corollary 1.1.6 the Webster connection is the transverse (Levi-Civita) connection on a Sasakian manifold.

Curvatures (usually called pseudo hermitian curvatures) associated with the Webster connection are studied by various authors [**?**] [**?**], in view of the above disscussion, they are the same as the transverse curvatures which are defined as curvatures of the transverse metric g^D, e.g.

$$R^D(X,Y)Z = \nabla^D_X \nabla^D_Y Z - \nabla^D_Y \nabla^D_X Z - \nabla^D_{[X,Y]} Z$$

All though not always a Riemanian submersion, the O'Neill formula [**?**] applied to the transverse curvatures. Recall the O'Neil tensors:

$$T_V W = (\nabla_{V^T} W^D)^T + (\nabla_{V^T} W^T)^D$$

$$A_V W = (\nabla_{V^D} W^D)^T + (\nabla_{V^D} W^T)^D$$

where $V, W \in TM$ and the superscript ${}^T, {}^D$ means projection to the Reeb direction and the subbundle D respectively.

Recall also the O'Neill formula

$$K(X,Y) = K^D(X,Y) - 3|A_X Y|^3$$

$$Ric(X) = Ric^D(X) - 2|A_X|^2 - |T_X|^2 + g(\nabla_X N, X)$$

$$s = s^D + s^T - |A|^2 - |T|^2 - |N|^2 - 2\delta^D N$$

As $\nabla_T T = 0$ by definition, the Reeb vector field T is geodesic, and it follows that the O'Neill T-tensor vanishes [**?**]. The mean-curvature vector field along the leaves $N = \nabla_T T$ also vanishes. For $X, Y \in D$ being horizontal vectors, we know [**?**]

$$A_X Y = \tfrac{1}{2}([X, Y])^T$$

Hence, $A_X JX = 1$, $A_X Y = 0$ if $g(JX, Y) = 0$, and $|A_X| = 1$, $|A| = 2n$. Now by the O'Neil formula and Corollary 1.1.6, let $\dot{K}$, $\dot{Ric}$ and $\dot{s}$ be the pseudo-hermitian curvatures,

Proposition 1.1.7 The Sasakian curvatures and the pseudo-hermitian curvatures are related as follows

$$K(X, JX) = \dot{K}(X, JX) - 3 \quad \text{for } X \in D \tag{1.10}$$

$$K(X, Y) = \dot{K}(X, Y) \quad \text{for } X, Y \in D, g(JX, Y) = 0 \tag{1.11}$$

$$Ric(X) = \dot{Ric}(X) - 2, \quad \text{for } X \in D \tag{1.12}$$

$$s = \dot{s} - 2n \tag{1.13}$$

1.2 Curvature properties

In this section, we collect some curvature properties of a Sasakian manifold. These are developed from the the perspective of a contact metric structure in the literature, see e.g. Blair [**?**], Boyer and Galicki [**?**]. With the help of the Webster connection, these are evident. A key point is $\nabla_X T = JX$.

Proposition 1.2.1

$$\nabla_X JY = J\nabla_X Y - g(X,Y)T \text{ for } X, Y \in D$$

Proof.

$$\begin{aligned}
\nabla_X JY &= \dot{\nabla}_X JY - g(JX, JY)T \\
&= J\dot{\nabla}_X Y - g(JX, JY)T \\
&= J\nabla_X Y - g(X,Y)T
\end{aligned}$$

Proposition 1.2.2

$$R(X,Y)T = 0, X, Y \in D$$

$$R(X,T)Y = -g(X,Y)T, X, Y \in D$$

Proof. We might assume X, Y are unit vectors.

$$
\begin{aligned}
R(X,Y)T &= \nabla_X \nabla_Y T - \nabla_Y \nabla_X T - \nabla_{[X,Y]} T \\
&= \nabla_X JY - \nabla_Y JX - J[X,Y] \\
&= J\nabla_X Y - g(X,Y)T - J\nabla_Y X + g(Y,X)T - J[X,Y] = 0
\end{aligned}
$$

For $\forall Z \in D$ we have $g(R(X,T)Y,Z) = g(R(Z,Y)T,X) = 0$, hence $R(X,T)Y$ has no component in D. The Reeb component is

$$
\begin{aligned}
g(R(X,T)Y,T) &= g(\nabla_X \nabla_T Y - \nabla_T \nabla_X Y - \nabla_{[X,T]} Y, T) \\
&= (Xg(\nabla_T Y, T) - g(\nabla_T Y, \nabla_X T)) - (Tg(\nabla_X Y, T) - g(\nabla_X Y, \nabla_T T)) \\
&\quad - ([X,T]g(Y,T) - g(Y, \nabla_{[X,T]} T)) \\
&= -g(\nabla_T Y, JX) + g(Y, J[X,T]) \\
&= g(Y, \nabla_T JX) + g(\nabla_X JY, T) + g(JY, \nabla_X T) \\
&= -g(Y,X) + g(-g(X,Y)T, T) + g(Y,X) = -g(X,Y)
\end{aligned}
$$

Corollary 1.2.3 Let $dimM = 2n+1$, then $Ric(T) = 2n$.

Proof. Let $\{e_i\}$ be an orthonormal basis for D. Using the previous proposition, $Ric(T) = \sum_i R(T, e_i, e_i, T) = 2n$

The sectional curvature $K(X,JX)$ for$X \in D$ is called the Φ-sectional curvature. On a Sasakian manifold, the whole curvature tensor is determined by the Φ-sectional cruvature. A Sasakian manifold with constant sectional curvature is called a Sasakian space form. They are classified by Tanno [**?**]: there are precisly three Sasakian space forms up to transverse homothety (see next section)

Proposition 1.2.4 Let $M(c)$ be a $2n+1$ dimensional, complete, simply connected Sasakian manifold of constant Φ-sectional curvature c. Then $M(c)$ is one of the following:

(1) If $c > -3$, $M(c)$ is isomorphic to $S^{2n+1}(c)$.

(2) If $c = -3$, $M(c)$ is isomorphic to $R^{2n+1}(-3)$.

(3) If $c < -3$, $M(c)$ is isomorphic to $B^n_{\mathbb{C}}(k) \times \mathbb{R}$, where $c = k - 3$.

1.3 Type I deformations

There are several ways to deform a Sasakian structure. One natrual way is to rescale the metric via the contact form. Let (M, η, D, J) be a Sasakian structure. We might define a new 1-form on M by $\tilde{\eta} = e^f \eta$, where f is a smooth function on M. Now the associate CR distribution $\tilde{D} = ker\tilde{\eta} = D$ does not change. We also have

$$d\tilde{\eta}|_D = e^f(df \wedge \eta + d\eta)|_D = e^f d\eta|_D$$

If we set $\tilde{J} = J$ on $\tilde{D} = D$, then the metric is partially conformally related, i.e. $\tilde{g}|_{\tilde{D}} = e^f g|_D$.

Definition 1.3.1 A type I deformation of a Sasakian structure (M, η, J) is a rescale $\tilde{\eta} = e^f \eta$ and $\tilde{J} = J$, so that $(M, \tilde{\eta}, \tilde{J})$ is again Sasakian.

Take an othornomal basis $\{X_i, Y_i = JX_i\}$ for the bundle D with the background metric $g|_D$. We can write the Reeb vector field $\tilde{T} = \mu T + a^i X_i + b^i Y_i$. Then,

$$1 = \tilde{\eta}(\tilde{T}) = e^f \eta(\mu T) = \mu e^f$$

So, $\mu = e^{-f}$, moreover, for any vector field V, we have

$$0 = d\tilde{\eta}(\tilde{T}, V) = de^f \wedge \eta(\tilde{T} \wedge V) + e^f d\eta(\tilde{T} \wedge V)$$

Let $V = X_i$ and $V = Y_i$ respectively, we get $a^i = \frac{1}{2} Y_i(e^{-f})$, $b^i = -\frac{1}{2} X_i(e^{-f})$, hence

$$\tilde{T} = e^{-f}(T - \frac{1}{2} J \nabla_b f) \tag{1.14}$$

where the sub-gradient $\nabla_b f = \nabla f - T(f)T$.

$\tilde{T}$ is also required to preserves the structure $\tilde{J} = J$. For any $X \in D$,

$$L_{\tilde{T}} JX - JL_{\tilde{T}} X = [\tilde{T}, JX] - J[\tilde{T}, X] = 0$$

Its component in the Reeb direction is,

$$\eta([\tilde{T}, JX] - J[\tilde{T}, X]) = e^{-f}\tilde{\eta}([\tilde{T}, JX]) = -2e^{-f}d\tilde{\eta}(\tilde{T}, JX) = 0$$

Write $\tilde{T} = e^{-f}T + H$, where $H = \frac{1}{2}J\nabla_b(e^{-f}) \in D$. Because T preserves J, we have

$$([e^{-f}T, JX] - J[e^{-f}T, X])^D = e^{-f}([T, JX] - J[T, X])^D = 0$$

hence, we should have

$$([H, JX] - J[H, X])^D = (\nabla_H JX - \nabla_{JX}H - J\nabla_H X + J\nabla_X H)^D = 0$$

By (**??**)

$$(\nabla_H JX - J\nabla_H X)^D = \dot{\nabla}_H JX - J\dot{\nabla}_H X = 0$$

Hence, $\tilde{T}$ preserves J if and only if, for any $X, Y \in D$

$$g(\nabla_{JX}H - J\nabla_X H, Y) = 0$$

$$
\begin{aligned}
g(\nabla_{JX}H,Y) &= JXg(H,Y) - g(H,\nabla_{JX}Y) \\
&= -\frac{1}{2}(JXg(\nabla_b e^{-f},JY) - g(\nabla_b e^{-f},J\nabla_{JX}Y)) \\
&= -\frac{1}{2}(JX(JY(e^{-f})) - \nabla_{\nabla_{JX}JY}e^{-f}) \\
&= -\frac{1}{2}\nabla^2_{JX,JY}e^{-f}
\end{aligned}
\tag{1.15}
$$

Similarly, $g(J\nabla_X H,Y) = -\frac{1}{2}\nabla^2_{X,Y}e^{-f}$,

Hence, we have

Proposition 1.3.2 $\tilde{\eta} = e^f\eta$ is a type I deformation if and only if $\nabla^2_{JX,JY}e^{-f} = \nabla^2_{X,Y}e^{-f}$ for any $X,Y \in D$.

Apparently, constants satisfy the above relation. When f is a constant, the deformation is called a transverse homothety [?]. In this case, $\tilde{\eta} = a\eta$, $\tilde{T} = a^{-1}T$, and $\tilde{g} = ag + (a^2 - a)\eta \otimes \eta$.

If $\psi \in \mathfrak{CR}(D,J)$, the CR transformation group, then the push forward 1-form $\psi_*\eta = e^f\eta$ for some function f. Notice that $\psi_*D = D$ and $\psi_*J = J$, the push forward structure $(M,\psi_*\eta,\psi_*J)$ is a type I deformation of (M,η,J). A type I deformation arises in this way from a CR transformation is called a pseudo-conformal rescaling [?]. Note that not all type I deformations arises in this way, in particular, transverse homotheties with constant $a \neq 1$ are never pseudo-conformal rescalings.

1.4 Moduli of type I deformations

A Sasakian manifold (M, η, J) is natrually a contact manifold. Let $\mathfrak{con}(M, D)$ be the Lie algebra of the contactomorphism group $\mathfrak{Con}(M, D)$. Denote by $\mathfrak{con}^+(M, D)$ the set of elements in $\mathfrak{con}(M, D)$ that is everywhere transverse to D. With a background η being chosen,

$$\mathfrak{con}^+(M, D) = \{V | \eta(V) > 0, L_V D = D\}$$

For $V \in \mathfrak{con}^+(M, D)$, Let $\mu = 1/\eta(V)$ and $\tilde{\eta} = \mu\eta$. Then, $\tilde{\eta}(V) = 1$. Since V preserves the contact distribution, we have $L_V \tilde{\eta} = \alpha\tilde{\eta}$ for some smooth function α, but,

$$\alpha = (L_V \tilde{\eta})(V) = d(\tilde{\eta}(V))(V) + d\tilde{\eta}(V, V) = 0$$

Hence, $d\tilde{\eta} \neg V = L_V \tilde{\eta} = 0$. So V is the Reeb vector field of $\tilde{\eta}$.

On the other hand, if V is a Reeb vector field of some $\tilde{\eta} = \mu\eta$ with some positive function μ, then $L_V \tilde{\eta} = 0$, hence V belongs to $\mathfrak{con}^+(M, D)$. So, we have

$$\mathfrak{con}^+(M, D) = R^+(M, D) \tag{1.16}$$

where $R^+(M, D)$ is all the transverse Reeb fields that is compactible with D.

Given a background contact form η, $R^+(M, D)$ is 1-1 correspondent to the set of smooth functions by setting $\tilde{\eta} = e^f \eta$. Hence on a Sasakian manifold, the set of type I deformations C_I is identified with a subset of $R^+(M, D)$ that consists of those Reeb vector fields that preserves the complex structure J. Let $\mathfrak{cr}(M, D, J)$ be the Lie algebra of the CR group, then $C_I = R^+(M, D) \cap \mathfrak{cr}(M, D, J)$, using (**??**) and Proposition 3.2, we have

Proposition 1.4.1

$$C_I = \mathfrak{con}^+(M, D) \cap \mathfrak{cr}(M, D, J) \doteq \mathfrak{cr}^+(M, D, J) \tag{1.17}$$

$$C_I = \{f \in C(M) | \nabla^2_{JX,JY} e^{-f} = \nabla^2_{X,Y} e^{-f}, X, Y \in D\}$$

$\mathfrak{cr}^+(M, D, J)$ is the set of CR vector fields that are everywhere transverse to D. Thus it is not hard to see that the set of type I deformations C_I is a convex open cone in $\mathfrak{cr}(D, J)$.

We are interested in the set of type I deformations mod out the effect of pseudo-conformal transformations. Fix a background Sasakian structure (η, T). Let $\psi \in \mathfrak{CR}(D, J)$ be a pseudo-conformal transformation. Then the type I deformation (pseudo conformal rescaling) arises from ψ corresponds to $\psi_* T$ via (**??**). Under the correspondence (**??**), the pseudo conformal transformation acts by the differential map. This correspond to the adjoint action of the group $\mathfrak{CR}(D, J)$ on its Lie algebra $\mathfrak{cr}(D, J)$. Let $\xi \in \mathfrak{cr}^+(D, J) = C_I$, then $\eta(\psi_* \xi) = \psi^* \eta(\xi) = e^f \eta(\xi) > 0$. So $Ad(\psi)\xi \in C_I$, i.e. the set of type I deformations are invariant under pseudo conformal transformations. So,

Definition 1.4.2 The moduli set $\mathfrak{t}^+(D, J) = \mathfrak{cr}^+(D, J)/\mathfrak{CR}(D, J)$ is called the Sasaki cone.

We defined the automorphism group of the Sasakian structure as $\mathfrak{Aut}(\eta) = \mathfrak{CR}(D, J) \cap \mathfrak{Con}(\eta)$. Apparently $\mathfrak{Aut}(\eta) \subset \mathfrak{Iso}(M, g)$. Let $\psi \in \mathfrak{CR}(D, J)$ be a CR transformation, if $\psi \in \mathfrak{Aut}(\eta)$, then $\psi_* \eta = \eta$; ψ gives rise to the trivial type I deformation. Also, ψ acts trivially on the set of type I deformations. Conversly, if ψ correspond to the trivial type I deformation, then it preserves the 1-form η, hence $\psi \in \mathfrak{Aut}(\eta)$. We arrrived at,

Corollary 1.4.3 $\mathfrak{Aut}(\eta) = \mathfrak{CR}(D, J)$ iff all pseudo-conformal transformations correspond to the trivial type I deformation. In that case, the Sasaki cone is equal to the set of type I deformations.

1.5 Curvature change formulas

In this section we calculate how curvature changes under a type I deformation.

Let (M, η, T, J, D, g) be a Sasakian manifold, and $(M, \tilde{\eta}, \tilde{T}, J, D, \tilde{g})$ be a type I deformation with $\tilde{\eta} = e^f \eta$. Let ∇ and $\tilde{\nabla}$ be the Sasakian connections respectively.

Proposition 1.5.1 For $X, Y \in D$, the Sasakian connections $\tilde{\nabla}$ and ∇ are related by:

$$\tilde{\nabla}_X Y = \nabla_X Y + \tfrac{1}{2}(f_X Y + f_Y X - f_{JX} JY - f_{JY} JX - g(X,Y)\nabla_b(f)) \quad (1.18)$$

$$\tilde{\nabla}_X \tilde{T} = JX \quad (1.19)$$

$$\tilde{\nabla}_{\tilde{T}} X = [\tilde{T}, X] + JX \quad (1.20)$$

We denote $X(f)$, $X(Y(f))$ etc. by f_X, f_{XY}

Proof. Only the first relation needs to be verified, the other two follows from the (**??**) , (**??**)

By Korzul's formula, we have, for $X, Y, Z \in D$

$$2g((\nabla_X Y), Z) = Xg(Y,Z) + Yg(X,Z) - Zg(X,Y) \\ + g([X,Y],Z) - g([X,Z],Y) - g([Y,Z],X) \tag{1.21}$$

$$2\tilde{g}((\tilde{\nabla}_X Y), Z) = X\tilde{g}(Y,Z) + Y\tilde{g}(X,Z) - Z\tilde{g}(X,Y) \\ + \tilde{g}([X,Y],Z) - \tilde{g}([X,Z],Y) - \tilde{g}([Y,Z],X) \tag{1.22}$$

We calculate that

$$X\tilde{g}(Y,Z) = X(e^f g(Y,Z)) = e^f f_X g(Y,Z) + e^f Xg(Y,Z) \tag{1.23}$$

and

$$\begin{aligned} &\tilde{g}([X,Y],Z) - e^f g([X,Y],Z) \\ &= \tilde{g}([X,Y]^{\tilde{D}} - [X,Y]^D, Z) \\ &= \tilde{g}(([X,Y] - \tilde{\eta}[X,Y]\tilde{T}) - ([X,Y] - \eta[X,Y]T), Z) \\ &= \tilde{g}(2d\tilde{\eta}(X,Y)\tilde{T} - 2d\eta(X,Y)T), Z) \\ &= -e^f d\eta(X,Y) g(J\nabla_b f, Z) \\ &= e^f f_{JZ} g(JX, Y) \end{aligned} \tag{1.24}$$

Multiply (**??**) by e^f, and substract from (**??**), using (**??**) and (**??**), we have,

$$g((\tilde{\nabla}_X Y)^{\tilde{D}}, Z) = g((\nabla_X Y)^D, Z) + \tfrac{1}{2}(f_X g(Y,Z) + f_Y g(X,Z) - f_Z g(X,Y))$$
$$+ \tfrac{1}{2}(f_{JZ} g(JX,Y) - f_{JY} g(JX,Z) - f_{JX} g(JY,Z)) \tag{1.25}$$

So,

$$\begin{aligned}(\tilde{\nabla}_X Y)^{\tilde{D}} &= (\nabla_X Y)^D + \tfrac{1}{2}(f_X Y + f_Y X - f_{JX} JY - f_{JY} JX) \\ &- \tfrac{1}{2}(g(X,Y)\nabla_b f + g(JX,Y) J\nabla_b f)\end{aligned} \tag{1.26}$$

Lastly, the Reeb component is

$$\begin{aligned}(\tilde{\nabla}_X Y)^{\tilde{T}} &= -\tilde{g}(JX,Y)\tilde{T} = -g(JX,Y)(T - \tfrac{1}{2}J\nabla_b f) \\ &= (\nabla_X Y)^T + \tfrac{1}{2}g(JX,Y)J\nabla_b f\end{aligned} \tag{1.27}$$

where we use the fact that $\eta(\nabla_X Y) = -g(JX,Y)$ and (**??**)

now the result follows combining (**??**) and (**??**).

Let the Riemaniann curvature tensor for ∇ and $\tilde{\nabla}$ be respectively, R and $\tilde{R}$. Next, we calculate $\tilde{R}(X,Y,X,Y)$ for any horizontal vectors X and Y.

Proposition 1.5.2 Suppose $X, Y \in D$ such that $g(X,X) = g(Y,Y) = 1$ and

$g(Y,X) = g(Y,JX) = 0$, then,

$$\tilde{R}(X,Y,X,Y) = e^f R(X,Y,X,Y) \\ + \tfrac{1}{4} e^f (\nabla^2_{X,X} f + \nabla^2_{Y,Y} f + \nabla^2_{JX,JX} f + \nabla^2_{JY,JY} f + |\nabla_b f|^2_g) \tag{1.28}$$

Proof. Firstly, note that $\nabla_X Y \in D$ since $g(\nabla_X Y, T) = -g(Y, \nabla_X T) = -g(Y, JX) = 0$. Likewise, $\nabla_Y X, [X,Y] \in D$, thus (**??**) applies succesively, we have explicitly

$$\begin{aligned}\tilde{\nabla}_X \tilde{\nabla}_Y X &= \tilde{\nabla}_X (\nabla_Y X + \tfrac{1}{2}(f_X Y + f_Y X - f_{JX} JY - f_{JY} JX)) \\ &= \nabla_X \nabla_Y X + \tfrac{1}{2}(f_{\nabla_Y X} X + f_X \nabla_Y X - f_{J\nabla_Y X} JX - f_{JX} J \nabla_Y X) \\ &\quad + \tfrac{1}{2}(f_{XX} Y + f_X \tilde{\nabla}_X Y + f_{XY} X + f_Y \tilde{\nabla}_X X) \\ &\quad - \tfrac{1}{2}(f_{X,JX} JY + f_{JX} \tilde{\nabla}_X JY + f_{X,JY} JX + f_J Y \tilde{\nabla}_X JX)\end{aligned}$$

$$\begin{aligned}\tilde{\nabla}_Y \tilde{\nabla}_X X &= \tilde{\nabla}_Y (\nabla_X X + f_X X - f_{JX} JX - \tfrac{1}{2} \nabla_b f) \\ &= \nabla_Y \nabla_X X + \tfrac{1}{2}(f_{\nabla_X X} Y + f_Y \nabla_X X - f_{J\nabla_X X} JY - f_{JY} J \nabla_X X) \\ &\quad - \tfrac{1}{2} g(Y, \nabla_X X) \nabla_b f + (f_{YX} X + f_X \tilde{\nabla}_Y X) \\ &\quad - (f_{Y,JX} JX + f_{JX} \tilde{\nabla}_Y JX) - \tfrac{1}{2} \tilde{\nabla}_Y (\nabla_b f)\end{aligned}$$

$$\tilde{\nabla}_{[X,Y]}X = \nabla_{[X,Y]}X + \tfrac{1}{2}(f_{[X,Y]}X + f_X[X,Y])$$
$$- \tfrac{1}{2}(f_{J[X,Y]}JX + f_{JX}J[X,Y]) - \tfrac{1}{2}g([X,Y],X)\nabla_b f$$

By definition $\tilde{R}(X,Y)X = \tilde{\nabla}_X\tilde{\nabla}_Y X - \tilde{\nabla}_Y\tilde{\nabla}_X X - \tilde{\nabla}_{[X,Y]}X$, carefully combine terms in the previous expressions, using properties of Sasakian connection and Proposition 1.2.1, we have

$$\begin{aligned}
\tilde{R}(X,Y)X = R(X,Y)X &+ \tfrac{1}{2}f_Y(\tilde{\nabla}_X - \nabla_X)X - \tfrac{1}{2}f_X(\tilde{\nabla}_Y - \nabla_Y)X \\
&- \tfrac{1}{2}f_{JY}(\tilde{\nabla}_X - \nabla_X)JX + \tfrac{1}{2}f_{JX}(\tilde{\nabla}_Y - \nabla_Y)JX - \tfrac{1}{2}f_{JX}(\tilde{\nabla}_X - \nabla_X)JY \\
&+ \tfrac{1}{2}(\nabla^2_{X,X}f)Y - \tfrac{1}{2}(\nabla^2_{Y,X}f)X - \tfrac{1}{2}(\nabla^2_{X,JX}f)JY \\
&+ \tfrac{1}{2}(\nabla^2_{Y,JX}f)JX - \tfrac{1}{2}(\nabla^2_{X,JY}f)JX \\
&+ \tfrac{1}{2}f_{JY}T + \tfrac{1}{2}f_T JY + \tfrac{1}{2}\tilde{\nabla}_Y(\nabla_b f)
\end{aligned}$$

Note that $R(X,Y,X,T) = 0$ by Propostion 2.2, the terms on the right hand side of the above equation is in D, except the last line. Hence, taking inner product with Y with respect to $\tilde{g}$, using $\tilde{g}|_D = e^f g|_D$,

$$\begin{aligned}
\tilde{R}(X,Y,X,Y) = e^f R(X,Y,X,Y) &- \tfrac{1}{4}e^f f_Y^2 - \tfrac{1}{4}e^f f_X^2 + \tfrac{1}{4}e^f f_{JY}^2 + \tfrac{1}{2}e^f\nabla^2_{X,X}f \\
&+ \tfrac{1}{2}\tilde{g}(f_{JY}T,Y) + \tfrac{1}{2}\tilde{g}(\tilde{\nabla}_Y(\nabla_b f),Y)
\end{aligned}$$

The last 2 terms are further calculated as follows,

$$\begin{aligned}\tilde{g}(f_{JY}T, Y) &= f_{JY}\tilde{g}(e^f\tilde{T} + \tfrac{1}{2}J\nabla_b f, Y) = \tfrac{1}{2}f_{JY}\tilde{g}(J\nabla_b f, Y)\\ &= \tfrac{1}{2}e^f f_{JY}g(J\nabla_b f, Y) = -\tfrac{1}{2}e^f f_{JY}^2\end{aligned}$$

$$\begin{aligned}\tilde{g}(\tilde{\nabla}_Y(\nabla_b f), Y) &= Y(\tilde{g}(\nabla_b f, Y) - \tilde{g}(\nabla_b f, \tilde{\nabla}_Y Y)\\ &= Y(e^f g(\nabla_b f, Y)) - e^f g(\nabla_b f, \tilde{\nabla}_Y Y)\\ &= Y(e^f f_Y) - e^f g(\nabla_b f, \nabla_Y Y + f_Y Y - f_{JY}JY - \tfrac{1}{2}\nabla_b f)\\ &= e^f(\nabla^2_{Y,Y}f + f_{JY}^2 + \tfrac{1}{2}|\nabla_b f|_g^2)\end{aligned}$$

where we have noticed that $\tilde{\nabla}_Y Y \in D$ and (**??**) is used.

We arrived at,

$$\begin{aligned}\tilde{R}(X, Y, X, Y) = e^f R(X, Y, X, Y) + \tfrac{1}{4}e^f(-f_Y^2 - f_X^2 + f_{JY}^2 + f_{JX}^2)\\ + \tfrac{1}{2}e^f(\nabla^2_{X,X}f + \nabla^2_{Y,Y}f) + \tfrac{1}{4}e^f|\nabla_b f|_g^2\end{aligned}$$

finally, by Proposition 1.3.2, we may write $\nabla^2_{XX}f - f_X^2 + f_{JX}^2 = \nabla^2_{JXJX}f$, and the proposition is proved.

Proposition 1.5.3 Suppose $X \in D$ such that $g(X,X) = 1$, then

$$\tilde{R}(X,JX,X,JX) = e^f R(X,JX,X,JX) + e^f(2\nabla^2_{X,X}f + 2\nabla^2_{JX,JX}f + |\nabla_b f|^2_g + 3(e^f - 1)) \tag{1.29}$$

Proof. The calculation is similar to the previous proposition, only that now both $\nabla_X X$ and $[X,JX]$ do not lie in D. In order to utilize (**??**), we need to write $\nabla_X X = (\nabla_X X)^D + T$ and $[X,JX] = ([X,JX])^D - 2T$.

$$\begin{aligned}\tilde{\nabla}_X\tilde{\nabla}_{JX}X &= \tilde{\nabla}_X((\nabla_{JX}X)^D + T + f_X JX + f_{JX}X)\\ &= \nabla_X\nabla_{JX}X + \tfrac{1}{2}(f_{\nabla_{JX}X}X + f_X\nabla_{JX}X - f_{J\nabla_{JX}X}JX - f_{JX}J\nabla_{JX}X)\\ &\quad - \tfrac{1}{2}g(X,\nabla_{JX}X)\nabla_b f - \nabla_X T - \tfrac{1}{2}(f_T X + f_X T) + \tilde{\nabla}_X T\\ &\quad + \tfrac{1}{2}(f_{XX}JX + f_X\tilde{\nabla}_X JX + f_{XJX}X + f_{JX}\tilde{\nabla}_X X)\end{aligned}$$

$$\begin{aligned}\tilde{\nabla}_{JX}\tilde{\nabla}_X X &= \nabla_{JX}\nabla_X X + \tfrac{1}{2}(f_{\nabla_X X}JX + f_{JX}\nabla_X X + f_{J\nabla_X X}X + f_X J\nabla_X X)\\ &\quad - \tfrac{1}{2}g(JX,\nabla_X X)\nabla_b f + (f_{JX,X}X + f_X\tilde{\nabla}_{JX}X)\\ &\quad - (f_{JX,JX}JX + f_{JX}\tilde{\nabla}_{JX}JX) - \tfrac{1}{2}\tilde{\nabla}_{JX}(\nabla_b f)\end{aligned}$$

$$
\begin{aligned}
\tilde{\nabla}_{[X,JX]}X &= \tilde{\nabla}_{[X,JX]^D}X - 2\tilde{\nabla}_T X = \tilde{\nabla}_{[X,JX]^D}X + 2\nabla_T X + 2(\nabla_X T - \tilde{\nabla}_X T) \\
&= \nabla_{[X,JX]}X + \tfrac{1}{2}(f_{[X,Y]}X + f_X[X,Y] + f_T X + f_X T) \\
&\quad - \tfrac{1}{2}(f_{J[X,Y]}JX + f_{JX}J[X,Y]) - \tfrac{1}{2}g([X,Y],X)\nabla_b f + 2JX - 2\tilde{\nabla}_X T
\end{aligned}
$$

Carefully combine the above, we get

$$
\begin{aligned}
\tilde{R}(X,JX)X = R(X,JX)X &+ f_{JX}(\tilde{\nabla}_X - \nabla_X)X + f_{JX}(\tilde{\nabla}_{JX} - \nabla_{JX})JX \\
&+ (\nabla^2_{X,X}f)JX + (\nabla^2_{JX,JX}f)JX \\
&- 2f_X T - 2f_T X + \tfrac{1}{2}\tilde{\nabla}_{JX}(\nabla_b f) - 3JX + 3\tilde{\nabla}_X T
\end{aligned}
$$

As in the previous proposition, taking inner product with JX with respect to $\tilde{g}$, notice that the first line of the right hand side of the above express lies in D.

$$
\begin{aligned}
\tilde{R}(X,JX,X,JX) = e^f R(X,JX,X,JX) &- e^f f_{JX}^2 + e^f\nabla^2_{X,X}f + e^f\nabla^2_{JX,JX}f \\
&- e^f f_X^2 + \tfrac{1}{2}e^f(\nabla^2_{JX,JX}f + f_X^2 + \tfrac{1}{2}|\nabla_b f|^2) - 3e^f + 3\tilde{g}(\tilde{\nabla}_X T, JX)
\end{aligned}
$$

The last term is further calculated as follows,

$$
\begin{aligned}
\tilde{g}(\tilde{\nabla}_X T, JX) &= \tilde{g}(\tilde{\nabla}_X(e^f\tilde{T} + \tfrac{1}{2}J\nabla_b f), JX) \\
&= \tilde{g}(e^f f_X \tilde{T} + e^f\tilde{\nabla}_X\tilde{T} + \tfrac{1}{2}\tilde{\nabla}_X(J\nabla_b f), JX) \\
&= e^f\tilde{g}(JX, JX) + \tfrac{1}{2}(\tilde{\nabla}_X\tilde{g}(J\nabla_b f, JX) - \tilde{g}(J\nabla_b f, \tilde{\nabla}_X JX)) \\
&= e^{2f} + \tfrac{1}{2}X(e^f f_X) - \tfrac{1}{2}\tilde{g}(J\nabla_b f, J\tilde{\nabla}_X X - e^f\tilde{T}) \\
&= e^{2f} + \tfrac{1}{2}X(e^f f_X) - \tfrac{1}{2}e^f g(\nabla_b f, \tilde{\nabla}_X X) \\
&= e^{2f} + \tfrac{1}{2}e^f(f_X^2 + f_{XX}) - \tfrac{1}{2}g(\nabla_b f, \nabla_X X + f_X X - f_{JX}JX - \tfrac{1}{2}\nabla_b f) \\
&= e^{2f} + \tfrac{1}{2}e^f\nabla^2_{XX}f + \tfrac{1}{2}e^f f_{JX}^2 + \tfrac{1}{4}|\nabla_b f|^2
\end{aligned}
$$

We finally arrived at,

$$
\begin{aligned}
\tilde{R}(X, JX, X, JX) = e^f R(X, JX, X, JX) + e^f(-\tfrac{1}{2}f_X^2 + \tfrac{1}{2}f_{JX}^2) \\
+ e^f(\tfrac{5}{2}\nabla^2_{X,X}f + \tfrac{3}{2}\nabla^2_{JX,JX}f + 3(e^f - 1) + |\nabla_b f|_g^2)
\end{aligned}
$$

Use the relation $\nabla^2_{XX}f - f_X^2 = \nabla^2_{JXJX}f - f_{JX}^2$ (Proposition 3.2), we see the above formula is equivalent to the proposition.

Corollary 1.5.4 For g-unit vectors $X, Y \in D$ such that $g(Y, X) = g(Y, JX) = 0$, we have sectional curvautres,

$$
\tilde{K}(X, Y) = e^{-f}(K(X, Y) - \tfrac{1}{4}\nabla^2_{X,X}f - \tfrac{1}{4}\nabla^2_{Y,Y}f - \tfrac{1}{4}\nabla^2_{JX,JX}f - \tfrac{1}{4}\nabla^2_{JY,JY}f - \tfrac{1}{4}|\nabla_b f|_g^2) \tag{1.30}
$$

$$\tilde{K}(X,JX) = e^{-f}(K(X,JX) - 2\nabla^2_{X,X}f - 2\nabla^2_{JX,JX}f - |\nabla_b f|^2 - 3(e^f - 1)) \tag{1.31}$$

Proof. Note that $\tilde{K}(X,Y) = \frac{\tilde{R}(X,Y,Y,X)}{|X\wedge Y|_{\tilde{g}}} = e^{-2f}\tilde{R}(X,Y,Y,X)$

Corollary 1.5.5 Let $dimM = 2n+1$, then the Ricci curvatures are related by,

$$\begin{aligned} \tilde{Ric}(X) = Ric(X) - \tfrac{n+2}{2}(\nabla^2_{X,X}f + \nabla^2_{JX,JX}f) - \tfrac{n+1}{2}|\nabla_b f|^2|X|^2 \\ - \tfrac{1}{2}(\Delta_b f)|X|^2 - 2(e^f - 1)|X|^2) \end{aligned} \tag{1.32}$$

The scalar curvatures are related by,

$$\tilde{s} = e^{-f}(s - (2n+2)\Delta_b f - (n+1)n|\nabla_b f|^2 - 2n(e^f - 1)) \tag{1.33}$$

The sub-Laplacian $\Delta_b f = \Delta f - \nabla^2_{T,T} f$

Proof. Take the trace, using (**??**) , (**??**), and Proposition 2.2.

In the special case of a transverse homothety, i.e. when f is a constant function, we have much simpler relations summarized in the corollary below.

Corollary 1.5.6 [**?**] Let $\tilde{\eta} = a\eta$ be a transverse homothety, then $\tilde{T} = a^{-1}T$, and

$(1)\tilde{\nabla} - \nabla = (a-1)(J \otimes \eta + \eta \otimes J)$

$(2)\tilde{K}(X,Y) = a^{-1}K(X,Y)$ for $X, Y \in D, g(X,Y) = g(JX,Y) = 0$

$(3)K(X,JX) = a^{-1}(K(X,JX)+3) - 3$ for $X \in D$

$(4)\tilde{Ric} = Ric - 2(a-1)g + 2(a-1)(an+n+1)\eta \otimes \eta$

$(5)\tilde{s} = a^{-1}(s+2n) - 2n$

1.6 CR groups of certain Sasakian manifolds

1.6.1 Sasakian space forms

Recall from Section 2 that Sasakian space forms are Sasakain manifolds with constant Φ-sectional curvature, i.e. $K(X,JX)$ is constant for any $X \in D$. Suppose M has constant Φ-sectional curvature $K(X,JX) = c$. Let ϕ be a pseudo-conformal tranformation, and let $\tilde{\eta} = \phi_* \eta = e^f \eta$. The Sasakian structure on $(\tilde{M}, \tilde{\eta})$ is the push forward, hence it has the same curvature properties, i.e. $\tilde{K}(X,JX) = c$. Now utilize (**??**), we get, for any $X \in D$,

$$c = e^{-f}(c - 2\nabla^2_{X,X}f - 2\nabla^2_{JX,JX}f - |\nabla_b f|^2 - 3(e^f - 1))$$

Let X run over an orthonormal basis of D, summing up, we get,

$$n(c+3)(1-e^f) = 2\Delta_b f + n|\nabla_b f|^2 \tag{1.34}$$

If $c < -3$, at the minimal of f we have $\Delta_b f \leq 0$ and $\nabla_b f = 0$. (**??**) then implies $1 - e^f \leq 0$, thus $f_{min} \geq 0$. Similarly, at the maximal of f, (**??**) implies $1 - e^f \geq 0$, thus $f_{max} \leq 0$. Hence, $e^f = 1$

If $c = -3$, the left hand side of (**??**) is 0. If we integrate (**??**) over M, we get $|\nabla_b f| = 0$, thus $X(f) = 0$ for all $X \in D$. As for the Reeb direction T, since $\eta([X, JX]) = -d\eta(X, JX) = 1$, we can write $T = -[X, JX] + Y$, for some $Y \in D$, and hence $T(f) = JX(X(f)) - X(JX(f)) + Y(f) = 0$. So, f is constant. Finally, the volume of Sasakian structure doesn't change under a pullback, so

$$0 = \int d\tilde{vol} - \int dvol = \int (e^{(n+1)f} - 1) dvol$$

Hence the rescaling factor $e^f = 1$.

Summing up the above disscusion, we have,

Theorem 1.6.1 Let M be a compact Sasakian space forms with constant Φ-sectional curvautre c, then

(1) If $c < -3$ or

(2) If $c = -3$

then M has no non trival pseudo-conformal rescaling.

In view of Corollary 1.4.3, we have

Corollary 1.6.2 Let M be as stated in Theorem, then $\mathfrak{CR}(M, D, J) = \mathfrak{Aut}(M, \eta)$.

Remark. for the simply connected sasakian space forms with $c > -3$, they are isomorphic to transverse homotheties of the standard sphere. The underlying CR structures of there manifolds are the same as the standard sphere. It is known that the CR transformation group of th standard $(2n+1)$-sphere is $SU(n+1,1)$ which is non-compact. But the Sasakian Automorphism groups of these sasakain space forms are subgroups of the isometry group $Iso(M, g)$, which is compact. So we don't expect a similar result as Theorem 6.1 for the case $c > -3$.

1.6.2 Constant scalar curvature

Let (M, η) be a compact Sasakian manifold with constant scalar curvature $s = c$. Let $(M, \tilde{\eta})$ be a pseudo-conformal rescaling of (M, η), where $\tilde{\eta} = e^f \eta$. $(M, \tilde{\eta})$ still has constant scalar curvature $\tilde{s} = c$.

According to (**??**)

$$\tilde{s} = e^{-f}(s - (2n+2)\Delta_b f - (n+1)n|grad_b f|^2 - 2n(e^f - 1))$$

This implies now,

$$(1 - e^f)(c + 2n) = (2n + 2)\Delta_b f + (n + 1)n|\nabla_b f|^2 \tag{1.35}$$

By a similar argument as in the previous section,

(1) If $c < -2n$, then $f \geq 0$ at its minimal, and $f \leq 0$ at its maximal, hence $f = 0$. Thus the pseudo-conformal rescaling is trivial.

(2) If $c = -2n$, then the left hand side of (**??**) is 0. Integrate both sides, we deduce that $\nabla_b f = 0$. This further implies $\nabla f = 0$, hence f is constant. Finally, $f = 1$ as the volume does not change under a pseudo-conformal rescaling.

Theorem 1.6.3 Let M be a compact Sasakian manifold with constant scalar curvautre c, then
(1) If $c < -2n$ or
(2) If $c = -2n$
then M has no non trival pseudo-conformal deformation. And thus $\mathfrak{CR}(M, D, J) = \mathfrak{Aut}(M, \eta)$

By (**??**) compact sasakian manifolds whose transverse metric has zero scalar curvture satisfy Theorem 6.3(2). Sasakian manifolds whose transverse metric with a constant scalar curvature which is negative satisfy Theorem 6.3(1). In particular, For compact Sasakian manfolds whose transverse metric being a (Kähler) Einstein metric of nonpositive type, we always have $\mathfrak{CR}(M, D, J) = \mathfrak{Aut}(M, \eta)$.

Charles Boyer and Krzysztof Galicki ([**?**] p 262 has proved that when the transverse Ricci curvature is nonpositive, the Lie algebra of the automorphism group is generated by the Reeb vector field, $\mathfrak{aut}(\eta) = \{T\}$. Combining this with the previous result, we have

Corollary 1.6.4 Let (M, D, J) be a compact CR manifold. If (M, D, J) admits a Sasakian structure (η, T) whose transverse metric has constant scalar curvature and nonpositive ricci curvature, then the Lie algebra of the CR group is generated by the Reeb vector field, i.e. $\mathfrak{cr}(D, J) = \{T\}$. And, (M, D, J) admits only one Sasakian struture up to transverse homothety, i.e. $(a\eta, a^{-1}T)$.

Proof. Note that any Reeb vector field of a sasakian structure generates CR-automorphisms, hence lies in $\mathfrak{cr}(D, J)$, hence the compactible Reeb vector field is unique up to scaling.

Theorem 1.6.5 Let (M, η) be a compact Sasakian manifold. Suppose its transverse ricci curvature is nonpositive and the (transverse) scalar curvature is constant, then (M, η) does not admit any non trivial type I deformation, i.e. All type I deformations are given by transverse homothety.

Proof. Let $\tilde{\eta} = e^f \eta$ be a type I deformation. By (**??**), $\tilde{T} = e^{-f}(T - \frac{1}{2}\nabla_b f)$. Then, by Corollary 1.6.4, $\nabla_b f \in \{T\}$, hence $\nabla_b f = 0$. This further implies $\nabla f = 0$. Hence f is a constant.

Remark. An interesting related result of F. Belgun [**?**] says any 3-dimensional compact sperical CR manifold that admits a Sasakian struture has no type I deformation besides transverse homothety. Note that those manifolds are circle bundles over a Riemanian surface with positive genus.

1.7 Type I deformations of Sasaki-Einstein manifolds

Let M be a Sasaki-Einstein manifold. let $\tilde{\eta} = e^f \eta$ be a type I deformation. It was conjectured by K. Galicki and C. Boyer that any type I deformation that transfer into another Sasaki-Einstein metric is a pseudo-conformal rescaling. In other words, only the trivial element in the Sasaki cone transfers a Sasaki-Einstein metric to a Sasaki-Einstein metric.

Theorem 1.7.1 [?] A non-trivial transverse homothety does not relate two Sasaki-Einstein metric.

Proof.. By Proposition 2.3, on a $2n+1$ dimensional Sasakian manifold, $Ric(T) = 2n$. Hence when the metric is Einstein, we have $Ric = 2ng$, the Einstein constant is always $2n$. Hence scalar curvature is a constant $2n(2n+1)$. Let $\tilde{\eta} = a\eta$ be a transverse homothety, Then Corollary 1.5.6(5) implies $a = 1$.

For a general type I deformation, recall the formula (**??**), plug in with$\tilde{Ric} = Ric = 2ng$, we get that, if two Sasaki-Eistein are related by a type I deformation, then

$$\nabla^2_{X,X} f + \nabla^2_{JX,JX} f = \frac{1}{n+2}(4(n+1)(1-e^f) - (n+1)|\nabla_b f|^2 - \Delta_b f) g(X,X)$$

Define

$$S(X,Y) = \nabla^2_{X,Y} f + \nabla^2_{JX,JY} f$$

$$\Lambda(X,Y) = \nabla^2_{X,Y} f - \nabla^2_{JX,JY} f$$

Then the above formula implies $S(X,Y)$ is a multiple of the metric, say $S(X,Y) = \beta g(X,Y)$. Let $\{e_i, Je_i\}_{i=1}^n$ be an othonormal basis for the subbundle D, then $S(e_i, e_i) = \beta$, hence

$$\Delta_b f = \sum_i S(e_i, e_i) = n\beta = \frac{n}{n+2}(4(n+1)(1-e^f) - (n+1)|\nabla_b f|^2 - \Delta_b f)$$

Simplying, we get

$$2n(1-e^f) = \Delta_b f + \frac{n}{2}|\nabla_b f|^2$$

$$\beta = \frac{1}{n}\Delta_b f = 2(1-e^f) - \frac{1}{2}|\nabla_b f|^2$$

Summing up the above and Proposition 1.3.2, we have

Proposition 1.7.2 A function e^f relates two Sasaki-Eistein by a type I deformation, if and only if for any $X, Y \in D$

$$(1) S(X,Y) = \beta g(X,Y), \beta = 2(1-e^f) - \frac{1}{2}|\nabla_b f|^2 \tag{1.36}$$

$$(2) \Lambda(X,Y) = X(f)Y(f) - JX(f)JY(f)$$

Next, we confirm the conjecture of K.Galicki and C.Boyer in the case of the standard sphere.

Theorem 1.7.3 Let (S, η) be the standard sphere; $\eta = e^f \eta$ be a type I deformation, such that $(\tilde{S}, \tilde{\eta})$ is Einstein. Then this type I deformation arises from a pseudo-conformal transformation. i.e. there is a CR transformation $\phi : S \to \tilde{S}$, such that $\tilde{\eta} = \phi_* \eta$.

Proof.. Combining (**??**), (**??**), (**??**), we get

$$\tilde{K}(X, Y) - 1 = e^{-f}(K(X, Y) - 1)$$

$$\tilde{K}(X, JX) - 1 = e^{-f}(K(X, JX) - 1)$$

Also, for 2-planes containing the Reeb directions, by Proposition 1.2.1, we have $\tilde{K}(T, X) = 1$

These relations implies $(\tilde{S}, \tilde{\eta})$ still has constant sectional curvature 1, and is again the standard sphere. Hence the standard metric is the only Einstein metric within the type I deformation class. Moreover, according to the classification of Sasakian space forms by Tanno [**?**], $(\tilde{S}, \tilde{\eta})$ is isomorphic to the standard Sasakian structure on the sphere. Hence there is a diffeomorphism from the $\phi : S \to \tilde{S}$, such that $\tilde{\eta} = \phi_* \eta$, and $\tilde{D} = \phi_* D$, $\tilde{J} = \phi_* J$. But as a type I deformation, $\tilde{D} = D$, $\tilde{J} = J$, thus ϕ is a CR automorphism.

Chapter 2

Products of Sasakian manifolds

2.1 Contact metric structures

Definition 2.1.1 A contact structure on a $2n+1$ dimensional manifold M is a one form η such that $\eta \wedge (d\eta)^n$ is a nondegenerate top form on M. The one form η is called the contact form

As indicated in the definition, $\eta \wedge (d\eta)^n$ is a volume form, and hence the manifold M is oriented. A vetor field called the Reeb vector field T can be uniquely determined by the following equations $\eta(T) = 1$, $d\eta(T, \cdot) = 0$.

Let $D \subset TM$ be the distribution that is anhiliated by η, then $d\eta$ is a sympletic form for the vector bundle D over M. It is then easy to find local 1-forms $\{\mu^i\}_{1 \leq i \leq n}$ and $\{\nu^i\}_{1 \leq i \leq n}$, such that $d\eta = \sum \mu^i \wedge \nu^i$ (In fact, by a contact version of Darboux theorem, there exist local charts $(x^1, \dots, x^n, y^1, \dots, y^n, z)$ so that $d\eta = \sum dx^i \wedge dy^i)$. By definition of T, it is apparent that $\mu^i(T) = \nu^i(T) = 0$, and hence $\{\eta, \mu^i, \nu^i\}$ forms a basis for T^*M. Let the dual be $\{T, X_i, Y_i\}$, then D is spanned by $\{X_i, Y_i\}_{1\leq i\leq n}$. It is easy to see the rank n sub-distribution spanned by ${\mu^i}_{1\leq i\leq n}$ is always integrable by Frobenius theorem. In fact, any rank $2n$ distribution (of a $2n+1$ dimensional manifold) contains rank n sub-distributions that are integrable. On the other hand, Let $E \subset D$ be a integrable sub-distribution, then we must have $[E, E] \subset E$, hence for any $V, W \in E$, $0 = \eta[V, W] = d\eta(V, W)$, which implies E is isotropic and hence its rank is no more than n. Hence, D contains no integrable sub-distribution of rank $n+1$. A hyperplane distribution that has no integrable sub-distribution of more than half of its rank is said to be maximally non-integrable.

For the reason of the above, sometimes a contact structure (in the wider sense) on a $2n+1$ manifold is defined as the existence of a maximally non-integrable rank $2n$ distribution D. Equivalently, a contact structure in the wider sense, is a family of local contact forms η_i on an atlas, such that $\eta_i = f_{ij}\eta_j$ on the overlaps for some nonvanishing functions f_{ij}. When both M and D are orientable, the contact structure in the wider sense is in fact a contact structure (in the restricted sense) since a nonvanishing global section of the line bundle T^*M/D^* give rise to a global contact form on M.

As the bundle D over M is sympletic with sympletic form $d\eta$, there exist compactible almost complex struture J and metric g, i.e. $g(\cdot,) = d\eta(\cdot, J\cdot)$. One can extend J to a tensor on M by requiring $J(T) = 0$, likewise, extending g to be a metric on M by requiring $g(T, T) = 1$ and $g(T, D) = 0$.

Definition 2.1.2 The quadruple (M, η, J, g) is called a contact metric structure.

It is apparent that the triple (M, D, J) forms an almost CR structure. This CR structure is nondegenerate and strictly pseudo convex, as it has a Levi form $d\eta$ which is positive definite

An almost Kähler structure is associated with a contact metric struture (M, η, J, g). Namely, let $C(M) = M \times \mathbb{R}^+$ be equipped with the cone metric $\bar{g} = r^2 g + 4dr^2$, where r is the coordinate of the $\mathbb{R}$ factor. We can define a almost complex structure on the cone as $\bar{J}(V, a\Psi) = (J(V) - 2aT, \frac{1}{2}\eta(V)\Psi)$, where $\Psi = r\frac{\partial}{\partial r}$ is the Euler vector field. It is easy to see the almost complex structure $\bar{J}$ so defined is compactible with the cone metric $\bar{g}$. the associated 2-form is $r^2 d\eta + 2r dr \wedge \eta = d(r^2\eta)$, which is exact, and hence sympletic. We concluded that the cone $(C(M), \bar{g}, \bar{J}, d(r^2\eta))$ is a non-compact almost Kähler manifold. The contact manifold M is identified with the $r = 1$ slice of $\bar{M}$. Note that a contact metric structure by a dilation is no longer contact metric, other slices with $r \neq 1$ is not identified with M

Definition 2.1.3 (M, g, J, η) is said to be Sasakian, if the cone $(C(M), \bar{g}, \bar{J}, d(r^2\eta))$ is Kähler, i.e. if $\bar{J}$ is integrable.

By the Nirenberg-Newlander Theorem, $\bar{J}$ is integrable if and only if the Nijenhaus tensor $N_{\bar{J}}(U, V)$ vanishes. By tensoriality and symmetry, as any vector $V = f_1 X + f_2 T + f_3 \Psi$, where $X \in D$, the vanishing of the whole Nijenhaus tensor is equivalent to the vanishing of the following $N_{\bar{J}}(X, Y)$, $N_{\bar{J}}(X, T)$, $N_{\bar{J}}(X, \Psi)$, $N_{\bar{J}}(T, T)$, $N_{\bar{J}}(T, \Psi)$, and $N_{\bar{J}}(\Psi, \Psi)$. The last three of these are apparently zero.

$$N_{\bar{J}}(X, Y) = -[X, Y] + [JX, JY] - J([JX, Y] + [X, JY]) = N_J(X, Y) \quad (2.1)$$

$$\begin{aligned} N_{\bar{J}}(X, T) &= -[X, T] + [JX, \frac{1}{2}\Psi] - J([JX, T] + [X, \frac{1}{2}\Psi]) \\ &= L_T X + J L_T J X = -(L_T J) J X \end{aligned} \quad (2.2)$$

$$N_{\bar{J}}(X, \Psi) = N_{\bar{J}}(X, 2JT) = 2J(N(X, T)) = 2(L_T J) X \quad (2.3)$$

Thus, we have

Proposition 2.1.4 A contact metric structure (M, η, J, g) is Sasakian if and only if

(1)The almost CR struture (D, J) is integrable. i.e. $N_J(X, Y) = 0$ for any $X, Y \in D$

(2)The Reeb vector field T is an infinitesmal CR transformation. i.e. $L_T J = 0$

By the definition of the Reeb vector field, we have $L_T\eta = d(\eta(T)) - d\eta \neg T = 0$. Hence $L_T(d\eta) = 0$, and thus (2) is equivalent to $L_T g = 0$. i.e T is a killing vector field. A contact metric structure that satisfied (2) is usually called K-contact.

A standard example of Sasakian manifold is the odd dimensional sphere $S^{2n-1} \subset \mathbb{C}^n$. The contact form is given by $\eta = \sum y_i dx_i - x_i dy_i$. Its The Reeb vector field is given by $T = \sum y_i \frac{\partial}{\partial x_i} - x_i \frac{\partial}{\partial y_i}$. metric g and J is inherited from $\mathbb{C}^n$.

A famous example of complex manifold that doesn't admit Kähler metric is the Hopf manifold $(\mathbb{C}^n \setminus \{0\})/\{z \sim 2z\}$. Topologically, this is $S^{2n-1} \times S^1$. We can generalize this construction by considering the action $r \to 2r$ on the cone over Sasakian manifolds.

Definition 2.1.5 Let M be a Sasakian manifold, and let $C(M) = M \times \mathbb{R}^+$ be the metric cone as defined above. Then $C(M)/\{r \sim 2r\}$ is called the quotient cone over M.

Topologically, the quotient cone is just $M \times S^1$. Note that $C(M)$ is Kähler, and it is a covering of the quotient cone. Hence the quotient cone is Locally conformally Kähler.

2.2 Products of contact metric structures

In the section, we consider the product of two contact metric manifolds, namely $(M_1, \eta_1, J_1, g_1, D_1)$ and $(M_2, \eta_2, J_2, g_2, D_2)$. The product $M = M_1 \times M_2$ is even dimensional. It carries a family of almost complex structures I_{ab} defined by $I_{ab}(X_1) = JX_1$, $I_{ab}(X_2) = JX_2$, $I_{ab}(T_1) = -\frac{a}{b}T_1 + \frac{1}{b}T_2$, and $I_{ab}(T_2) = \frac{a^2+b^2}{b}T_1 + \frac{a}{b}T_2$, where $X_1 \in D_1, X_2 \in D_2$ and T_1, T_2 are the Reeb vector fields for each component.

The associate metric G_{ab} is specified by the following: Let $E = span\{T_1, T_2\}$, then $TM = D_1 + D_2 + E$, the three components are orthogonal under G_{ab} and that $G_{ab}|_{D_1}(X_1, X_1) = g_1(X_1, X_1)$, $G_{ab}|_{D_2}(X_2, X_2) = g_2(X_2, X_2)$, $G_{ab}|_E(T_1, T_1) = 1$, $G_{ab}|_E(T_2, T_2) = a^2 + b^2$, and $G_{ab}|_E(T_1, T_2) = a$. This gives the product M an almost hermitian structure. The associate 2-form is $\omega = d\eta_1 + d\eta_2 + b\eta_1 \wedge \eta_2$.

In particular, when $a = 0, b = 1$, This is the standard product, where

$$G = g_1 + g_2$$

$$\omega = d\eta_1 + d\eta_2 + \eta_1 \wedge \eta_2$$

$$I(V_1, V_2) = J_1(V_1) + J_2(V_2) - \eta_2(V_2)T_1 + \eta_1(V_1)T_2$$

The integrability of I_{ab} depends on the vanishing of the Nijenhause tensor. As any vector $V \in TM$ can be written as $V = f_1X_1 + f_2T_1 + f_3X_2 + f_4T_2$ where f_i are smooth functions, $X_1 \in D_1$ and $X_2 \in D_2$, by tensorialty and symmetry, It is easy to see we only need to check whether the following vanishes: $N_I(X_1, Y_1)$, $N_I(X_2, Y_2)$, $N_I(X_1, T_1)$, $N_I(X_1, T_2)$, $N_I(X_2, T_1)$ and $N_I(X_2, T_2)$

we have

$$\begin{aligned} N_I(X_1, Y_1) &= -[X_1, Y_1] + [J_1X_1, J_1Y_1] - J_1([J_1X_1, Y_1] + [X_1, J_1Y_1]) \\ &= N_{J_1}(X_1, Y_1) \end{aligned} \tag{2.4}$$

$$
\begin{aligned}
N_I(X_1,T_1) &= -[X_1,T_1] + [J_1X_1, -\frac{a}{b}T_1 + \frac{1}{b}T_2] - J_1([J_1X_1,T_1] \\
&\quad + [X_1, -\frac{a}{b}T_1 + \frac{1}{b}T_2]) \\
&= L_{T_1}X_1 + \frac{a}{b}L_{T_1}J_1X_1 - \frac{a}{b}J_1L_{T_1}X_1 + J_1L_{T_1}J_1X_1 \\
&= (L_{T_1}J_1)(\frac{a}{b}X_1 - J_1X_1)
\end{aligned} \tag{2.5}
$$

$$
\begin{aligned}
N_I(X_1,T_2) &= -[X_1,T_2] + [J_1X_1, \frac{a^2+b^2}{b}T_1 + \frac{a}{b}T_2] \\
&\quad - J_1([J_1X_1,T_2] + [X_1, \frac{a^2+b^2}{b}T_1 + \frac{a}{b}T_2]) \\
&= -\frac{a^2+b^2}{b}L_{T_1}J_1X_1 + \frac{a^2+b^2}{b}J_1L_{T_1}J_1X_1 \\
&= \frac{a^2+b^2}{b}(L_{T_1}J_1)(J_1X_1)
\end{aligned} \tag{2.6}
$$

Note that both (**??**) and (**??**) are equivalent to $L_{T_1}J_1 = 0$.

Similarily, the vanishing of $N_I(X_2,Y_2)$, $N_I(X_2,T_1)$ and $N_I(X_2,T_2)$ are equivalent to $N_{J_2}(X_2,Y_2) = 0$ and $L_{T_2}J_2 = 0$. Combining with Proposition 2.1.4, we have

Proposition 2.2.1 [**?**] The Product $(M_{a,b}, I_{a,b}, G_{a,b})$ is a hermitian manifold if and only if both the two contact metric factors (M_1,η_1,J_1,g_1) and (M_2,η_2,J_2,g_2) are Sasakian manifolds.

The construction in this section may be considered as a generalization of the famous Calabi-Eckmann manifolds $S^{2p+1} \times S^{2q+1}$. As mentioned in last section, the standard odd dimensional spheres are Sasakiam manifolds. The Calabi-Eckmann manifolds are hermitian manifolds that carries no Kähler structures.

2.3 The locally conformally Kähler condition

As seen in the previous sections, the metric cone over a contact metric manifold and the product of two contact metric manifolds share the same type of integrability conditions. In this section, we see while the quotient cone are always locally conformally Kähler, However, the generalized Calabi-Eckmann manifolds are always not.

Let ω be the standard sympletic form on $\mathbb{C}^n$, it is easy to see that the mapping $\Lambda^1 \to \Lambda^{2n-1}$ defined by $\alpha \longrightarrow \alpha \wedge \omega^{n-1}$ is a bijection. Hence under this mapping, any $(2n-1)$ form on a complex manifold determines a 1-form.

Definition 2.3.1 Let M be a n dimmensional hermitian manifold with fundamental 2-form ω. The associated Lee form is the unique 1-form ϕ, such that $d\omega \wedge \omega^{n-2} = \phi \wedge \omega^{n-1}$.

Apparently, when M is Kähler, the Lee form vanishes. If M is locally conformally Kähler, then there is locally a function e^f, such that $e^f\omega$ is closed, thus,

$$0 = d(e^f\omega) = e^f(df \wedge \omega + d\omega)$$

This implies that $d\omega \wedge \omega^{n-2} = df \wedge \omega^{n-1}$, hence the Lee form $\phi = df$, which means the Lee form is a closed form. In particular, if M is globally conformally Kahher, then ϕ is an exact 1-from.

Proposition 2.3.2 Let (M_1, η_1) and (M_2, η_2) be sasakian manifolds of dimension $2n_1 + 1$ and $2n_2 + 1$ respectively, the the Lee form of the hermitian manifold $M_{a,b} = M_1 \times M_2$, whose hermitian structure are giving in Section 2, is $\phi = b(\frac{n_1}{n_1+n_2}\eta_2 - \frac{n_2}{n_1+n_2}\eta_1)$

Proof. As in Section 2.1, we can take local coframes $(\eta_1, \alpha_1^i, \beta_1^i)$ and $(\eta_2, \alpha_2^i, \beta_2^i)$, so that $d\eta_1 = \sum \alpha_1^i \wedge \beta_1^i$ and $\eta_2 = \sum \alpha_2^i \wedge \beta_2^i$. Write $\Omega = \sum \alpha_1^i \wedge \beta_1^i + \sum \alpha_2^i \wedge \beta_2^i$, and let $n = n_1 + n_2 + 1$ be the complex dimension of $M_{a,b}$, then the fundamental form of $M_{a,b}$ is

$$\omega = \Omega + b(\eta_1 \wedge \eta_2)$$

$$d\omega = b(d\eta_1 \wedge \eta_2 - \eta_1 \wedge d\eta_2) = b(\sum \alpha_1^i \wedge \beta_1^i \wedge \eta_2 - \sum \alpha_2^j \wedge \beta_2^j \wedge \eta_1)$$

Then,

$$
\begin{aligned}
& d\omega \wedge \omega^{n-2} \\
&= b(\sum \alpha_1^i \wedge \beta_1^i \wedge \eta_2 - \sum \alpha_2^j \wedge \beta_2^j \wedge \eta_1) \wedge (\Omega^{n-2} + (n-2) b\eta_1 \wedge \eta_2 \wedge \Omega^{n-3}) \\
&= b(\sum \alpha_1^i \wedge \beta_1^i \wedge \eta_2 - \sum \alpha_2^j \wedge \beta_2^j \wedge \eta_1) \wedge \Omega^{n-2}
\end{aligned}
$$

Note that $n-1 = n_1 + n_2$ is the highest power of Ω to not be 0, it is not hard to see that, so for any i or j,

$$
\begin{aligned}
(\alpha_1^i \wedge \beta_1^i) \wedge \Omega^{n-2} &= \frac{1}{n-1} \Omega^{n-1} \\
(\alpha_2^j \wedge \beta_2^j) \wedge \Omega^{n-2} &= \frac{1}{n-1} \Omega^{n-1}
\end{aligned}
$$

So,

$$
d\omega \wedge \omega^{n-2} = b(\frac{n_1}{n-1} - \frac{n_2}{n-1}) \wedge \Omega^{n-1}
$$

On the other hand, let $\phi = b(\frac{n_1}{n_1+n_2}\eta_2 - \frac{n_2}{n_1+n_2}\eta_1)$, then

$$
\begin{aligned}
\phi \wedge \omega^{n-1} &= b(\frac{n_1}{n-1}\eta_2 - \frac{n_2}{n-1}\eta_1) \wedge (\Omega^{n-1} + (n-1) b\eta_1 \wedge \eta_2 \wedge \Omega^{n-2}) \\
&= b(\frac{n_1}{n-1}\eta_2 - \frac{n_2}{n-1}\eta_1) \wedge \Omega^{n-1}
\end{aligned}
$$

It is apparent that this Lee form ϕ is not closed, hence

Corollary 2.3.3 The product $M_{a,b}$ is not locally conformally Kähler.

We might think of S^1 as a degenerated Sasakian manifold, and consider the product of a Sasakian manifold and S^1, i.e. $M_{a,b} = M_1 \times S^1$. All the previous calculations and arguments still apply. In the case $a = 0, b = 2$, the induced complex structure on this product is equivalent to that of the quotient cone $\bar{C}(M) = M_1 \times S^1$. However, the metric on the quotient cone is related to the warp product metric, and it is locally conformally Kähler. This lead us to consider whether there is a simple way of 'warping', so that the product of two Sasakian manifold becomes locally conformally Kähler as in the case of one of the component being degenerated S^1. The answer is negative.

Proposition 2.3.4 On the product $M_1 \times M_2$ the local 2-form $\omega = f d\eta_1 + g d\eta_2 + h\eta_1 \wedge \eta_2$ is not a closed form, for any smooth function f, g, h defined locally on $M_1 \times M_2$

Proof. Take local coframes $(\eta_1, \alpha_1^i, \beta_1^i)$ and $(\eta_2, \alpha_2^i, \beta_2^i)$, so that $d\eta_1 = \sum \alpha_1^i \wedge \beta_1^i$ and $\eta_2 = \sum \alpha_2^i \wedge \beta_2^i$. The dual frames are (T_1, X_i^1, Y_i^1) and (T_2, X_i^2, Y_i^2) where T_1 and T_2 are Reeb vector fields. Suppose ω is close, we have

$$0 = d\omega = df \wedge d\eta_1 + dg \wedge d\eta_2 + dh \wedge \eta_1 \wedge \eta_2 + hd\eta_1 \wedge \eta_2 - h\eta_1 \wedge d\eta_2$$

Evaluate at $T_1 \wedge T_2 \wedge U$, where U is any linear combination of $X_i^1, Y_i^1, X_i^2, Y_i^2$, we see then $dh(U) = 0$. Hence, $dh = a\eta_1 + b\eta_2$. Take derivative of this, we get,

$$0 = ddh = da \wedge \eta_1 + ad\eta_1 + db \wedge \eta_2 + bd\eta_2$$

Evaluate at, say $X_i^1 \wedge Y_i^1$, we see $a = 0$, similarly, $b = 0$, so $dh = 0$. Without loss of generality, set $h = 1$, and so we can rewrite $d\omega$ as

$$\begin{aligned} 0 = d\omega &= df \wedge d\eta_1 + dg \wedge d\eta_2 + d\eta_1 \wedge \eta_2 - \eta_1 \wedge d\eta_2 \\ &= (df - \eta_2) \wedge d\eta_1 + (dg - \eta_1) \wedge d\eta_2 \end{aligned}$$

Evaluate this at $T_2 \wedge X_i^1 \wedge Y_i^1$, we get $T_2(f) = 1$; at $T_1 \wedge X_i^1 \wedge Y_i^1$, get $T_1(f) = 0$; Evaluate at $X_i^2 \wedge X_i^1 \wedge Y_i^1$ and also $Y_i^2 \wedge X_i^1 \wedge Y_i^1$, get $X_i^2(f) = Y_i^2(f) = 0$, so

$$df = \eta_2 + \sum a_i \alpha_1^i + b_i \beta_1^i$$

Take derivative again, and evaluate at $X_i^2 \wedge Y_i^2$ sees $0 = 1$. contradiction.

2.4 The absence of Kähler metrics

In this section, we show that the quotient cone over a Sasakian manifold does not admit any Kähler structure. We also show under very mild assumption, that the product of two Sasakian manifolds does not admit a symmpletic form, thus cannot admit any Kähler metric.

Firstly, we quote a well known theorem on the betti numbers of Sasakian manifolds. This was developed by Tachibana [**?**], Blair and Goldberg [**?**].

Lemma 2.4.1 Let M be a compact Sasakian manifold of dimension $2n+1$. If k is an odd number and that $1 \leq k \leq n$, then the k^{th} betti number is even; If k is an even number and that $n+1 \leq k \leq 2n$, then the k^{th} betti number is even.

Proposition 2.4.2 Let M be a compact Sasakian manifold, the quotient cone $\bar{C}(M) = (M \times \mathbb{R}^+)/\{r \sim 2r\}$ is a compact complex manifold that does not admit any Kähler metric.

Proof. We already know the quotient cone is compact complex. Using Lemma 2.4.1 and the Kuneth formula, we conclude that the first betti number of the quotient cone is odd, and hence topologically cannot admit any Kähler metric.

Lemma 2.4.3 Let (M, η, g, J) be a compact K-contact manifold and α be a harmonic 1-from, then $\alpha(T)$=0, where T is the Reeb vector field.

Proof. Write $\alpha = \beta + f\eta$, where $\beta(T) = 0$. Since T is a Killing field and α

is harmonic, we have

$$0 = L_T\alpha = d(\alpha(T)) = df$$

Hence f is constant, and thus $d\beta + fd\eta = 0$, by Stocks theorem, we have

$$0 = \int_M d(\beta \wedge \eta \wedge (d\eta)^{n-1}) = -\int_M f\eta \wedge (d\eta)^n$$

hence $f = 0$, so $\alpha(T) = \beta(T) = 0$

In fact, there is a more general result for Sasakian manifold due to Tachibana [**?**].

Lemma 2.4.4 Let M be a compact Sasakian manifold of dimension $2n+1$. Let α be a harmonic k-form $(1 \leq k \leq n)$, then $\alpha \neg T = 0$.

Now, we consider the product of Sasakian manifolds.

Proposition 2.4.5 Let M and N be compact, simply connected, Sasakian manifolds, then the product $M \times N$ does not admit any Kähler metric.

Proof. We will prove that $M \times N$ has no sympletic form, so consequently cannot admit any Kähler metric. Suppose it does admit a sympletic form ω, then $[\omega]$ is an element of $H^2(M \times N)$. By the Kunneth formula $H^2(M \times N) = H^2(M) \oplus H^2(N) \oplus H^1(M) \otimes H^1(N)$. Since M and N are simply connected, the second cohomology has the form $[\omega] = \pi_1^*[\omega_1] + \pi_2^*[\omega_2]$, where $[\omega_1] \in H^2(M)$ and $[\omega_2] \in H^2(N)$. Let the dimension of M and N be respectively $2p+1$ and $2q+1$, then dimension of the product is $2n = 2(p+q+1)$

$$[\omega]^n = (\pi_1^*[\omega_1] + \pi_2^*[\omega_2])^{p+q+1} = 0$$

since $[\omega_1]^{p+1} = 0$, $[\omega_2]^{q+1} = 0$. However, as a sympletic form ω^n is a non-zero top form, and hence $[\omega]^n \neq 0$. Contradiction.

Corollary 2.4.6 Let M and N be compact, Sasaki-Einstein manifolds, then the product $M \times N$ is a hermitian Einstein manifold that does not admit any Kähler metrics.

Proof. It is well known that the Einstein constant of a Sasaki-Einstein manifold is $2n$ where $2n+1$ being the dimension of the underlying manifold. Hence M and N are compact manifolds with positive Ricci curvature, their first cohomology vanishes by a well known theorem of Bochner. Now the same lines of argument as Proposition 4.5 goes through, so the product $M \times N$ does not admit any Kähler metric. On the other hand, the complex structure $J_{a,b}$ where $a = 0$ and $b = \sqrt{\frac{n}{m}}$ is a integrable complex structure on $M \times N$. The metric $g = mg_m + ng_n$ is an Einstein metric on $M \times N$ which is compactible with $J_{a,b}$

A Sasakian manifold M is called η-Einstein, if its Ricci tensor $Ric_g = \lambda_1 g + \lambda_2 \eta \otimes \eta$ for some functions λ_1, λ_2. Since on a Sasakian manifold we always have $Ric(T,T) = 2n$, we see that $\lambda_1 + \lambda_2 = 2n$. Using the contracted Bianchi identity [**?**], it is also not hard to see that λ_1, λ_2 must be constant when the dimension of M is greater than 3. When $\lambda_1 > -2$, An η-Einstein manifold essentially carries Sasaki-Einstein metrics [**?**]. Namely, by a transverse homothety, the quadruple $(M, \lambda_1 \eta, J, \lambda_1 g + (\lambda_1^2 - \lambda_1)\eta \otimes \eta)$ is Sasaki-Einstein. Thus, product of pairs of this class of η-Einstein Sasakian manifolds provide even more examples of Einstein Hermitian manifolds that has on Kähler metrics.

This is very different compared to the dimension 4 story. In [**?**], C.LeBrun recently proved that a compact hermitian Einstein complex surface must be Kähler Einstein, unless the metric is the Page metric or the Chen-LeBrun-Weber metric [**?**] . The two exceptions are nontheless conformally Kähler. Thus all compact hermitian Einstein complex surface must carry a Kähler structure compactible with its underlying complex structure.

Using Lemma 2.4.4, we can prove non existence of a symplectic form on the product under milder hypothesis.

Proposition 2.4.7 Let M and N be compact Sasakian manifolds, Then the product $M \times N$ doesn't admit any Kähler metric, provided that at least one of the components has dimension greater than 3.

Proof. Supppose ω is the harmonic 2-form that represents the cohomology class of a sympletic 2-form on the product. Then by Kunneth theorem, ω can be written as

$$\omega = \omega_1 + \omega_2 + \sum \alpha_i \cup \beta_i$$

where $\omega_1 \in \mathcal{H}^2(M)$, $\omega_2 \in \mathcal{H}^2(N)$, $\alpha_i \in \mathcal{H}^1(M)$ and $\beta_i \in \mathcal{H}^1(N)$

Let the dimension of M and N be respectively $2p + 1$ and $2q + 1$ and $n = (p + q + 1)$ is the complex dimension of the product. Then ω^n is a sum of terms of the form $\omega_1^{l_1} \wedge \omega_2^{l_2} \wedge \Pi(\alpha_i \wedge \beta_i)$, which are top forms. However by Lemma 4.4, if $p + q > 2$, the interior product of any terms of this form with the Reeb vetors fields T_1 and T_2 is 0. Thus all these terms are 0, and it follows $\omega^n = 0$. Contradict with the fact that ω represents the cohomology class of a sympletic form.

In the case that $p = q = 1$ and both M and N has non trivial first cohomology. Let α and β be a harmonic 1-form on M and N respectively, then the 2-form $\omega = *\alpha + *\beta + \alpha \wedge \beta$ is closed, and that $\omega^3 = \alpha \wedge *\alpha \wedge \beta \wedge *\beta$ is the volume form on the standard product. Thus the product does admit a sympletic form.

Bibliography

[1] A.Besse. Einstein manifolds. *Classics in Mathematics*, 2008.

[2] A.Newlander and L.Nirenberg. Complex analytic coordinates in almost complex manifolds. *Ann. of Math*, 65:391–404, 1957.

[3] B.L.Reinhart. Foliatied manifolds with bundle-like metrics. *Ann. of Math*, 69(2):119–132, 1959.

[4] B.Watson. New examples of strictly almost Kahler manifolds. *Proc. Amer. Math. Soc.*, 88, 1983.

[5] C.Boyer and K.Galicki. Sasakian geometry. *Oxford Mathematical Monographs*, 2008.

[6] K.Galicki C.Boyer and S.Simanca. Canonical Sasakian metrics. *Comm. Math. Phys.*, 279(3):705–733, 2008.

[7] C.LeBrun. On Einstein, Hermitian 4-manifolds. *arXiv:1010.0238v1*.

[8] D.E.Blair. Rienannian geometry of contact and sympletic manifolds. *Progress in Math.*, 203, 2002.

[9] D.E.Blair and S.I.Goldberg. Topology of almost contact manifolds. *J. Differential Geometry*, 1:347–354, 1967.

[10] F.Belgun. Normal CR structures on compact 3-manifolds. *Math. Z.*, 238(3):441–460, 2001.

[11] J.H.Park J.C.Lee and K.Sekigawa. Hermitian structures on the product of Sasakian manifolds. *arXiv:110.1295v1*.

[12] J.M.Lee. Pseudo Einstein structures on CR manifolds. *Amer. Jour. Math.*, 110(1):157–178, 1988.

[13] Molino. Riemanian foliations. *Progress in Mathematics*, 73, 1988.

[14] M.Stanton. Intrinsic connections for Levi metrics. *Manuscripta Math.*, 75(4):349–364, 1992.

[15] N.Tanaka. A differential geometric study on strongly pseudo-convex manifolds. *Kinokuniya Company Ltd. Tokyo*, 1975.

[16] O'Neill. The fundamental equations of a submersion. *Mich. Math. J.*, 13:459–469.

[17] S.M.Webster. Pseudohermitian structures on a real hypersurface. *J. Differential Geometry*, 3:25–41, 1978.

[18] S. Tachibana. On harmonic tensors in compact Sasakian spaces. *Tohoku. Math. J.*, 17:271–284, 1965.

[19] Tanno. Harmonic forms and betti numbers of certain contact Riemanian manifolds. *J.Math.Soc.Japan*, 19:308–316, 1967.

[20] Tanno. The topology of contact Riemanian manifolds. *Illinois. J. Math.*, 12:700–717, 1968.

[21] Tanno. Sasakian manifold with constant ϕ-holomorphic sectional curvature. *Tohoku Math. Jour.*, 21(2):501–507, 1969.

[22] Tanno. Geodesic flows on c_l-manifolds and Einstein metrics on $s^3 \times s^2$. *North-Holland, Amsterdam*, pages 283–292, 1979.

[23] C.LeBrun X.Chen and B.Webber. On conformally Kahler, Einstein manifolds. *J. Amer. Math. Soc.*, 21:1137–1168, 2008.