Teichmüller Space of a Once Punctured Disk: Complex Coordinates on the Space of Abelian Differentials and the Takhtajan-Zograf Metric

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Abstract of the Dissertation

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We construct an embedding of $T(\mathbb{Z})$, the Teichmüller space of $\mathbb{Z}$—a Fuchsian group with a single parabolic generator—into the space of Nehari bounded abelian differentials on the Riemann surface $M = \mathbb{Z}\backslash\mathbb{H}$. This "pre-Bers embeddig" is much simpler, and yet it still draws many parallels with the Bers embedding, such as the existence of a linear local right inverse. This gives rise to new complex coordinates which are compatible with the Bers coordinates. The differentials of the pre-Bers and the Bers embedding belong to a one parameter family of operators on the space of Beltrami differentials on M, which behave nicely with the Takhtajan-Zograf metric on $T(\mathbb{Z})$.

Contents

List of Symbols

$[\mu]$	Equivalence class of μ, $\Phi(\mu) = [\mu]$, page 6
β	Bers embedding, $\beta([\mu]) = S(w^{\mu})$, page 7
$\dot{w}[\mu]$	$\frac{d}{dt}(w_{t\mu})_{t=0}$, page 20
η	$\eta(\phi)(z) = (z - \overline{z})\phi(\overline{z})$, page 20
$\mathbb{Z}$	Subgroup of $PSL(2, \mathbb{R})$ generated by $z \to z + 1$, page 9
$\Gamma(s)$	Gamma function, $\Gamma(s) = \int_0^\infty y^s e^{-y} dy$, page 28
$\langle \mu, \nu \rangle$	Takhtajan-Zograf pairing, $\int_\sqcap \mu\overline{\nu} d^2z$, page 28
Λ	Ahlfors-Weill local section, $\Lambda(\phi)(z) = -\frac{1}{2}\rho^{-1}(z)\phi(\overline{z})$, page 7
$\mathcal{N}(\mathbb{H})$	Infinitesimally trivial Beltrami differentials , page 7
ρ	Hyperbolic metric, $\rho(z) = -(z - \overline{z})^{-2}$, page 5
$\mathbb{H}$	Upper-half plane, page 5
$\mathbb{L}$	Lower-half plane, page 5
$\mathcal{D}$	Univalent functions on $\mathbb{L}$ with quasiconformal extensions to $\mathbb{C}$, fixing $0, 1, \infty$, page 13
$\mathcal{D}(\mathbb{Z})$	$f \in \mathcal{D}$ such that $f(z + 1) = f(z) + 1$, page 13
$\Omega^{-1,1}(\mathbb{H})$	Bounded harmonic Beltrami differentials, $\frac{1}{2}(z - \overline{z})^2\phi(\overline{z})$, page 7
$\Phi[\mu]$	$\Phi[\mu](z) = \frac{1}{2}(\dot{w}[\mu] + i\dot{w}[i\mu])$, page 20
Ψ	$\Psi(\phi) = \phi' - \frac{1}{2}\phi^2$, page 10
$\sqcap$	Fundamental domain of $\mathbb{Z}$, $[0, 1] \times [0, \infty)$, page 28

Θ Lift of the Pre-Bers Embedding, $\Theta(\mu) = \theta([\mu])$, page 10

θ Pre-Bers Embedding, $\theta([\mu]) = \frac{(w^\mu)''}{(w^\mu)'}$, page 10

Φ Quotient map from $L^\infty(\mathbb{H}, G)_1 \to T(G)$, page 6

Φ_β Lift of the Bers embedding, page 7

$A^1_\infty(\mathbb{L})$ Nehari bounded abelian differentials, $(z - \overline{z})\phi(\overline{z}) \in L^\infty(\mathbb{H})$, page 10

$A^1_\infty(\mathbb{L})_r$ Ball of radius r centered at 0 in $A^1_\infty(\mathbb{L})$, page 10

e_n $e_n(z) = e^{2\pi i n \overline{z}}$, page 28

$F[\mu]$ $F[\mu](z) = \frac{1}{2}(\dot{w}[\mu] - i\dot{w}[i\mu])$, page 20

$H^{-1,1}(\mathbb{H})$ Pre-Bers harmonic differentials, $(z - \overline{z})\phi(\overline{z}) \in L^\infty(\mathbb{H})$, page 20

$Hol(D)$ Holomorphic functions on D, page 7

$L^\infty(D)$ Bounded measurable functions on D, page 5

$L^\infty(D)_r$ Ball of radius r centered at 0 in $L^\infty(D)$, page 5

N Pre-Bers projection, $N\mu(z) = -\frac{2}{\pi}(z - \overline{z}) \int_{\mathbb{H}} \frac{\mu(w)}{(w-\overline{z})^3} d^2w$, page 24

N_s $g_s y^{-s} P(y^s P)$, page 28

P Bers projection, $(P\mu)(z) = -\frac{3(z-\overline{z})^2}{\pi} \int_{\mathbb{H}} \frac{\mu(u)}{(u-\overline{z})^4} d^2u$, page 7

$R(\nu, \mu)$ Differential of $(R_\mu)^{-1}$ at μ in the direction of ν, page 8

R_μ Right composition by w_μ, page 8

S Schwarzian derivative, $S(f) = \left(\frac{f''}{f'}\right)' - \frac{1}{2}\left(\frac{f''}{f'}\right)^2$, page 7

$T(G)$ Teichmüller Space of group G, page 6

w^μ Solution to the Beltrami equation extended by 0, fixing 0,1,∞, page 6

w_μ Solution to the Beltrami equation extended by reflection, fixing 0,1,∞, page 6

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Chapter 0

Introduction

0.1 History and Background

In this dissertation, we define $T(G)$—the Teichmüller space of a Fuchsian group G—to be a quotient of the unit ball centered at 0 in the Banach space of bounded measurable $(-1,1)$-differentials on $\mathbb{H}$. We are particularly interested in the case where $G = \mathbb{Z}$, the group generated by $z \mapsto z+1$. Since $\mathbb{Z}\backslash\mathbb{H} \simeq \mathbb{D} - \{0\}$, we say that $T(\mathbb{Z})$ is the Teichmüller space of a once punctured disk. We are also interested in the case where $G = \{1\}$, the trivial group. $T(1)$ is called the universal Teichmüller space. This space is universal in the sense that every Teichmüller space $T(G)$ embeds complex analytically into $T(1)$.

The identification of $T(1)$ with a subgroup of quasisymmetric maps on $\mathbb{R}$ gives $T(1)$ a group structure via composition of quasisymmetric mappings. The right multiplication is a biholomorphic map, but the left muliplication is not even continuous (c.f. [6]). While this does not induce a group structure on $T(G)$ for a general G, it is shown in [13] that a suitable choice of normalization makes $T(\mathbb{Z})$ a subgroup of $T(1)$

Interpreting $T(G)$ as a moduli space of marked surfaces, one defines the Bers fiber space over $T(G)$ as the universal cover of the tautological fiber bundle $\mathcal{T}(G)$ over $T(G)$. For $G = \{1\}$, $F(1) = \mathcal{T}(1)$, and this space is known as the universal Teichmüller curve. $\mathcal{T}(1)$ was extensively studied by Teo in [12], where it is shown that

1. There is a "pre-Bers embedding" of $\mathcal{T}(1)$ into the space of Nehari bounded abelian differentials on $\mathbb{D}$, which induces the usual complex structure.

2. Conformal welding gives an identification of $\mathcal{T}(1)$ with a subgroup of quasisymmetric mappings of S^1 fixing 1. The right multiplication on the induced group structure induced on $\mathcal{T}(1)$ is holomorphic.

3. Velling's second variation of spherical areas defines a unique right invariant Kähler metric on $\mathcal{T}(1)$, which pulls back to the Kirillov metric on
$$S^1\backslash \mathit{Diff}^+(S^1) \hookrightarrow S^1\backslash Homeo_{qs}(S^1) \simeq \mathcal{T}(1).$$

In [5], Bers proved that if G and $\dot{G}$ are such that $G\backslash\mathbb{H}$ and $(\dot{G}\backslash\mathbb{H}) - \{a\}$ are conformally equivalent, then there is a biholomorphism between $T(G)$ and $F(\dot{G})$. In particular, this gives an isomorphism between $T(\mathbb{Z})$ and $\mathcal{T}(1)$.

The Bers isomorphism between $T(\mathbb{Z})$ and $\mathcal{T}(1)$ gives a very interesting phenomenon in which $T(\mathbb{Z})$, a complex embedded submanifold of $T(1)$, is at the same time the total space of a complex fiber bundle over $T(1)$.

$$\begin{array}{ccc} T(\mathbb{Z}) & \simeq & \mathcal{T}(1) \\ & & \downarrow \\ T(\mathbb{Z}) & \hookrightarrow & T(1) \end{array}$$

To be more precise, let us denote by $T_D(G)$ the Teichmüller space modeled on the domain D, and similarly for $\mathcal{T}_D(G)$. Let $p : \mathbb{H} \to \mathbb{D} \setminus \{0\}$, where $p(z) = e^{2\pi i z}$. This induces the isomorphism between $T_{\mathbb{H}}(\mathbb{Z})$ and $\mathcal{T}_{\mathbb{D}}(1)$, so the diagram becomes

$$\begin{array}{ccc} T_{\mathbb{H}}(\mathbb{Z}) & \simeq & \mathcal{T}_{\mathbb{D}}(1) \\ & & \downarrow \\ T_{\mathbb{D}}(\Gamma_0) & \hookrightarrow & T_{\mathbb{D}}(1) \end{array} \qquad \begin{array}{ccc} [0] \neq [w\frac{\overline{k(p(z))}}{k(p(z))}] & \longleftarrow\!\shortmid & ([0], w) \\ & & \downarrow \\ [0] & \mapsto & [0] \end{array}$$

where Γ_0 is a subgroup of $PSU(1,1)$ generated by a single parabolic element. The isomorphism between $T_{\mathbb{H}}(\mathbb{Z})$ and $T_{\mathbb{D}}(\Gamma_0)$ can be characterized as a pull-back on the Beltrami differentials by a Möbius transformation mapping $\mathbb{Z}$ to Γ_0 (which in turn, would map $\mathbb{H}$ to $\mathbb{D}$). Under the Bers isomorphism,

$$\mathcal{T}_{\mathbb{D}}(1) \ni ([0], w) \mapsto [w\frac{\overline{k(p(z))}}{k(p(z))}] \in T_{\mathbb{H}}(\mathbb{Z})$$

where $k(z) = z/(1+z)^2$, a rotation of the Koebe function.

Teo studied the case $G = \mathbb{Z}$ and $\dot{G} = \{1\}$ in [13], where it is shown that the Bers isomorphism in this case is actually a group isomorphism, and the pull-back of the Velling-Kirillov metric on $\mathcal{T}(1)$ to $T(\mathbb{Z})$ at $[0]$ is given by

$$\langle \mu, \nu \rangle_{TZ} = \int_{\sqcap} \mu(z)\overline{\nu}(z) d^2 z. \tag{1}$$

Here $\mu, \nu \in \Omega^{-1,1}(\mathbb{H}, \mathbb{Z}) \simeq T_{[0]}T(\mathbb{Z})$, and $\sqcap$ is the vertical strip $[0,1] \times [0, \infty)$.

This form of the Velling-Kirillov metric has a striking resemblance to a metric discovered by Takhtajan and Zograf in [10] for $T_{g,n}$, the Teichmüller space of finite conformal type (g, n). The Takhtajan-Zograf metric on $T_{g,n}$ is given by

$$\langle \mu, \nu \rangle_{TZ} = \int_X \mu(z)\overline{\nu}(z)E(z,2)\rho(z)d^2z. \tag{2}$$

Here $\mu, \nu \in \Omega^{-1,1}(X) \simeq T_{[X]}T_{g,n}$, ρ is the hyperbolic metric, and $E(z,s)$ is the Eisenstein-Maass series for the uniformizing Fuchsian group G (i.e. $X \simeq G\backslash\mathbb{H}$). One can think of the Eisenstein-Maass series for $\mathbb{Z}$ to be $E(z,s) = (\text{Im } z)^s$. Substituting this expression for E into (2) gives (1). Takhtajan and Zograf showed that (2) is a Kähler metric. Likewise, Teo's work in [13] shows that (1) defines a (unique) right invariant Kähler metric on $T(\mathbb{Z})$.

The properties of Takhtajan-Zograf metric in the finite conformal case has been studied by Obitsu, who showed that it was incomplete in [8]. He also studied its asymptotic behaviors in relation to the Weil-Petersson metric through the investigation of the properties of the Eisenstein-Maass series in [9]. It was conjectured by Obitsu in [8] that the metric has negative sectional curvature, but currently nothing is known.

0.2 This Dissertation

Here, we briefly discuss the organization and the content of this dissertation.

Chapter 1 is a review already known results. The main purpose of this chapter is to review some basic Teichmüller theory and to fix notations.

Chapter 2 introduces a "pre-Bers embedding" of $T(\mathbb{Z})$ into the space of abelian differentials. By identifiying $T(\mathbb{Z})$ with a normalized subspace of conformal mappings of $\mathbb{L}$, we define the map θ as the logarithmic derivative of the derivative. This map is injective on $T(\mathbb{Z})$, and its image lies in the Banach space of Nehari bounded holomorphic 1-differentials of norm less than 6. The image of θ contains a ball of some radius α centered at 0. We will also show that this map is holomorphic and compute its derivative.

Chapter 3 uses the embedding from chapter 2 to construct a coordinate chart on $T(\mathbb{Z})$. We start by constructing a linear local right inverse from a small ball in $A^1_\infty(\mathbb{L}, \mathbb{Z})$ into the subspace $H^{-1,1}(\mathbb{H}, \mathbb{Z})$ of pre-Bers harmonic Beltrami differentials, similiar to the Ahlfors-Weill Local section for the Bers embedding. Using this right inverse, we construct complex coordinate charts on $T(\mathbb{Z})$ based on $A^1_\infty(\mathbb{L},\mathbb{Z})$ and verify its compatibility with the complex structure induced by the Bers embedding. We end the chapter with some variational formulas for $H^{-1,1}(\mathbb{H},\mathbb{Z})$ which closely parallel those of $\Omega^{-1,1}(\mathbb{H})$.

Chapter 4 introduces a 1-parameter family of projection operators on $L^\infty(\mathbb{H},\mathbb{Z})$ of the form

$$N_s = c_s E(*,-s)PE(*,s)P,$$

where s is a parameter, c_s a normalizing constant, P the Bers projection, and $E(z,s) = (\mathrm{Im}\ z)^s$, which may be thought of as the Eisenstein-Maass series for the group $\mathbb{Z}$. We show that $N_sN_t = N_s$ and $\langle N_s\mu, N_t\nu\rangle = \langle N_k\mu, N_l\nu\rangle$ up to a multiplicative constant that only depends on the parameters. We pay a special attention to N_2, which has the property $\langle N_2\mu, N_2\nu\rangle = \langle N_2\mu, \nu\rangle$ as P does with respect to the Weil-Petersson pairing.

Chapter 5 uses the tools from preceding chapters to study the properties of Takhtajan-Zograf metric on $T(\mathbb{Z})$. The material in this chapter is work that is still in progress and will need further investigation. The main conjecture is an expression for the first derivative of the Takhtajan-Zograf metric in the coordinates discussed in Chapter 3:

$$\frac{\partial g_{\mu\bar{\nu}}}{\partial \kappa}(0) = \int_{\sqcap} N_2\mu(z)\overline{\nu(z)}N_2\kappa(z)d^2z, \tag{3}$$

where $\mu,\nu,\kappa \in H^{-1,1}(\mathbb{H},\mathbb{Z})$. This expression, if valid, gives an indepedent proof of the fact that Takhtajan-Zograf metric on $T(\mathbb{Z})$ is Kähler.

Chapter 1

Preliminary Teichmüller Theory

Here we present some necessary facts from Teichmüller theory. For details, see [3, 6, 7].

1.1 Definition of $T(G)$.

Let $\mathbb{H} = \{z \in \mathbb{C} : \text{Im } z > 0\}$, $\mathbb{L} = \{z \in \mathbb{C} : \text{Im } z < 0\}$, and $\rho(z) = -(z-\overline{z})^{-2}$ denote the hyperbolic metric of constant curvature -4 on $\mathbb{H}$ (or $\mathbb{L}$). The subscripts z and $\overline{z}$ will always stand for the partial derivatives $\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)$ and $\frac{\partial}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$, unless otherwise stated.

Let G be a torsion free discrete subgroup of $PSL(2,\mathbb{R})$. A measurable function $\sigma : D \to \mathbb{C}$ on a domain D is said to be a (q,r)-form if $\sigma(z) = \sigma(g(z))g'(z)^q\overline{g'(z)}^r$ for all $g \in G$. The $(-1,1)$-forms are the Beltrami differentials. If σ is a Beltrami differential, then $|\sigma|$ is a measurable real valued function. We denote by $L^\infty_{(-1,1)}(D,G)$ the complex Banach space of Beltrami differentials on D with the norm $||\sigma||_\infty = ess\sup|\sigma|$. For $r > 0$, we let $L^\infty_{(-1,1)}(D,G)_r = \{\mu \in L^\infty_{(-1,1)}(D,G) : ||\mu||_\infty < r\}$. In general, for any normed vector space V and $r > 0$, we denote by V_r the elements of V of norm less than r.

We give two definitions of $T(G)$.

Given $\mu \in L^\infty_{(-1,1)}(\mathbb{H},G)_1$ we define an extension $\hat{\mu}$ to $\mathbb{C}$ via reflection:

$$\hat{\mu}(z) = \begin{cases} \mu(z), & \text{Im } z > 0 \\ \overline{\mu(\overline{z})}, & \text{Im } z < 0, \end{cases} \tag{1.1}$$

so that

$$\hat{\mu}(z) = \overline{\hat{\mu}(\overline{z})}. \tag{1.2}$$

Consider the unique quasiconformal mapping of the (extended) plane solving the Beltrami equation

$$f_{\overline{z}} = \hat{\mu} f_z \tag{1.3}$$

which fixes $0, 1$ and ∞. We will denote the solution to (1.3) by w_μ.

Due to the reflection symmetry (1.2), w_μ satisfies

$$w_\mu(z) = \overline{w_\mu(\overline{z})}. \tag{1.4}$$

It follows from (1.4) that w_μ preservers $\mathbb{R}$. Since w_μ is orientation preserving, the domains $\mathbb{H}$, $\mathbb{L}$ are preserved as well.

We could also extend μ by

$$\check{\mu}(z) = \begin{cases} \mu(z), & \operatorname{Im} z > 0 \\ 0, & \operatorname{Im} z < 0. \end{cases} \tag{1.5}$$

Again, consider the unique quasiconformal mapping of the (extended) plane solving the Beltrami equation

$$f_{\overline{z}} = \check{\mu} f_z \tag{1.6}$$

which fixes $0, 1$ and ∞. We will denote the solution by w^μ. Note that w^μ is conformal on $\mathbb{L}$.

If $f = w_\mu$ (or w^μ), the condition $\mu = \mu \circ g \frac{\overline{g'}}{g'}$ for $g \in G$ implies that both $f \circ g$ and f are solutions to (1.3) (or (1.6)). By uniqueness, $f \circ g \circ f^{-1} \in PSL(2, \mathbb{C})$.

Definition 1.1.1 (Model A)**.** *For $\mu, \nu \in L^\infty_{(-1,1)}(\mathbb{H}, G)_1$, define an equivalence relation $\mu \sim_A \nu$ given by $w_\mu|_\mathbb{R} = w_\nu|_\mathbb{R}$. The Teichmüller space of G is defined as*

$$T(G) = L^\infty_{(-1,1)}(\mathbb{H}, G)_1 / \sim_A .$$

Definition 1.1.2 (Model B)**.** *For $\mu, \nu \in L^\infty_{(-1,1)}(\mathbb{H}, G)_1$, define an equivalence relation $\mu \sim_B \nu$ given by $w^\mu|_\mathbb{L} = w^\nu|_\mathbb{L}$. The Teichmüller space of G is defined as*

$$T(G) = L^\infty_{(-1,1)}(\mathbb{H}, G)_1 / \sim_B .$$

$w_\mu|_\mathbb{R} = w_\nu|_\mathbb{R}$ if and only if $w^\mu|_\mathbb{L} = w^\nu|_\mathbb{L}$, so these two definitions are equivalent. The set $T(G)$ is a topological space with the quotient topology induced from $L^\infty_{(-1,1)}(\mathbb{H}, G)_1$. We will denote the quotient map by Φ, and for every $\mu \in L^\infty_{(-1,1)}(\mathbb{H}, G)_1$ set $\Phi(\mu) = [\mu] \in T(G)$.

1.2 The Bers Embedding and the Ahlfors-Weill Local Section

For σ a $(p,0)$-differential, we define

$$||\sigma||_{p,\infty} = ||\rho^{-p}\sigma||_\infty \tag{1.7}$$

and $A^p_\infty(\mathbb{L}, G)$ to be the complex Banach space of holomorphic $(p,0)$-differentials with the norm given by (1.7).

Let Φ_β be defined by

$$L^\infty_{(-1,1)}(\mathbb{H}, G)_1 \ni \mu \mapsto w^\mu|_{\mathbb{L}} = f \mapsto S(f) \in A^2_\infty(\mathbb{L}, G), \tag{1.8}$$

where $S(f) = \left(\frac{f''}{f'}\right)' - \frac{1}{2}\left(\frac{f''}{f'}\right)^2$, the Schwarzian derivative.

The image of Φ_β is contained in $A^2_\infty(\mathbb{L}, G)_6$ and contains $A^2_\infty(\mathbb{L}, G)_2$. Φ_β descends to a map β to $T(G)$ known as the Bers embedding, and it induces a complex structure on $T(G)$.

The Ahlfors-Weill local section, given by

$$\Lambda(\phi)(z) = -\frac{1}{2}\rho^{-1}(z)\phi(\overline{z}) \tag{1.9}$$

is a right inverse to Φ_β on $A^2_\infty(\mathbb{L}, G)_2$. Λ maps $A^2_\infty(\mathbb{H}, G)$ into

$$\Omega^{-1,1}(\mathbb{H}, G) = \{\mu \in L^\infty |\ \mu(z) = \frac{(z-\overline{z})^2}{2}\phi(\overline{z}), \phi \in A^2_\infty(\mathbb{H}, G)\},$$

the space of harmonic Beltrami differentials.

The derivative of $\Lambda \circ \Phi_\beta : L^\infty_{(-1,1)}(\mathbb{H}, G)_1 \to \Omega^{-1,1}(\mathbb{H}, \mathbb{Z})_1$ at 0 is a projection operator on $L^\infty_{(-1,1)}(\mathbb{H}, G)$ into $\Omega^{-1,1}(\mathbb{H}, G)$, which can be written explicitly as

$$(P\mu)(z) = -\frac{3(z-\overline{z})^2}{\pi}\int_{\mathbb{H}} \frac{\mu(u)}{(u-\overline{z})^4} d^2u. \tag{1.10}$$

We will refer to P as the Bers projection.

The kernel of P is the space of infinitesimally trivial Beltrami differentials:

$$\mathcal{N}(\mathbb{H}, G) = \left\{ \mu \in L^\infty_{(-1,1)}(\mathbb{H}, G) : \iint\limits_D \mu\, \phi\, d^2z = 0 \text{ for all } \phi \in A^2_1(\mathbb{H}, G) \right\}, \tag{1.11}$$

where D is a fundamental domain of G in $\mathbb{H}$, and $A^2_1(\mathbb{H}, G)$ is the space of holomorphic $(2,0)$-differentials which are integrable over a fundamental domain (in particular, D).

The decomposition

$$L^\infty_{(-1,1)}(\mathbb{H},G)=\mathcal{N}(\mathbb{H},G)\oplus\Omega^{-1,1}(\mathbb{H},G) \tag{1.12}$$

identifies the holomorphic tangent space $T_{[0]}T(G)=L^\infty_{(-1,1)}(\mathbb{H},G)/\mathcal{N}(\mathbb{H},G)$ at the origin of $T(G)$ with the Banach space $\Omega^{-1,1}(\mathbb{H},G)$.

The Banach space $\Omega^{-1,1}(\mathbb{H},G)$ is not separable whenever $G\backslash\mathbb{H}$ has an ideal boundary.

1.3 Right Translation Map

Let $[\mu]\in T(G)$. For any representative $\mu\in[\mu]$, let $G_\mu=w_\mu\circ G\circ w_\mu^{-1}$. For any $g\in G$, we have that $g_\mu=w_\mu\circ g\circ w_\mu^{-1}$ is a Möbius transformation. Therefore, g_μ is completely determined by its restriction to $\mathbb{R}$, and so it depends only on the equivalence class of μ. Furthermore, since w_μ preserves $\mathbb{H}$ and $\mathbb{L}$, it follows that G_μ is a Fuchsian group independent of the choice of the representative μ. Let $H_\mu=w_\mu(\mathbb{H})$.

For any $\mu\in L^\infty_{(-1,1)}(\mathbb{H},G)_1$, let $R_\mu:L^\infty_{(-1,1)}(H_\mu,G_\mu)_1\to L^\infty_{(-1,1)}(\mathbb{H},G)_1$, so that if $R_\mu(\kappa)=\nu$, then $w_\kappa\circ w_\mu=w_\nu$. Equivalently,

$$\kappa=\left(\frac{\nu-\mu}{1-\bar{\mu}\nu}\,\frac{(w_\mu)_z}{(\overline{w}_\mu)_{\bar{z}}}\right)\circ w_\mu^{-1}. \tag{1.13}$$

We will also use the notation $\kappa=\nu*\mu^{-1}$. R_μ descends to a map $R_{[\mu]}:T(G_\mu)\to T(G)$ (by restricting to $\mathbb{R}$), and it is a biholomorphic map of complex manifolds. ([7]) (Similarly, one defines $R^\mu:L^\infty_{(-1,1)}(H^\mu,G^\mu)_1\to L^\infty_{(-1,1)}(\mathbb{H},G)_1$ by using w^μ instead of w_μ.)

For every $\mu\in L^\infty_{(-1,1)}(\mathbb{H},G)_1$, we have the following identification of tangent spaces:

$$T_{[\mu]}T(G)=(DR_{[\mu]})_{[0]}\left(T_{[0]}T(G_\mu)\right)\simeq T_{[0]}T(G_\mu)\simeq\Omega^{-1,1}(\mathbb{H},G_\mu). \tag{1.14}$$

Let $\nu\in L^\infty_{(-1,1)}(\mathbb{H},G)$. This defines a vector field $\frac{\partial}{\partial\nu}$ on $L^\infty_{(-1,1)}(\mathbb{H},G)_1$ by $\frac{\partial}{\partial\nu}(\mu)=\frac{d}{dt}\Big|_{t=0}(\mu+\nu t)\in T_\mu L^\infty_{(-1,1)}(\mathbb{H},G)_1$. For $\nu\in\Omega^{-1,1}(\mathbb{H},G)$, the vectorfield $\frac{\partial}{\partial\nu}$ can be pushed down to a vector field $\frac{\partial}{\partial\epsilon^\nu}$ on the open subset $U_0\subset T(G)$ corresponding to $A^2_\infty(\mathbb{L},G)_2$. Using (1.14), we represent the tangent vector $\frac{\partial}{\partial\epsilon^\nu}([\mu])\in T_{[\mu]}T(G)$ by

$$\frac{\partial}{\partial\epsilon^\nu}([\mu])=PR(\nu,\mu)\in\Omega^{-1,1}(\mathbb{H},G_\mu) \tag{1.15}$$

where

$$R(\nu,\mu) = \frac{d}{dt}\bigg|_{t=0} (\mu+\nu t) * \mu^{-1} = \left(\frac{\nu}{1-|\mu|^2}\,\frac{(w_\mu)_z}{(\overline{w}_\mu)_{\overline{z}}}\right) \circ w_\mu^{-1} \tag{1.16}$$

and μ is the unique harmonic representative of $[\mu]$.

From here on, we're exclusively going to be looking at the case where $G = \{1\}$ or $G = \mathbb{Z}$, where $\mathbb{Z} = \langle z \mapsto z+1\rangle$. Since $g' = 1$ for all $g \in G$, we will simply identify all the differentials with functions on $\mathbb{H}$ (or $\mathbb{L}$).

The following statement was originally given and proved in [13].

Lemma 1.3.1. *Let $\mu \in L^\infty(\mathbb{H},\mathbb{Z})_1$. If $G = \mathbb{Z}$, then $G_\mu = \mathbb{Z}$.*

Proof. Let $\gamma(z) = z+1$. For $\mu \in L^\infty(\mathbb{H},\mathbb{Z})_1$, we have $\gamma_\mu \in PSL(2,\mathbb{R})$. Since γ is parabolic, and w_μ fixes ∞, γ_μ is parabolic and fixes ∞. So $\gamma_\mu(z) = z + b$. Furthermore,

$$\begin{aligned}
\gamma_\mu \circ w_\mu(z) &= w_\mu(z+1)\\
\gamma_\mu \circ w_\mu(0) &= w_\mu(1)\\
\gamma_\mu(0) &= 1.
\end{aligned}$$

It follows that $\gamma_\mu(z) = z+1$, and $G_\mu = \langle z \mapsto \gamma_\mu\rangle = \langle z \mapsto z+1\rangle = \mathbb{Z}$. □

In particular, this show that $T(\mathbb{Z})$ is a subgroup of $T(1)$, and for $[\mu] \in T(\mathbb{Z})$, $R_{[\mu]}$ defines a right multiplication on $T(\mathbb{Z})$.

Chapter 2

Embedding of $T(\mathbb{Z})$ into $A^1_\infty(\mathbb{L}, \mathbb{Z})$

We define the Banach space of Nehari bounded abelian differential to be

$$A^1_\infty(\mathbb{L}, \mathbb{Z}) = \{\phi \in Hol(\mathbb{L}) : \sup |(z - \overline{z})\phi(z)| < \infty\},$$

with the norm $||\phi||_1 = \sup |(z - \overline{z})\phi(z)|$.

In this chapter, we will prove the following theorem.

Theorem 2.0.1. *Let $\Theta : L^\infty(\mathbb{H}, \mathbb{Z}) \to A^1_\infty(\mathbb{L}, \mathbb{Z})$ be given by $[\mu] \mapsto \frac{d}{dz} \log(w^\mu|'_{\mathbb{L}})$. Then,*

(a) Θ descends to an injective map on $T(\mathbb{Z})$.

(b) $\Theta(L^\infty(\mathbb{H}, \mathbb{Z})) \subset A^1_\infty(\mathbb{L}, \mathbb{Z})_6$.

(c) There is a positive number α such that $A^1_\infty(\mathbb{L}, \mathbb{Z})_\alpha \subset \Theta(L^\infty(\mathbb{H}, \mathbb{Z}))$.

(d) Θ is holomorphic, and its derivative at μ is given by

$$D_\mu \Theta(\nu)(z) = -\frac{2}{\pi}(w^\mu)'(z) \int_{\mathbb{H}} \frac{\nu(u)(w^\mu_z)(u))^2}{(w^\mu(u) - w^\mu(z))^3} d^2u, \qquad (2.1)$$

where $\nu \in L^\infty(\mathbb{H}, \mathbb{Z})$.

Throughout the rest of this dissertation, we will denote

$$\theta(h) = \frac{h_{zz}}{h_z} = \frac{d}{dz} \log h_z,$$

and

$$\Psi(\phi) = \phi_z - \frac{1}{2}\phi^2,$$

so that if S denotes the Schwarzian derivative, we have $S(h) = \Psi(\theta(h))$.

2.1 Existence and Uniqueness

In this section, we will prove that part (a) of the main theorem in fact holds for $T(1)$. The fact that Θ descends to θ on $T(1)$ is trivial, so it is sufficient to show that θ is injective. This will be done by proving an existence and uniqueness statement for θ analogous to the Schwarzian derivative.

Let $\mathcal{D}$ denote the space of univalent functions on $\mathbb{L}$ which admits a quasiconfomal extension to $\mathbb{C}$ fixing $0, 1, \infty$. The identification $T(1) \simeq \mathcal{D}$ is given by $[\mu] \mapsto w^{\mu}|_{\mathbb{L}}$.

Lemma 2.1.1. *Let ϕ be a holomorphic function on a simply connected domain A in the complex plane. Then there is a locally injective holomorphic function f in A such that*

$$\theta(f) = \phi. \tag{2.2}$$

The solution is unique up to a post composition by a Möbius transformation fixing ∞.

The uniqueness part of Lemma 2.1.1 shows that θ is injective on $\mathcal{D}$. The existence part will become important in the proof of part (c). (See section 2.3)

Note 2.1.1. *Let $\mathcal{T}(1)$ the be universal Teichmüller curve over $T(1)$. Unlike when $T(1)$ is modeled on the unit disk, the map θ is not injective as a map from $\mathcal{T}(1)$ to $A^1_\infty(\mathbb{H})$ (c.f. [12].) In fact, the map collapses the fibers of $\mathcal{T}(1)$ over $T(1)$. This can be most directly seen by identifying $\mathcal{T}(1)$ with the space of univalent function on $\mathbb{H}$ that admit a quasiconformal extension with fixed points 0 and ∞.*

Proof of Lemma 2.1.1. We rewrite (2.2) as

$$v' - \phi v = 0, \tag{2.3}$$

where $v = f'$. Let $z_0 \in A$, and let $\phi(z) = \sum_{n\geq 0} a_n(z - z_0)^n$.

If $v(z) = \sum_{n\geq 0} c_n(z - z_0)^n$ is to be a solution to (2.3), then

$$(n + 1)c_{n+1} = \sum_{k=0}^{n} a_{n-k}c_k. \tag{2.4}$$

If $c_0 = 0$, then we get that $c_n = 0$ for all n. Set $c_0 = 1$.

Let $r, M > 0$ be chosen such that $|a_n| < M/r^n$, and $rM < 1$. Such numbers necessarily exist since

$$\frac{\phi^{(n)}(z_0)}{n!} = \frac{1}{2\pi i}\int_{C_R} \frac{\phi(w)}{(w-z_0)^{n+1}}dw \tag{2.5}$$

$$|a_n| \leq \frac{M(R)}{R^n}, \tag{2.6}$$

where C_R is the circle of some fixed radius R centered at z_0, and $M(R) = \sup_{|w|=R}|\phi(z)|$. By the maximum principle, $M(r)$ is an increasing function of r, and so for $r < R$, we have $|a_n| < M(r)/r^n \leq M(R)/r^n$.

Let $M(R) = M$ and choose r sufficiently small so that $rM < 1$. Then,

$$(n+1)|c_{n+1}| \leq \sum_{k=0}^{n} \frac{M}{r^{n-k}}|c_k| < \frac{1}{r^{n+1}}\sum_{k=0}^{n}|c_k|r^k. \tag{2.7}$$

Using $c_0 = 1$, the induction hypothesis $|c_k| < 1/r^k$ for $k = 1, 2, \ldots, n$, and (2.7), we have

$$|c_{n+1}| < \frac{1}{r^{n+1}},\ n \geq 0. \tag{2.8}$$

It follows that $v(z) = \sum_{n\geq 0} c_n(z-z_0)^n$ is holomorphic on $|z - z_0| < r$, and solves the equation (2.3). Furthermore, since A is simply connected, analytic continuation and the monodromy theorem gives us a global solution. We also have that if $v(z_0) = 0$ at some point z_0, then $v \equiv 0$ on A. So it follows that either f is locally injective or constant.

Uniqueness follows directly from the relation:

$$\theta(f \circ g) = \theta(f) \circ g\ g' + \theta(g). \tag{2.9}$$

Suppose that g is a solution to the equation $\theta(g) = \phi$. If h is any other solution, we have (locally)

$$0 = \theta(h) - \theta(g) = \theta(h \circ g^{-1}) \circ g\ g'. \tag{2.10}$$

Since $\theta(h \circ g^{-1}) = 0$, it follows (for instance, from setting $\phi = 0$ in (2.3)) that $h(z) = \alpha g(z) + \beta$ for some α and β. □

2.2 Global Bound

In this section, we will prove that part (b) of the main theorem holds for $T(1)$.

We denote the space $A^k_\infty(\mathbb{L})$ the Banach space of holomorphic functions on $\mathbb{L}$ with the norm $||\phi||_k = \sup_{z\in\mathbb{L}}|(z-\overline{z})^k\phi(z)|$.

Proposition 2.2.1. *Let h be a univalent function on $\mathbb{L}$. Then,*

$$\sup_{z\in\mathbb{L}} |(Im\ z)\theta(h)(z)| \leq 3. \tag{2.11}$$

This bound is sharp.

Proof. Let $\lambda(z) = i\frac{z-1}{z+1}$, $a, b \in \mathbb{R}$ with $a > 0$. For any univalent function h of the lower half-plane, set

$$g(z) = \frac{h(a\lambda(z) + b) - h(-ai + b)}{2iah'(-ai + b)}. \tag{2.12}$$

Then, g is a univalent function on the unit disk with $g(0) = 0$, $g'(0) = 1$. By Bieberbach's estimate,

$$|c_2| = \left|\frac{g''(0)}{2}\right| \leq 2. \tag{2.13}$$

On the other hand, one can show that $g''(0) = -2 + 2ia\theta(h)(-ai + b)$. It follows from (2.13) that

$$| - 1 + ia\theta(h)(-ai + b)| \leq 2, \tag{2.14}$$

and triangle inequality gives us the desired inequality.

$h(z) = z^{-2}$ achieves the extremal value. $\square$

Note 2.2.1. *$z \mapsto z^2$ on $\mathbb{L}$ is not an extremal function for $||\theta(h)||_1 = 6$, but it is still an extremal function for (2.13). Both z^2 and z^{-2} are extremal functions for the Bers embedding. (These maps are rotations of the Koebe function on the unit disk pulled back to the lower half-plane, then composed with a Möbius transformation.)*

2.3 The Image of Θ Contains an Open Ball

Let $\mathcal{D}(G) \subset \mathcal{D}$ with the property $f \circ \gamma = \gamma_f \circ f$ for $\gamma \in G$, where $\gamma_f \in PSL(2, \mathbb{C})$. For any G, the space $T(G)$ can be identified with $\mathcal{D}(G)$ by $[\mu] \mapsto w^\mu|_{\mathbb{L}}$. For $G = \mathbb{Z}$, if $\gamma(z) = z + 1$, then γ_f is parabolic and fixes ∞. The other two fixed points of f forces $\gamma_f = \gamma$, so $f(z + 1) = f(z) + 1$. In short,

$$\mathcal{D}(\mathbb{Z}) = \{f \in \mathcal{D}|f(z + 1) = f(z) + 1\}. \tag{2.15}$$

From Proposition 2.2.1, we know that θ on $\mathcal{D}(\mathbb{Z})$ maps into $A^1_\infty(\mathbb{L}, \mathbb{Z})$. We will prove part (c) of the main theorem by showing that $A^1_\infty(\mathbb{L}, \mathbb{Z})_\alpha \subset \theta(\mathcal{D}(\mathbb{Z}))$ for some $\alpha > 0$.

First, we start with a lemma.

Lemma 2.3.1. *Let $\Psi(\phi) = \phi_z - \frac{1}{2}\phi^2$. If $\phi \in A_1(\mathbb{L})$ with $||\phi||_1 < \delta$, then*

$$||\Psi(\phi)||_2 \leq 4\delta + \frac{1}{2}\delta^2. \tag{2.16}$$

The proof of this lemma will be given later in this section.

Proposition 2.3.1. *Let $\theta : \mathcal{D}(\mathbb{Z}) \to A_1(\mathbb{L}, \mathbb{Z})$ be given by*

$$\theta(f) = \frac{f''}{f'}. \tag{2.17}$$

Then, the image of θ contains an open ball centered at 0.

The group $\mathbb{Z}$ is used here for the first time.

Proof. For any normed vector space V, denote by V_r the the ball of radius r centered at 0 in V. Let α be the positive root of $4\delta + \frac{1}{2}\delta^2 = 2$. By Lemma 2.3.1, $\Psi(A^1_\infty(\mathbb{L}, \mathbb{Z})_\alpha) \subset A^2_\infty(\mathbb{L}, \mathbb{Z})_2$. We will show that $A^1_\infty(\mathbb{L}, \mathbb{Z})_\alpha$ lies in the image of θ.

Let $\phi \in A^1_\infty(\mathbb{L}, \mathbb{Z})_\alpha$. By the existence part of Lemma 2.1.1, there is a locally injective holomorphic function f such that $\theta(f) = \phi$. Since $S(f) \in A^2_\infty(\mathbb{L}, \mathbb{Z})_2$, by the Ahlfors-Beurling extension theorem (cf. [6, Theorem II.5.1]), f is injective and admits a quasconformal extension. By the $\mathbb{Z}$-invariance of ϕ, $z \mapsto f(z+1)$ is also a solution which is injective with a quasiconformal extension. The uniqueness part of Lemma 2.1.1 gives

$$f(z+1) = af(z) + b. \tag{2.18}$$

We will show that $a = 1$. If $a \neq 1$, then take z_0 such that $f(z_0) \neq 0$. Repeated application of (2.18) gives

$$f(z_0 + n) = a^n f(z_0) + b\left(\frac{1 - a^n}{1 - a}\right), \tag{2.19}$$

and taking $n \to \pm\infty$ in (2.19) gives

$$f(\infty) = \infty = \frac{b}{(1-a)} \tag{2.20}$$

It follows that $a = 1$, and thus $f(z+1) = f(z) + b$. The injectivity of f forces $b \neq 0$.

Let $g(z) = (f(z) - f(0))/b$. Then, $g \in \mathcal{D}(\mathbb{Z})$, and $\theta(g) = \theta(f) = \phi$. It follows that $A^1_\infty(\mathbb{L}, \mathbb{Z})_\alpha \subset \theta(\mathcal{D}(\mathbb{Z}))$. ∎

Corollary 2.3.1. *Let $\phi \in A^1_\infty(\mathbb{L}, \mathbb{Z})_\alpha$. If $\theta(f) = \phi$, then $f(\infty) = \infty$.*

Note 2.3.1. *Currently, it is not known whether the $\mathbb{Z}$-invariance is a necessary condition for Proposition 2.3.1. If Corollary 2.3.1 is true for $\mathcal{D}$, then Theorem 2.0.1 would hold for $T(1)$.*

Proof of Lemma 2.3.1. Suppose that $\phi \in A_1(\mathbb{L})$, so that $||\phi(z)||_1 < \delta$. Let C be a circle of radius R centered at z for $R < |\text{Im } z|$. Then,

$$\phi'(z) = \frac{1}{2\pi i}\int_C \frac{\phi(w)}{(w-z)^2}dw \tag{2.21}$$

$$|\phi'(z)| \leq \frac{\delta}{4\pi}\int_C \frac{|dw|}{|\text{Im } w||w-z|^2} \tag{2.22}$$

This integral can be computed exactly using residues:

$$\int_C \frac{|dw|}{|\text{Im } w||w-z|^2} = \frac{1}{R^2}\int_0^{2\pi} \frac{d\theta}{\sin(\theta) + R^{-1}\text{Im } z} \tag{2.23}$$

$$= \frac{2}{R^2}\int_{|w|=1} \frac{dw}{w^2 + 2i(R^{-1}\text{Im } z)w - 1} \tag{2.24}$$

$$= \frac{2\pi}{R^2\sqrt{(R^{-1}\text{Im } z)^2 - 1}}. \tag{2.25}$$

Therefore, we get the following estimate:

$$|\phi'(z)| \leq \frac{\delta}{2R\sqrt{(\text{Im } z)^2 - R^2}}. \tag{2.26}$$

The right hand side of (2.26) achieves its minimum at $R = (\sqrt{2})^{-1}|\text{Im } z|$, and so it follows that

$$|\phi'(z)| \leq \frac{\delta}{(\text{Im } z)^2}, \tag{2.27}$$

and by triangle inequality, we have

$$4(\text{Im } z)^2|\Psi(\phi)| \leq 4(\text{Im } z)^2(|\phi'| + \frac{1}{2}|\phi|^2|) \leq 4\delta + \frac{1}{2}\delta^2. \tag{2.28}$$

□

Note 2.3.2. *This is a modification of the proof of Theorem A.2. in [12].*

Note 2.3.3. *Using limiting rectangles instead of a circle, it is possible to improve the estimate to $\frac{8}{\pi}\delta + \frac{1}{2}\delta^2$. I believe that the sharp estimate is $2\delta + \frac{1}{2}\delta^2$.*

2.4 Holomorphicity and the Derivative

In this section we will prove part (d) of the main theorem. Let $\Theta : L^\infty(\mathbb{H})_1 \to A^1_\infty(\mathbb{L})$ be given by $\mu \mapsto \theta(w^\mu|_{\mathbb{L}})$. Proposition 2.2.1 says that the image of Θ is contained in a ball of radius 6 at 0. We will introduce a shift of base point by some $\alpha \in L^\infty(\mathbb{H})_1$, then show that the shifted map Θ^α also satisfies a global bound. The global bound will be used to show that Θ^α is holomorphic at every point. Its directional derivative at 0 will be computed, which corresponds to the directional derivative of Θ at α. This

The exposition closely follows the corresponding discussion for the Bers embedding in [7].

2.4.1 Shift of Base Point

For $\alpha \in L^\infty(\mathbb{H})_1$, set $D_1 = w^\alpha(\mathbb{L})$, $D_2 = w^\alpha(\mathbb{H})$. Let ρ_α be such that $\rho_\alpha(w)|dw|^2 = \rho(z)|dz|^2$, where $w = w^\alpha(z)$. Let $A^1_\infty(D_1)$ be the Banach space of holomorphic functions on D_1 with the norm $||f|| = \sup_{D_1} |\sqrt{\rho_\alpha} f|$, and

$$\begin{aligned}(w^\alpha)_1^* : A^1_\infty(D_1) &\to A^1_\infty(\mathbb{L})\\ f &\mapsto f \circ w^\alpha (w^\alpha)'.\end{aligned}$$

This map is an isometry.

If $\mu \in L^\infty(D_2)_1$, we define w^μ to be the solution to the Beltrami equation with coefficient μ extended to all of $\mathbb{C}$ by 0 on D_1 normalized by fixing $0, 1, \infty$. (See (1.6)) Note that $w^\mu \circ w^\alpha$ is a quasiconformal mapping which is conformal on $\mathbb{L}$ fixing $0, 1, \infty$. And so, there exists a $\kappa \in L^\infty(\mathbb{H})_1$ so that $w^\mu \circ w^\alpha = w^\kappa$. Explicitly,

$$\mu = \left(\frac{\kappa - \alpha}{1 - \kappa\overline{\alpha}} \frac{(w^\alpha)_z}{\overline{w^\alpha}_{\bar{z}}} \right) \circ (w^\alpha)^{-1}. \tag{2.29}$$

Let $R^\alpha(\mu) = \kappa$. (Compare with (1.13).)

For $\mu \in L^\infty(D_2)_1$, we define $\Theta^\alpha(\mu) = \theta(w^\mu|_{D_1})$. Note that $\Theta^0 = \Theta$.

Proposition 2.4.1. $\Theta^\alpha : L^\infty(D_2)_1 \to A^1_\infty(D_1)_{12}$.

Proof.

$$\begin{aligned}\Theta^0 \circ R^\alpha(\mu) &= \theta(w^\mu \circ w^\alpha) &(2.30)\\ &= \theta(w^\mu) \circ w^\alpha (w^\alpha)' + \theta(w^\alpha) &(2.31)\\ \Theta^0 \circ R^\alpha(\mu) &= (w_1^\alpha)^* \Theta^\alpha(\mu) + \Theta^0(\alpha). &(2.32)\end{aligned}$$

It follows from Proposition 2.2.1 and triangle inequality that

$$||\Theta^\alpha(\mu)||_1 = ||(w_1^\alpha)^*\Theta^\alpha(\mu)||_1 \le ||\Theta^0(R^\alpha(\mu))||_1 + ||\Theta^0(\alpha)||_1 \le 12. \tag{2.33}$$

□

In summary, we have the following commutative diagram.

$$\begin{tikzcd} L^\infty(D_2)_1 \arrow[r, "R^\alpha"] \arrow[d, "\Theta^\alpha"] & L^\infty(\mathbb{H})_1 \arrow[d, "\Theta"] \\ A^1_\infty(D_1) \arrow[r, "f\mapsto(w^\alpha)^*_1 f+\Theta(w^\alpha)"] & A^1_\infty(\mathbb{L}) \end{tikzcd}$$

2.4.2 Proof of Holomorphicity

We will show that $\Theta^\alpha : L^\infty(D_2) \to A^1_\infty(D_1)$ is holomorphic at every point. Let $\epsilon > 0$ be such that if $|t| < 5\epsilon$, then $|\mu + t\nu| \le 1$. Fix some z on D_1, and let

$$g(t) = (\rho_\alpha)^{-1}(z)\left(\frac{\Theta^\alpha(\mu+t\nu)(z) - \Theta^\alpha(\mu)(z)}{t}\right). \tag{2.34}$$

It is enough to show that $|g(t_1) - g(t_2)|$ is uniformly bounded over all $z \in D_1$ by $|t_1 - t_2|$.

Let

$$h(t) = (\rho_\alpha(z))^{-1}\theta^\alpha(w^{\mu+t\nu})(z) \tag{2.35}$$

so that

$$g(t) = \frac{h(t) - h(0)}{t}. \tag{2.36}$$

Both g and h are holomorphic on $|t| < 5\epsilon$ (c.f. [1]), and $|h(t)| \le 12$ by (2.4.1).

For $|t| < 3\epsilon$, we have

$$h(t) - h(0) = \frac{t}{2\pi i}\int_{|s|=4\epsilon} \frac{h(s)}{(s-t)s} ds \tag{2.37}$$

$$|h(t) - h(0)| \le \frac{|t|}{2\pi}\int_{|s|=4\epsilon} \frac{|h(s)|}{||s|-|t|||s|}|ds| \le \frac{12}{\epsilon}|t| \tag{2.38}$$

$$|g(t)| \le \frac{12}{\epsilon} \tag{2.39}$$

For $|t_1|, |t_2| < \epsilon$, we have

$$g(t_1)-g(t_2)=\frac{t_1-t_2}{2\pi i}\int_{|s|=2\epsilon}\frac{g(s)ds}{(s-t_1)(s-t_2)} \tag{2.40}$$

$$|g(t_1)-g(t_2)|\leq\frac{|t_1-t_2|}{2\pi}\int_{|s|=2\epsilon}\frac{|g(s)||ds|}{(|s|-|t_1|)(|s|-|t_2|)}\leq\frac{|t_1-t_2|}{2\pi}\int_0^{2\pi}|g(s)|\frac{2}{\epsilon}d\theta. \tag{2.41}$$

Putting (2.36) into (2.41) then using (2.39), we get

$$|g(t_1)-g(t_2)|\leq\frac{|t_1-t_2|}{2\pi}\int_0^{2\pi}\frac{24}{\epsilon^2}d\theta\leq\frac{24|t_1-t_2|}{\epsilon^2}. \tag{2.42}$$

Taking the supremum over z, we get

$$\left|\left|\frac{\Theta^\alpha(\mu+t_1\nu)-\Theta^\alpha(\mu)}{t_1}-\frac{\Theta^\alpha(\mu+t_2\nu)-\Theta^\alpha(\mu)}{t_2}\right|\right|_1\leq\frac{24|t_1-t_2|}{\epsilon^2}. \tag{2.43}$$

This shows that Θ^α is holomorphic at μ.

2.4.3 The Derivative of Θ

Proposition 2.4.2. *Let $\mu,\nu\in L^\infty(\mathbb{H})$ with $||\mu||_\infty<1$. For $z\in\mathbb{L}$,*

$$D_\mu\Theta(\nu)(z)=-\frac{2}{\pi}(w^\mu)'(z)\int_{\mathbb{H}}\frac{\nu(u)((w^\mu)_z(u))^2}{(w^\mu(u)-w^\mu(z))^3}d^2u. \tag{2.44}$$

Proof. From section 2.4.2, we know that the derivative exists.

$$\left.\frac{d}{dt}\right|_{t=0}\Theta^\alpha(t\nu)=\left.\frac{d}{dt}\right|_{t=0}\frac{(w^{t\nu})''}{(w^{t\nu})'}=(\dot{w}^\nu)'', \tag{2.45}$$

where $\dot{w}^\nu=\left.\frac{d}{dt}\right|_{t=0}w^{t\nu}$. Here we use the fact that $[\frac{d}{dz},\frac{d}{dt}]=0$ (c.f. [1]).

From [4, Chapter V Section C Theorem 5],

$$\dot{w}^\nu(z)=-\frac{1}{\pi}\int_{D_2}\nu(w)\left(\frac{1}{w-z}+\frac{z-1}{w}-\frac{z}{w-1}\right)d^2w \tag{2.46}$$

$$(\dot{w}^\nu)''(z)=-\frac{2}{\pi}\int_{D_2}\frac{\nu(w)}{(w-z)^3}d^2w. \tag{2.47}$$

It follows from (2.45) and (2.47) that

$$D_0\Theta^\alpha(\nu)(z)=-\frac{2}{\pi}\int_{D_2}\frac{\nu(w)}{(w-z)^3}d^2w. \tag{2.48}$$

(2.32) gives

$$\Theta(\mu+\nu t)=\Theta(\mu)+(w^\mu)_1^*\Theta^\mu(R_\mu^{-1}(\mu+\nu t)) \tag{2.49}$$

Taking the derivative at $t=0$ in (2.49), we get

$$(D_\mu\Theta)(\nu)=(w_1^\mu)^*D_0\Theta^\mu R(\nu,\mu) \tag{2.50}$$

where

$$R(\nu,\mu)=\frac{d}{dt}\bigg|_{t=0}R_\mu^{-1}(\mu+\nu t)=\left(\frac{\nu}{1-|\mu|^2}\frac{w^\mu}{\overline{w^\mu}_{\bar z}}\right)\circ(w^\mu)^{-1}. \tag{2.51}$$

Using (2.48) and (2.50) and by change of variables $w=w^\mu(u)$, we get

$$(D_\mu\Theta)(\nu)(z)=-\frac{2}{\pi}(w^\mu)'(z)\int_{D_2}\left(\frac{\nu}{1-|\mu|^2}\frac{w_z^\mu}{\overline{w^\mu}_{\bar z}}\right)\circ(w^\mu)^{-1}(w)\frac{d^2w}{(w-w^\mu(z))^3} \tag{2.52}$$

$$=-\frac{2}{\pi}(w^\mu)'(z)\int_{\mathbb{H}}\frac{\nu(u)(w_z^\mu(u))^2}{(w^\mu(u)-w^\mu(z))^3}d^2u. \tag{2.53}$$

□

Corollary 2.4.1. *For $\mu\in L^\infty(\mathbb{H})$,*

$$D_0\Theta(\nu)(z)=-\frac{2}{\pi}\int_{\mathbb{H}}\frac{\mu(u)}{(u-z)^3}d^2u. \tag{2.54}$$

Note 2.4.1. *Following the same notations as above, for the Bers embedding we have $(\Phi_\beta)^\alpha: L^\infty(D_2)_1\to A_\infty^2(D_1)_{12}$ with*

$$D_0(\Phi_\beta)^\alpha(\nu)(z)=-\frac{6}{\pi}\int_{D_2}\frac{\nu(w)}{(w-z)^4}d^2w. \tag{2.55}$$

It's clear that

$$D_0(\Phi_\beta)^\alpha(\nu)(z)=D_0\Theta^\alpha(\nu)'(z)=(\dot w^\nu)'''(z), \tag{2.56}$$

and since $D_0(\Phi_\beta)^\alpha(\nu)(z)=0$ if and only if $(\dot w^\nu)'(z)=0$, it follows that

$$\ker D_0(\Phi_\beta)^\alpha(\nu)(z)=\ker D_0\Theta^\alpha(\nu)(z),$$

or equivalently,

$$\ker D_\mu\Phi_\beta(\nu)(z)=\ker D_\mu\Theta(\nu)(z)$$

for all $\mu,\nu\in L^\infty(\mathbb{H})$ with $||\mu||_\infty<1$. Since Φ_β descends to β on $T(1)$ and β is an injective immersion on $T(1)$, it follows that θ is an injective immersion on $T(1)$ as well.

Chapter 3

New Complex Coordinates

We define the space of pre-Bers harmonic Beltrami differentials as follows:

$$H^{-1,1}(\mathbb{H},\mathbb{Z}) = \{\mu \in L^\infty(\mathbb{H},\mathbb{Z}) : \mu(z) = (z-\overline{z})\phi(\overline{z}) \text{ where } \phi \in A^1_\infty(\mathbb{L},\mathbb{Z})\}.$$

For $\mu \in L^\infty(\mathbb{H})$, let $\dot{w}[\mu] = \frac{d}{dt}\Big|_{t=0} w_{t\mu}$, and $F[\mu] = \frac{1}{2}(\dot{w}[\mu] - i\dot{w}[i\mu])$ and $\Phi[\mu] = \frac{1}{2}(\dot{w}[\mu] + i\dot{w}[i\mu])$. Then,

$$F[\mu](z) = -\frac{1}{\pi}\int_{\mathbb{H}} \mu(u)\left(\frac{1}{(u-z)} + \frac{z-1}{u} - \frac{z}{u-1}\right) d^2z \tag{3.1}$$

$$\Phi[\mu](z) = -\frac{1}{\pi}\int_{\mathbb{H}} \overline{\mu(u)}\left(\frac{1}{(\overline{u}-z)} + \frac{z-1}{\overline{u}} - \frac{z}{\overline{u}-1}\right) d^2z \tag{3.2}$$

This chapter contains one of the main results of this dissertation, which is given below:

Theorem 3.0.1. *Let $\eta : A^1_\infty(\mathbb{L},\mathbb{Z}) \to H^{-1,1}(\mathbb{H},\mathbb{Z}) \subset L^\infty(\mathbb{H},\mathbb{Z})$ be given by*

$$\eta(\phi)(z) = (z-\overline{z})\phi(\overline{z}).$$

Then,

- *(a) η is a right inverse to Θ on $A^1_\infty(\mathbb{L},\mathbb{Z})_\alpha$, where α is the same constant from Theorem 2.0.1(c).*
- *(b) For $p \in T(\mathbb{Z})$, let $U_p = R_p(U_0)$, where U_0 is the image of $A^1_\infty(\mathbb{L},\mathbb{Z})_\alpha$ in η. Then, $(U_p, \theta \circ R_p^{-1})$ form a complex coordinate chart on $T(\mathbb{Z})$ that is compatible with the Bers coordinates.*
- *(c) For $\mu \in H^{-1,1}(\mathbb{H},\mathbb{Z})$, $F[\mu] = (z-\overline{z})\overline{\Phi[\mu]} + \overline{\Phi[\mu]}$.*

Strictly speaking $H^{-1,1}(\mathbb{H}, \mathbb{Z})$ is looks like a subspace of $(-\frac{1}{2}, \frac{1}{2})$-differentials. While it may be possible to make sense of this identification of the $(-\frac{1}{2}, \frac{1}{2})$-differentials with the $(-1, 1)$-differentials in an invariant way by using the Eisenstein-Maass series for the group G, doing this in a meaningful way is beyond the scope of this dissertation.

3.1 Local Right Inverse

In this section, we will prove part (a) of the main theorem. (Compare with the Ahlfors-Weill Local section for the Bers embedding in section 1.2.)

Proposition 3.1.1. *Let $\eta(\phi)(z) = (z - \overline{z})\phi(\overline{z})$. If $||\phi||_1 < \alpha$, then $\Theta(\eta(\phi)) = \phi$.*

Proof. Let $\phi \in A^1_\infty(\mathbb{L}, \mathbb{Z})_\alpha$, where α is as in the proof of Proposition 2.3.1. Then by Proposition 2.3.1, there is $f \in \mathcal{D}(\mathbb{Z})$ so that $\theta(f) = \phi$. We define

$$F(z) = \begin{cases} f(\overline{z}) + (z - \overline{z})f'(\overline{z}), & z \in \mathbb{H} \\ f(z), & z \in \mathbb{L} \end{cases} \tag{3.3}$$

Since f admits a quasiconformal extension, it follows that F is a continuous map on the extended complex plane, which is injective on $\overline{\mathbb{L}}$. On $\mathbb{H}$, we have

$$F_{\bar{z}} = f'(\bar{z}) - f'(\bar{z}) + (z - \overline{z})f''(\overline{z}) = (z - \overline{z})f''(\bar{z}) \tag{3.4}$$

$$F_z = f'(\bar{z}). \tag{3.5}$$

Let $\mu_F = F_{\bar{z}}/F_z$. Then,

$$\mu_F = \frac{F_{\bar{z}}}{F_z} = (z - \bar{z})\frac{f''(\bar{z})}{f'(\bar{z})} = (z - \overline{z})\phi(\overline{z}) = \eta(\phi). \tag{3.6}$$

Since $||\mu_F||_\infty = ||\phi||_1 = \alpha = 2\sqrt{5} - 4 < 1$, it follows that F is local homeomorphism of the sphere onto itself, and therefore must be globally injective. F is quasiconformal.

By Corollary 2.3.1, we have $F(\infty) = \infty$. And so

$$w^{\mu_F} = \frac{F(z) - F(0)}{F(1) - F(0)} = F(z). \tag{3.7}$$

Therefore,

$$\Theta(\eta(\phi)) = \theta(w^{\mu_F}|_{\mathbb{L}}) = \theta(f) = \phi. \tag{3.8}$$

$\square$

Corollary 3.1.1. *The composition*

$$(\eta \circ D_0\Theta)(\mu) = -\frac{2}{\pi}(z - \overline{z}) \int_{\mathbb{H}} \frac{\mu(w)}{(w - \overline{z})^3} d^2w, \tag{3.9}$$

defines a projection operator from $L^\infty(\mathbb{H}, \mathbb{Z})$ *into* $H^{-1,1}(\mathbb{H}, \mathbb{Z})$.

Example 3.1.1. *Let* $f(z) = z^n$. *Then,*

$$\theta(f)(z) = \frac{n-1}{z},$$

so $||\theta(f)||_1 = 2|n-1|$. *For* $|n-1| < 1/2$, *we have*

$$\eta(\theta(f))(z) = (z - \overline{z})\frac{n-1}{\overline{z}} \in L^\infty(\mathbb{H})_1.$$

In that case, F *defined as in* (3.3) *is a quasiconformal mapping. We don't need to use the smaller constant* α, *since* f *already has a continuous extension to the real line. The condition* $|n-1| < 1/2$ *ensures that the extension is injective.*

On the other hand,

$$S(f)(z) = \frac{1-n^2}{2z^2}.$$

Since $||S(f)(z)||_2 = 2|(1-n^2)|$, f *admits an Ahlfors-Weill extension whenever* $|n^2 - 1| < 1$.

This example shows that there are Beltrami differentials with representatives in $H^{-1,1}(\mathbb{H})$ *but not in* $\Omega^{-1,1}(\mathbb{H})$ *and vice-versa.*

3.2 Complex Structure and Compatibility

In this section we will prove part (b) of the main theorem.

Set $V_0 = \Phi \circ \eta(A^1_\infty(\mathbb{L}, \mathbb{Z})_\alpha) \subset T(\mathbb{Z})$. For $p \in T(\mathbb{Z})$, let $V_p = R_p(V_0)$, and

$$h_p : V_p \to A^1_\infty(\mathbb{L}, \mathbb{Z})_\alpha$$

be given by $h_p = \theta \circ R_p^{-1}$.

Proposition 3.2.1. (h_p, V_p) *form a complex coordinate chart on* $T(\mathbb{Z})$.

Proof. First, note that $p \in V_p$, and so $\bigcup_{p \in T(\mathbb{Z})} V_p = T(\mathbb{Z})$. Let $q \in V_p$. Since $h_p^{-1} = R_p \circ \Phi \circ \eta$, we have

$$h_q \circ h_p^{-1} = \theta \circ R_q^{-1} \circ R_p \circ \Phi \circ \eta = \theta \circ R_r \circ \Phi \circ \eta \tag{3.10}$$
$$= \theta \circ \Phi \circ R_{\tilde{r}} \circ \eta \tag{3.11}$$
$$= \Theta \circ R_{\tilde{r}} \circ \eta, \tag{3.12}$$

where $r = p * q^{-1}$, and $\tilde{r}$ is a representative of r in $L^\infty(\mathbb{H}, \mathbb{Z})_1$. This shows that $h_q \circ h_p^{-1}$ is a composition of holomorphic maps. □

Let us denote by $\mathcal{C}$ the complex structure defined by the coordinate chart in Proposition 3.2.1.

Proposition 3.2.2. *$\mathcal{C}$ is compatible with the complex structure of the Bers coordinates.*

The Bers coordinates are defined as follows. Let $\Phi(\Lambda(A^2_\infty(\mathbb{L}, \mathbb{Z})_2)) = U_0 \subset T(\mathbb{Z})$. For $p \in T(\mathbb{Z})$, $U_p = R_p(U_0)$, and

$$b_p : U_p \to A^2_\infty(\mathbb{L}, \mathbb{Z})_2$$

with $b_p = \beta \circ R_p^{-1}$. To prove Proposition 3.2.2, it is sufficient to check that the maps

$$h_p \circ b_p^{-1} : b_p(U_p \cap V_p) \subset A^2_\infty(\mathbb{L}, \mathbb{Z}) \to h_p(U_p \cap V_p) \subset A^1_\infty(\mathbb{L}, \mathbb{Z})$$

and

$$b_p \circ h_p^{-1} : h_p(U_p \cap V_p) \subset A^1_\infty(\mathbb{L}, \mathbb{Z}) \to b_p(U_p \cap V_p) \subset A^2_\infty(\mathbb{L}, \mathbb{Z})$$

are holomorphic.

By definition,

$$h_p \circ b_p^{-1} = \theta \circ R_p^{-1} \circ R_p \circ \Phi \circ \Lambda = \theta \circ \Phi \circ \Lambda \tag{3.13}$$
$$= \Theta \circ \Lambda, \tag{3.14}$$

which is a composition of holomorphic mappings.

The inverse is given by

$$b_p \circ h_p^{-1} = \beta \circ R_p^{-1} \circ R_p \circ \Phi \circ \eta \tag{3.15}$$
$$= \beta \circ \Phi \circ \eta. \tag{3.16}$$

If $\phi \in A^1_\infty(\mathbb{L}, \mathbb{Z})$, then $\theta(w^{\eta(\phi)}|_{\mathbb{H}}) = \phi$. So

$$b_p \circ h_p^{-1}(\phi) = \beta \circ \Phi \circ \eta(\phi) = S(w^{\eta(\phi)}|_{\mathbb{H}}) = \Psi(\theta(w^{\eta(\phi)}|_{\mathbb{H}})) = \Psi(\phi). \tag{3.17}$$

Lemma 3.2.1. $\Psi : A_1(\mathbb{H}) \to A_2(\mathbb{H})$ *is holomorphic.*

Proof. The proof of Lemma 2.3.1 shows that $\phi \mapsto \phi_z$ is continuous at 0. Since this map is linear, it is continuous everywhere and holomorphic.

The map $\phi \mapsto \phi^2$ is obviously continuous and holomorphic with Fréchet derivative at ϕ along ψ given by $2\phi\psi$, since

$$\left(\frac{(\phi+\epsilon\psi)^2-\phi^2}{\epsilon} - 2\psi\phi\right) = \epsilon\psi^2 \tag{3.18}$$

$$\left|\left|\frac{(\phi+\epsilon\psi)^2-\phi^2}{\epsilon} - 2\psi\phi\right|\right|_2 = \epsilon||\psi^2||_2 = \epsilon(||\psi||_1)^2. \tag{3.19}$$

Therefore, $\Psi(\phi) = \phi_z - \frac{1}{2}\phi^2$ is holomorphic with Fréchet derivative

$$D\Psi_\phi(\psi) = \psi_z - \psi\phi.$$

□

Since Ψ is holomorphic, it follows that $b_p \circ h_p^{-1}$ is holomorphic.

3.3 Variational Formulas

In this section, we will prove part (c) of the main theorem. In fact, we will prove a more general statement.

Proposition 3.3.1. *Let* $\mu \in H^{-1,1}(\mathbb{H})$*. Then,*

$$F[\mu] = (z-\overline{z})\overline{\Phi[\mu]'} + \overline{\Phi[\mu]}. \tag{3.20}$$

Note 3.3.1. *For* $\mu \in \Omega^{-1,1}(\mathbb{H})$*, we have*

$$F[\mu] = \frac{(z-\overline{z})^2}{2}\overline{\Phi[\mu]''} + (z-\overline{z})\overline{\Phi[\mu]'} + \overline{\Phi[\mu]}. \tag{3.21}$$

So the formulas relating F and Φ for harmonic and pre-Bers harmonic Beltrami differentials are identical except for the missing the quadratic term in (3.20)*. (c.f. [2])*

It is known that $\dot{w}[\mu]$ extends to a continuous function on all of $\mathbb{C}$. The normalization of $w_{t\mu}$ forces and $w[\mu](x) = 0$ for $x = 0, 1$ and $z^{-2}w[\mu](z) \to 0$ as $z \to \infty$. We also have for $z \in \mathbb{H}$, that $F[\mu]_{\bar{z}} = \mu$ and $\Phi[\mu]_{\bar{z}} = 0$.

Let N be an operator on $L^\infty(\mathbb{H})$, given by

$$N\mu(z) = -\frac{2}{\pi}(z-\overline{z})\int_{\mathbb{H}} \frac{\mu(w)}{(w-\overline{z})^3} d^2w. \tag{3.22}$$

By differentiating Φ twice, we see that the following statement holds:

Proposition 3.3.2. *For $\mu \in L^\infty(\mathbb{H})$,*

$$N\mu = (z-\overline{z})\overline{\Phi[\mu]''}. \tag{3.23}$$

N and P are related as follows:

Corollary 3.3.1. *For $\mu \in L^\infty(\mathbb{H})$,*

$$P\mu = \frac{(z-\overline{z})^2}{2}\left(\frac{N\mu}{(z-\overline{z})}\right)_{\bar{z}} \tag{3.24}$$

(3.23) is the key identity needed in the proof of Proposition 3.3.1.

Proposition 3.3.3. *Let N be defined as* (3.22)*. Then,*

(a) $\ker N = \mathcal{N}(\mathbb{H})$.

(b) Im $N \subset L^\infty(\mathbb{H})$.

(c) $N^2 = N$.

Note 3.3.2. *When restricted to the subspace $L^\infty(\mathbb{H}, \mathbb{Z})$, $N = \eta \circ D_0\Theta$ and Proposition 3.3.3 follows trivially.*

Proof. For Proposition 3.3.3(a), since $\mathcal{N}(\mathbb{H}) = \ker \Phi$, it follows from Proposition 3.3.2 that $\mathcal{N}(\mathbb{H}) \subset \ker N$. Also, from Corollary 3.3.1, we have $\ker N \subset \ker P = \mathcal{N}(\mathbb{H})$.

Proposition 3.3.3(b) is shown as follows:

$$|N\mu(z)| \leq \frac{4|y|}{\pi}\int_{\mathbb{H}} \frac{|\mu(w)|}{(w-\overline{z})^3} d^2w \leq \frac{4|y|||\mu||_\infty}{\pi}\int_0^\pi \int_{|y|}^\pi \frac{r dr d\theta}{r^3} = 4||\mu||_\infty \tag{3.25}$$

For Proposition 3.3.3(c), let $\mu(z) = (z-\overline{z})\overline{\phi(z)}$ and further suppose that $\phi(z)$ is real analytic on $\mathbb{R}$. Let $D_R = \{z \in \mathbb{H} : |z| < R\}$ and C_R the boundary of D_R in $\mathbb{C}$. Then,

$$\int_{D_R} \frac{(u-\overline{u})\overline{\phi(u)}}{(u-\overline{z})^3} d^2u = \int_{D_R} \left(\frac{1}{(u-\overline{z})^2} + \frac{\overline{z}-\overline{u}}{(u-\overline{z})^3} \right) \overline{\phi(u)} \frac{d\overline{u} \wedge du}{2i} \tag{3.26}$$
$$= \int_{C_R} \left(\frac{1}{(u-\overline{z})} + \frac{\overline{z}-\overline{u}}{2(u-\overline{z})^2} \right) \overline{\phi(u)} \frac{d\overline{u}}{2i} \tag{3.27}$$
$$= -\frac{1}{4i} \int_{C_R} \frac{\overline{\phi(\overline{u})}}{u-\overline{z}} du + O(R^{-1}) \tag{3.28}$$
$$= -\frac{\pi}{2} \overline{\phi(z)} + O(R^{-1}) \tag{3.29}$$

If ϕ is not real analytic on $\mathbb{R}$, then use $\phi(z - i\epsilon)$, $\epsilon > 0$ in place of $\phi(z)$ in the above computation then let $\epsilon \to 0$. □

Proof of Proposition 3.3.1. Since $w[\mu]_{\bar{z}} = (z-\overline{z})\overline{\Phi[\mu]''}$, we have

$$w[\mu] = (z-\overline{z})\overline{\Phi[\mu]'} + \overline{\Phi[\mu]} + f,$$

where f is analytic on $\mathbb{L}$. From Proposition 3.3.3(b), we have that $|\Phi[\mu]''| = O(y^{-1})$ so it follows that $|y\Phi[\mu]'| \to 0$ as $y \to 0$ and

$$f|_{\mathbb{R}} = w[\mu] - \overline{\Phi[\mu]}.$$

Therefore, f is continuous on $\mathbb{R}$ with $f(0) = 0$ and $f(1) = 0$.

We have

$$0 = \text{Im } (w[\mu]|_R) = \text{Im } \left(\overline{\Phi[\mu]} + f\right)\Big|_{\mathbb{R}} = \text{Im } (-\Phi[\mu] + f)|_{\mathbb{R}},$$

so it follows that the function $g = f - \Phi[\mu]$ is real on $\mathbb{R}$. Since g is a holomorphic function on $\mathbb{H}$, g extends to an entire function by the Schwarz reflection principle.

Im $z^{-2}g$ = Im $z^{-2}w[\mu] \to 0$ as $z \to \infty$. Combined with the fact that $g(0) = 0$ and $g(1) = 0$, we have that $g(z) = a(z - z^2)$ for some real number a.

It follows that

$$w[\mu] = (z-\overline{z})\overline{\Phi[\mu]'} + \overline{\Phi[\mu]} + \Phi[\mu] + a(z - z^2). \tag{3.30}$$

Since $z^{-2}(z-\overline{z})\overline{\Phi[\mu]'} \to 0$ as $z \to \infty$, this forces $a = 0$. □

The following are some interesting consequences of Proposition 3.3.1

Corollary 3.3.2. *Let $\mu \in H^{-1,1}(\mathbb{H})$. Then,* *(a)* $w[\mu]_z = 2Re\ \Phi[\mu]$,

(b) $\frac{d}{dt}\Big|_{t=0} w^*_{t\mu} d^2z = 4Re\ (\Phi[\mu]')d^2z$,

(c) $\frac{d}{dt}\Big|_{t=0} w^*_{t\mu}(\rho(z)d^2z) = 0$,

(d) $\frac{d}{dt}\Big|_{t=0} \frac{(\overline{w_{t\mu}})_{\bar{z}}}{(w_{t\mu})_z} = 0$.

Proof. For Corollary 3.3.2(a), take the z-derivative of (3.20) to get

$$F[\mu]_z = \overline{\Phi[\mu]'} \tag{3.31}$$

$$w[\mu]_z = \overline{\Phi[\mu]'} + \Phi[\mu]' = 2\mathrm{Re}\ \Phi[\mu]. \tag{3.32}$$

For Corollary 3.3.2(b), we have

$$\frac{d}{dt}\bigg|_{t=0} w^*_{t\mu} d^2z = (|(w_{t\mu})_z|^2 - |(w_{t\mu})_{\bar{z}}|^2)d^2z = |(w_{t\mu})_z|^2(1 - t^2|\mu|^2)d^2z \tag{3.33}$$

$$= w[\mu]_z + \overline{w[\mu]}_{\bar{z}} = 2\mathrm{Re}\ (w[\mu]_z) = 4\mathrm{Re}\ (\Phi[\mu]')d^2z. \tag{3.34}$$

For Corollary 3.3.2(c), we have

$$\frac{d}{dt}\bigg|_{t=0} w^*_{t\mu}(\rho(z)d^2z) = \frac{d}{dt}\bigg|_{t=0} \frac{w^*_{t\mu}d^2z}{(w_{t\mu} - \overline{w_{t\mu}})^2} \tag{3.35}$$

$$= \left(\frac{4\mathrm{Re}\ (\Phi[\mu]')}{(z-\overline{z})^2} + \frac{-2(w[\mu] - \overline{w[\mu]})}{(z-\overline{z})^3}\right) d^2z \tag{3.36}$$

$$= \left(\frac{2(\Phi[\mu]' + \overline{\Phi[\mu]'})}{(z-\overline{z})^2} + \frac{-2(z-\overline{z})(\Phi[\mu] + \overline{\Phi[\mu]'})}{(z-\overline{z})^3}\right) d^2z \tag{3.37}$$

$$= 0. \tag{3.38}$$

For Corollary 3.3.2(d), we have

$$\frac{d}{dt}\bigg|_{t=0} \frac{(\overline{w_{t\mu}})_{\bar{z}}}{(w_{t\mu})_z} = \frac{\overline{w[\mu]}_{\bar{z}} - w[\mu]_z}{1} = 0 \tag{3.39}$$

□

Note 3.3.3. *Corollary 3.3.2(c) also holds when* $\mu \in \Omega^{-1,1}(\mathbb{H})$*, which is a classical result by Ahlfors in [2, Lemma 2]*

Chapter 4

Family of Operators

In this chapter, we discuss a one-parameter family of projection operators on $L^\infty(\mathbb{H}, \mathbb{Z})$. We shall see in Chapter 5 how these operators may be used to study the properties of the Takhtajan-Zograf metric on $T(\mathbb{Z})$.

Let $\sqcap = [0,1] \times [0, \infty)$, a fundamental domain of the group $\mathbb{Z}$. Let $L^2(\sqcap)$ be the space of measurable functions that are square integrable on $\sqcap$ with respect to the Euclidean area. We will refer to the L^2-inner product as the Takhtajan-Zograf pairing (TZ-pairing) which will be denoted as $\langle \mu, \nu \rangle = \int_\sqcap \mu \overline{\nu}$. Let $e_n(z) = e^{2\pi i n \bar{z}}$. With a slight abuse of notation, we will denote the function $z \mapsto (\text{Im } z)^s$ by y^s. Finally, let $\Gamma(s) = \int_0^\infty y^s e^{-y} dy$.

Theorem 4.0.1. *Let P be the Bers projection, and $N_s = g_s y^{-s} P(y^s P)$, where $g_s = \frac{4}{\Gamma(3-s)\Gamma(3+s)}$. Then,*

(a) For $0 \leq s < 2$, $N_s : L^\infty(\mathbb{H}, \mathbb{Z}) \to L^\infty(\mathbb{H}, \mathbb{Z})$.

(b) $N_t N_s = N_t$ for all $0 \leq s < 2$ and $0 \leq t \leq 2$.

(c) $\ker N_t = \mathcal{N}(\mathbb{H}, \mathbb{Z})$ for $0 \leq t \leq 2$.

(d) The following is true for all $s, t \in [0, 2]$.

1. *$N_t : L^\infty(\mathbb{H}, \mathbb{Z}) \to L^2(\sqcap)$. Furthermore, $y^s N_t \mu \in L^\infty(\mathbb{H}, \mathbb{Z})$ for all $s > 0$.*
2. *$\langle N_s \mu, N_t \nu \rangle = g_{s,t} \langle P\mu, P\nu \rangle$, where $g_{s,t} = \frac{\Gamma(5-s-t)}{6\Gamma(3-s)\Gamma(3-t)}$.*

(e) $\langle N_2 \mu, \nu \rangle = \langle N_2 \mu, N_2 \nu \rangle$.

The motivation for looking at such operators is as follows. For $\nu \in \Omega^{-1,1}(\mathbb{H}, G)$ and $\mu \in L^\infty(\mathbb{H}, G)$, the Weil-Petersson pairing is given by

$$\langle \mu, \nu \rangle_{WP} = \int_D \mu(z)\nu(z)\rho(z)d^2z,$$

where D is a fundamental domain of G. The Bers projection P is self-adjoint with respect to the Weil-Petersson pairing in the sense that

$$\langle P\mu, P\nu \rangle_{WP} = \langle \mu, P\nu \rangle_{WP},$$

for any $\mu, \nu \in L^\infty(\mathbb{H})$ whenever both sides are convergent. (This follows directly from the fact that $\mu - P\mu \in \mathcal{N}(\mathbb{H}, G)$ for all $\mu \in L^\infty(\mathbb{H}, G)$.)

One could obtain an analogous operator for the TZ-pairing as follows: for $\mu, \nu \in L^\infty(\mathbb{H}, \mathbb{Z})$,

$$\begin{aligned}\langle P\mu, P\nu \rangle &= -4\langle P\mu, y^2 P\nu \rangle_{WP} \\ &= -4\langle P\mu, Py^2 P\nu \rangle_{WP} \\ &= \langle \mu, y^{-2}Py^2P\nu \rangle.\end{aligned}$$

Up to a multiplicative constant, the operator $y^{-2}Py^2P$ corresponds to N_2 in Theorem 4.0.1. Unfortunately, $N_2\mu$ does not map $L^\infty(\mathbb{H})$ to $L^\infty(\mathbb{H})$, so it does not correspond to a differential of a coordinate map. However, part (d) of the theorem allows us to relate the TZ-pairings of N_2 with $N_1 = N$ or $N_0 = P$, which correspond to the differentials of the pre-Bers and the Bers embedding, respectively.

Theorem 4.0.1 will be proven in the subsequent sections. The majority of the proofs rely heavily on the properties of Fourier coefficients of the harmonic Beltrami differentials, which we will prove as we go along.

4.1 N_s is bounded.

In this section, we will prove part (a) of the main theorem by proving a slightly stronger statement.

Proposition 4.1.1. *Let $\mu \in L^\infty(\mathbb{H})$. Then, $y^{-s}P(y^s\mu) \in L^\infty(\mathbb{H})$ for $s \in [0, 2)$.*

Proof. Let $z = x + iy$ and $w = u + iv$. Then,

$$\left| \int_{\mathbb{H}} \frac{v^s \mu(w)}{(w - \overline{z})^4} d^2w \right| \leq ||\mu||_\infty \int_{\mathbb{H}} \frac{v^s}{|w - \overline{z}|^4} d^2w \tag{4.1}$$

$$\int_{\mathbb{H}} \frac{v^s}{|w-\overline{z}|^4} d^2w \leq \int_0^{\pi} \int_y^{\infty} \frac{|v-y|^s}{r^4} r dr d\theta = \int_0^{\pi} \int_y^{\infty} r^{s-3} |\sin(\theta) - \frac{y}{r}|^s dr d\theta \tag{4.2}$$

Since $0 < y \leq r$, we have $|\sin\theta - \frac{y}{r}| \leq 1$, and so for $s \geq 0$, we have $|\sin\theta - \frac{y}{r}|^s \leq 1$.

$$\int_{\mathbb{H}} \frac{v^s}{|w-\overline{z}|^4} d^2w \leq \pi \int_y^{\infty} r^{s-3} dr, \tag{4.3}$$

which converges when $s < 2$. So we have the following estimate

$$y^{2-s} \int_{\mathbb{H}} \frac{v^s}{|w-\overline{z}|^4} d^2w \leq \frac{\pi}{2-s} \tag{4.4}$$

□

For any $\mu \in L^{\infty}(\mathbb{H})$, we have $P\mu \in L^{\infty}(\mathbb{H})$. Applying Proposition 4.1.1 to $P\mu$ gives $y^{-s}Py^sP\mu \in L^{\infty}(\mathbb{H})$.

4.2 Overwriting Property

To prove part (b) of the main theorem, it is sufficient to prove the following statement:

Proposition 4.2.1. *Let $\mu \in L^{\infty}(\mathbb{H}, \mathbb{Z})$ and $s \in [0, 2)$, Then,*

$$PN_s\mu = P\mu. \tag{4.5}$$

We will prove (4.2.1) by comparing the Fourier expansions of $P\mu$ and $PN_s\mu$. In the statement of Proposition 4.2.1 the extremal value $s = 2$ has been omitted to ensure that $N_s\mu \in L^{\infty}(\mathbb{H}, \mathbb{Z})$. Nevertheless, the expansion of $N_s\mu$ will be valid even when $s = 2$.

The Fourier expansion of $N_s\mu$ is given as follows.

Proposition 4.2.2. *Suppose that $\mu(z) = y^2 \sum_{n\geq 1} c_n e_n(z)$ and $s \in [0, 2]$. Then,*

$$N_s\mu(z) = \frac{2}{\Gamma(3-s)(4\pi)^s} y^{2-s} \sum_{n\geq 1} \frac{c_n}{n^s} e_n(z). \tag{4.6}$$

Example 4.2.1. *Let* $\mu(z) = y^2 \sum_{n\geq 1} ne_n = y^2 k(e_1(z))$*, where* $z \mapsto k(z)$ *is the Koebe function on* $\mathbb{D}$*. Then,* $\mu \in L^\infty(\mathbb{H}, \mathbb{Z})$*. Putting* $s = 2$ *and* $c_n = n$ *into* (4.6)*, it is easy to see that* $N_2\mu \notin L^\infty(\mathbb{H}, \mathbb{Z})$*.*

We need some preliminary results to obtain the expression (4.6).

Lemma 4.2.1. *Let* $\mu \in L^\infty(\mathbb{H}, \mathbb{Z})$*. Then, the Bers projection* P *has the expansion:*

$$P\mu(z) = 32\pi^3 y^2 \sum_{n\geq 1} n^3 \langle \mu, e_n \rangle e_n(z). \tag{4.7}$$

To apply the expression (4.7) in the computation of $y^{-s}P(y^s P\mu)$, we need the following proposition.

Proposition 4.2.3. *For any* $s > 0$*,* $y^s P\mu \in L^\infty(\mathbb{H})$*.*

Lemma 4.2.1 and Proposition 4.2.3 will be proven later in this section.

Proof of Proposition 4.2.2 using Lemma 4.2.1 and Proposition 4.2.3:
Suppose that $\mu(z) = y^2 \sum_{n\geq 1} c_n e_n(z)$ with $\mu \in L^\infty(\mathbb{H}, \mathbb{Z})$. Then, $P\mu = \mu$, and so applying (4.7), we get

$$P(y^s P\mu) = 32\pi^3 y^2 \sum_{n\geq 1} n^3 \langle y^{2+s} \sum_{m\geq 1} c_m e_m, e_n \rangle e_n(z). \tag{4.8}$$

$\langle y^s P\mu, e_n \rangle$ is absolutely convergent, so by Fubini's theorem and uniform convergence we get

$$\langle y^{2+s} \sum_{m\geq 1} c_m e_m, e_n \rangle = c_n \int_0^\infty y^{2+s} e^{-4\pi n y} dy = c_n \frac{\Gamma(3+s)}{(4\pi n)^{3+s}}. \tag{4.9}$$

Putting (4.9) into (4.8) gives

$$y^{-s}P(y^s P\mu) = h_s y^{2-2} \sum_{n\geq 1} \frac{c_n}{n^s} e_n(z), \tag{4.10}$$

where $h_s = 32\pi^3 \Gamma(3+s)/(4\pi)^{3+s}$. Therefore,

$$N_s\mu(z) = g_s h_s y^{2-s} \sum_{n\geq 1} \frac{c_n}{n^s} e_n(z),$$

with $g_s h_s = 2(4\pi)^{-s}/\Gamma(3-s)$. $\square$

Proof of Proposition 4.2.1.

Suppose that $\mu(z) = y^2 \sum_{n\geq 1} c_n e_n(z)$, so that $P\mu = \mu$. By putting (4.6) into (4.7), we obtain

$$P(N_s\mu)) = \frac{64\pi^3}{\Gamma(3-s)(4\pi)^s} y^2 \sum_{n\geq 1} n^3 \langle y^{2-s} \sum_{m\geq 1} \frac{c_m}{m^s} e_m, e_n \rangle e_n(z). \tag{4.11}$$

The integral $\langle y^{2-s} \sum \frac{c_m}{m^s} e_m, e_n \rangle$ converges absolutely. Using Fubini's theorem and uniform convergence, we obtain

$$\langle y^{2-s} \sum_{m\geq 1} \frac{c_m}{m^s} e_m, e_n \rangle = \int_0^\infty y^{2-s} e^{-4\pi n y} dy = \frac{\Gamma(3-s)}{(4\pi n)^{3-s}}. \tag{4.12}$$

And so,

$$P(N_s\mu) = \frac{(4\pi)^{3-s}\Gamma(3-s)}{\Gamma(3-s)(4\pi)^{3-s}} y^2 \sum_{n\geq 1} n^3 \frac{c_n}{n^s} \frac{1}{n^{3-s}} e_n(z) \tag{4.13}$$

$$= y^2 \sum_{n\geq 1} c_n e_n(z) = P\mu \tag{4.14}$$

□

The remainder of this section will be used to prove Lemma 4.2.1 and Proposition 4.2.3.

Proof of Lemma 4.2.1. Let $\mu \in L^\infty(\mathbb{H}, \mathbb{Z})$. The lemma basically follows from rearranging the integral in the following way:

$$\int_{\mathbb{H}} \frac{\mu(w)}{(w-\overline{z})^4} d^2w = \sum_{n\in\mathbb{Z}} \int_{\sqcap} \frac{\mu(w)}{(w-\overline{z}+n)^4} d^2w. \tag{4.15}$$

The integral is absolutely convergent as one can see from the proof of Proposition 4.1.1 with $s = 0$, so

$$\int_{\mathbb{H}} \frac{\mu(w)}{(w-\overline{z})^4} d^2w = \int_{\sqcap} \sum_{n\in\mathbb{Z}} \frac{\mu(w)}{(w-\overline{z}+n)^4} d^2w. \tag{4.16}$$

By classical complex analysis, we have

$$\sum_{n\in\mathbb{Z}} \frac{1}{(u-n)^4} = \frac{8\pi^4}{3} \sum_{n\geq 1} n^3 e^{2\pi i n u}$$

for Im $u > 0$. So

$$\int_{\mathbb{H}} \frac{\mu(w)}{(w-\overline{z})^4} d^2w = \int_{\sqcap} \sum_{n\geq 1} \mu(w) n^3 e^{2\pi i n(w-\overline{z})} d^2w. \tag{4.17}$$

The expression on the right hand side of (4.17) is absolutely convergent for Im $z > 0$, so we obtain the desired expression. $\square$

The following lemma will have many applications so we state it here.

Lemma 4.2.2. *Let $\mu(z) = y^2 \sum_{n\geq 1} c_n e_n(z) \in L^\infty(\mathbb{H}, \mathbb{Z})$. Then,*

$$\sum_{n\geq 1} \frac{|c_n|^2}{n^{s+4}} \tag{4.18}$$

is convergent whenever $s > 0$.

Proof. Let $s > 0$. Then,

$$\int_D |\mu(z)|^2 y^{s-1} e^{-4\pi y} d^2z \leq \frac{(||\mu||_\infty)^2 \Gamma(s)}{(4\pi)^s} < \infty. \tag{4.19}$$

By Fubini's theorem and uniform convergence, the integral on the left hand side can be evaluated as follows

$$\sum_{n,m} c_n \overline{c_m} \int_0^\infty \int_0^1 y^{s+3} e^{2\pi i(n-m)x} e^{-2\pi(n+m+2)y} dx dy = \sum_{n\geq 1} |c_n|^2 \int_0^\infty y^{s+3} e^{-4\pi(n+1)y} dy \tag{4.20}$$

$$= \sum_{n\geq 1} \frac{|c_n|^2 \Gamma(s+4)}{(4\pi(n+1))^{s+4}}. \tag{4.21}$$

By limit comparison, the series $\sum_{n\geq 1} \dfrac{|c_n|^2}{n^{s+4}}$ is convergent.

$\square$

Note 4.2.1. *This is a direct proof of remark 4.2 in [13].*

Corollary 4.2.1. *Let $\mu \in \infty$. Then, for $y > 1$,*

$$|P\mu| \leq C' e^{-2\pi y}. \tag{4.22}$$

Proof. From Lemma 4.2.2, we see that for n sufficiently large, $|c_n| \leq Cn^3$ for all $n > M$. So

$$\left|\sum_{n\geq 1} c_n e_n\right| \leq \sum_{n\geq 1} |c_n| e^{-2\pi n y} \leq \sum_{n\geq 1}^{M} |c_n| e^{-2\pi n y} + C \sum_{n>M} n^3 e^{-2\pi n y}, \tag{4.23}$$

and

$$\sum_{n\geq M+1} n^3 e^{-2\pi n y} = \frac{e^{-2\pi(M+1)y}\left(e^{-4\pi y} + 4e^{-2\pi y} + 1\right)}{\left(e^{-2\pi y} - 1\right)^4} \leq \frac{6e^{-2\pi(M+1)y}}{\left(e^{-2\pi} - 1\right)^4}. \tag{4.24}$$

It follows that for $y > 1$, we have

$$|P\mu| < C' e^{-2\pi y} \tag{4.25}$$

for some constant C'. □

And finally, we prove Proposition 4.2.3:

Proof of Proposition 4.2.3. For $0 \leq y \leq 1$, $0 \leq y^s \leq 1$, so

$$|y^s P\mu| \leq |P\mu| \leq ||P\mu||_\infty.$$

For $y > 1$, we know $|y^s P\mu|$ is bounded from Corollary 4.2.1. □

4.3 Kernel of N_t

The following proposition is an immediate consequence of Proposition 4.2.1.

Proposition 4.3.1.

(a) $\ker P = \ker N_t$ *for* $t \in [0, 2)$.

(b) $\ker P \subset \ker N_2$

To prove part (c) of the main theorem, one only has to show that $\ker N_2 \subset \ker P$. This will be done by proving the following statement:

Proposition 4.3.2. *Let* $\mu \in L^\infty(\mathbb{H}, \mathbb{Z})$. *Then,*

$$\Phi[\mu]_z = \overline{N_2\mu}. \tag{4.26}$$

Since $\ker \Phi = \ker P = \mathcal{N}(\mathbb{H})$, (4.26) implies that $\ker P \subset \ker N_2$.

We prove Proposition 4.3.2 by comparing the Fourier expansions of Φ and N_2. From Proposition 4.2.2, we have

$$N_2\mu = \frac{1}{8\pi^2} \sum_{n \geq 1} \frac{c_n}{n^2} e_n. \tag{4.27}$$

Proposition 4.3.3. *For $\mu \in L^\infty(\mathbb{H}, \mathbb{Z})$,*

$$\Phi[\mu](z) = -\int_\sqcap \mu(w)(\cot(\pi(\overline{w} - z)) - \cot(\pi\overline{w}))d^2w. \tag{4.28}$$

Proof. Let $\mu \in L^\infty(\mathbb{H}, \mathbb{Z})$, and

$$\Phi[\mu] = -\frac{1}{\pi} \int_{\mathbb{H}} \overline{\mu(w)} R(\overline{w}, z) d^2w, \tag{4.29}$$

so that $R(w, z) = \frac{1}{w-z} + \frac{z-1}{w} - \frac{z}{w-1}$. This integral is absolutely convergent, so

$$\Phi[\mu] = -\frac{1}{\pi} \int_\sqcap \sum_{n \in \mathbb{Z}} \overline{\mu(w)} R(\overline{w}, z+n) d^2u. \tag{4.30}$$

Using classical complex analysis, one can show that

$$\sum_{n \in \mathbb{Z}} R(\overline{w}, z+n) = \pi(\cot(\pi(\overline{w} - z)) - \cot(\pi\overline{w})). \tag{4.31}$$

$\square$

Proof of Proposition 4.3.2. Taking the derivative of (4.28) gives

$$\Phi[\mu]'(z) = -\pi \int_\sqcap \mu(w) \csc^2(\pi(\overline{w} - z)) d^2w = 4\pi \int_\sqcap \mu(w) \sum_{n \geq 1} n e^{2\pi n(w - \overline{z})} d^2w. \tag{4.32}$$

Since

$$\int_\sqcap \sum_{n \geq 1} |\mu(w) n e^{2\pi n(w-\overline{z})}| d^2w \leq 2\pi ||\mu||_\infty \int \sum n e^{-2\pi n(v+y)} dv = ||\mu||_\infty \sum n e^{-2\pi y} \tag{4.33}$$

is convergent when $y > 0$, we can apply Fubini's theorem to (4.32) to obtain

$$\Phi[\mu]'(z) = 4\pi \sum_{n\geq 1} \int_{\sqcap} \mu(w) n e^{2\pi n(w-\bar{z})} d^2 w \tag{4.34}$$

$$= 4\pi \sum_{n\geq 1} n\langle \mu, e_n\rangle e_n(z). \tag{4.35}$$

Putting $\mu(z) = y^2 \sum_{n\geq 1} c_n e_n$ into (4.34) gives the desired result. $\square$

4.4 Square Integrability and the Invariance of the TZ-pairing

In this section we will prove part (d) of the main theorem. First, we prove

Proposition 4.4.1. *If $\mu \in \infty$ and $t \in [0, 2]$, then $|y^k N_t \mu| \in L^\infty(\mathbb{H}, \mathbb{Z})$ for any $k > 0$.*

Proof. The proof follows exactly that of Proposition 4.2.3, except using N_t in place of P. One sees that Corollary 4.2.1 holds for N_t by simply noting that

$$\sum_{n\geq 1} \frac{|c_n|}{n^s} e^{-2\pi n y} < \sum_{n\geq 1} |c_n| e^{-2\pi n y}$$

in (4.23) $\square$

Proposition 4.4.2. *For $\mu \in L^\infty(\mathbb{H}, \mathbb{Z})$ and $s, t \in [0, 2]$, we have*

(a) $y^{-s} P y^s P\mu \in L^2(\sqcap)$ for $0 \leq s \leq 2$.

(b) $\langle N_s \mu, N_t \mu\rangle = g_{s,t}\langle P\mu, P\nu\rangle$, where $g_{s,t} = \frac{\Gamma(5-s-t)}{6\Gamma(3-s)\Gamma(3-t)}$.

The first part is proven using Lemma 4.2.2. The second part follows from direct computation.

Proof of Proposition 4.4.2(a). By Proposition 4.2.3, we have that for $\mu \in L^\infty(\mathbb{H}, \mathbb{Z})$ and $k > 0$, that $y^{-s+\frac{k}{2}} P y^s P\mu \in L^\infty(\mathbb{H}, \mathbb{Z})$. Therefore,

$$\int_{\sqcap} |y^{-s} P y^s P\mu(z)|^2 (y^k e^{-4\pi k y}) d^2 z \leq \frac{C}{4\pi k}, \tag{4.36}$$

where $C = (||y^{-s+\frac{k}{2}} P y^s P\mu||_\infty)^2$. Using Fubini's theorem and uniform convergence, we get

$$\langle y^{2-s}\sum_{n\geq 1}\frac{c_n}{n^s}e_n, y^{2-s}\sum_{n\geq 1}\frac{c_n}{n^s}e_n(y^k e^{-4\pi k y})\rangle = \sum_{n\geq 1}\int y^{4-2s+k}\frac{|c_n|^2}{n^{2s}}e^{-4\pi(n+k)y} \tag{4.37}$$

$$= \sum_{n\geq 1}\frac{\Gamma(5-2s+k)|c_n|^2}{n^{2s}(4\pi(n+k))^{5-2s+k}}. \tag{4.38}$$

Note that

$$\sum_{n\geq 1}\frac{\Gamma(5-2s+k)|c_n|^2}{n^{2s}(4\pi(n+k))^{5-2s+k}}$$

is non-decreasing as $k\to 0$. By the monotone convergence theorem, we obtain

$$\lim_{k\to 0}\langle y^{2-s}\sum_{n\geq 1}\frac{c_n}{n^s}e_n, y^{2-s}\sum_{n\geq 1}\frac{c_n}{n^s}e_n(y^k e^{-4\pi k y})\rangle = \frac{\Gamma(5-2s)}{(4\pi)^{5-2s}}\sum_{n\geq 1}\frac{|c_n|^2}{n^5} \tag{4.39}$$

The quantity on the right hand side of (4.39) is convergent by Lemma 4.2.2. Noting that $|y^{-s}Py^sP\mu(z)|^2(y^k e^{-4\pi k y})$ is non-dreasing as $k\to 0$, we apply the monotone convergence once again to get

$$||y^{2-s}\sum_{n\geq 1}\frac{c_n}{n^s}e_n||^2 = \frac{\Gamma(5-2s)}{(4\pi)^{5-2s}}\sum_{n\geq 1}\frac{|c_n|^2}{n^5} \tag{4.40}$$

$\square$

Proof of Proposition 4.4.2(b). If $\mu(z) = y^2\sum_{n\geq 1}c_n e_n$, then by Proposition 4.2.2, we have

$$\langle N_s\mu, (N_t\mu)e^{-4\pi k y}\rangle = \langle y^{2-s}\sum_{n\geq 1}\frac{c_n}{n^s}e_n, y^{2-t}\sum_{n\geq 1}\frac{c_n}{n^t}e_n e^{-4\pi k y}\rangle \tag{4.41}$$

$$= \sum_{n\geq 1}\int y^{4-s-t}\frac{|c_n|^2}{n^{s+t}}e^{-4\pi(n+k)y} \tag{4.42}$$

$$= \sum_{n\geq 1}\frac{\Gamma(5-s-t)|c_n|^2}{n^{s+t}(4\pi(n+k))^{5-s-t}}. \tag{4.43}$$

The series in (4.43) is non-decreasing as $k\to 0$. So using monotone convergence, we get

$$\lim_{k\to 0}\langle y^{2-s}\sum_{n\geq 1}\frac{c_n}{n^s}e_n, y^{2-t}\sum_{n\geq 1}\frac{c_n}{n^t}e_n e^{-4\pi k y}\rangle = \frac{\Gamma(5-s-t)}{(4\pi)^{5-s-t}}\sum_{n\geq 1}\frac{|c_n|^2}{n^5}. \tag{4.44}$$

Applying the dominated convergence theorem, we get

$$\langle y^{2-s}\sum_{n\geq 1}\frac{c_n}{n^s}e_n, y^{2-t}\sum_{n\geq 1}\frac{c_n}{n^t}e_n\rangle = \frac{\Gamma(5-s-t)}{(4\pi)^{5-s-t}}\sum_{n\geq 1}\frac{|c_n|^2}{n^5}. \tag{4.45}$$

Putting everything together, we get:

$$\langle N_s\mu, N_t\mu\rangle = \frac{4\Gamma(5-s-t)}{(4\pi)^5\Gamma(3-s)\Gamma(3-t)}\sum_{n\geq 1}\frac{|c_n|^2}{n^5} \tag{4.46}$$

$$= \frac{4\Gamma(5-s-t)}{(4\pi)^5\Gamma(3-s)\Gamma(3-t)}\frac{(4\pi)^5}{\Gamma(5)}\sum_{n\geq 1}\langle P\mu, P\mu\rangle \tag{4.47}$$

$$= g_{s,t}\langle P\mu, P\mu\rangle. \tag{4.48}$$

□

4.5 Self-Adjointness of N_2

In this section, we will prove part (e) of the main theorem.

Proposition 4.5.1. *For $\mu, \nu \in L^\infty(\mathbb{H}, \mathbb{Z})$,*

$$\langle N_2\mu, N_2\nu\rangle = \langle N_2\mu, \nu\rangle. \tag{4.49}$$

The equation (4.49) can be thought of a statement of self-adjointness of the extremal operator N_2 in the sense that it is an “orthogonal projection” of $L^\infty(\mathbb{H}, \mathbb{Z})$ onto the “orthogonal complement” of $\mathcal{N}(\mathbb{H}, \mathbb{Z})$ with respect to the TZ-pairing (see Equation (4.34).) This of course, does not make literal sense since the image of N_2 does not lie in $L^\infty(\mathbb{H}, \mathbb{Z})$. (See Example 4.2.1.) However, on the subspace $L^\infty(\mathbb{H}, \mathbb{Z}) \cap Hol(\mathbb{H}, \mathbb{Z})$, N_2 does act as a projection operator. In fact, pulling back to the punctured disk by the logarithm shows that the integral kernel of N_2 is just the Bergman kernel function of weight 1.

Proposition 4.5.2. *For any $\mu \in L^\infty(\mathbb{H}, \mathbb{Z})$ and $\kappa \in \ker P$, $\langle N_2\mu, \kappa\rangle = 0$.*

Proof of Proposition 4.5.1 using Proposition 4.5.2.

By Proposition 4.5.2, we have

$$\langle \nu - P\nu, N_2\mu\rangle = 0 \tag{4.50}$$

$$\langle \nu, N_2\mu\rangle = \langle P\nu, N_2\mu\rangle. \tag{4.51}$$

Using Proposition 4.4.2(b), we obtain the desired equality as follows.

$$\langle N_2\kappa, N_2\mu\rangle = g(2,2)\langle P\kappa, P\mu\rangle = \frac{g(2,2)}{g(2,0)}\langle P\kappa, N_2\mu\rangle = \frac{g(2,2)}{g(2,0)}\langle \kappa, N_2\mu\rangle = \langle \kappa, N_2\mu\rangle. \tag{4.52}$$

□

To show Proposition 4.5.2, it is sufficient to show that $N_2\mu \in L^1(\sqcap)$ for $\mu \in L^\infty(\mathbb{H}, \mathbb{Z})$. Then, the orthogonality will follow from (1.11).

Lemma 4.5.1. *For $\mu \in L^\infty(\mathbb{H}, \mathbb{Z})$,*

$$\int_\sqcap |N_2\mu|^2 e^{2ky} d^2z < \infty \tag{4.53}$$

for all $0 \leq k < 2\pi$.

Proof. We divide up $\sqcap = R_1 \cup R_2$, where $R_1 = ([0,1] \times [0,1])$ and $R_2 = [0,1] \times [0,\infty)$.

From Proposition 4.2.3, we have that for $s > 0$, $y^s e^{ky} N_2\mu \in L^\infty(R_1)$. Since R_1 has finite area, $\int_{R_1} |y^s H\mu|^2 e^{2ky} d^2z < \infty$.

Suppose that $\mu(z) = y^2 \sum_{n\geq 1} c_n e_n$. Then by Proposition 4.2.2,

$$N_2\mu(z) = \frac{1}{8\pi^2}\sum_{n\geq 1} \frac{c_n}{n^2} e_n,$$

and using the estimate $|c_n| \leq n^3$ for $n > M$, we get

$$8\pi^2 |N_2\mu(z)| \leq \sum_{n\geq 1} \frac{|c_n|}{n^2} e^{-2\pi n y} \leq \sum_{n=1}^{M} \frac{|c_n|}{n^2} e^{-2\pi n y} + \frac{e^{-2\pi(M+1)y}}{(1-e^{-2\pi y})}. \tag{4.54}$$

It follows from this estimate that on R_2, we have $|N_2\mu(z)| < Ce^{-2\pi y}$, so

$$\int_{R_2} |N_2\mu|^2 e^{2ky} dy \leq C \int_{R_2} e^{(-4\pi+2k)y} dy. \tag{4.55}$$

This integral is convergent as long as $k < 2\pi$.

Finally, let

$$f_s = \begin{cases} |y^s N_2\mu|^2, & z \in R_1 \\ |N_2\mu|^2, & z \in R_2 \end{cases}. \tag{4.56}$$

f_s is non-decreasing as $s \to 0$, $f_s \to |N_2\mu|^2$, and

$$\int f_s e^{ky} \to C' \sum_{n \geq 1} \frac{|c_n|^2}{n^4(4\pi n - 2k)} \tag{4.57}$$

for some constant C', which is convergent by limit comparison. □

Proposition 4.5.3. $N_2\mu \in L^1(\sqcap)$.

Proof. By Hölder's inequality, we have

$$\left(\int_{\sqcap} |N_2\mu| d^2z\right)^2 \leq \left(\int_{\sqcap} |N_2\mu|^2 e^{2ky} d^2z\right)\left(\int_{\sqcap} e^{-2ky} d^2z\right). \tag{4.58}$$

□

Since $N_2\mu$ is anti-holomorphic and $L^1(\sqcap)$, it follows from (1.11) that

$$\langle \nu, N_2\mu \rangle = 0 \tag{4.59}$$

for $\nu \in \mathcal{N}(\mathbb{H}, \mathbb{Z})$.

Chapter 5

Conclusion and Conjecture

The contents of this chapter are work that is still in progress.

In this chapter, we will use the tools from chapters 3 and 4 to study the Takhtajan-Zograf metric on $T(\mathbb{Z})$.

Definition 5.0.1. *We define the Takhtajan-Zograf metric to be the right invariant metric h on $T(\mathbb{Z})$ with its value at $[0] \in T(\mathbb{Z})$ given by*

$$h_{\mu\overline{\nu}}([0]) = \langle \mu, \nu \rangle, \tag{5.1}$$

where $\mu, \nu \in \Omega^{-1,1}(\mathbb{H}, \mathbb{Z})$. μ, ν represent tangent vectors at $T_{[0]}T(\mathbb{Z})$, and $\langle *, * \rangle$ is the TZ-pairing. (See chapter 4).

Theorem 4.0.1(d) tells us that h is convergent.

For $\mu, \nu, \kappa \in \Omega^{-1,1}(\mathbb{H}, \mathbb{Z})$ with $||\kappa||_\infty < 1$, μ, ν representing tangent vectors at $T_{[\kappa]}T(\mathbb{Z})$, the Takhtajan-Zograf metric can be represented in the following way

$$h_{\mu\overline{\nu}}(\kappa) = \langle PR(\mu, \kappa), PR(\nu, \kappa) \rangle, \tag{5.2}$$

where R is defined as in section 1.3.

One could also define a right invariant metric g on $T(\mathbb{Z})$ by representing the tangent vectors of $T(\mathbb{Z})$ using $H^{-1,1}(\mathbb{H}, \mathbb{Z})$ instead of $\Omega^{-1,1}$ by

$$g_{\mu\overline{\nu}}([0]) = \langle \mu, \nu \rangle, \tag{5.3}$$

where $\mu, \nu \in H^{-1,1}(\mathbb{H}, \mathbb{Z})$. And again, convergence is guaranteed by Theorem 4.0.1(d).

Similar to (5.2), for $\mu, \nu, \kappa \in H^{-1,1}(\mathbb{H}, \mathbb{Z})$ with $||\kappa||_\infty < \alpha$, μ, ν representing tangent vectors in $T_{[\kappa]}T(\mathbb{Z})$, we get

$$g_{\mu\overline{\nu}}(\kappa) = \langle NR(\mu, \kappa), NR(\nu, \kappa) \rangle. \tag{5.4}$$

Proposition 5.0.4. *Let N be as in* (3.22)*, and N_s as in Theorem 3.0.1. Then for $\mu \in L^\infty(\mathbb{H},\mathbb{Z})$, $N\mu = N_1\mu$.*

Proof. By Proposition 3.3.3(a) and Theorem 4.0.1(c), we have that $\ker N = \ker N_1 = \ker P$. So it is sufficient to check the equality on $\Omega^{-1,1}(\mathbb{H},\mathbb{Z})$. Let $\mu(z) = y^2 \sum_{n\geq 1} c_n e_n$. One can obtain the Fourier expansion of $N\mu$ using Proposition 3.3.2 and Proposition 4.2.2 for $s = 0$. The resulting series is equal to the one in Proposition 4.2.2 for $s = 1$. $\square$

In other words, $\eta \circ D\Theta_0 = N_1$.

Proposition 5.0.5. $g = 3h$

Proof. Since g and h are both right-invariant, it is sufficient to show that the equality holds at $p = [0]$. If $\mu, \nu \in \Omega^{-1,1}(\mathbb{H}, G)$ represent tangent vectors on $T_{[0]}T(\mathbb{Z})$, then $N\mu, N\nu \in H^{-1,1}(\mathbb{H},\mathbb{Z})$ represent the same vectors in the θ-coordinates. Then the statement of the proposition follows directly from Theorem 4.0.1(d) by putting $s = t = 1$. $\square$

By Theorem 4.0.1(d) one can freely change the projection operator N or P in Definition 5.0.1 with any of the N_t, in particular with N_2. Using the pre-Bers harmonic Beltrami differentials allows us to work with simpler variation formulas as we will see below.

Conjecture 5.0.1. *Let $\mu, \nu, \kappa \in H^{-1,1}(\mathbb{H},\mathbb{Z})$. Then,*

$$\frac{\partial g_{\mu\overline{\nu}}}{\partial \kappa}(0) = 4\int_{\sqcap} N_2\mu(z)\ \overline{\nu}(z)\ N_2\kappa(z)\ d^2z, \tag{5.5}$$

where

$$\frac{\partial g_{\mu\overline{\nu}}}{\partial \kappa}(0) = \frac{\partial}{\partial \epsilon}\bigg|_{\epsilon=0} g_{\mu\overline{\nu}}(\epsilon\kappa). \tag{5.6}$$

Using Theorem 4.0.1:

$$\begin{aligned}
\frac{1}{2}g_{\mu\overline{\nu}}(\kappa) &= \frac{1}{2}\langle NR(\mu,\kappa), NR(\nu,\kappa)\rangle && (5.7)\\
&= \langle N_2R(\mu,\kappa), N_2R(\mu,\kappa)\rangle = \langle N_2R(\mu,\kappa), R(\nu,\kappa)\rangle && (5.8)\\
&= \int_{\sqcap} N_2R(\mu,\kappa)\circ w_\kappa(z)\overline{\nu(z)}((\overline{w_\kappa})_{\bar z}(z))^2 d^2z && (5.9)
\end{aligned}$$

Ignoring all convergence issues, replacing κ with $\epsilon\kappa$ then taking the derivative at $\epsilon = 0$ gives

$$\frac{1}{2}\frac{\partial g_{\mu\overline{\nu}}}{\partial\kappa}(0) = \int_{\sqcap}\left(\left.\frac{\partial}{\partial\epsilon}\right|_{\epsilon=0} N_2 R(\mu,\epsilon\kappa)\circ w_{\epsilon\kappa}\right)\overline{\nu(z)}d^2z + 2\int_{\sqcap} N_2\mu\ \overline{\nu}\ (\overline{\Phi[\kappa]'})d^2z \tag{5.10}$$

By Proposition 4.3.2, we can put $\overline{\Phi[\kappa]'} = N_2\kappa$. And with the following conjecture, we get the desired expression.

Conjecture 5.0.2. *Let $\mu,\kappa \in H^{-1,1}(\mathbb{H},\mathbb{Z})$. Then,*

$$\left.\frac{\partial}{\partial\epsilon}\right|_{\epsilon=0} N_2 R(\mu,\epsilon\kappa)\circ w_{\epsilon\kappa} = 0 \tag{5.11}$$

There are some computations using the integral kernel of N_2 (see Proposition 4.3.3) that suggest eq. (5.11), but it has been omitted here.

Note 5.0.1. *Conjecture 5.0.2 is a complete analogue of the first part of [11, Prop 7.1], where it is shown that*

$$\left.\frac{\partial}{\partial\epsilon}\right|_{\epsilon=0} N_0 R(\mu,\epsilon\kappa)\circ w_{\epsilon\kappa}\frac{(\overline{w_{\epsilon\kappa}})_{\bar{z}}}{(w_{\epsilon\kappa})_z} = 0 \tag{5.12}$$

for $\mu,\kappa \in \Omega{-1,1}(\mathbb{H})$. The factor $\frac{(\overline{w_{\epsilon\kappa}})_{\bar{z}}}{(w_{\epsilon\kappa})_z}$ can be omitted in (5.11) *because of Corollary 3.3.2(d).*

The expression in (5.13) is obviously symmetric in μ and κ, which implies that g is Kähler. It also has the advantage that it is simpler than the formula in proof of Lemma 3 in [10], which contains the Green's function for the operator $\left(\Delta_0 + \frac{1}{2}\right)$.

Using integration by parts on (5.13), one can arrive at a more symmetric form for the first derivative of the Takhtajan-Zograf metric:

Conjecture 5.0.3. *For $\mu,\nu,\kappa \in H^{-1,1}(\mathbb{H},\mathbb{Z})$,*

$$\frac{\partial g_{\mu\overline{\nu}}}{\partial\kappa}(0) = 4\int_{\sqcap} N_2\mu(z)\ \overline{N_2\nu(z)}\ N_2\kappa(z)\ d^2z. \tag{5.13}$$

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