Hyperhähler $4n$-manifolds with n commuting Quaternionic Killing fields

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Joseph Malkoun

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Joseph Malkoun

We, the dissertation committee for the above candidate for the Doctor of Philosophy degree, hereby recommend acceptance of this dissertation.

Claude LeBrun – Dissertation Advisor
Professor, Department of Mathematics

H. Blaine Lawson – Chairperson of Defense
Professor, Department of Mathematics

Michael Anderson
Professor, Department of Mathematics

Martin Roček
Professor, Department of Physics

This dissertation is accepted by the Graduate School.

Charles Taber
Interim Dean of the Graduate School

Abstract of the Dissertation

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We consider a hyperkähler $4n$-manifold M admitting n commuting quaternionic (real) Killing fields X^1,...,X^n which are pointwise quaternionically linearly independent, and such that the first $n-1$ of them, namely X^1,..., X^{n-1} are further assumed to be triholomorphic. We show that such spaces fall into 2 categories, depending on whether ∇X^n has a vanishing self-dual component or not. In the first case, we show that such manifolds M can be obtained by the Hitchin-Karlhede-Lindström-Roĉek ansatz for hyperkahler $4n$-manifolds with n commuting triholomorphic Killing fields. In the second case, we obtain a canonical form for the n vector fields

X^1,...,X^n in special coordinates. Moreover, a Kähler potential Ω for a compatible complex structure I is shown to satisfy some symmetries, as well as a system of non-linear second order PDE's coming from the symplectic Monge-Ampere equations.

In the process of obtaining this result, we also obtain local necessary and sufficient conditions for a (smooth) real vector field X to be quaternionic Killing on a hyperkähler $4n$-manifold M.

Our study is completely local, and is a generalization of the Boyer and Finley work for self-dual Ricci-flat 4-manifolds with a Killing field.

To my family.

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Chapter 1

Basic Concepts (Complex, Kähler and Hyperkähler Manifolds)

We assume that the reader is familiar with the basic facts about smooth manifolds and real differential geometry. We begin by recalling some basic definitions and fixing the notation.

A *topological n-manifold* M is a second-countable Hausdorff topological space which is locally Euclidean. By locally Euclidean, we mean that each point of M has a neighborhood which is homeomorphic to an open subset of $\mathbb{R}^n$ with its usual topology.

If, in addition, M is equipped with a smooth atlas $(\mathcal{U}, (\varphi_U))$, it is said to be a *smooth n-manifold*. A smooth atlas $(\mathcal{U}, (\varphi_U))$ consists of an open cover $\mathcal{U}$ of M and a collection of homeomorphisms (φ_U) indexed by $U \in \mathcal{U}$, where $\varphi_U : U \to U'$ is a homeomorphism from U onto an open subset $U' \subseteq \mathbb{R}^n$, such that the transition maps $\varphi_{U_2} \circ \varphi_{U_1}^{-1} : U_1' \to U_2'$ are smooth (i.e. C^∞) and have smooth inverses, for all U_1, $U_2 \in \mathcal{U}$.

The presence of a *smooth structure*, which is an equivalence class of smooth atlases, where two smooth atlases are equivalent if and only if they are compatible in an obvious sense (if their union is a smooth atlas), allows us to define the tangent space $T_m(M)$ at a chosen point $m \in M$. Moreover, the set-theoretic disjoint union of the $T_m(M)$'s as m varies over M, has a natural induced smooth structure from that of M, and forms a smooth $2n$-dimensional manifold called the tangent bundle of M, and denoted by $T(M)$.

A *complex m-manifold* is a topological $2m$-manifold together with a complex atlas, consisting of an open cover and a collection of homeomorphisms onto open subsets of $\mathbb{C}^m$, such that the transition mappings are biholomorphic.

There is an alternative way to describe complex manifolds, via the so-called almost complex structures. If M is a smooth $2m$-dimensional manifold, a smooth section I of $End(T(M)) := T^*(M) \otimes T(M)$ satisfying $I^2 = -\mathrm{Id}$ is called an *almost complex structure*. We call the pair (M, I) an *almost complex manifold*. Examples of almost complex manifolds include complex manifolds (if z^j's are local complex coordinates, write $z^j = x^j + iy^j$, x^j, y^j real coordinates, define $I : \partial_{x^j} \mapsto \partial_{y^j}$ and check that it is well-defined). This leads us to ask, what condition(s) do we need to impose on I to ensure that (M, I) admits a complex atlas compatible with I? The answer is provided by the Newlander-Nirenberg theorem ([1]).

Definition 1.0.1. *The* Nijenhuis tensor N_I *of an almost complex structure is*

$$N_I(X, Y) = [X, Y] + I[IX, Y] + I[X, IY] - [IX, IY],$$

where X, Y are vector fields.

Theorem 1.0.2 (Newlander-Nirenberg)**.** *An almost complex manifold (M, I) admits a compatible complex atlas if and only if the Nijenhuis tensor N_I of I vanishes identically.*

If there is a complex atlas compatible with I, we say that I is *integrable*. Thus, the Newlander-Nirenberg theorem says that I is integrable if and only if its Nijenhuis tensor is identically zero. We note that in that case, there is up to equivalence a unique complex atlas compatible with I, because if a diffeomorphism from an open subset of $(\mathbb{R}^{2m}, I)$ with its natural almost complex structure and coordinates $\mathbf{z}$ to another open subset of $(\mathbb{R}^{2m}, I)$ with coordinates $\mathbf{w}$ preserves I, then it naturally corresponds to a holomorphic mapping $\mathbf{z} = f(\mathbf{w})$ (strictly speaking it corresponds to this holomorphic mapping and its complex conjugate mapping $\overline{\mathbf{z}} = \overline{f}(\overline{\mathbf{w}})$, which is antiholomorphic, but we choose the holomorphic mapping).

If I is integrable, we refer to it simply as a *complex structure*, and refer to (M, I) as a *complex manifold*. Thus a complex manifold can be described in two equivalent ways, namely with a complex atlas, or with an integrable almost complex structure I, both up to equivalence.

If a complex manifold (M, I) is equipped with a smooth metric g such that

1. $g(IX, IY) = g(X, Y)$ for all vector fields X and Y, and

2. the 2-form ω_I defined by $\omega_I(X, Y) = g(IX, Y)$ is closed,

then we refer to (M, g, I) as a *Kähler m-manifold* (or simply Kähler manifold, omitting the complex dimension m). We have

Proposition 1.0.3. *If (M, g, I) is a Kähler manifold, then $\nabla_{LC} I = 0$, where ∇_{LC} is the Levi-Civita connection of g.*

It thus follows from the proposition that the (restricted) holonomy group of a Kahler m-manifold is a Lie subgroup of $\mathrm{U}(m)$.

We are now ready to define hyperkähler manifolds. A smooth $4n$-manifold M, together with a smooth metric g and three complex structures I, J and K such that $K = IJ = -JI$, and such that g is Kähler with respect to each of these 3 complex structures (I, J and K), is said to be *hyperkähler*. We will refer to (M, g, I, J, K) as a *hyperkähler manifold*.

We remark that the (restricted) holonomy group of a hyperkähler $4n$-manifold (M, g, I, J, K) is a Lie subgroup of $\mathrm{Sp}(n)$.

Chapter 2

Moment Maps and Construction of Quotients

2.1 In Symplectic Geometry

Let (M, ω) be a symplectic $2m$-manifold (i.e. M is a smooth $2m$-manifold and ω is a closed non-degenerate 2-form on M). Assume there is a k-dimensional Lie group G acting freely on M and preserving ω. The naive quotient M/G may not even be even dimensional, and therefore does not carry a symplectic structure induced from ω. There is however a construction of a "quotient" due to Marsden and Weinstein ([2]) which constructs a $2m - 2k$ dimensional symplectic manifold from (M, ω) and the symplectic group action of G. We now describe this procedure.

Let $0 \neq \xi \in \mathfrak{g}$, and let X be the corresponding vector field. More precisely, if $\gamma(t)$ is the unique one-parameter subgroup of G such that $\gamma'(0) = \xi$, then given $m \in M$, we define

$$X_m = \frac{d}{dt}(\gamma(t).m)\bigg|_{t=0}.$$

Since the action of G preserves ω, it follows that $\mathcal{L}_X(\omega) = 0$, where $\mathcal{L}_X$ denotes the Lie derivative with respect to X.

We recall *Cartan's magic formula.* If Y is any smooth vector field, and β is a differential k-form, then

$$\mathcal{L}_Y(\beta) = (d \circ \iota_Y + \iota_Y \circ d)(\beta),$$

where ι_Y denotes inner contraction with the vector field Y.

Back to our setting, it follows therefore that

$$0 = \mathcal{L}_X(\omega) = d(\iota_X(\omega)),$$

(since ω is closed). Hence $\iota_X(\omega)$ is closed. We assume further that it is exact, so that there is a function h_X on M such that

$$dh_X = \iota_X(\omega).$$

The function h_X is only defined up to a constant (assuming M is connected).

Next, we choose a basis, ξ_1, ..., ξ_k of $\mathfrak{g}$, this corresponds to vector fields X_1,...,X_k, and k functions h_{X_1}, ..., h_{X_k}, each defined up to a constant. We then extend h linearly, and thus get a map

$$h : M \to \mathfrak{g}^*.$$

If we can choose the constants in such a way as to make the map h G-equivariant (the adjoint action of G on its lie algebra $\mathfrak{g}$ is assumed), we call such an h a *moment map.* A moment map is defined up to the addition of a constant element of $\mathfrak{g}^*$ which is fixed by the action of G.

Having a moment map h of the group action of G, we remark that, by the G-equivariance property, G acts on $h^{-1}(0)$. If 0 is a regular value of the moment map h, then by the implicit function theorem, $h^{-1}(0)$ to be a smooth $(2m-k)$ dimensional submanifold.

We further assume that the action of G on $h^{-1}(0)$ is such that $\tilde{M} := h^{-1}(0)/G$ is a smooth (Hausdorff) $(2m-2k)$ dimensional manifold (with the quotient topology).

Theorem 2.1.1 (Marsden-Weinstein)**.** *If (M, ω) is a $2m$ dimensional symplectic manifold, with a k-dimensional Lie group acting symplectically on M, and if $h : M \to \mathfrak{g}^*$, then there is a unique 2-form $\tilde{\omega}$ on $\tilde{M}$ whose pullback to $h^{-1}(0)$ via the natural projection $h^{-1}(0) \to \tilde{M}$ is the restriction of ω to $h^{-1}(0)$. Moreover, $\tilde{\omega}$ is closed and non-degenerate, i.e. a symplectic form on $\tilde{M}$.*

The symplectic manifold $(\tilde{M}, \tilde{\omega})$ is known as the *Marsden-Weinstein reduction* of (M, ω) by the group action of G (sometimes the word quotient is used instead of reduction, but this construction must not be confused with the ordinary quotient of a smooth manifold by a smooth and free Lie group action!).

2.2 In Kähler Geometry

In this section, we let (M, I, g) be a Kähler m-manifold (see 1 for some quick definitions). We further assume that there is a k-dimensional Lie group G acting freely on M by biholomorphic isometries (it preserves I and g).

Thus, if $0 \neq \xi \in \mathfrak{g}$ and X is the corresponding vector field generated via the action of G (see 2.1), we then have that $\mathcal{L}_X(\omega_I) = 0$, so that

$$0 = d(\iota_X(\omega_I)),$$

by Cartan's magic formula. The 1-form $\iota_X(\omega_I)$ is closed, and we assume further that it is exact (which would be the case if for instance $H^1(M, \mathbb{R}) = 0$), in other words, that there is a smooth function h_X on M such that

$$dh_X = \iota_X(\omega_I).$$

Just like for the symplectic case (see 2.1), we get a smooth map $h : M \to \mathfrak{g}^*$, defined up to addition of a constant element of $\mathfrak{g}^*$. If we further assume that h is G-equivariant (with G acting on its dual Lie algebra $\mathfrak{g}^*$ via the co-adjoint action), we call such an h a *moment map* on M for the action of G.

Also, similar to 2.1, G acts on $h^{-1}(0)$. We further assume that 0 is a regular value of the moment map h, so that $h^{-1}(0)$ is a real smooth submanifold of M of dimension $2m - k$. Then, by the Marsden-Weinstein symplectic reduction (see 2.1), $\tilde{M} := h^{-1}(0)/G$ (which we assume to be a smooth Hausdorff manifold of dimension $2m - 2k$) inherits a symplectic form $\tilde{\omega}$ from ω_I.

Moreover, $h^{-1}(0)$ inherits the metric $\mathrm{res}(g)$ obtained by restricting g to it, and $\tilde{M}$ inherits a metric $\tilde{g}$ from that restriction, by a standard procedure, since G acts by isometries. The latter can be thus summarized: in order to define $\tilde{g}_{\tilde{m}}(\tilde{Y}_1, \tilde{Y}_2)$, where $\tilde{Y}_1$, $\tilde{Y}_2$ are tangent vectors to $\tilde{M}$ at $\tilde{m} \in \tilde{M}$, we just choose a lift $m \in f^{-1}(0)$ of $\tilde{m}$ and two tangent vectors Y_1 and Y_2 which project down to $\tilde{Y}_1$ and $\tilde{Y}_2$ respectively, and which are orthogonal to the vertical subspace of $T_m(h^{-1}(0))$ spanned by the generating vectors of the action of G (see figure 2.1), and we define

$$\tilde{g}_{\tilde{m}}(\tilde{Y}_1, \tilde{Y}_2) := \mathrm{res}(g)_m(Y_1, Y_2).$$

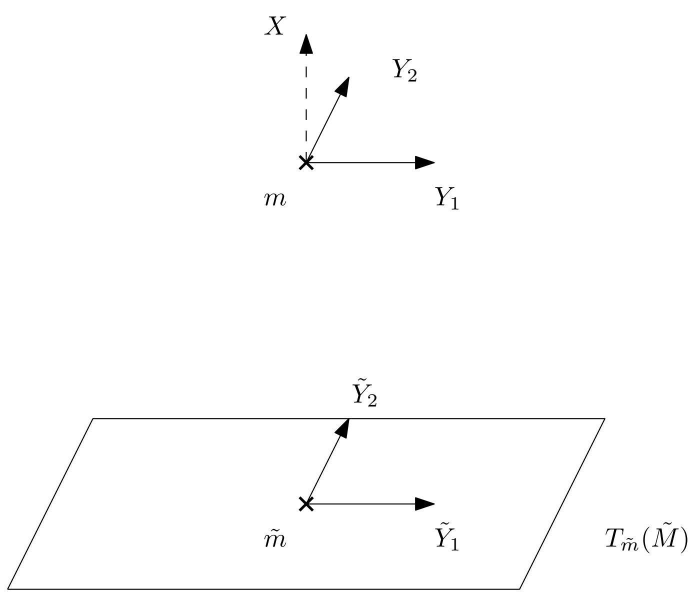

Figure 2.1: Quotient metric under an isometric Lie group action. In this figure, X denotes a generating vector of the action of G, and X is both orthogonal to Y_1 and Y_2

One can check that this is independent of the choices of lifts involved.

As a generalization of the Marsden-Weinstein symplectic reduction to the Kähler case, in [3], the authors show the following theorem:

Theorem 2.2.1 ([3]). *If (M, g, ω) is a Kähler m-manifold, with a (real) k-*

dimensional group G acting by Kähler isometries (i.e. preserving g and ω), and if $h : M \to \mathbb{R}$ is the moment map of the action of G, then $\tilde{M} = h^{-1}(0)/G$, is Kähler, when endowed with the quotient metric $\tilde{g}$, the Marsden-Weinstein symplectic form $\tilde{\omega}$ and the complex structure $\tilde{I}$ given by

$$\tilde{\omega}(-,-) = \tilde{g}(\tilde{I}-,-).$$

Proof. For a proof of this result, please refer to [3]. □

2.3 In Hyperkähler Geometry

The reduction of a hyperkähler manifold ([3]) by a Lie group action is similar to the reduction of a Kähler manifold, and so we will omit some details.

Let (M, g, I, J, K) denote a hyperkähler $4n$-manifold admitting an isometric and triholomorphic free action of a Lie group G of dimension k. In this setting, for a given (non-zero) generating vector field X of G, we get 3 functions h^1_X, h^2_X and h^3_X corresponding to I, J and K respectively. Collecting the functions together, we get a map

$$h : M \to \mathfrak{g}^* \otimes \mathbb{R}^3,$$

which we assume to be G-equivariant, where G acts on $\mathfrak{g}^*$ by the co-adjoint action and acts trivially on $\mathbb{R}^3$. We denote by $\tilde{M}$ the quotient $f^{-1}(0)/G$. We then have

Theorem 2.3.1 ([3])**.** *If (M, g, I, J, K) is a hyperkähler $4n$ dimensional man-*

ifold, with a (real) k dimensional Lie group acting on M by triholomorphic isometries, and if $h : M \to \mathfrak{g}^* \otimes \mathbb{R}^3$ is the corresponding moment map of the action of G, then the $4(n-k)$-dimensional manifold $\tilde{M} = h^{-1}(0)/G$ is hyperkähler, when endowed with the quotient metric $\tilde{g}$, and the three symplectic forms $\tilde{\omega}_1$, $\tilde{\omega}_2$ and $\tilde{\omega}_3$ arising from the Marsden-Weinstein reductions of ω_1, ω_2 and ω_3. The complex structures $\tilde{I}_i$, for $1 \leq i \leq 3$ of $\tilde{M}$ are defined by

$$g(\tilde{I}_i X, Y) = \tilde{\omega}_i(X, Y).$$

Proof. We let

$$h^+ = h^2 + ih^3.$$

It follows from the construction of the moment maps h^i that dh^+ is of type $(1,0)$ with respect to I and hence that h^+ is holomorphic with respect to I. Hence $(h^+)^{-1}(0)$ is a complex manifold of dimension $2l - k$ which is moreover Kähler with respect to the restriction of (g, I), such that G acts on it by holomorphic isometries (since h is G-equivariant). We apply the Kähler reduction construction to $(h^+)^{-1}(0)$, and we get that

$$\tilde{M} = h^{-1}(0)/G = ((h^+)^{-1}(0) \cap (h^1)^{-1}(0))/G$$

is Kähler with respect to $(\tilde{g}, \tilde{I})$. Repeating the argument for J and K instead of I shows that $(\tilde{M}, \tilde{g}, \tilde{I}, \tilde{J}, \tilde{K})$ is hyperkähler.

This trick of specializing one complex structure (say I) in the natural S^2 of complex structures on a hyperkähler manifold will be useful to us later.

We remark before leaving this topic that

$$\omega^+ = \omega^2 + i\omega^3$$

is a holomorphic form, with respect to I, of type $(2,0)$ (in fact, it is even covariantly constant with respect to the Levi-Civita connection ∇_{LC} of g). $\square$

Chapter 3

Twistor Spaces of Hyperkähler manifolds

3.1 Introduction

Sir Roger Penrose introduced twistor theory in the 1970's (cf. [4] and [5]) for the case of an antiselfdual Lorentzian Einstein 4-manifold. One of his aims was to unify gravity and quantum theory. The Riemannian version of Penrose's twistor theory is due to Atiyah, Hitchin and Singer ([6]). From a purely mathematical point of view, one associates to a conformal 4-manifold (M, c) an almost complex 6-manifold $(Z, \underline{\mathrm{I}})$, which is diffeomorphic to an S^2 bundle over M. The almost complex structure $\underline{\mathrm{I}}$ on Z encodes the conformal structure c. Moreover, c is antiselfdual if and only if $\underline{\mathrm{I}}$ is integrable, in which case $(Z, \underline{\mathrm{I}})$ becomes a complex 3-manifold. A general philosophy in twistor theory is to encode as many equations and operators of interest to physicists and/or geometers (Dirac, wave, Laplace...) using the $\bar{\partial}$-operator on the twistor space.

In [7] and [8], Simon Salamon developed a twistor theory of quaternionic manifolds (see however the much earlier work of J. Wolf in [9]). In [3], N.J. Hitchin, A. Karlhede, U. Lindström and M. Roĉek specialized and developed that theory to hyperkähler manifolds, and then applied it to the case of hyperkähler manifolds with toric symmetry. Hyperkähler manifolds can be viewed as a generalization of Ricci-flat anti-selfdual Riemannian 4-manifolds.

In this section, we review the twistor theory of hyperkähler $4n$-manifolds, as found in [3].

3.2 The Twistor Space of a Hyperkähler Manifold

Let (M, g, I, J, K) be a hyperkähler $4n$-manifold. We let $Z = M \times S^2$, as a real smooth $4n+2$-manifold. The sphere S^2 has a natural complex structure, when thought of as the Riemann sphere $\mathbb{P}^1$. Using stereographic projection, S^2 can be covered by 2 open sets U and $\tilde{U}$, both homeomorphic to $\mathbb{C}$, with coordinates ζ and $\tilde{\zeta}$ respectively, such that $\tilde{\zeta} = 1/\zeta$. In terms of ζ, the coordinates (x, y, z) of a point in $S^2 \subseteq \mathbb{R}^3$ are

$$(x, y, z) = \left(\frac{\zeta\bar{\zeta} - 1}{1 + \zeta\bar{\zeta}}, \frac{\zeta + \bar{\zeta}}{1 + \zeta\bar{\zeta}}, \frac{i(\bar{\zeta} - \zeta)}{1 + \zeta\bar{\zeta}} \right).$$

However, we find it more natural to use the conjugate complex structure I_{S^2} on S^2. One motivation for this, is that stereographic projection induces the opposite orientation on S^2 from the usual one (given by an outward normal

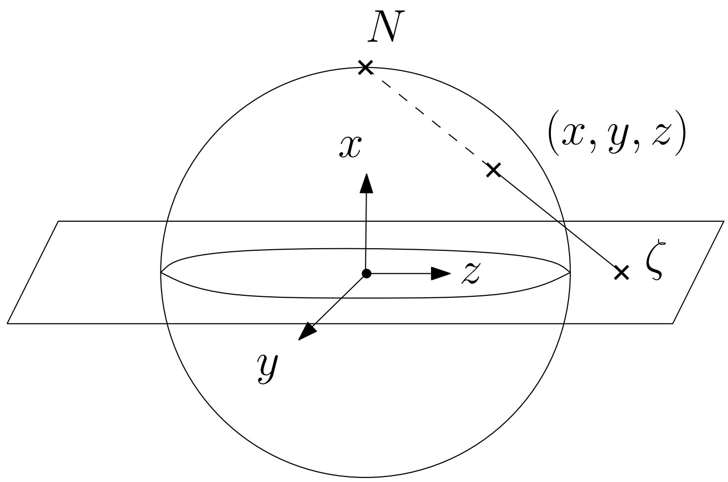

Figure 3.1: The steregraphic projection of (x, y, z) from N is the point ζ on the equatorial plane.

vector). With respect to I_{S^2}, the roles of ζ and $\bar{\zeta}$ are interchanged, and we get, instead of the previous formula:

$$(a, b, c) = \left(\frac{\zeta\bar{\zeta} - 1}{1 + \zeta\bar{\zeta}}, \frac{\zeta + \bar{\zeta}}{1 + \zeta\bar{\zeta}}, \frac{i(\zeta - \bar{\zeta})}{1 + \zeta\bar{\zeta}} \right).$$

We then define in $T(Z)$ the following almost complex structure

$$\underline{\mathrm{I}} = \left(\frac{\zeta\bar{\zeta} - 1}{1 + \zeta\bar{\zeta}} I + \frac{\zeta + \bar{\zeta}}{1 + \zeta\bar{\zeta}} J + \frac{i(\zeta - \bar{\zeta})}{1 + \zeta\bar{\zeta}} K, I_{S^2} \right) \tag{3.2.1}$$

where I_{S^2} is the conjugate complex structure in $T(S^2)$ from the one induced by stereographic projection.

Proposition 3.2.1. $\underline{I}$ *is integrable.*

Proof. We apply the Newlander-Nirenberg theorem, which says that an almost complex structure is integrable if and only if the ideal generated by the $(1, 0)$-forms (in the exterior algebra) is closed under d. This is a complex version of the Frobenius theorem.

We need to figure out what are the $(1,0)$-forms θ with respect to $\underline{\mathrm{I}}$ ($\underline{\mathrm{I}}\theta = -i\theta$). Let φ be a $(0,1)$-form with respect to I ($I\varphi = i\varphi$), and set

$$\theta = \varphi - \zeta K\varphi.$$

We claim that θ is of type $(1,0)$ for $\underline{\mathrm{I}}$. Indeed, we have

$$\begin{aligned} I\theta &= i\varphi + i\zeta K\varphi \\ J\theta &= iK\varphi - i\zeta\varphi \\ K\theta &= K\varphi + \zeta\varphi. \end{aligned}$$

From these and 3.2.1, it follows that $\underline{\mathrm{I}}\theta = -i\theta$. Moreover, it is clear that the map $\varphi \mapsto \theta$ is injective. Thus, by a simple dimension count, the θs together with $d\zeta$ gives us a complete basis for the $(1,0)$ forms for $\underline{\mathrm{I}}$ on Z. It is clear that $d\zeta$ is closed. We then compute

$$d\theta = dx^i \wedge \nabla_{x^i}(\varphi - \zeta K\varphi) - d\zeta \wedge K\varphi.$$

where the x^is are local coordinates on M. But $\nabla_{x^i}\underline{\mathrm{I}} = 0$, so that

$$\underline{\mathrm{I}}\nabla_{x^i}(\varphi - \zeta K\varphi) = -i\nabla_{x^i}(\varphi - \zeta K\varphi).$$

Thus we see that $d\theta$, as well as $d\zeta$ are in the ideal generated by the $(1,0)$ forms for $\underline{\mathrm{I}}$, therefore $\underline{\mathrm{I}}$ is integrable. $\square$

.

What we have done so far is encode the information about I, J and K in the complex structure $\underline{\mathrm{I}}$ in $T(Z)$. We remark that the projection $p : Z \to \mathbb{C}\mathcal{P}^1$ is holomorphic, since $p^*(d\zeta)$ is of type $(1,0)$ on Z. A point $m \in M$ corresponds to a holomorphic section of p whose image we denote by P_m, and is known as the *twistor line* of m.

We need to find the normal bundle of $\mathcal{N}$ of a twistor line P_m. This is defined as

$$\mathcal{N} = T'(Z)|_{P_m}/T'(P_m),$$

where $T'(Z)$, $T'(P_m)$ denote the holomorphic tangent bundles of Z and P_m respectively. We remark that the underlying real vector bundle of $\mathcal{N} \simeq P_m \times T_m$ (with $T_m \simeq \mathbb{C}^{2k}$) is trivial, but as a holomorphic vector bundle, it is not, as we shall see shortly.

We represent I, J and K on T_m by

$$I = \begin{pmatrix} i1_n & 0 \\ 0 & -i1_n \end{pmatrix} \tag{3.2.2}$$

$$J = \begin{pmatrix} 0 & 1_n \\ -1_n & 0 \end{pmatrix} \tag{3.2.3}$$

$$K = \begin{pmatrix} 0 & i1_n \\ i1_n & 0 \end{pmatrix}. \tag{3.2.4}$$

Hence

$$\underline{\mathrm{I}} = \frac{1}{1+\zeta\bar{\zeta}} \begin{pmatrix} -i(1-\zeta\bar{\zeta}) & 2\bar{\zeta} \\ -2\zeta & i(1-\zeta\bar{\zeta}) \end{pmatrix}. \tag{3.2.5}$$

Therefore the $-i$ eigenvectors of $\underline{\mathrm{I}}$ are of the form

$$\begin{pmatrix} v \\ -i\zeta v \end{pmatrix}.$$

If we express $\underline{\mathrm{I}}$ with respect to $\tilde{\zeta}$, we get that the $-i$ eigenvectors of $\underline{\mathrm{I}}$ are of the form

$$\begin{pmatrix} i\tilde{\zeta} w \\ w \end{pmatrix}.$$

Hence $v = i\tilde{\zeta} w$, so that $\mathcal{N}^* \simeq \mathbb{C}^{2n}(-1)$, from which we deduce that

$$\mathcal{N} \simeq \mathbb{C}^{2n}(1). \tag{3.2.6}$$

Proposition 3.2.2. *The form $\omega_+ = \omega_2 + i\omega_3$ is holomorphic with respect to I (in fact, it is covariantly constant) of type $(2,0)$.*

Proof. The holomorphic part of the statement is clear since the ω_is are covariantly constant. It remains to check that ω_+ is of type $(2,0)$.

$$\begin{aligned} \omega_+(IX,Y) &= \omega_2(IX,Y) + i\omega_3(IX,Y) \\ &= g(JIX,Y) + ig(KIX,Y) \\ &= -g(KX,Y) + ig(JX,Y) \\ &= i\omega_+(X,Y). \end{aligned}$$

$$\begin{aligned}
\omega_+(X, IY) &= g(JX, IY) + ig(KX, IY) \\
&= -g(KX, Y) + ig(JX, Y) \\
&= i\omega_+(X, Y).
\end{aligned}$$

□

Next, we prove that there is a holomorphic symplectic form on the fibres $F_\zeta = p^{-1}(\zeta)$ of $p : Z \to \mathbb{CP}^1$ which varies holomorphically with respect to ζ. But before we do that, We need the following lemma.

Lemma 3.2.3. *If $I\psi = -i\psi$, then $\underline{I}(\zeta + K)\psi = -i(\zeta + K)\psi$.*

Proof.

$$\begin{aligned}
I(\zeta\psi + K\psi) &= -i\zeta\psi + iK\psi \\
J(\zeta\psi + K\psi) &= -i\zeta K\psi - i\psi \\
K(\zeta\psi + K\psi) &= \zeta K\psi - \psi,
\end{aligned}$$

from which it follows that $\underline{\mathrm{I}}(\zeta\psi + K\psi) = -i(\zeta\psi + K\psi)$, using 3.2.1. □

There is a local coframe ψ_α on M, for $1 \leq \alpha \leq 2n$, such that

$$\omega_2 + i\omega_3 = \sum_{i=1}^{n} \psi_i \wedge \psi_{n+i}.$$

Next, we need to make $\omega_2 + i\omega_3$ vary (holomorphically) with ζ, so, using the previous lemma, we consider

$$\omega = \sum_{i=1}^{n} (\zeta + K)\psi_i \wedge (\zeta + K)\psi_{n+i}, \tag{3.2.7}$$

so that

$$\omega = \sum_{i=1}^{n} (\psi_i \wedge \psi_{n+i})\zeta^2 + \sum_{i=1}^{n} (\psi_i \wedge K\psi_{n+i} + K\psi_i \wedge \psi_{n+i})\zeta + \sum_{i=1}^{n} K\psi_i \wedge K\psi_{n+i}.$$

We compute, suppressing the summation sign

$$\begin{aligned}
&(\psi_i \wedge K\psi_{n+i} + K\psi_i \wedge \psi_{n+i})(X, Y) \\
&= -\psi_i(X)\psi_{n+i}(KY) - \psi_i(KX)\psi_{n+i}(Y) + \psi_i(Y)\psi_{n+i}(KX) + \psi_i(KY)\psi_{n+i}(X) \\
&= -\omega_+(X, KY) - \omega_+(KX, Y) \\
&= -g(JX, KY) - ig(KX, KY) - g(JKX, Y) - ig(K^2X, Y) \\
&= -2g(IX, Y) \\
&= -2\omega_1(X, Y).
\end{aligned}$$

We also have

$$\begin{aligned}
&(K\psi_i \wedge K\psi_{n+i})(X, Y) \\
&= \omega_+(KX, KY) \\
&= g(JKX, KY) + ig(K^2X, KY) \\
&= -g(JX, Y) + ig(KX, Y) \\
&= -(\omega_2 - i\omega_3)(X, Y).
\end{aligned}$$

Hence we have the following formula for ω:

$$\omega = (\omega_2 + i\omega_3)\zeta^2 - 2\omega_1\zeta - (\omega_2 - i\omega_3). \tag{3.2.8}$$

Hence, we see that $\omega \in H^0(Z, \mathcal{O}(\Lambda^2 T_F^*(2)))$, where T_F is the vertical subbundle of the map $p : Z \to \mathbb{CP}^1$ (in other words, $T_F = \ker(p_*)$, with $p_* : T(Z) \to T(\mathbb{CP}^1)$). We are denoting by $\mathcal{O}(n)$ the bundle $p^*(\mathcal{O}(n))$, by a slight abuse of notation. Moreover, ω restricts to a holomorphic symplectic form on each fiber F_ζ of the map p.

There is one last structure on the twistor space Z, namely a *real structure* τ, which is defined as the following map from Z to itself

$$\tau(m, \zeta) = (m, -1/\bar{\zeta}), \tag{3.2.9}$$

where we have used that Z is diffeomorphic to $M \times S^2$. Thus τ fixes the M factor and is the antipodal map on the S^2 factor. It follows from the definition of $\underline{\mathrm{I}}$ that τ is an antiholomorphic automorphism of Z (i.e. τ_* anti-commutes with $\underline{\mathrm{I}}$ and $\tau^2 = \mathrm{Id}$).

It turns out that we have encoded enough information about the hyperkähler manifold M as holomorphic (and antiholomorphic) data on its twistor space Z to reconstruct M from the twistorial data. More precisely, we have the following theorem:

Theorem 3.2.4. *Let Z be $2n+1$ dimensional complex manifold such that*

1. *there is a holomorphic map $p : Z \to S^2$ such that Z is a holomorphic fibre bundle over S^2,*

2. *Z has a family of holomorphic sections P of p, each with normal bundle $\mathcal{N}_P \simeq \mathbb{C}^{2n}(1)$,*

3. *there is a an $\omega \in H^0(Z, \mathcal{O}(\Lambda^2 T_F^*(2)))$ which restricts to a holomorphic*

symplectic form on each fibre $F_\zeta = p^{-1}(\zeta)$*,*

4. *there is an antiholomorphic involution* τ *on* Z *(*$\tau^2 = \mathrm{Id}$*), such that* $\sigma \circ p = p \circ \tau$*, where* σ *is the antipodal map on* S^2*, and* τ *is compatible with* ω*. What this last statement means is that if* P *is a twistor line (a holomorphic section of* p *with normal bundle* $\mathcal{N}_P \simeq \mathbb{C}^{2n}(1)$*), then* τ *induces a complex antilinear map* $\tau_* : H^0(P, \mathcal{O}(\mathcal{N}_P)) \to H^0(\tau(P), \mathcal{O}(\mathcal{N}_{\tau(P)}))$*, and if* X *and* Y *are two elements of* $H^0(P, \mathcal{O}(\mathcal{N}_P))$*, then* $\omega(\tau_* X, \tau_* Y) = -(1/\bar{\zeta}^2)\overline{\omega(X,Y)}$*.*

Proof. A theorem of Kodaira guarantees that if S is a submanifold of a complex manifold Z of normal bundle $\mathcal{N}_S$, and if $H^1(S, \mathcal{O}(\mathcal{N}_S)) = 0$, then there is an complex analytic m-dimensional family of deformations of S in Z, where m is the dimension of $H^0(S, \mathcal{O}(\mathcal{N}_S))$. If S' is a deformation of S in Z corresponding to point s' in the paremeter space B, then the Kodaira-Spencer deformation theory allows us to identify

$$T_{s'}(B) \simeq H^0(S', \mathcal{O}(\mathcal{N}_{S'}))$$

where $\mathcal{N}_{S'}$ is the normal bundle of S' in Z.

We apply this theorem to a twistor line P. Since

$$H^1(P, \mathcal{O}(\mathcal{N}_P)) = H^1(P, \mathcal{O}(\mathbb{C}^{2n}(1))) = 0 = H^1(P, \mathcal{O}(1)) \otimes \mathbb{C}^{2n},$$

it follows that there is a complex analytic $4n$-dimensional parameter family of deformations of P in Z, whose parameter space we denote by $M_{\mathbb{C}}$.

We note that τ induces an antiholomorphic map which we also denote by τ

from $M_{\mathbb{C}}$ to itself. The fixed point set of τ is a real $4n$-dimensional submanifold $M \subset M_{\mathbb{C}}$.

Next, we need to define a hyperkähler structure (g, I, J, K) on M whose twistor space is Z.

We remark that

$$\begin{aligned}
H^0(P, \mathcal{O}(\mathcal{N})) &= H^0(P, \mathcal{O}(\mathcal{N}(-1))) \otimes H^0(P, \mathcal{O}(1)) \\
&= H^0(P, \mathcal{O}(\mathbb{C}^{2n})) \otimes H^0(P, \mathcal{O}(1)) \\
&\simeq \mathbb{C}^{2n} \otimes \mathbb{C}^2.
\end{aligned}$$

From the definition of $\omega \in H^0(Z, \mathcal{O}(\Lambda^2 T_F^*(2)))$, it follows that ω can be viewed as a symplectic form on $H^0(P, \mathcal{O}(\mathcal{N}(-1)))$. Moreover, there is a natural symplectic form $\tilde{\omega}$ on $H^0(P, \mathcal{O}(1))$ defined by

$$\tilde{\omega}(a_1 + b_1\zeta, a_2 + b_2\zeta) = a_1 b_2 - b_1 a_2, \tag{3.2.10}$$

where a_i, b_i are complex numbers. Therefore

$$g_{\mathbb{C}} = \omega \otimes \tilde{\omega} \tag{3.2.11}$$

is a complex metric on $M_{\mathbb{C}}$ (a nondegenerate pairing between $T'(M_{\mathbb{C}})$ and $T'^*(M_{\mathbb{C}})$). Thus, if

$$x + \zeta y \in H^0(P, \mathcal{O}(\mathcal{N}_P)), \tag{3.2.12}$$

with x and y in $H^0(P, \mathcal{O}(\mathcal{N}(-1))) \simeq \mathbb{C}^{2n}$, we have

$$g(x + \zeta y, x + \zeta y) = 2\omega(x, y). \tag{3.2.13}$$

We still need to make use of the real structure τ in order to construct a *real* metric on M.

A *real structure* t on a complex vector space V is a complex anti-linear involution of V ($t(\lambda v) = \bar{\lambda} t(v)$ and $t^2 = \mathrm{Id}$). A closely related concept is that of a *quaternionic structure* j on V, which is a complex anti-linear map from V to itself ($j(\lambda v) = \bar{\lambda} j v$) such that $j^2 = -\mathrm{Id}$. We remark that if V and W are endowed with two quaternionic structures j_V and j_W, then $V \otimes W$ is endowed with a real structure, namely $j_V \otimes j_W$.

We remark that, up to sign, there is but one quaternionic strucure $\tilde{j}$ on $H^0(P_m, \mathcal{O}(1))$ covering the antipodal map $\sigma(\zeta) = -1/\bar{\zeta}$

$$\tilde{j}(a + b\zeta) = \bar{b} - \bar{a}\zeta.$$

The real structure τ induces a unique quaternionic structure j on $H^0(P_m, \mathcal{O}(\mathcal{N}_{P_m}(-1)))$ such that $\tau = j \otimes \tilde{j}$. □

Writing an element of $H^0(P_m, \mathcal{O}(\mathcal{N}_{P_m}))$ as $x + y\zeta$, where x and y are elements of $H^0(P_m, \mathcal{O}(\mathcal{N}_{P_m}(-1))) \simeq \mathbb{C}^{2n}$, we have

$$\tau(x + y\zeta) = jy - jx\zeta,$$

so that for a *real* twistor line P_m corresponding to a point $m \in M$, the *real* vectors at m correspond to elements of $T_m \otimes \mathbb{C} \simeq H^0(P_m, \mathcal{O}(\mathcal{N}_{P_m}))$ of the form

$$X = x - (jx)\zeta, \tag{3.2.14}$$

with $x \in H^0(P_m, \mathcal{O}(\mathcal{N}(-1)))$. We thus define the metric g on M by

$$g(X, X) = -2\omega(x, jx). \tag{3.2.15}$$

The compatibility of ω with τ and its nondegeneracy ensure that g is either positive definite, or negative definite (assuming that M is connected). Without loss of generality, we can assume that g is positive definite, and thus is a metric on M (otherwise, we just multiply g by -1).

Suppose that X vanishes at some ζ_0. Then $x = (jx)\zeta_0$, and we have

$$g(X, X) = -2\omega((jx)\zeta_0, jx) = 0,$$

which implies that X vanishes since g is positive-definite. Hence given a real twistor line P_m, for any given ζ, one can identify a neighborhood of $m \in M$ with a neighborhood in $F_\zeta = p^{-1}(\zeta)$ of the intersection of P_m with F_ζ.

We remind the reader that a real vector $X(\zeta)$ at P_m is of the form

$$X(\zeta) = x - (jx)\zeta,$$

with $x \in H^0(P_m, \mathcal{O}(\mathcal{N}(-1)))$. Letting $y = -jx$, we can alternatively write it as

$$X(\zeta) = jy + y\zeta.$$

So if we identify M (locally) with the fiber F_∞, the vector $X(\zeta)$ gets identified with $X(\infty) = y$. We define I on the real tangent vector y to be simply multiplication by i, and J to be left multiplication by ij

$$I(y) = iy,\ J(y) = ij(y) \text{ and } K(y) = -j(y).$$

We note that I and J anticommute because j is complex anti-linear (the reader is probably wondering why not define J to by left multiplication by j; this is just to be consistent with our formula for $I(\zeta)$). It is clear that I is integrable, since (M, I) is locally identified with the fibre F_∞ as a complex manifold. We now check that J and K are also integrable. This is because J and K correspond to left multiplication by i after we identify M locally with F_1 and F_{-i} respectively. We do the computation only for J (the other is similar). We have

$$X(1) = jy + y,$$

so

$$iX(1) = ijy + iy = ijy + j(ijy),$$

from which we see that y becomes ijy, which is precisely how J acts. Hence J is integrable, and similarly, so is K. It remains only to check that the Kähler forms ω_1, ω_2 and ω_3 are closed. We do this only for ω_1 (the other two are similar). Upon restricting ω on a fibre F_ζ, this defines a symplectic 2-form on

M which we denote by φ_ζ. If

$$X_1 = jy_1 + y_1\zeta$$
$$X_2 = jy_2 + y_2\zeta.$$

Hence

$$\varphi_i(X_1, X_2) = \omega(jy_1 + iy_1, jy_2 + iy_2)$$
$$\varphi_{-i}(X_1, X_2) = \omega(jy_1 - iy_1, jy_2 - iy_2),$$

from which we deduce that

$$\frac{1}{2}(\varphi_i - \varphi_{-i})(X_1, X_2) = i\omega(jy_1, y_2) + i\omega(y_1, jy_2) = -\omega_1(X_1, X_2).$$

Hence ω_1 is closed (since the φ_ζs are symplectic forms on M and therefore closed). This finishes the proof that (M, g, I, J, K) is hyperkähler.

Chapter 4

The Legendre Transform

In this chapter, we review the Legendre transform construction of [3]. Briefly, the idea is the following. If M has n commuting triholomorphic Killing fields, then the moment map induces a holomorphic map from the twistor space Z of M onto the total space of the holomorphic bundle $\mathbb{C}^n(2)$ over $\mathbb{CP}^1$. Thus, due to $\mathbb{C}^n$-equivariance, we may view Z as a principal fibre bundle over $\mathbb{C}^n(2)$ with group the additive group $\mathbb{C}^n$. This setup, naturally leads to a description of Z as the gluing of two copies of $\mathbb{C}^{2n+1}$ using a holomorphic symplectomorphism whose hamiltonian H is a holomorphic function of $n+1$ complex variables. Then, one expresses the Kähler potential K with respect to I as a Legendre transform of a holomorphic function F, which in turn is given by a contour integral expression in terms of the hamiltonian H. The contour integral expression for F is equivalent to F satisfying a system of linear elliptic second order PDEs, and is a higher dimension analogue of the classical Whittaker formula [10], which represents harmonic functions on $\mathbb{R}^3$ as a contour integral.

Assume that (M, g, I, J, K) is a hyperkähler manifold admitting an action of the abelian group $\mathbb{R}^n$ by triholomorphic isometries. This gives an action of $\mathbb{R}^n$ by biholomorphic transformations of the twistor space Z preserving ω. We assume that this action extends to a biholomorphic and free action of the complexification $\mathbb{C}^n$ of $\mathbb{R}^n$ on Z, also preserving ω.

The moment map of this complex symplectic action of $\mathbb{C}^n$ on Z can be viewed as a holomorphic map $\mu : Z \to Y$, where Y is the total space of $\mathbb{C}^n(2)$ over $\mathbb{CP}^1$. Moreover μ is $\mathbb{C}^n$-equivariant, and $\mathbb{C}^n$ acts trivially on Y since $\mathbb{C}^n$ is abelian. Thus, one can view Z as a principal fibre bundle over Y with group the additive group $\mathbb{C}^n$.

The complex manifold $\mathbb{C}^n(2)$ can be covered with two sets of coordinates (ζ, η^i) and $(\tilde{\zeta}, \tilde{\eta}^i)$ $(1 \leq i \leq n)$, on $\pi^{-1}(U)$ and $\pi^{-1}(\tilde{U})$ respectively ($\pi : \mathbb{C}^n(2) \to \mathbb{CP}^1$ the bundle projection), satisfying the following transition relations:

$$\eta^i = \zeta^2 \tilde{\eta}^i, \quad \tilde{\zeta} = 1/\zeta, \tag{4.0.1}$$

for $\zeta \neq 0$. As for the complex manifold Z, we need two sets of n additional coordinates ξ^i and $\tilde{\xi}^i$ to describe it. We then have

$$\tilde{\xi}^i = \xi^i + f^i(\eta^j, \zeta), \quad \eta^i = \zeta^2 \tilde{\eta}^i, \quad \tilde{\zeta} = 1/\zeta, \tag{4.0.2}$$

for $\zeta \neq 0$. In these coordinates, the group action is generated by the ∂_{ξ^i}s, and the η^is form the moment map μ, so that the symplectic form ω along the fibres is

$$\omega = \sum_i d\xi^i \wedge d\eta^i = \zeta^2 \sum_i d\tilde{\xi}^i \wedge d\tilde{\eta}^i \pmod{d\zeta}, \tag{4.0.3}$$

the last equality coming from the fact that ω is $\mathcal{O}(2)$-valued. Comparing with

4.0.2, we see that

$$\sum_{i,j} \frac{\partial f^i}{\partial \eta^j} d\eta^j \wedge d\eta^i = 0.$$

Hence $\sum_i f^i d\eta^i$ is closed with respect to d_η, so we can find a holomorphic function $H(\eta^i, \zeta)$ such that

$$f^i = \frac{\partial H}{\partial \eta^i} \quad (1 \leq i \leq n). \tag{4.0.4}$$

We remark that the symplectic transformation 4.0.2 patching together the two copies of $\mathbb{C}^{2n+1}$ has the following symplectic vector field

$$\sum_i \frac{\partial H}{\partial \eta^i} \frac{\partial}{\partial \xi^i}$$

corresponding to the Hamiltonian H.

We now set out to calculate the real structure τ. It should cover $\tau(\zeta) = -1/\bar{\zeta}$, and respect the transition relations 4.0.2. With these requirements, τ is uniquely determined, up to sign, or a change of coordinates (by say multiplying some coordinates by i), and is given by

$$\tau(\zeta) = -\frac{1}{\bar{\zeta}}, \quad \tau(\eta^i) = -\frac{\bar{\eta}^i}{\bar{\zeta}^2}, \quad \tau(\xi^i) = -\bar{\xi^i}. \tag{4.0.5}$$

A holomorphic section of $p : Z \to \mathbb{CP}^1$ is mapped holomorphically by μ to a holomorphic section of the bundle projection $\pi : Y \to \mathbb{CP}^1$. We thus set forth to determine the holomorphic sections of π. These are given by holomorphic functions $\eta^i(\zeta)$ and $\tilde{\eta}^i(\tilde{\zeta})$ satisfying

$$\eta^i(\zeta) = \zeta^2 \tilde{\eta}^i(1/\zeta), \tag{4.0.6}$$

for $\zeta \neq 0$. It then follows that each η^i is quadratic in ζ:

$$\eta^i = a^i + b^i\zeta + c^i\zeta^2, \tag{4.0.7}$$

for $1 \leq i \leq n$. These are images of the (complex) twistor lines. Images of the real twistor lines are

$$\eta^i = -\bar{z}^i - x^i\zeta + z^i\zeta^2. \tag{4.0.8}$$

Hence the projections of the $4n$-parameter family of complex (real) twistor lines in Z via μ are the $3n$-parameter family of complex (real) holomorphic sections of $\pi : Y \to \mathbb{CP}^1$ given by 4.0.7 (respectively 4.0.8). Thus we expect that the complex (real) twistor lines projecting to a fixed complex line in 4.0.8 form a complex (real) n-parameter family. Such twistor lines are given by holomorphic functions $\xi(\zeta)$ and $\tilde{\xi}(\tilde{\zeta})$ satisfying

$$\tilde{\xi}^i(\frac{1}{\zeta}) = \xi^i(\zeta) + \frac{\partial H}{\partial \eta^i}(\eta^1(\zeta), \ldots, \eta^n(\zeta), \zeta), \tag{4.0.9}$$

where $\eta^i(\zeta) = z^i - x^i\zeta - \bar{z}^i\zeta^2$, for a fixed $(x^i, z^i) \in \mathbb{R}^n \times \mathbb{C}^n$. We expand in power series

$$\tilde{\xi}^i(1/\zeta) = \sum_{n=0}^{\infty} a_n^i \zeta^{-n} \quad \text{and} \quad \xi^i(\zeta) = \sum_{n=0}^{\infty} b_n^i \zeta^n. \tag{4.0.10}$$

So if we expand

$$\frac{\partial H}{\partial \eta^i}(\eta^1(\zeta), \ldots, \eta^n(\zeta), \zeta)$$

in Laurent series in ζ, we are forced to assign the terms corresponding to negative powers of ζ to the a_n^is ($n \geq 1$) and those corresponding to positive powers of ζ to the b_n^is ($n \geq 1$). We have however an n-parameter freedom in assigning the constant terms, and this gives us the "missing" n-parameter family of twistor curves projecting to the same line in Y. More precisely, if C is a simple counterclockwise contour around $\zeta = 0$ (and not passing through ∞), we then have

$$a_0^i - b_0^i = \frac{1}{2\pi i} \int_C \frac{\partial H}{\partial \eta^i} \frac{d\zeta}{\zeta}. \tag{4.0.11}$$

Moreover, the reality condition forces

$$a_0^i = -\overline{b_0^i}. \tag{4.0.12}$$

We let $u^i = \tilde{\xi}^i(0)$, so that we have

$$z^i = \tilde{\eta}^i(0) \quad \text{and} \quad u^i = \tilde{\xi}^i(0), \tag{4.0.13}$$

which are holomorphic with respect to $I = \underline{\mathrm{I}}(\infty)$. We need to determine the functions x^i with respect to these coordinates z^j and u^j. We define

$$F(x^i, z^i, \bar{z}^i) = \frac{1}{2\pi i} \int_C H(\eta^1(\zeta), \ldots, \eta^n(\zeta), \zeta) \frac{d\zeta}{\zeta^2}, \tag{4.0.14}$$

where $\eta^i(\zeta) = -\bar{z}^i - x^i\zeta + z^i\zeta^2$. Differentiating with respect to x^i

$$\frac{\partial F}{\partial x^i}\int_C \frac{\partial H}{\partial \eta^i}(-\zeta)\frac{d\zeta}{\zeta^2} = -(a_0^i - b_0^i).$$

But

$$u^i = \tilde{\xi}^i(0) = a_0^i,$$

and

$$b_0^i = -\overline{a_0^i} = -\bar{u}^i,$$

so that

$$\frac{\partial F}{\partial x^i} = -(u^i + \bar{u}^i). \tag{4.0.15}$$

This gives the x^is implicitly in terms of the $u^j + \bar{u}^j$, z^j and $\bar{z}^j$. For

$$\omega = \sum_i d\xi^i \wedge d\eta^i,$$

it follows from 4.0.3 and 5.0.6 that

$$\tau^*(\omega) = \frac{\bar{\omega}}{\bar{\zeta}^2}, \tag{4.0.16}$$

while for

$$\varphi = -(\omega_2 - i\omega_3) - 2\omega_1\zeta + (\omega_2 + i\omega_3)\zeta^2, \tag{4.0.17}$$

the effect of τ on it is

$$\tau^*(\varphi) = -\frac{\bar{\varphi}}{\bar{\zeta}^2}. \tag{4.0.18}$$

In order to correct this, we can replace the η^j by $i\eta^j$. Hence, the Kähler form

ω_1 is determined by the ζ coefficient in

$$
\begin{aligned}
i\omega &= i\sum_j d\xi^j \wedge d\eta^j \\
&= i\sum_i (db_0^j + db_1^j\tilde{\zeta} + \ldots) \wedge (-d\bar{z}^j - dx^j\tilde{\zeta} + \ldots) \\
&= i\sum_j (-d\bar{u}^j + db_1^j\tilde{\zeta} + \ldots) \wedge (-d\bar{z}^j - dx^j\tilde{\zeta} + \ldots) \\
&= i\sum_j d\bar{u}^j \wedge d\bar{z}^j + i\sum_j (d\bar{u}^j \wedge dx^j - db_1^j \wedge d\bar{z}^j)\zeta + \ldots
\end{aligned}
$$

Hence

$$\omega_1 = \frac{i}{2}\sum_j (db_1^j \wedge d\bar{z}^j - d\bar{u}^j \wedge dx^j). \tag{4.0.19}$$

But

$$b_1^j = -\frac{1}{2\pi i}\int_C \frac{\partial H}{\partial \eta^j}\frac{d\zeta}{\zeta^2} = \frac{\partial F}{\partial \bar{z}^j}, \tag{4.0.20}$$

so that

$$\omega_1 = \frac{i}{2}\sum_j (d(\frac{\partial F}{\partial \bar{z}^j}) \wedge d\bar{z}^j - d\bar{u}^j \wedge dx^j). \tag{4.0.21}$$

Let

$$K = F - x^j\frac{\partial F}{\partial x^j}. \tag{4.0.22}$$

Note that we are suppressing the summation symbol to alleviate the notation.

Then

$$
\begin{aligned}
\overline{\partial}K &= \frac{\partial F}{\partial \bar{z}^j}d\bar{z}^j + \frac{\partial F}{\partial x^j}\overline{\partial}x^j - \overline{\partial}x^j\frac{\partial F}{\partial x^j} - x^j\overline{\partial}\left(\frac{\partial F}{\partial x^j}\right) \\
&= \frac{\partial F}{\partial \bar{z}^j}d\bar{z}^j - x^j\overline{\partial}\left(\frac{\partial F}{\partial x^j}\right)
\end{aligned}
$$

so that

$$\partial\overline{\partial}K = \partial\left(\frac{\partial F}{\partial \bar{z}^j}\right) \wedge d\bar{z}^j - \partial x^j \wedge \overline{\partial}\left(\frac{\partial F}{\partial x^j}\right) - x^j \partial\overline{\partial}\left(\frac{\partial F}{\partial x^j}\right).$$

Using the equation $\partial F/\partial x^j = -(u^j + \bar{u}^j)$, we arrive at

$$\partial\overline{\partial}K = \partial\left(\frac{\partial F}{\partial \bar{z}^j}\right) \wedge d\bar{z}^j + \partial x^j \wedge d\bar{u}^j. \tag{4.0.23}$$

Comparing with 4.0.21, and since ω_1 is of type $(1,1)$ in holomorphic coordinates with respect to I, it follows that $K/2$ is the Kähler potential of ω_1, where K is given by 4.0.22 and is the Legendre transform of F, and F is given by

$$F(x^i, z^i, \bar{z}^i) = \frac{1}{2\pi i} \int_C H(\eta^1(\zeta), \ldots, \eta^n(\zeta), \zeta) \frac{d\zeta}{\zeta^2}. \tag{4.0.24}$$

Chapter 5

The Boyer and Finley Equation

In this chapter, we review the derivation of the Boyer and Finley equation, as contained in [11]. The authors first consider a complex selfdual Einstein 4-manifold using the formalism of complex $\mathscr{H}$ spaces; more precisely, in this formalism, you have 2 sets of special complex coordinates q^1, q^2 and $\tilde{q}^1$, $\tilde{q}^2$, and the metric is determined by a smooth complex-valued function Ω, referred to as a *potential*. There is gauge freedom in the choice of the special coordinates q^A and $\tilde{q}^A$, as well as in the choice of potential. The authors first determine this gauge freedom, and then consider a vector field K satisfying the Killing equation, which implies a differential relation between Ω and K. There is also another constraint on K, namely that the antiselfdual part of ∇K must be constant. Imposing then a reality condition on the complex selfdual Einstein 4-manifold implies that there are two cases to consider, either the antiselfdual part of ∇K is zero (i.e. ∇K is selfdual), or it is nonzero. They consider each case separately, and use the gauge freedom to simplify the differential relation between Ω and K (what they called the *master equation*). The first case turns out to correspond to the Gibbons and Hawking ansatz, where Ω is determined by a solution to the Laplace equation on (an open subset of) $\mathbb{R}^3$. On the other hand, in the second case, the master equation can be simplified using the gauge freedom and a Legendre transform to the so-called Boyer and Finley equation on (an open subset of) $\mathbb{R}^3$:

$$F_{q\bar{q}} + (e^F)_{JJ} = 0, \tag{5.0.1}$$

where q, $\bar{q}$ and J are coordinates on $\mathbb{R}^3$ (q complex and J real).

Let $M_{\mathbb{C}}$ be a complex selfdual Einstein 4-manifold. Then there exist special coordinates q^1, q^2 and $\tilde{q}^1$, $\tilde{q}^2$ and a smooth complex-valued function Ω of these coordinates, such that in these coordinates the metric takes the form

$$g = 2P_{AB} dq^A \odot \tilde{q}^B, \quad P_{AB} = \Omega_{q^A \tilde{q}^B}, \tag{5.0.2}$$

and Ω satisfies:

$$P^{AB} P_{AB} = 2, \tag{5.0.3}$$

where raising and lowering of indices A, B is done via a constant (with respect to these coordinates) skew-symmetric rank 2 spinor field ϵ_{AB}, satifying $\epsilon_{12} = 1$:

$$\psi_A = \psi^B \epsilon_{BA} \quad \varphi^A = \epsilon^{AB} \varphi_B. \tag{5.0.4}$$

Normalized symmetrization (respectively skew-symmetrization) is denoted by parentheses (respectively square brackets). For instance,

$$\psi_{(AB)C} = \frac{1}{2}(\psi_{ABC} + \psi_{BAC}),$$
$$\psi_{[ABC]} = \frac{1}{3!}(\psi_{ABC} - \psi_{BAC} + \psi_{BCA} - \psi_{CBA} + \psi_{CAB} - \psi_{ACB}).$$

A useful trick when dealing with 2-spinors is the following

$$\psi_{[AB]} = \frac{1}{2}\psi_C{}^C\epsilon_{AB}. \tag{5.0.5}$$

The equation 5.0.3 is the *Monge-Ampere equation*, which also goes by the name the *Heavenly equation* in Plebanski's Ω-formalism. The converse is also true, in the sense that if Ω is a complex-valued solution to the Monge-Ampere equation, then the metric g given by 5 is a complex selfdual Einstein 4-manifold.

There is also a very simple prescription to obtain a *real* selfdual Einstein 4-manifold from $M_{\mathbb{C}}$ by imposing the following reality conditions:

$$\tilde{q}^A = \overline{q^A} = \bar{q}^A, \quad \Omega \text{ real}. \tag{5.0.6}$$

Moreover, any real Riemannian selfdual Einstein 4-manifold admits complex coordinates q^1, q^2 and their complex conjugates, together with a potential Ω, which is now a real-valued function, such that the metric is given by 5 with $\bar{q}^B$ replacing $\tilde{q}^B$ and Ω satisfies the Monge-Ampere equation 5.0.3.

We now go back to the case of a (complex) $\mathscr{H}$-space $M_{\mathbb{C}}$, by which we mean a complex selfdual Einstein 4-manifold (with respect to a complex holomorphic metric). We let

$$\Sigma = dq_A \wedge dq^A = 2dq^1 \wedge dq^2, \tag{5.0.7}$$

$$\tilde{\Sigma} = d\tilde{q}_A \wedge d\tilde{q}^A = 2d\tilde{q}^1 \wedge d\tilde{q}^2. \tag{5.0.8}$$

Then Σ and $\tilde{\Sigma}$ are two closed antiselfdual 2-forms on $M_{\mathbb{C}}$. Moreover, the 2-form $P_{AB} dq^A \wedge d\tilde{q}^B$ is also closed (this follows from the torsion being 0), so that we have a basis of the antiselfdual 2-forms on $M_{\mathbb{C}}$ consisting of closed 2-forms, namely

$$S^{\dot{A}\dot{B}} = (P_{AB} dq^A \wedge d\tilde{q}^B, \Sigma, \tilde{\Sigma}). \tag{5.0.9}$$

We wish to investigate changes of coordinates of the form

$$q'^R = q'^R(q^A, \tilde{q}^B) \tag{5.0.10}$$

$$\tilde{q}'^S = q'^S(q^A, \tilde{q}^B), \tag{5.0.11}$$

such that the metric g is still given by in the primed coordinates and with Ω replaced by its transformed version Ω'. We also require that the new $S'^{\dot{A}\dot{B}}$ still consists of closed 2-forms, and that it be obtainable from $S^{\dot{A}B}$ by

$$S'^{\dot{R}\dot{S}} = l^{\dot{R}}{}_{\dot{A}} l^{\dot{S}}{}_{\dot{B}} S^{\dot{A}\dot{B}}, \tag{5.0.12}$$

where $l^{\dot{R}}{}_{\dot{A}}$ is a smooth function on $M_{\mathbb{C}}$ with values in $SL_2(\mathbb{C})$. In order to preserve the closedness of the 2-forms in $S^{\dot{A}\dot{B}}$, the function $l^{\dot{R}}{}_{\dot{A}}$ must be a constant element of $SL_2(\mathbb{C})$. The solution to all these requirements is that such coordinate changes are of the form

$$dq'^R = (d^R{}_A)(\Delta dq^A - i\tau\Omega_{q_A\tilde{q}^B} d\tilde{q}^B) \tag{5.0.13}$$

$$d\tilde{q}'^S = (\tilde{d}^S{}_B)(\tilde{\Delta} d\tilde{q}^B + i\tilde{\tau}\Omega_{q^A q_{\tilde{B}}} dq^A), \tag{5.0.14}$$

where $d^R{}_A$, $\tilde{d}^S{}_B$ are arbitrary smooth functions with values in $SL_2(\mathbb{C})$, and Δ, $\tilde{\Delta}$, τ, $\tilde{\tau}$ are constants satisfying

$$\Delta\tilde{\Delta} + \tau\tilde{\tau} = 1. \tag{5.0.15}$$

The 2 by 2 matrix-valued functions

$$l^{\dot{R}}{}_{\dot{A}} = \begin{pmatrix} \Delta & i\tau \\ i\tilde{\tau} & \tilde{\Delta} \end{pmatrix}, \tag{5.0.16}$$

$$l^R{}_A = d^R{}_A. \tag{5.0.17}$$

are used to transform dotted and undotted indices respectively. Finally, we would like to determine how the potential Ω transforms. Consider the Monge-Ampere equation

$$(\partial^A\tilde{\partial}^B\Omega)(\partial_A\tilde{\partial}_B\Omega) = 2. \tag{5.0.18}$$

It follows from this equation that there exist smooth functions F and $\tilde{F}$ such that

$$(\tilde{\partial}^B\Omega)(\partial_A\tilde{\partial}_B\Omega) = q_A + \partial_A F \tag{5.0.19}$$

$$(\partial^A\Omega)(\partial_A\tilde{\partial}_B\Omega) = \tilde{q}_B + \tilde{\partial}_B\tilde{F} \tag{5.0.20}$$

Then Ω transforms according to

$$\Omega' = \Omega + i\left(\frac{\tilde{\Delta}\tau(\tilde{F}+P) - \Delta\tilde{\tau}(F+\tilde{P})}{\Delta\tilde{\Delta} - \tau\tilde{\tau}}\right) + A(q) + \tilde{A}(\tilde{q}), \tag{5.0.21}$$

where A, respectively $\tilde{A}$, is a function of the q variables, respectively $\tilde{q}$ variables, only, and P and $\tilde{P}$ are some functions of all variables.

Next, the authors consider a vector field

$$K = L^A \partial_A + \tilde{L}^A \tilde{\partial}_A \tag{5.0.22}$$

satisfying the Killing equation

$$\nabla_{(a} K_{b)} = 0, \tag{5.0.23}$$

for a, b ranging over all 4 complex coordinates q^A and $\tilde{q}^A$. After some work, the authors show that the Killing equation implies that there are constants b_0 and $\tilde{b}_0$, as well as functions $H = H(q^A)$ and $\tilde{H} = \tilde{H}(\tilde{q}^A)$, such that

$$K\Omega = -\tilde{b}_0 F - b_0 \tilde{F} + H + \tilde{H}, \tag{5.0.24}$$

which is a necessary equation that Ω and K must satisfy if K is a Killing field. There is another condition that needs to be satisfied though, namely that

$$l^{\dot{A}\dot{B}} = \nabla_A{}^{\dot{A}} K_B{}^{\dot{B}} \epsilon^{AB} \tag{5.0.25}$$

is constant. This *integrability condition* comes from the fact that an arbitrary

Killing field K on a Riemannian manifold must satisfy

$$\nabla_a \nabla_b K_c = R_{bcad} K^d, \tag{5.0.26}$$

where R_{bcad} is the Riemann curvature tensor. Both the left-hand side and right-hand side of the previous equation are skew-symmetric in b and c. We then equate the anti-selfdual part with respect to b and c on both sides to get the integrability condition 5.0.25; indeed, the anti-selfdual part of the right-hand side vanishes because $M_{\mathbb{C}}$ is a selfdual Ricci-flat (complex) 4-manifold. Using the Killing equation 5.0.23 and the integrability condition 5.0.25, the authors then show that there exists a constant c_0 and functions $\zeta = \zeta(q^A)$ and $\tilde{\zeta} = \tilde{\zeta}(\tilde{q}^A)$ such that the components of the Killing field K are of the form

$$L^A = b_0 \partial^A(\Omega) - \frac{1}{2} i c_0 q^A + \partial^A \zeta \tag{5.0.27}$$

$$\tilde{L}^A = \tilde{b}_0 \tilde{\partial}^A(\Omega) + \frac{1}{2} i c_0 \tilde{q}^A + \tilde{\partial}^A \tilde{\zeta}. \tag{5.0.28}$$

Then the authors focus on the real case, obtained by imposing Ω to be real and replacing the tilde by the complex conjugate, so that $\tilde{q}$ and $\tilde{b}_0$ become $\bar{q}$ and $\bar{b}_0$, and so on. Under this reality assumption, consider

$$l^{\dot{A}\dot{B}} = -2 \begin{pmatrix} b_0 & ic_0 \\ ic_0 & \bar{b}_0 \end{pmatrix}, \quad c_0 \text{ real.} \tag{5.0.29}$$

We have two cases to consider:

case 1: the determinant of $l^{\dot{A}\dot{B}}$ is 0, in which case $l^{\dot{A}\dot{B}}$ vanishes, which means that ∇K is selfdual.

case 2: the determinant of $l^{\dot{A}\dot{B}}$ is nonzero or, in other words, the antiselfdual part of ∇K is nonvanishing.

In case 1, using the "gauge" freedom determined towards the beginning of this section, the authors simplify the Monge-Ampere equation and then, using a Legendre transform, simplify it further to the 3-dimensional Laplace equation. In other words, a selfdual Ricci-flat 4-manifold admitting a selfdual Killing field can be obtained from the Gibbons-Hawking ansatz, which is what Jones and Tod had proved in [12].

In case 2, again by using the "gauge" freedom and then making use of a Legendre-like transform, the authors arrive at the *Boyer and Finley* equation [11] (which goes by the name the SU(∞) Toda lattice equation in the physics literature):

$$F_{q\bar{q}} + (e^F)_{JJ} = 0, \tag{5.0.30}$$

where q, $\bar{q}$ and J are coordinates on $\mathbb{R}^3$ (q complex and J real).

Before closing, we make the following 2 remarks. In [13] (1991), Claude LeBrun showed that any Kähler scalar-flat 4-manifold with Killing field comes (locally) from a pair of functions (u, V) where u is a solution to the Boyer and Finley equation, while V is a solution of its linearization (at u).

In arXiv:hep-th/0609071v1 (2006), Paul Tod considered the case of an anti-selfdual Einstein 4-manifold with nonzero cosmological constant, and showed that such manifolds are also determined by a solution to the Boyer and Finley equation, just like for the case of a selfdual Einstein 4 manifold with 0 cosmological constant, which was considered by Boyer and Finley.

Chapter 6

Hyperkähler Manifolds and Quaternionic Killing fields

6.1 Special Coordinates

In this section, we extend the formalism used in the Boyer and Finley work [11] (Plebansky's Ω formalism) in the case of a self-dual Ricci-flat 4-manifold to the case of a hyperkähler $4n$-manifold (which are generalizations of anti-self-dual Ricci-flat 4-manifolds).

We choose a complex structure, say I, in the S^2 of compatible complex structures on a hyperkähler $4n$-manifold M. Then, denoting by ω_1, ω_2 and ω_3 the Kähler 2-forms corresponding to I, J and K respectively, the complex 2-form

$$\omega_+ = \omega_2 + i\omega_3 \tag{6.1.1}$$

is a complex holomorphic (in fact covariantly constant) symplectic 2-form with respect to I. We now apply the holomorphic version of Darboux's theorem, which guarantees the existence of local holomorphic coordinates (with respect to I) q^A for A going from 1 to $2n$ such that ω_+ is locally of the form

$$\omega_+ = \epsilon_{AB} dq^A \wedge dq^B = 2 \sum_{k=1}^{n} dq^k \wedge dq^{n+k}, \tag{6.1.2}$$

where

$$(\epsilon_{AB}) = \begin{pmatrix} 0_n & 1_n \\ -1_n & 0_n \end{pmatrix} \tag{6.1.3}$$

In these local coordinates, the metric g takes the form

$$P_{A\bar{B}} dq^A \odot d\bar{q}^{\bar{B}},$$

where

$$P_{A\bar{B}} = \Omega_{q^A \bar{q}^{\bar{B}}},$$

and Ω is a smooth real function. Moreover, the condition $J^2 = -\operatorname{Id}$ implies the following equation

$$P_{A\bar{U}} P_B{}^{\bar{U}} = \epsilon_{AB}. \tag{6.1.4}$$

We will refer to this equation as the *symplectic Monge-Ampere equation* (SMA equation for short). The raising and lowering of indices in this formalism is done via ϵ_{AB}, and not via the metric $P_{A\bar{B}}$, using the following conventions

$$\psi_B = \psi^A \epsilon_{AB}, \quad \varphi^A = \epsilon^{AB} \varphi_B.$$

We also remark that ${\epsilon_A}^B$ behaves like the identity, namely

$$\psi_A = {\epsilon_A}^B \psi_B, \quad \varphi^B = \varphi^A {\epsilon_A}^B.$$

The following is a basis of the (complexified) self-dual 2-forms consisting of closed forms

$$S^{\dot{A}\dot{B}} = (\omega_+, P_{A\bar{B}} dq^A \wedge d\bar{q}^{\bar{B}}, \overline{\omega_+}).$$

We wish to investigate the freedom involved in a choice of special coordinates on M. We remark that instead of choosing I, we could have chosen any other complex structure in the S^2 of compatible complex structures on M. Thus, any other new special coordinates q'^R, $\bar{q}'^{\bar{S}}$ can be obtained from the old special coordinates q^A, $\bar{q}^{\bar{B}}$ using transformations

$$\begin{aligned} q'^R &= q'^R(q^A, \bar{q}^{\bar{B}}) \\ \bar{q}'^{\bar{S}} &= q'^R(q^A, \bar{q}^{\bar{B}}), \end{aligned}$$

whose Jacobian is of the form

$$dq'^R = ({d^R}_A)(\Delta dq^A - i\tau {P^A}_{\bar{B}}\, d\bar{q}^{\bar{B}}) \tag{6.1.5}$$

$$d\bar{q}'^{\bar{S}} = ({\bar{d}^{\bar{S}}}_{\bar{B}})(\bar{\Delta} d\bar{q}^{\bar{B}} + i\bar{\tau} {P_A}^{\bar{B}} dq^A), \tag{6.1.6}$$

where $({d^R}_A)$ is symplectic,

$${d^R}_A \epsilon_{RS} {d^S}_B = \epsilon_{AB}, \tag{6.1.7}$$

, and Δ and τ are constants satisfying

$$\Delta\bar{\Delta} + \tau\bar{\tau} = 1. \tag{6.1.8}$$

Under this transformation, we remark that

$$S'^{\dot{R}\dot{S}} = l^{\dot{R}}{}_{\dot{A}} l^{\dot{S}}{}_{\dot{B}} S^{\dot{A}\dot{B}}, \tag{6.1.9}$$

where

$$(l^{\dot{R}}{}_{\dot{A}}) = \begin{pmatrix} \Delta & i\tau \\ i\bar{\tau} & \bar{\Delta} \end{pmatrix}. \tag{6.1.10}$$

In other words, $(l^{\dot{R}}{}_{\dot{A}}) \in \mathrm{SU}(2)$.

We go back to the SMA system, which implies that

$$\partial_{[B}(\partial^{\bar{R}}\Omega\partial_{A]}\bar{\partial}_{\bar{R}}\Omega) = \epsilon_{AB},$$

so that there exists a smooth function F such that

$$\partial^{\bar{R}}\Omega\partial_A\bar{\partial}_{\bar{R}}\Omega = -q_A + \partial_A F, \quad \partial^D\Omega\partial_D\bar{\partial}_{\bar{B}}\Omega = -\bar{q}^{\bar{B}} + \bar{\partial}_{\bar{B}}\bar{F}. \tag{6.1.11}$$

We find that Ω transforms as

$$\Omega' = \Omega + i\left(\frac{\bar{\Delta}\tau(\bar{F}+P) - \Delta\bar{\tau}(F+\bar{P})}{\Delta\bar{\Delta} - \tau\bar{\tau}}\right) + A(q') + \bar{A}(\bar{q}'), \tag{6.1.12}$$

where A, respectively $\bar{A}$, is a function of the q' variables, respectively $\bar{q}'$ variables, only, and $\bar{P}$ satisfy

$$\partial'_A\bar{\partial}'_{\bar{R}}\bar{P} = i\bar{\tau}\Delta d_A{}^C\bar{d}_{\bar{R}}{}^{\bar{U}}\bar{\partial}_{\bar{U}}(P_C{}^{\bar{T}}\bar{\partial}_{\bar{T}}F) - i\tau\bar{\Delta}d_A{}^C\bar{d}_{\bar{R}}{}^{\bar{T}}\partial_C(P^D{}_{\bar{T}}\partial_D F). \quad (6.1.13)$$

The operators ∂'_A and $\bar{\partial}'_{\bar{R}}$ are defined by

$$\partial'_A = (d_A{}^C)(-\bar{\Delta}\partial_C + i\bar{\tau}P_C{}^{\bar{U}}\bar{\partial}_{\bar{U}}) \quad (6.1.14)$$

$$\bar{\partial}'_{\bar{R}} = (\bar{d}_{\bar{R}}{}^{\bar{T}})(-\Delta\bar{\partial}_{\bar{T}} - i\tau P^D{}_{\bar{T}}\partial_D) \quad (6.1.15)$$

These are the kind of transformations we will consider. They rotate the complex structures but preserve the local form of ω_+.

We remark that the vector fields ∂'_A and $\bar{\partial}'_{\bar{R}}$ are assumed to be coordinate vector fields, so that the Lie bracket of any two of them is assumed to vanish. This implies that

1. the functions $d_A{}^C$ are holomorphic,
2. the functions $P_{A\bar{U}}\bar{d}_{\bar{R}}{}^{\bar{U}}$ are holomorphic,
3. $d_{[A|}{}^C\partial_C d_{|B]}{}^D = 0$,
4. $d_A{}^C\partial_C P^D{}_{\bar{T}} = P^C{}_{\bar{T}}\partial_C d_A{}^D$.

6.2 Quaternionic Killing fields

Let M be a hyperkähler $4n$-manifold.

Definition 6.2.1. *A real vector field X on M is said to be* Killing *if*

$$\mathcal{L}_X(g) = 0,$$

in other words if X preserves the metric g.

Definition 6.2.2. *A real vector field X on M is said to be* quaternionic *if*

$$\mathcal{L}_X \Gamma(M, \mathcal{V}) \subseteq \Gamma(M, \mathcal{V}),$$

where $\mathcal{V}$ is the real rank 3 bundle spanned by ω_k, for $k = 1, ..., 3$. In other words $\mathcal{V}$ is the bundle of self-dual 2-forms on M.

We also need the following lemma:

Lemma 6.2.3. *There exist locally a smooth function $F = F(q, \bar{q})$ such that*

$$\bar{\partial}^{\bar{R}} \Omega \partial_A \bar{\partial}_{\bar{R}} \Omega = -q_A + \partial_A F. \tag{6.2.1}$$

The proof of the lemma is straightforward and follows from the SMA system of equations.

We now prove the following theorem:

Theorem 6.2.4. *If X is a real quaternionic Killing field, then locally, in the special coordinates q^A and $\bar{q}^{\bar{B}}$ constructed in the previous section, X is of the form*

$$X = L^A \partial_A + \bar{L}^{\bar{B}} \bar{\partial}_{\bar{B}}, \tag{6.2.2}$$

where

$$L^A = b\partial^A\Omega - \frac{1}{2}icq^A + \partial^A\zeta \tag{6.2.3}$$

$$,\bar{L}^{\bar{B}} = \bar{b}\bar{\partial}^{\bar{B}}\Omega + \frac{1}{2}ic\bar{q}^{\bar{B}} + \bar{\partial}^{\bar{B}}\bar{\zeta}, \tag{6.2.4}$$

for some $\zeta = \zeta(q)$, $b \in \mathbb{C}$ and $c \in \mathbb{R}$ and the following master equation *holds*

$$X\Omega = -\bar{b}F - b\bar{F} + H + \bar{H}, \tag{6.2.5}$$

for some $H = H(q)$ and $\bar{H} = \bar{H}(\bar{q})$. Conversely, if X is a real vector field on a hyperkähler manifold M of the form above in local special coordinates and the master equation is satisfied, then X is quaternionic Killing.

Proof. The Killing equation in special coordinates q^A, $\bar{q}^{\bar{B}}$, with A and $\bar{B}$ going from 1 to $2n$, splits into the following equations

$$P_{(A}{}^{\bar{U}}\partial_{B)}\bar{L}_{\bar{U}} = 0 \tag{6.2.6}$$

$$P^{D(\bar{A}}\bar{\partial}^{\bar{B})}L_D = 0 \tag{6.2.7}$$

$$\partial_A(P_{D\bar{B}}L^D) + \bar{\partial}_{\bar{B}}(P_{A\bar{U}}\bar{L}^{\bar{U}}) = 0. \tag{6.2.8}$$

The first two equations are complex conjugates of each other, and their solution can be shown to be (see Appendix A):

$$L^A = E^A{}_B\partial^B\Omega + J^A, \tag{6.2.9}$$

where $E_{AB} = E_{AB}(q)$, $J^A = J^A(q)$ and $E_{AB} = -E_{BA}$.

Next, we have a constraint coming from the equation

$$\nabla_a \nabla_b X_c = R_{bca}{}^d X_d, \tag{6.2.10}$$

which holds in fact for any real Killing vector field X.

Moreover, on a hyperkähler manifold M, there exist complex bundles E and H of rank $2n$ and 2 respectively corresponding to the standard representation of $\mathrm{Sp}(n)$ on $\mathbb{C}^{2n}$ and the trivial representation of $\mathrm{Sp}(n)$ on $\mathbb{C}^2$ respectively. The bundles E and H are each equipped with a symplectic 2-form, ϵ_E and ϵ_H. In addition to these symplectic structures, each of them is equipped with a quaternionic structure, j_E and j_H respectively. We note that the complexified tangent bundle $T_{\mathbb{C}}(M)$ is isomorphic to $E \otimes H$, and the bundle of self-dual 2-forms is S^2H. By that we actually mean $(\epsilon_E) \otimes S^2H \subseteq \Lambda^2 E \otimes S^2 H$, but we will often omit references to the ϵ's for simplicity.

It can be shown that the curvature of a hyperkähler manifold M is a section of S^4E. Hence from the constraint equation 6.2.10, it follows that if X is a real Killing field on a hyperkähler manifold M, then

$$\nabla[(\nabla X)_{S^2H}] = 0. \tag{6.2.11}$$

We remark that

$$\mathcal{V} = (\omega_1, \omega_2, \omega_3),$$

where

$$\omega_1 = iP_{A\bar{R}}dq^A \wedge d\bar{q}^{\bar{R}} \tag{6.2.12}$$

$$\omega_2 = \frac{1}{2}(\epsilon_{AB}dq^A \wedge dq^B + \epsilon_{\bar{R}\bar{S}}d\bar{q}^{\bar{R}} \wedge d\bar{q}^{\bar{S}}) \tag{6.2.13}$$

$$\omega_3 = \frac{i}{2}(\epsilon_{AB}dq^A \wedge dq^B - \epsilon_{\bar{R}\bar{S}}d\bar{q}^{\bar{R}} \wedge d\bar{q}^{\bar{S}}). \tag{6.2.14}$$

The condition

$$\mathcal{L}_X(\omega_2) \in \Gamma(M, \mathcal{V})$$

implies, after a short computation, that

$$i(E_{AC}P^C{}_{\bar{R}} - \bar{E}_{\bar{R}\bar{T}}P_A{}^{\bar{T}}) \text{ is a real function times } P_{A\bar{R}} \tag{6.2.15}$$

$$2\partial_{[A}E_{B]C}\partial^C\Omega - 2E_{C[B}\partial_{A]}\partial^C\Omega + 2\partial_{[A}J_{B]} \text{ is a complex function times } \epsilon_{AB}. \tag{6.2.16}$$

Similarly, the condition

$$\mathcal{L}_X(\omega_3) \in \Gamma(M, \mathcal{V})$$

implies, similarly, that

$$E_{AC}P^C{}_{\bar{R}} + \bar{E}_{\bar{R}\bar{T}}P_A{}^{\bar{T}} \text{ is a real function times } P_{A\bar{R}} \tag{6.2.17}$$

$$i(2\partial_{[A}E_{B]C}\partial^C\Omega - 2E_{C[B}\partial_{A]}\partial^C\Omega + 2\partial_{[A}J_{B]}) \text{ is a complex function times } \epsilon_{AB}. \tag{6.2.18}$$

In particular, we have

$$E_{AC}P^C{}_{\bar{R}} \text{ is a complex function times } P_{A\bar{R}},$$

so that, contracting with ${P_B}^{\bar{R}}$ on both sides, and making use of the SMA system, we get that

$$E_{AB} = f\epsilon_{AB},$$

for some holomorphic $f = f(q)$. In order to gain more information about f, we make use of the constraint 6.2.11, which implies, in particular that

$$\nabla[(P_{C\bar{R}}\bar{\nabla}_{\bar{S}}L^C)d\bar{q}^{\bar{R}} \wedge d\bar{q}^{\bar{S}} + (P_{A\bar{U}}\nabla_B\bar{L}^{\bar{U}})dq^A \wedge dq^B]_{S^2H} = 0.$$

But

$$P_{C\bar{R}}\bar{\nabla}_{\bar{S}}L^C = -f\epsilon_{\bar{R}\bar{S}},$$

since f and J^C are holomorphic. Hence the constraint 6.2.11 implies that f is constant, say $f = -b$, where $b \in \mathbb{C}$. Hence

$$E_{BC} = -b\epsilon_{BC}$$
$$L^A = b\partial^A\Omega + J^A.$$

Using these equations and equations 6.2.16 and 6.2.18, we conclude that there is a holomorphic function h such that

$$2\partial_{[A}J_{B]} = h\epsilon_{AB}. \tag{6.2.19}$$

We go back to the constraint 6.2.11, which implies also that

$$\nabla[\nabla_A(P_{D\bar{R}}L^D) + \bar{\nabla}_{\bar{R}}(P_{A\bar{U}}\bar{L}^{\bar{U}})]_{S^2H} = 0.$$

After a short computation, this implies that

$$\partial_D J^D + \bar{\partial}_{\bar{U}} \bar{J}^{\bar{U}} = id,$$

where $d \in \mathbb{R}$. Hence the left-hand side is real, while the right-hand side is pure imaginary, so that both sides vanish. Hence $\partial_D J^D$ is pure a pure imaginary function. This in turn implies that h is both holomorphic and pure imaginary, so that

$$h = -ic, \tag{6.2.20}$$

where $c \in \mathbb{R}$. Thus we have from 6.2.19 that

$$\partial_{[A} J_{B]} = -i\frac{c}{2}\epsilon_{AB}. \tag{6.2.21}$$

Solving this equation yields that there is locally a holomorphic function $\zeta = \zeta(q)$ such that

$$J_B = -i\frac{c}{2}q_B + \partial_B \zeta, \tag{6.2.22}$$

and therefore,

$$L^A = b\partial^A \Omega - i\frac{c}{2}q^A + \partial^A \zeta. \tag{6.2.23}$$

One can also see that the master equation comes from equation 6.2.8.

Conversely, if X is a real vector field on a hyperkähler manifold M with L^A of the above form, and such that the master equation holds, then it is clear that X is Killing and that $\mathcal{L}_X\omega_2$ and $\mathcal{L}_X\omega_3$ are sections of $\mathcal{V}$. So it remains only to check that $\mathcal{L}_X\omega_1$ is a section of $\mathcal{V}$, which can easily be done, and follows from the master equation, as can be checked. Indeed, the condition $\mathcal{L}_X\omega_1$ is a section of $\mathcal{V}$ if

$$\partial_A \bar{\partial}_{\bar{R}}(b\bar{F} + \bar{b}F + X(\Omega)) \text{ is a real function times } \partial_A \bar{\partial}_{\bar{R}}\Omega. \tag{6.2.24}$$

But the master equation tells us precisely that the left-hand side of the previous equation vanishes, and the converse is thus shown to be true; the real vector field X on a hyperkähler manifold M of the form above in local special coordinates and satisfying the master equation is thus quaternionic Killing. □

6.3 Application to hyperkähler $4n$-manifolds with n commuting quaternionic Killing fields

Theorem 6.3.1. *Let $X^1,\dots,X^n$ be n commuting quaternionic Killing fields on a hyperkähler $4n$-manifold M such that the first $n-1$ of them, $X^1,\dots,X^{n-1}$ are triholomorphic. If $(\nabla X^n)_{S^2H}$ vanishes at some point $p \in M$, then it vanishes in a neighborhood of p. Thus we have two cases: in case 1, $(\nabla X^n)_{S^2H}$ vanishes locally, and in case 2, $(\nabla X^n)_{S^2H}$ is nonzero locally.*

Consider first case 1. This is the case of [3]. There are special local coordinates q^j, p^j and $\bar{q}^j$, $\bar{p}^j$, for j going from 1 to n, such that

$$\omega_+ = \omega_2 + i\omega_3 = \sum_{j=1}^{n} dq^j \wedge dp^j \tag{6.3.1}$$

and the vector fields X^j are locally of the form

$$X^j = \partial_{q^j} + \partial_{\bar{q}^j}, \quad (1 \le j \le n). \tag{6.3.2}$$

Moreover, the Kähler potential Ω in these special coordinates can be chosen such that $X^j(\Omega) = 0$, for j going from 1 to n. We then let $u^j = q^j + \bar{q}^j$ and $v^j = i(\bar{q}^j - q^j)$, and write

$$\Omega(q, p, \bar{q}, \bar{p}) = K(v, p, \bar{p}). \tag{6.3.3}$$

Then using matrix notation, H satisfies

$$1_n = K_{vv}^T K_{p\bar{p}} - K_{pv}^T K_{v\bar{p}}, \tag{6.3.4}$$

$$K_{p\bar{p}}^T K_{v\bar{q}} = K_{v\bar{p}}^T K_{p\bar{p}} \tag{6.3.5}$$

$$K_{vv}^T K_{pv} = K_{pv}^T K_{vv}. \tag{6.3.6}$$

After a Legendre transform F of K with respect to the real variables v^j, the equations above reduce to

$$F_{VV} + F_{p\bar{p}} = 0 \tag{6.3.7}$$

$$F_{Vp} \text{ is symmetric.} \tag{6.3.8}$$

Case 1 is the very well known Hitchin-Karlhede-Lindström-Roček result [3].

As for case 2, dimension 4 (i.e. $n = 1$) was already considered by Boyer and Finley. We assume that $n > 1$. After possibly rotating the hyperkähler structure, there exist special coordinates q^j, p^j and $\bar{q}^j$, $\bar{p}^j$, for j going from 1 to n, such that

$$\omega_+ = \omega_2 + i\omega_3 = \sum_{j=1}^{n} dq^j \wedge dp^j, \tag{6.3.9}$$

and moreover,

$$X^{\hat{j}} = \partial_{q^{\hat{j}}} + \partial_{\bar{q}^{\hat{j}}} \quad (1 \leq \hat{j} \leq n-1) \tag{6.3.10}$$

$$X^n = 2i(\sum_{k=1}^{n} p^k \partial_{p^k} - \sum_{k=1}^{n} \bar{p}^k \partial_{\bar{p}^k}) + \partial^A \eta \partial_A + \bar{\partial}^{\bar{A}} \bar{\eta} \bar{\partial}_{\bar{A}}, \tag{6.3.11}$$

where $\eta = \eta(q,p)$ is a holomorphic function satisfying

$$[X^{\hat{j}}, \partial^A \eta \partial_A] = 0. \tag{6.3.12}$$

In addition, the local Kähler potential Ω can be chosen such that $X^j(\Omega) = 0$, for j going from 1 to n. Then, in addition to the equations $X^j(\Omega) = 0$, for $1 \leq j \leq n$, Ω satisfies using matrix notation the symplectic Monge-Ampere equations:

$$1_n = \Omega^T_{q\bar{q}} \Omega_{p\bar{p}} - \Omega^T_{p\bar{q}} \Omega_{q\bar{p}} \tag{6.3.13}$$

$$\Omega^T_{q\bar{q}} \Omega_{p\bar{q}} = \Omega^T_{p\bar{q}} \Omega_{q\bar{q}}, \tag{6.3.14}$$

$$\Omega^T_{p\bar{p}} \Omega_{q\bar{p}} = \Omega^T_{q\bar{p}} \Omega_{p\bar{p}} \tag{6.3.15}$$

Case 2 is a (partial) generalization of the Boyer and Finley [11] work in dimension 4 (we say "partial" because it might be possible to simplify the equations further using a Legendre transform perhaps).

Proof. We remark first that a quaternionic Killing field X which satisfies $(\nabla X)_{S^2H} = 0$ has $b = c = 0$ and takes the form

$$X = \partial^A \zeta \partial_A + \text{ c.c.},$$

in special local coordinates q^A and $\bar{q}^{\bar{B}}$ (with A and $\bar{B}$ going from 1 to $2n$), where $\zeta = \zeta(q)$ (c.c. is an abbreviation for complex conjugate). Thus we see that such a quaternionic Killing field X is triholomorphic.

We consider first case 1. We choose any special local coordinates q^A, $\bar{q}^{\bar{B}}$ at first (say the q's are holomorphic for I). Then there are holomorphic functions $\zeta^j = \zeta^j(q)$ such that

$$X^j = \partial^A \zeta^j \partial_A + \text{ c.c.}. \tag{6.3.16}$$

We make two additional assumptions. First, the vector fields X^1, IX^1, JX^1, KX^1,...,X^n, IX^n, JX^n and KX^n are assumed to be pointwise linearly independent (so that they form a local frame). Second,

$$\omega_\alpha(X^j, X^k) = 0,$$

for $1 \le j, k \le n$ and $1 \le \alpha \le 3$.

We then define some new coordinates, $p^j = \zeta^j$. By the Carathéodory-Jacobi-Lie extension of the Darboux symplectic lemma, there are n additional holomorphic coordinates (for I) q^j, with j going from 1 to n such that

$$\omega_+ = \omega_2 + i\omega_3 = \sum_{j=1}^{n} dq^j \wedge dp^j. \tag{6.3.17}$$

In these coordinates q^j,p^j and their c.c., we have locally

$$X^j = \partial_{q^j} + \text{ c.c.}. \tag{6.3.18}$$

We then consider the master equations

$$X^j(\Omega) = H^j + \bar{H}^j.$$

We would like to absorb the H^j's and their complex conjugates in Ω. We can do this by a proper choice of A, i.e. by replacing Ω by

$$\Omega' = \Omega + A + \bar{A},$$

where A is holomorphic and satisfies

$$A_{q^j} = H^j.$$

There exists such an A locally provided

$$(H^j)_{q^k} = (H^k)_{q^j},$$

which is garanteed by the fact that

$$[X^j, X^k] = 0.$$

Dropping the prime in Ω', we have shown that by a proper choice of local special coordinates and of Ω, the master equations can be written simply as

$$X^j(\Omega) = (\partial_{q^j} + \partial_{\bar{q}^j})(\Omega) = 0, \qquad (6.3.19)$$

for j going from 1 to n.

We then consider the SMA equations:

$$1_n = \Omega_{q\bar{q}}^T \Omega_{p\bar{p}} - \Omega_{p\bar{q}}^T \Omega_{q\bar{p}} \tag{6.3.20}$$

$$\Omega_{q\bar{q}}^T \Omega_{p\bar{q}} = \Omega_{p\bar{q}}^T \Omega_{q\bar{q}}, \tag{6.3.21}$$

$$\Omega_{p\bar{p}}^T \Omega_{q\bar{p}} = \Omega_{q\bar{p}}^T \Omega_{p\bar{p}}, \tag{6.3.22}$$

which reduce to

$$1_n = \Omega_{vv}^T \Omega_{p\bar{p}} - \Omega_{pv}^T \Omega_{v\bar{p}}, \tag{6.3.23}$$

$$\Omega_{p\bar{p}}^T \Omega_{v\bar{q}} = \Omega_{v\bar{p}}^T \Omega_{p\bar{p}} \tag{6.3.24}$$

$$\Omega_{vv}^T \Omega_{pv} = \Omega_{pv}^T \Omega_{vv}, \tag{6.3.25}$$

using $u^j = q^j + \bar{q}^j$ and $v^j = i(\bar{q}^j - q^j)$. Indeed, the master equations for instance become simply $\Omega_{u^j} = 0$, in these new coordinates.

Next, we make a Legendre transform

$$F = \sum_{j=1}^{n} (v^j V^j) - K, \tag{6.3.26}$$

where $V^j = K_{v^j}$, we arrive at the Hitchin-Karlhede-Lindström-Roček ansatz for a hyperkähler $4n$-manifold M with n commuting triholomorphic vector fields X^1,...,X^n, namely

$$F_{VV} + F_{p\bar{p}} = 0 \tag{6.3.27}$$

$$F_{Vp} \text{ is symmetric.} \tag{6.3.28}$$

This finishes the study of case 1, which is well known [3].

We now move on to case 2. Let q^A, $\bar{q}^{\bar{A}}$ be local special coordinates, with A, $\bar{A}$ going from 1 to $2n$. Then we have that

$$\omega_+ = \epsilon_{AB} dq^A \wedge dq^B,$$

and, since the $X^{\hat{j}}$'s are triholomorphic ($1 \leq \hat{j} \leq n-1$), and X^n is quaternionic Killing with $(\nabla X^n)_{S^2H}$ nonzero, it follows that there exist holomorphic functions $\zeta^k = \zeta^k(q)$, for $1 \leq k \leq n$, such that

$$X^{\hat{j}} = \partial^A \zeta^{\hat{j}} \partial_A + \text{c.c.}, \tag{6.3.29}$$

$$X^n = (b\partial^A \Omega - i\frac{c}{2} q^A + \partial^A \zeta^n)\partial_A + \text{c.c.}. \tag{6.3.30}$$

If $b = 0$ we can rotate the hyperkähler structure to have $b' \neq 0$. So without loss of generality, we assume that $b \neq 0$. Then we make the following Ω transformation to absorb ζ^n and its complex conjugate

$$\Omega \mapsto \Omega' = \Omega + \frac{\zeta^n}{b} + \frac{\bar{\zeta}^n}{\bar{b}}. \tag{6.3.31}$$

Then, dropping the prime on Ω, we have that

$$X^n = (b\partial^A \Omega - i\frac{c}{2} q^A)\partial_A + \text{c.c.}. \tag{6.3.32}$$

We then rotate the hyperkähler structure to have $b' = 0$, and scale X^n to normalize c. We then obtain that there exist some holomorphic functions $\eta^{\hat{j}} = \eta^{\hat{j}}(q')$ ($1 \leq \hat{j} \leq n-1$) such that, in the new special local coordinates q'^A

and $\bar{q}'^{\bar{A}}$, we have, after dropping the primes

$$X^{\hat{j}} = \partial^A \eta^{\hat{j}} \partial_A + \text{ c.c.}, \tag{6.3.33}$$

$$X^n = i(q^A \partial_A - \bar{q}^{\bar{A}} \bar{\partial}_{\bar{A}}). \tag{6.3.34}$$

We digress a bit to consider dimension 4, i.e. $n = 1$. In that case, the local expression for X^n can be further simplified. We introduce first

$$q^A = \frac{1}{\sqrt{2}}(q, p),$$

so that

$$\epsilon_{AB} dq^A \wedge dq^B = dq \wedge dp.$$

We then introduce some new coordinates q', p' by

$$q' = qp \tag{6.3.35}$$

$$p' = \frac{p}{q}. \tag{6.3.36}$$

We then have

$$q^A \partial_A = 2q' \partial_{q'}. \tag{6.3.37}$$

After scaling $X = X^1$ by $\frac{1}{2}$, we get

$$X = i(q\partial_q - \bar{q}\partial_{\bar{q}}). \tag{6.3.38}$$

This is the simplest form for such a Killing field in dimension 4, as shown in [11]. Continuing along this line of thought ultimately leads to the Boyer and

Finley equation (see [11] or section 5).

We then go back to case 2 and assume from now on that the dimension is $4n$ with $n > 1$. Consider once more equations 6.3.33. There are $n-1$ holomorphic functions involved in the local expressions of the X^k's, namely the $\zeta^{\hat{j}}$'s. However we can do better than that.

We make the assumption that X^1, IX^1, JX^1, KX^1,...,IX^{n-1}, JX^{n-1}, KX^{n-1} are pointwise linearly independent, and that

$$\omega_\alpha(X^{\hat{j}}, X^{\hat{k}}) = 0,$$

for $1 \leq \alpha \leq 3$ and $1 \leq \hat{j}, \hat{k} \leq n-1$.

We introduce

$$p'^{\hat{j}} = \eta^{\hat{j}},$$

and then, by the Carathéodory-Jacobi-Lie extension of the Darboux symplectic lemma, whose hypotheses are satisfied by our assumption above, we can find a complete set of holomorphic coordinates for I_1, namely q'^k, p'^k, for k going from 1 to n, such that, dropping the primes, we have

$$X^{\hat{j}} = \partial_{q^j} + \partial_{\bar{q}^j}, \tag{6.3.39}$$

$$X^n = iq^A\partial_A + \partial^A\eta\partial_A + \text{c.c.}. \tag{6.3.40}$$

where η is some holomorphic function and

$$q^A = \frac{1}{\sqrt{2}}(q^1, \ldots, q^n, p^1, \ldots, p^n).$$

Hence, we can also write

$$X^n = 2ip^k\partial_{p^k} + \partial^A\eta'\partial_A + \text{c.c.}, \tag{6.3.41}$$

where $\eta' = \eta - i\sum_k q^k p^k$ is holomorphic and, moreover,

$$[X^{\hat{\jmath}}, \partial^A\eta'\partial_A] = 0. \tag{6.3.42}$$

The master equations are

$$X^{\hat{\jmath}}(\Omega) = H^{\hat{\jmath}} + \bar{H}^{\hat{\jmath}} \tag{6.3.43}$$

$$X^n(\Omega) = H^n + \bar{H}^n. \tag{6.3.44}$$

In order to simplify the master equations, we make use of the Ω transformation

$$\Omega \mapsto \Omega' + A + \bar{A},$$

where A satisfies

$$X^{\hat{\jmath}}(A) = H^{\hat{\jmath}} \tag{6.3.45}$$

$$X^n(A) = H^n. \tag{6.3.46}$$

A solution A of the above system exists locally provided

$$X^{\hat{\jmath}}H^{\hat{k}} = X^{\hat{\jmath}}H^{\hat{k}} \tag{6.3.47}$$

$$X^n H^{\hat{\jmath}} = X^{\hat{\jmath}}H^n, \tag{6.3.48}$$

which follows from the assumption that the X^k's are commuting.

The symplectic Monge-Ampere equations $P_{A\bar{U}} {P_B}^{\bar{U}} = \epsilon_{AB}$, can be written explicitly in our coordinates q^j, p^j and $\bar{q}^j$, $\bar{p}^j$ as

$$1_n = \Omega_{q\bar{q}}^T \Omega_{p\bar{p}} - \Omega_{p\bar{q}}^T \Omega_{q\bar{p}} \tag{6.3.49}$$

$$\Omega_{q\bar{q}}^T \Omega_{p\bar{q}} = \Omega_{p\bar{q}}^T \Omega_{q\bar{q}}, \tag{6.3.50}$$

$$\Omega_{p\bar{p}}^T \Omega_{q\bar{p}} = \Omega_{q\bar{p}}^T \Omega_{p\bar{p}}, \tag{6.3.51}$$

with the master equations $X^k(\Omega) = 0$, for k going from 1 to n. This finishes the proof of the theorem. $\square$

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Appendix A

Solving the equation $P^{E(\bar{A}}\bar{\partial}^{\bar{B})}L_E = 0$

In this appendix, we solve

$$P^{E(\bar{A}}\bar{\partial}^{\bar{B})}L_E = 0 \tag{A.0.1}$$

on a hyperkähler $4n$-manifold M. This equation comes from the Killing equation in special local coordinates q^A, $\bar{q}^{\bar{A}}$ (A and $\bar{A}$ going from 1 to $2n$).

On one hand, we have

$$2P_F{}^{\bar{A}}P^E{}_{(\bar{A}}\bar{\partial}_{\bar{B})}L_E = -\bar{\partial}_{\bar{B}}L_F + P_F{}^{\bar{A}}P^E{}_{\bar{B}}\bar{\partial}_{\bar{A}}L_E. \tag{A.0.2}$$

On the other hand, we have

$$2P^E{}_{\bar{B}}P_{(E|}{}^{\bar{A}}\bar{\partial}_{\bar{A}}L_{|F)} = -\bar{\partial}_{\bar{B}}L_F + P_F{}^{\bar{A}}P^E{}_{\bar{B}}\bar{\partial}_{\bar{A}}L_E. \tag{A.0.3}$$

Hence, we conclude that on a hyperkähler $4n$-manifold M, the equation A.0.1 is equivalent to the following equation

$$P_{(E|}{}^{\bar{A}}\bar{\partial}_{\bar{A}}L_{|F)} = 0. \tag{A.0.4}$$

Then we prove the following lemma

Lemma A.0.2. *The following Lie bracket vanishes*

$$[P_A{}^{\bar{C}}\bar{\partial}_{\bar{C}}, P_B{}^{\bar{D}}\bar{\partial}_{\bar{D}}] = 0. \tag{A.0.5}$$

Proof.

$$
\begin{aligned}
&(P_{[A|}{}^{\bar{C}}\bar{\partial}_{\bar{C}}P_{|B]}{}^{\bar{D}})\bar{\partial}_{\bar{D}} \\
&= (P_{[B|}{}^{\bar{C}}\bar{\partial}_{\bar{D}}P_{|A]\bar{C}})\bar{\partial}^{\bar{D}} \\
&= -(P_{[A|\bar{C}}\bar{\partial}_{\bar{D}}P_{|B]}{}^{\bar{C}})\bar{\partial}^{\bar{D}} \\
&= (P_{[A|}{}^{\bar{C}}\bar{\partial}_{\bar{C}}P_{|B]\bar{D}})\bar{\partial}^{\bar{D}} \\
&= -(P_{[A|}{}^{\bar{C}}\bar{\partial}_{\bar{C}}P_{|B]}{}^{\bar{D}})\bar{\partial}_{\bar{D}},
\end{aligned}
$$

from which the lemma follows. □

Now consider

$$P_A{}^{\bar{E}}\bar{\partial}_{\bar{E}}(P_B{}^{\bar{F}}\bar{\partial}_{\bar{F}}L_C). \tag{A.0.6}$$

It is skew in B and C from equation A.0.4 and symmetric in A and B from the previous lemma, hence it vanishes, so that we have proved that equation A.0.1 (which is equivalent to A.0.4) implies that

$$P_A{}^{\bar{E}}\bar{\partial}_{\bar{E}}(P_B{}^{\bar{F}}\bar{\partial}_{\bar{F}}L_C) = 0. \tag{A.0.7}$$

This implies that

$$P_B{}^{\bar{F}}\bar{\partial}_{\bar{F}}L_C = E_{BC}, \tag{A.0.8}$$

where E_{BC} is holomorphic and skew in B and C. This implies that

$$P^B{}_{\bar{H}}\, E_{BC} = -\bar{\partial}_{\bar{H}}L_C,$$

which can be rewritten as

$$\bar{\partial}_{\bar{H}}(\partial^B\Omega E_{BC}) = -\bar{\partial}_{\bar{H}}L_C.$$

Hence we conclude that L^A is of the form

$$L^A = E^A{}_B\partial^B\Omega + J^A, \tag{A.0.9}$$

where E_{AB} is holomorphic and skew in A and B, while J^A is holomorphic. Finally, plugging in the expression A.0.10 for L^A back in A.0.1 shows that it is indeed a solution.

We have proved:

Theorem A.0.3. *The general solution to A.0.1 is given by*

$$L^A = E^A{}_B\partial^B\Omega + J^A, \tag{A.0.10}$$

where E_{AB} is holomorphic and skew in A and B, while J^A is holomorphic.