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# GROUP NORMS AND THEIR DEGENERATION
IN THE STUDY OF
# PARALLELISM

A Dissertation Presented

by

**Pedro Antonio Ricardo Martín Solórzano Mancera**

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in Partial Fulfillment of the
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in

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# Abstract of the Dissertation

## Group norms and their degeneration in the study of parallelism

by

**Pedro Antonio Ricardo Martín Solórzano Mancera**

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Endowing total spaces of vector bundles over Riemannian manifolds with a Riemannian structure sets them within the realm of Gromov's Theory of Convergence. The particular choice of Riemannian metric is a generalization of the one studied by Sasaki on tangent bundles. In this work, the "static" and the "dynamic" properties of said bundles are studied.

Here "static" means the metric and differential geometric properties of the interplay between the Riemannian metrics of the base and the total space. Differential geometrically, the fibers are known to be flat and totally geodesic. Metrically, it is shown that their departure from convexity is controlled quite explicitly by the concept of *holonomic spaces*. A holonomic space is a triple $(V, H, L)$, where $V$ is a normed vector space, $H$ is a group of norm preserving linear maps, and $L$ is a group norm, together with a convexity assumption. In the particular geometric setting, $V$ is a fixed fiber of a vector bundle, $H$ is the holonomy group at that fiber, and $L$ is a geometric group norm, *the length-norm*, obtained by looking at the "smallest loop that generates a given holonomy element". The degenerations of these group-norms are fundamental to determining the "dynamic" properties. It is also seen that by restricting the class of maps to *geodesic Riemannian maps*, the Sasaki metric construction renders the tangent bundle a metric functor.

The "dynamic" perspective is to analyze the convergence of these metrics of Sasaki type under the Gromov-Hausdorff topology. A pre-compactness result is obtained under

the assumption of a uniform upper bound on rank. Furthermore, the limiting spaces possess a surprisingly rich structure.

Limits of Sasaki-type metrics are submetries over the limit of their bases and retain a notion of re-scaling and a compatible norm (understood here as a "distance to zero" of sorts). The fibers of which are conical are worst: in fact, their topology is that of a quotient of Euclidean space by a compact group of orthogonal transformations. This group, called *the wane group*, is essentially obtained by looking at limits of holonomy elements with waning length-norm; it depends on the base point, and thus the limits in general fail to be locally trivial. These groups will further play a rôle for the uniqueness problem of a limiting notion of parallelism, also introduced here.

The length-norm studied here had been overlooked before perhaps due to its lack of continuity with respect to the standard Lie group topology on the holonomy groups. However, tautologically, a group norm is continuous with respect to the metric topology induced by itself. This topology, seemingly artificial, also has some of the nice properties one should require a topological transformation group to have; even certain "wrong way" inheritance is exhibited.

Overall, this work dwells upon the interactions between the metric properties and the algebraic nature of vector bundles, as well as their possible degeneration in a limiting process.

*À ma Stephanie-Félicie Poterin du Motel,*
*qui j'eſpère rencontrer un jour.*

*To thoſe already departed*
*—wherever they may be—,*
*who will always guide me.*


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## Ackowledgements

The present work marks the conclusion of a path that I decided to follow not really knowing what I was getting into. And I was in for manifold surprises, hard times, and, after all, the opportunity to rediscover myself. I would not have accomplished all this if not for the infinite support that I have received from many.

From the start, the Mathematics Department in Stony Brook was always very welcoming and has been a home for me. I can say that I truly regard its members as family. I believe that pursuing a doctoral degree is not only about being smart or intelligent with regards to the topic in question; in my experience, the emotional component was also fundamental.

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# Introduction

> ...la géométrie euclidienne classique peut être considérée comme une magie; au prix d'une distorsion minime des apparences (le point étendue, la droite sans épaisseur...), le langage purement formel de la géométrie décrit adéquatement la réalité spatiale. En ce sens, on pourrait dire que *la géométrie est une magie qui réussit*. J'aimerais énoncer une réciproque: toute magie, dans la mesure où elle réussit, n'est-elle pas nécessairement une géométrie?
>
> *Stabilité structurelle et morphogénèse.*
> René Thom

The question of determining how do structures degenerate in a limiting process can sometimes say more about the structures themselves. The Gromov-Hausdorff convergence of Riemannian manifolds was introduced by Gromov in the late 1970's as a way to achieve this program. Unlike smooth convergence, limit spaces under Gromov-Hausdorff convergence need not be smooth or even Lipschitz. Adding conditions of uniform curvature bounds to the sequence of metrics one can control the regularity of the limit spaces to some extent. Work of Cheeger and Colding [11, 12, 13] showed that the regular set of limit spaces with Ricci curvature bounded below is dense and a $C^{1,\alpha}$ submanifold. Adding the stronger condition of one- or two-sided bounds on sectional curvature, there have been significantly stronger results; many structural results have been obtained by Cheeger [10], Cheeger, Fukaya, and Gromov [15], Yamaguchi [52], Shioya and Yamaguchi [44], as well as upcoming work of Rong [40]. Only by assuming both a sectional curvature and lower bound on volume or that the sequence is Einstein with a lower bound on injectivity radius does one obtain limits which are $C^{1,\alpha}$ manifolds, as seen by Anderson [3], and Anderson and Cheeger [4]. However, it should be noted that a common feature is to have certain assumptions on the curvature, injectivity radius, etc.

Vector bundles with metric connections (i.e. Euclidean bundles with compatible connections) have natural metrics Riemannian metrics on their total spaces called metrics of Sasaki-type (see Definition 3.28). These metrics were first introduced on tangent bundles by Sasaki [42] and for more general vector bundles by Benyounes, Loubeau, and Wood [5], where they introduce a two-parameter family of metrics that include a metric know to exist by the results of Cheeger and Gromoll [14]. These metrics coincide with the full classification of the natural metrics on tangent bundles given by Kowalski and Sekizawa [30]. It should be noted that the use of the word *natural* coincides with its usage in the classification of natural bundles given by Terng [47] as part of her doctoral dissertation; and thus the results stated here could be stated as certain continuity properties of these *natural* bundle functors. In Chapter 3 it is seen that in the case of the tangent bundle, the class of maps for which the Sasaki type metric yields a functor contains the Riemannian maps (a generalization of isometries, isometric immersions and submersions given by Fischer [18]) which have totally geodesic image and totally geodesic fibers.

The first explicit rendering of the Cheeger-Gromoll metric, together with a systematic study of the Sasaki metric was given by Musso and Tricerri [35]. Later, it has been developed by many, in particular by Abbassi and Sarih [1]. Most of the attention has been for the case of the tangent bundle. In this case, on the tangent bundle, $TM$, over a Riemannian manifold, $M$, with the standard connection, the Sasaki metric on $TM$ is uniquely defined so that $\pi : TM \to M$ be a Riemannian submersion where the horizontal lifts of curves are simply parallel translations along curves and, furthermore, that the individual tangent spaces be totally geodesic flats; i.e with the intrinsic distance, the fibers are isometric to Euclidean space. However, with the restricted metric, distances between points in a fiber may be achieved by paths that leave the fibers; in some cases even by horizontal paths, thus relating the problem to the semi-Riemannian context. The fibers with the restricted metrics are holonomic spaces, whose metrics depends on the holonomy group and the shortest lengths of curves representing each holonomy element (See Definition 3.31).

At any given point on a Riemannian manifold there are three pieces of information that interplay: the tangent space, as a normed vector space $V$; the holonomy group, as a subgroup $H$ of the isometry group of the fiber; and a group-norm $L$ on the holonomy group, given by considering the infimum

$$L(a) = \inf_{\gamma} \ell(\gamma) \tag{0.1}$$

of the lengths of the loops $\gamma$ that yield a given holonomy element $a$.

A *holonomic space* is a triplet $(V, H, L)$ consisting of a normed vector space $V$; a group $H$ of linear isometries of $V$; and a group-norm $L$ on such group; satisfying a local convexity property that relates them: For any element $u \in V$ there is a ball around it such that for any two elements $v, w$ in that ball the following inequality holds:

$$\|v-w\|^2 - \|av-w\|^2 \leq L^2(a) \tag{0.2}$$

for any element $a \in H$. See Definition 2.1.

By considering the following distance function, $d_L : V \times V \to \mathbb{R}$,

$$d_L(u,v) = \inf_a \sqrt{L^2(a) + \|au-v\|^2}, \tag{0.3}$$

one gets a modified metric-space structure on $V$ that sheds light on the definition of a holonomic space:

**Theorem A** (Theorem 2.9). *A triplet $(V,H,L)$ is a holonomic space if and only if $d_L$ is locally isometric to usual distance on $V$.*

The measure of nontriviality of a holonomic space is controlled by the *holonomy radius*, a continuous function on $V$ given by the supremum of the radii of balls for which the local convexity property is satisfied. This function is finite if and only if $H$ is nontrivial [Proposition 2.13].

Considering the holonomy radius at the origin already yields some information on the group-norm in the case when the normed vector space is actually an inner product space. Namely the following result.

**Theorem B.** *Given a holonomic space $(V,H,L)$, the identity map on $H$ is Lipschitz between the left invariant metrics on $H$ induced by $L(a^{-1}b)$ and $\sqrt{2\|a-b\|}$ respectively, where $\|\cdot\|$ stands for the operator norm. Moreover, the dilation is precisely the reciprocal of the holonomy radius $\rho_0$ at the origin of $V$.*

$$\sqrt{2\|a-b\|} \leq \frac{1}{\rho_0} L(a^{-1}b). \tag{0.4}$$

This is a consequence of Theorem 2.18 and Corollary 2.19 in Chapter 2.

Recall that a Sasaki-type metric $\mathrm{G}$ on a Euclidean vector bundle with compatible connections is given in terms of the connection map $\kappa : TE \to E$, uniquely determined by requiring that $\kappa(\sigma_* x) = \nabla^E_x \sigma$, as

$$\mathrm{G}(\xi,\eta) = g(\pi_*\xi, \pi_*\eta) + h(\kappa\xi, \kappa\eta), \tag{0.5}$$

for vectors $\xi, \eta \in TE$.

Given these considerations one gets the following result.

**Theorem C.** *Given a Euclidean vector bundle with a compatible connection over a Riemannian manifold, each point in the base space has a naturally associated holonomic space, with the fiber over that point being the underlying normed vector space.*

*Furthermore, if the total space is endowed with the corresponding Sasaki-type metric then the aforementioned modified metric-space structure coincides with the restricted metric on the fibers from the metric on the total space.*

This result is stated more precisely in Proposition 3.31, Theorem 3.53, and Theorem 3.56. The group-norm in Theorem C is given precisely by (0.1). The study of this group-norm was already hinted in the work of Tapp [46] and Wilkins [50].

This group-norm induces a new topological group structure on the holonomy group that makes the the group-norm continuous while retaining the continuity of the holonomy action (Lemma 2.6). It should be noted that with the standard topology (i.e. that of a Lie group) of the holonomy group, this group-norm is not even upper semicontinuous. Wilkins [50] had already noted this (an immediate example is to consider a metric that is flat in a neighborhood of a point and consider the group-norm associated at that point). He proved that if the Lie group topology is compact then —in the language of this report— the group-norm topology is bounded, which is a surprising result given that the group-norm topology is finer.

Tapp [46] defines a 'size' for a given holonomy element as an infimum over *acceptable* smooth metrics on the holonomy group (quoted here as Theorem 4.4). As such, he proved that holonomy 'size' and the length group-norm (0.1) are comparable up to a constant that depends only on the base space and the norm of the curvature (see Theorem 4.5). These results are discussed in more detail in Chapter 4.

Here it is only assumed that there is a sequence of Riemannian manifolds which converges in the pointed Gromov-Hausdorff sense to a limit space; the sequence of tangent bundles over those Riemannian manifolds —or more generally, an arbitrary sequence of vector bundles over the converging sequence of Riemannian manifolds— is then analyzed. Throughout this report, none of the usual uniform bounds on curvature, diameter, volume or injectivity radius are assumed. Only those properties which can be derived from the pointed Gromov-Hausdorff convergence of the base spaces are used.

It is worth noting that the results discussed here differ from the very interesting approach taken by Rieffel [39]. He introduces a Lipschitz seminorm of a very natural space of matrix-valued functions to control distances between vector bundles. In essence, he regards Euclidean vector bundles as a certain type of map into the space of self-adjoint idempotent matrices. In the case of vector bundles over smooth manifolds, this can be easily be seen as maps into a suitable Grassmannian. Under the assumption that two (compact) metric spaces be $\varepsilon$-close, he gives a correspondence between their vector bundles with control on their Lipschitz seminorm on any metric on their disjoint union that makes said spaces $\varepsilon$-Hausdorff close.

In Example 5.36, if a single compact $n$-dimensional Riemannian manifold is re-scaled so that it converges in the Gromov-Hausdorff sense to a single point, then the Gromov-Hausdorff limit of the tangent bundles endowed with their Sasaki metrics is homeomorphic to $\mathbb{R}^n/H$ where $H$ is the closure of the holonomy group of $M$ inside the orthogonal group $O(n)$. Intuitively, horizontal paths became so short under rescaling that in the limit, vectors related by a horizontal curve are no longer distinct points. In contrast, if one has a sequence of standard 2-dimensional flat tori collapsing to a circle (Example 5.40), then the limit of the tangent bundles is $S^1 \times \mathbb{E}^2$, where the fibers are Euclidean since the holonomy group is trivial in this setting.

Observe also that there are sequences of Riemannian manifolds which converge in the Gromov-Hausdorff sense whose tangent bundles do not converge (Example 5.40). Nevertheless a precompactness theorem for tangent bundles and other vector bundles can be obtained:

**Theorem D** (Theorem 5.14)**.** *Given a precompact collection of (pointed) Riemannian manifolds $\mathcal{M}$ and a positive integer $k$, the collection* $\mathrm{BWC}_k(\mathcal{M})$ *of vector bundles with metric connections of rank $\leq k$ endowed with metrics of Sasaki-type is also precompact. The distinguished point for each such bundle is the zero section over the distinguished point of their base.*

The assumption that the rank be bounded is easily satisfied for natural bundles over a convergent sequence of Riemannian manifolds (such as tangent bundles, cotangent bundles, or combinations thereof).

The notion of holonomic space metric introduced in Chapter 2 is used to analyze the convergence of these bundles by analyzing the fiberwise behavior of a convergent sequence of metrics of Sasaki-type. This approach proved to be quite useful in view of the following results.

**Theorem E** (Propositions 5.16, 5.17, and 5.18)**.** *For any sequence of Riemannian manifolds $\{(X_i, p_i)\}$ converging to $(X_\infty, x_\infty)$ consider a convergent family of bundles with metric connection $(E_i, h_i, \nabla_i)$ over it converging to $(E_\infty, y_\infty)$. Then there exist continuous maps $\pi_\infty : E_\infty \to X_\infty$, $\varsigma_\infty : X_\infty \to E_\infty$, $\mu_\infty : E_\infty \to \mathbb{R}$, and a subsequence, without loss of generality also indexed by $i$, such that:*

1. *the projection maps $\pi_i : E_i \to X_i$ converge to $\pi_\infty : E_\infty \to X_\infty$, which is also a submetry with equidistant fibers;*

2. *the zero section maps $\varsigma_i : X_i \to E_i$ converge to $\varsigma_\infty : X_\infty \to E_\infty$, which is also a isometric embedding;*

3. $\pi_\infty \circ \varsigma_\infty = id_{X_\infty}$;

4. *the maps $\mu_i : E_i \to \mathbb{R}$, given by*

$$\mu_i(u) = d_{E_i}(u, \varsigma_i \circ \pi_i(u)) = \sqrt{h_i(u,u)},$$

*converge to $\mu_\infty : E_\infty \to \mathbb{R}_{\geq 0}$ also given by*

$$\mu_\infty(y) = d_{E_\infty}(y, \varsigma_\infty \circ \pi_\infty(y));$$

5. *The scalar multiplications on $E_i$ converge to an $\mathbb{R}$–action on $E_\infty$ such that*

$$\mu_\infty(\lambda u) = |\lambda| \mu_\infty(u)$$

*For any $\varepsilon > 0$ and for any sequence $\{q_i\}$, $q_i \in X_i$, converging to $q \in X_\infty$,*

6. $\pi_i^{-1}(B_\varepsilon(q_i)) \xrightarrow{pt-GH} \pi_\infty^{-1}(B_\varepsilon(q))$;

7. $\pi_i^{-1}(q_i) \xrightarrow{pt-GH} \pi_\infty^{-1}(q)$.

As mentioned before, Example 5.36 already suggests that the holonomy group must play a significant rôle. With this in mind, the following result yields more information about fibers of the limiting map.

**Theorem F** (Theorem 5.19). *Let $\pi_i : E_i \to X_i$ be a convergent sequence of vector bundles with bundle metric and compatible connections $\{(E_i, h_i, \nabla_i)\}$, with limit $\pi : E \to X$. Then there exists a positive integer $k$ such that for any point $p \in X$ there exists a compact Lie group $G \leq O(k)$, that depends on the point, such that the fiber $\pi^{-1}(p)$ is homeomorphic to $\mathbb{R}^k/G$, i.e. the orbit space under the standard action of $G$ on $\mathbb{R}^k$.*

The group $G$ here is described explicitly in Theorem 5.5 and will be called the *wane group* at $x \in X$ because of another precise description of the fibers as $V/G_0$ given in Theorem 5.4 where $G_0$ is defined in terms of the metrics $d_i$ (on the fibers of the converging sequence of vector bundles) by essentially looking at sequences of holonomy elements with waning norm.

It is important to remark that the wane group G truly depends on the base point; thus the limit $\pi : E \to X$ need not be a fiber bundle. This occurs for example when a sequence of Riemannian manifolds converge smoothly everywhere except at a point and at the point they develop a conical singularity. In Chapter 3 such examples are produced.

The notion of *holonomy radius at a point* on a Riemannian manifolds is defined to be the holonomy radius at the origin of the corresponding tangent space. Since for metrics of Sasaki-type the fibers of the vector bundle in cosideration are totally geodesic and flat, it makes sense to consider the following definition.

**Definition.** *Consider a vector bundle $E$ with metric and connection over a Riemannian manifold. The* holonomy radius *of a point $p$ in the base is the largest $R > 0$ such that the restricted metric on $B_R(0_p) \cap E_p \subseteq E$ is Euclidean.*

For a more technical definition see Definition 2.2. Now, if one is willing to assume some further restriction on a convergent sequence of manifolds, a uniform lower bound on their holonomy radii yields the following.

**Theorem G** (Theorem 5.22)**.** *Let $\pi_i : E_i \to X_i$ be a convergent sequence of vector bundles with bundle metric and compatible connections $\{(E_i, h_i, \nabla_i)\}$, with limit $\pi : E \to X$. Suppose further that there exists a uniform positive lower bound for the holonomy radii of $\pi_i : E_i \to X_i$. Then the fibers of $\pi_\infty$ are vector spaces.*

This is somewhat surprising since one is only controlling the information near the origin.

Furthermore, there is a natural way to define a notion of parallelism on these limit spaces by considering horizontal curves.

**Definition.** *Given a submetry $\pi : Y \to X$, a curve in $Y$ is horizontal if and only if its length is equal to the length of its projection in $X$.*

The collection of horizontal curves over a given curve $\alpha$ in $X$ gives a relation between the fibers the endpoints of $\alpha$. For loops at a point, it follows that the set of parallel translates form a *-semigroup, which will be called the *Holonomy monoid* of $\pi$, because it generalizes the holonomy group, yet it is not necessarily a group.

In particular, the limit spaces $\pi : E \to X$ satisfy very nice properties summarized in the next result.

**Theorem H** (Corollary 6.7)**.** *Let $\pi_i : E_i \to X_i$ be a convergent sequence of vector bundles with bundle metric and compatible connections $\{(E_i, h_i, \nabla_i)\}$, with limit $\pi : E \to X$. Given any curve $\alpha : I \to X$ and a point $u \in \pi^{-1}(\alpha(0))$ there exists a parallel translate $\gamma$ of $\alpha$ with initial point $u$, furthermore the norm is constant along $\gamma$ and any re-scaling of $\gamma$ is also a parallel translate of $\alpha$.*

The non uniqueness of parallel translates is exactly encoded by the lack of invertibility of holonomy elements (see Theorem 6.12). Also, a necessary condition for having uniqueness for this weak notion of parallelism is given by whether the wane groups are conjugates of each other or not (see Theorem 6.13).

Some questions were left unanswered (e.g. that of sufficient conditions for uniqueness of parallel translates). In the final chapter of this report (Chapter 7), some questions, as leads for future directions, are posed. In particular, these are of two natures: static and dynamic.

On the static side, the question of computing the length-norm and the holonomy radius remains. The former is a question that relates to the isoperimetric problem (as seen in Section 3.7) and the latter ought to be related to curvature in a more direct way.

The dynamic considerations suggest that the wane groups detect the emergence of singularities. Can they not only detect but also distinguish them? If so, is there a stratification on the limit spaces (in terms of wane groups) such that the strata are smooth?

# Chapter 1

# Hors d'Œuvre

> Tudo quanto o homem expõe ou exprime é uma nota à margem de um texto apagado de todo. Mais ou menos, pelo sentido da nota, tiramos o sentido que havia de ser o do texto; mas fica sempre uma dúvida, e os sentidos possíveis são muitos.
>
> *Livro do Desassossego*
> Fernando Pessoa

This chapter is devoted to the introduction of all the terminology and the notation that will be used later on. Most of the topics treated here are, to a certain degree, elementary yet seemingly disparate. Nevertheless, they will intertwine in manifold ways. The study the metric structure of Riemannian spaces is now also very closely related to the study of length spaces in general. The notion of convergence of metric spaces will be a central one; however, even before considering limits, certain geometric properties can be translated into algebra through the study of groups the arise geometrically. These groups not only act naturally, but will also come equipped with metric structures themselves, namely through certain group-norms. Lastly, once limits are considered, some of this algebraic structures degenerate into weaker ones. As an act of justice —or by divine intervention— other structures appear.

**To the experienced reader:** The topics discussed here are the following: differential manifolds, group norms (i.e left-invariant metrics on topological groups), semimetrics (and the quotient by the identification of zero-distance-apart points), certain categorical properties of relations (i.e. subsets of cartesian products of sets, their compositions, etc.), and, lastly, the rudiments of the theory of convergence of metric spaces introduced by Gromov in the late 1970's.

## 1.1 The differential assumption.

The ultimate goal of this report is to describe certain limiting structures occurring on limits of Riemannian manifolds. Therefore the notion of a differential manifold will be freely used, as it is central to the results presented here. For the sake of completeness, a few remarks are presented here. The properties listed here are precisely those of interest in the sequel. A basic familiarity with the differentiability of functions between Euclidean spaces will be assumed.

**1.1 Definition.** A *differential manifold* is a paracompact Hausdorff locally Euclidean topological space $M$ endowed with a fix maximal atlas: A collection $\{(U, \varphi_U)\}$ of pairs where $U \subseteq M$ is open and $\varphi_U : U \to \mathbb{R}^n$ is a homeomorphism, such that for any $p \in M$ there exists $(U, \varphi)$ with $p \in U$ and such that whenever the corresponding open sets $U, V$ of two such pairs have non trivial intersection, the map $\varphi_U \circ \varphi_V^{-1}$ is a diffeomorphism when restricted to $\varphi_V(U \cap V)$.

If $M$ is connected it follows that $n$ is necessarily constant. Such constant is called the dimension of $M$. Henceforth, all differential manifolds will be assumed to be connected.

**1.2 Definition.** A *smooth map* between two differential manifolds $M$ and $N$ is a continuous map $\varphi : M \to N$ such that $\varphi_V \circ \varphi \circ \varphi_U^{-1}$ is smooth, whenever the latter composition makes sense.

**1.3 Definition.** A tangent vector at a point $p$ on a differential manifold $M$ is an equivalence class $\dot{\alpha}(0) := [\alpha]$ of curves $\alpha : (-\varepsilon, \varepsilon) \to M$, such that $\alpha(0) = p$ and such that for any $(U, \varphi_U)$ the maps $\varphi_U \circ \alpha$ agree to first order at 0.

The set of tangent vectors at a point $p$ is a vector space of the same dimension of the manifold $M$ and will be called the *tangent space* at $p$ and will be denoted by $M_p$ or $T_pM$.

**1.4 Definition.** Given a smooth map $\varphi : M \to N$ between differential manifolds, the *differential of $f$* is the linear map $f_* : T_pM \to T_{\varphi(p)}N$, given by $f_*[\alpha] = [f \circ \alpha]$.

The collection $TM$ of all tangent spaces over a given $n$-dimensional manifold is naturally a differential manifold of dimension $2n$ locally modeled by $(\pi_M^{-1}(U), (\varphi_U \circ \pi_M) \times (\varphi_U)_*)$, where $\pi_M[\alpha] = \alpha(0)$ is the *canonical projection*. The latter will necessarilly be a smooth map. The space $TM$ is called the *tangent bundle*. With this in mind, it follows that the map $f_*$ is smooth when regarded as a map between tangent bundles, thus yielding a natural[1] functor from the category of differential manifolds and smooth maps to itself. In this language it also follows that $\pi_M$ is a natural transformation.

The structures that render these spaces metric will be analyzed in Chapter 3.

[1] Naturality is a rigorous concept studied by Terng [47] in her doctoral dissertation.

## 1.2 Distances: metrics, et cetera.

The next primary concept is that of distance. Throughout this report, measuring distances between objects, be it points in a metric spaces, elements in a group, or even between spaces, can be described not only as means to an end but also as an end on its own. Because of this, the notion of (semi-) metric is quickly reviewed here.

**1.5 Definition.** Let $X$ be a set together with a function from $d : X \times X \to \mathbb{R}$. $(X,d)$ is a *semimetric space* if $d$ is

- nonnegative: $d(x,y) \geq 0$;
- symmetric: $d(x,y) = d(y,x)$;
- reflexive: $d(x,x) = 0$; and
- satisfies the triangle inequality $d(x,y) \leq d(x,z) + d(z,y)$.

It is called a metric space if it further satisfies

- the identity of indiscernibles, $d(x,y) = 0$ only if $x = y$.

This last condition is the only that is not immediately preserved under limits. Yet, even if under a limiting process indiscernibles arise, the following process identifies them without loosing any other information.

**1.6 Proposition** (see [51]). *Given a semimetric space $(X,d)$, let $x \sim y$ if $d(x,y) = 0$. Then $X' = X/\sim$ is a metric space with metric, $d'$,*

$$d'([x],[y]) = d(x,y) \tag{1.1}$$

*for any choice of representatives. Also, the canonical projection map is open and continuous with the quotient topology.*

## 1.3 Groups and their norms.

As mentioned in the introduction, groups have been an important tool to understanding geometric properties of spaces with shape. In return, in this section, a particular way of endowing groups with a geometric structure will be analyzed. Not only as entertainment for the souls of the mathematically oriented, but especially since it will be seen in the sequel that perhaps —per haps?— the most natural way to introduce a notion of distance for certain geometric groups (namely the holonomy groups, to be discussed in Chapter 3) is through the notion of group-norm.

**1.7 Definition.** Let $G$ be any group. A *group-norm* on $G$ is a function $N : G \to \mathbb{R}$ that satisfies the following properties.

1. Positivity: $N(A) \geq 0$.
2. Non-degeneracy: $N(A) = 0$ iff $A = id_V$.
3. Symmetry: $N(A^{-1}) = N(A)$.
4. Subadditivity ("Triangle inequality"): $N(AB) \leq N(A) + N(B)$.

**1.8 Example.** Let $V$ be a normed vector space and let $G$ be a subgroup of the group of norm preserving automorphisms of $V$. Then $N(A) = \|id_V - A\|$, the operator norm, is a group-norm.

**1.9 Example.** Let $f : [0, \infty) \to [0, \infty)$ be any non-decreasing subadditive function, $f(t+s) \leq f(t) + f(s)$, with $f(0) = 0$. Let $N : G \to \mathbb{R}$ be any group-norm on $G$. Then $f \circ N$ is also a group-norm on G.

**1.10 Proposition.** *A group G together with a group-norm N becomes a topological group with the left invariant metric induced by*

$$d(A, B) = N(A^{-1}B). \tag{1.2}$$

*Proof.* Left-invariance follows from the fact that $(CA)^{-1}CB = A^{-1}B$. Now, the map

$$(A, B) \mapsto A^{-1}B$$

is continuous since

$$\begin{aligned} d(A^{-1}B, C^{-1}D) &= N(B^{-1}AC^{-1}D) \\ &\leq N(A^{-1}B) + N(C^{-1}D) = d(A, B) + d(C, D) \\ &\leq \sqrt{2}\sqrt{d^2(A, B) + d^2(C, D)}. \end{aligned}$$

□

**1.11 Definition.** Given a group-norm $N$ on a group $G$, the topology generated by $N$ will be called *the $N$-topology* on $G$.

**1.12 Proposition.** *With the $N$-topology on $G$, the group-norm $N$ is continuous.*

*Proof.* This follows from the fact that $|N(A) - N(B)| \leq N(A^{-1}B)$, which in turn follows directly from the triangle inequality in Definition 1.7. $\square$

As seen, the notion of group-norm is completely equivalent to that of a left-invariant metric on a group. As usual, it can be also seen to be equivalent to right-invariant metrics. For more details on normed groups see the survey by Bingham and Ostaszewski [9].

## 1.4 Transformation groups.

Groups arise naturally from metric structures. The set of isometries (i.e. distances preserving maps) from a given metric space to itself is evidently closed under composition and set-theoretic inverses exist and are easily seen to be isometries as well.

This is the first section where different notions are seen to interact and intertwine. A technical lemma about certain compactness of isometry groups for very general metric spaces is reviewed, an a proposition will be proved that takes a metric spaces with a group acting by isometries and produces a new metric. This new metric, for the very particular case of the holonomy groups will be seen to be the guiding light to many of the original results reported in this dissertation.

**1.13 Definition.** An *isometric group action* consists of a triplet $(G, X, \varphi)$, where $G$ is an abstract group, $(X, d)$ a metric space and $\varphi : G \to \mathrm{Iso}(X, d)$ a group homomorphism. The *orbit* of a point $x \in X$, denoted by $G(x)$, is the equivalence class of all $y \in X$ such that $y = gx := \varphi(g)(x)$ for some $g \in G$. The space of equivalence classes is called *orbit space* and will be denoted by $G\backslash X$. If, furthermore, $G$ is a topological group and the map $(g, x) \mapsto gx$ is continuous then $(X, G)$ is a *transformation group*.

*1.14 Remark.* The quotient map from $X$ to $G\backslash X$ is an open continuous map with the quotient topology.

The following fact will be used in the sequel. It is a classical result of spaces of continuous maps with the compact open topology that the continuity of the evaluation map is equivalent, under some assumptions, to the continuity of the embedding (see Munkres [34]).

**1.15 Proposition.** *If $X$ is locally compact then a transformation group $(X, G)$ is equivalent to a continuous homomorphism $\varphi : G \to \mathrm{Iso}(X, d)$, where the codomain has the compact open topology.*

**1.16 Lemma** (see [27]. p. 47). *Let $(X,d)$ be a locally compact connected metric space and let $\{\varphi_i\}$ be a sequence of isometries of $(X,d)$. If there exists a point $x \in X$ such that $\{\varphi_i(x)\}$ converges, then there exists a subsequence $\{\varphi_{i_k}\}$ that converges to an isometry of $(X,d)$.*

With the hypotheses of the previous lemma, one has the following fact, which for future reference is included here.

**1.17 Proposition.** *Let $(X,G)$ be a transformation group where $X$ is locally compact and connected. Then $G\backslash X$ is a semimetric space with*

$$d(G(x),G(y)) = \inf_g d(x,gy) \tag{1.3}$$

*Furthermore, let $H$ be the closure of $\varphi(G)$ in the isometry group of $X$. Then there exists an isometry such that*

$$H\backslash X \cong (G\backslash X)/\sim. \tag{1.4}$$

*Proof.* All the properties of a semimetric are straightforward. Let $x \in X$, then its equivalence class $[x] \subseteq X$ in the right-hand side of the equation is the set

$$[x] = \{y : \exists\varepsilon > 0, \exists g \in G, d(x,gy) < \varepsilon\}.$$

This is equivalent to $H(y)$ for any fixed $y \in [x]$ since one can produce a sequence $\{g_i\}$ of isometries such that the sequence $\{g_i(y)\}$ converges to $x$; thus by Lemma 1.16 there is a $g \in H$ with $x = gy$. There is therefore a canonical bijection between both sides of the equation. It follows that it is an isometry since the metric on each side is defined to be the distance between equivalence classes as subsets of $X$ (cf. (1.1), (1.3)). □

**1.18 Corollary.** *Let $(X,G)$ be a transformation group where $X$ is locally compact and connected. The orbit space is a metric space if $\varphi(G)$ is a closed subgroup of* $\mathrm{Iso}(X,d)$.

Being closed is too strong of an assumption. In most cases, non-necessarily closed subgroups of $\mathrm{Iso}(X,d)$ will be of interest. In any event, the following result, which intertwines both the metric structure of $G$ and that of $X$ together.

**1.19 Proposition.** *Let $(G,d_G)$ be a metric topological group with left-invariant metric $d_G$. Let $(X,G)$ be a transformation group (no assumption on connectedness or local compactness). Then so is $(X\times G, G)$, where the action is given by $g(x,h) := (gx,gh)$. The quotient space $G\backslash(X\times G)$ is a metric space and is homeomorphic to $X$ under the map $x \mapsto G(x,e)$, the orbit of $(x,e)$. This induces a new metric on $X$ given by*

$$d'(x,y) = \inf_g \sqrt{d_G^2(e,g) + d^2(x,gy)}.$$

*Proof.* That $(X \times G, G)$ is a transformation group is immediate from the hypotheses. The second quotient, $G\backslash(X \times G)/\sim$, is a metric space by Prop 1.6. Let $[x,g] := [G(x,g)]$ stand for the equivalence class $(x,g)$ in the second quotient, regarded as a subset of $X \times G$. The map

$$x \mapsto [x,e] \tag{1.5}$$

is a injective; indeed, suppose that $[x,e] = [y,e]$. For all $\eta > 0$ there exists a group element $g$ such that

$$d(x,gy),\ d_G(e,g) < \eta.$$

Then, by continuity of the action, for every $\varepsilon > 0$ there exists $0 < \delta \leq \varepsilon/2$ such that for all $g$, with $d_G(g,e) < \delta$,

$$d(y,gy) < \varepsilon/2,$$

by equation (1.3). So by letting $\eta = \delta$,

$$d(x,y) \leq d(x,gy) + d(y,gy) < \varepsilon.$$

It is also onto since $[x,g] = [g^{-1}x, e]$. It is continuous since it is a composition of continuous maps and it is open since the quotient maps are open and the map $(x,g) \mapsto g^{-1}x$ is continuous, hence it is a homeomorphism. Since the map $x \mapsto G(x,e)$ is also bijective and continuous, it follows that $G\backslash X$ was already a metric space, as claimed. □

## 1.5 Relations: the usual [abstract] nonsense.

Semigroups and monoids, which are further equipped with an involutive anti-homomorphism are somewhat pervasive in mathematics; e.g. in complex-valued matrices (or the more general $C^*$ algebras), etc.

Even without any further assumption, the category of relations provides a fundamental example. Relations between sets are the weakest way to —never a better name was given to a mathematical concept— relate one set to another. If follows that relations can be composed (just as functions are) and reversed (without concerning oneself with satisfying the *vertical line test* for functions). The language of relations will be the correct language for describing the notion of parallelism that prevails even after passing to the limits considered in this report.

The following definitions and statements review these concepts. Their proofs are all elementary (cf. Freyd and Scedrov [20]).

1.20 **Definition.** Given sets $A$ and $B$, a *set-valued function* $f : A \dashrightarrow B$ (or equivalently a *relation*) is a subset of $f \subseteq A \times B$. It can equivalently be seen as the composition of the standard embedding $A \to 2^A$ with any function $2^A \to 2^B$.

*1.21 Remark.* Usual functions $f : A \to B$ can be regarded as set-valued functions by considering $a \mapsto \{f(a)\}$ or simply by identifying them with their graph (Of course, this is essentially a tautology).

**1.22 Definition.** Given two relations $f : A \dashrightarrow B$, $g : B \dashrightarrow C$, the composition $g \circ f : A \dashrightarrow C$ is defined in the usual way:

$$g \circ f(a) = \{c | \exists b \in f(a), c \in g(b)\}. \tag{1.6}$$

**1.23 Lemma.** *The composition is associative and for any $f : A \dashrightarrow B$,*

$$f \circ id_A = id_B \circ f = f \tag{1.7}$$

*Proof.* Elementary. □

**1.24 Definition.** Given $f : A \dashrightarrow B$ there exists a relation $f^* : B \dashrightarrow A$ given by

$$f^*(b) = \{a | b \in f(a)\}$$

It is worthwhile noticing that in the case of actual functions, $f^*$ coincides with the inverse, whenever the latter exists. Also, nothing prevents an element from having empty image.

**1.25 Lemma.** *Given two relations $f : A \dashrightarrow B$, $g : B \dashrightarrow C$,*

$$(g \circ f)^* = f^* \circ g^*. \tag{1.8}$$

*Furthermore,*

$$f^{**} = f. \tag{1.9}$$

*Proof.* Elementary. □

**1.26 Lemma.** *Given relations $f, g, h, k$ such that $f \circ h$, $k \circ f$, $g \circ h$ and $k \circ h$ exist, then if*

$$f \subset g \tag{1.10}$$

*then*

$$f \circ h \subseteq g \circ h \tag{1.11}$$

*and*

$$k \circ f \subseteq k \circ g. \tag{1.12}$$

The following fact summarizes the previous statements.

**1.27 Proposition.** *Given a set $X$, the set of relations, with the composition and subsumption given as before, is an ordered $^*$-semigroup with identity.*

**1.28 Proposition.** *A submonoid $\mathcal{H}$ of relations is a group if and only if for all $a \in \mathcal{H}$, $a^* = a^{-1}$*

*Proof.* The sufficiency is immediate since it prescribes the existence of an inverse, in particular it follows that for any $a \in \mathcal{H}$, $a$ is an invertible function. For the necessity, one first sees that because $aa^{-1} = id$, then $a$ is necessarily onto, i.e. that for all $y$ there exists $x$, namely any element in $a^{-1}(y)$, such that $a(x) = y$.

Furthermore, since the monoids are ordered (see Proposition 1.27), if for $a, b \in \mathcal{H}$ are such that $a \subseteq b$ then, by multiplication on both sides by $b^{-1}$ yields that

$$ab^{-1} \subseteq id \tag{1.13}$$

which in turn implies equality since $ab^{-1}$ must be surjective. Therefore, since

$$aa^*, a^*a \supseteq id, \tag{1.14}$$

equalities must hold as well. □

## 1.6 Gromov's Theory.

In the late 1970's Gromov [24] introduced a metric on the moduli space of compact metric spaces and with that a notion of convergence valid also for proper metric spaces.

In this section, the basic elements of Gromov's theory are introduced. Of these, the one that is the most powerful is the observation that one can think of limits as honest limits of points, which allows one to develop notions of convergence for families of continuous functions, whose limits will be continuous as well. That said, a particular form of the classical Arzelà-Ascoli theorem is stated and proved. This is a minor modification of previous results of Gromov [23] and Grove and Petersen [25].

**1.29 Definition** (Gromov [24])**.** Given two complete metric spaces $(X, d_X)$ and $(Y, d_Y)$, their *Gromov-Hausdorff* distance is defined as the following infimum.

$$d_{GH}(X,Y) = \inf \left\{ \varepsilon > 0 \;\middle|\; \begin{array}{ll} (1) & \exists d : (X \sqcup Y) \times (X \sqcup Y) \to \mathbb{R}, \text{ metric} \\ (2) & d|_{X\times X} = d_X, d|_{Y\times Y} = d_Y \\ (3) & \forall x \in X (\exists y \in Y, d(x,y) < \varepsilon) \\ (4) & \forall y \in Y (\exists x \in X, d(x,y) < \varepsilon) \end{array} \right\} \tag{1.15}$$

That is the infimum of possible $\varepsilon > 0$ for which there exists a metric on the disjoint union $X \sqcup Y$ that extends the metrics on $X$ and $Y$, in such a way that any point of $X$ is $\varepsilon$-close to some point of $Y$ and vice versa.

*1.30 Remark.* This is a generalization of the *Hausdorff distance* between subspaces of a fixed metric space $(Z,d)$. In this case, the distance $d_H^Z(X,Y)$, between subspaces $X,Y \subseteq Z$, is defined as follows.

$$d_H(X,Y) = \inf \left\{ \varepsilon > 0 \;\middle|\; \begin{array}{ll} (3) & \forall x \in X (\exists y \in Y, d(x,y) < \varepsilon) \\ (4) & \forall y \in Y (\exists x \in X, d(x,y) < \varepsilon) \end{array} \right\} \tag{1.16}$$

*1.31 Remark.* For compact metric spaces the assignment (1.15) is always finite, since

$$d_{GH}(X,Y) \leq \frac{1}{2} \max\{diam(X), diam(Y)\};$$

it may however be infinite if compactness is not assumed. This assignment is positive, symmetric and satisfies the triangle inequality (provided it makes sense). Two compact/complete spaces are zero distance apart if and only if they are isometric.

**1.32 Definition** (Gromov [24])**.** Let $X$ and $Y$ be metric spaces. For $\varepsilon > 0$, an *$\varepsilon$-isometry* from $X$ to $Y$ is a (possibly non-continuous) function $f : X \to Y$ such that:

1. for all $x_1, x_2 \in X$,
$$|d_X(x_1,x_2) - d_Y(f(x_1), f(x_2))| < \varepsilon; \text{ and} \tag{1.17}$$

2. for all $y \in Y$ there exists $x \in X$ such that
$$d_Y(f(x), y) < \varepsilon. \tag{1.18}$$

**1.33 Proposition** (Gromov [24])**.** *Let $X$ and $Y$ be metric spaces and $\varepsilon > 0$. Then,*

*1. if $d_{GH}(X,Y) < \varepsilon$ then there exists a $2\varepsilon$-isometry between them.*

*2. if there exists an $\varepsilon$-isometry form $X$ to $Y$, then $d_{GH}(X,Y) < 2\varepsilon$.*

Except for some set theoretical considerations, the collection of isometry classes of metric spaces, together with the Gromov-Hausdorff distance, $(\mathcal{M}, d_{GH})$ behaves like an extended metric space (i.e. allowing infinite values). When restricted to compact metric spaces, it is a metric space and, as such, yields a notion of convergence for sequences.

*1.34 Remark.* It was proved by Gromov [23] that if a sequence $\{X_i\}$ of compact metric spaces converges in the Gromov-Hausdorff sense to a compact metric space $X$, then there exists a metric on $X \sqcup \bigsqcup_i X_i$ for which the sequence $\{X_i\}$ converges in the Hausdorff sense. Because of this, it makes sense to say that a sequence of points $x_i \in X_i$ converge to a point $x \in X$.

*1.35 Remark.* In the setting of compact $X_i, Z$, a sequence of subspaces $X_i \subseteq Z$ converges to a subspace $X \subseteq Z$ if and only if:

1. for any convergent sequence $x_i \to x$, such that $x_i \in X_i$ for all $i$, it follows that $x \in X$; and

2. for any $x \in X$ there exists a convergent sequence $x_i \to x$, with $x_i \in X_i$.

**1.36 Definition** (Gromov [24])**.** Let $\{X_i\}$, $\{Y_i\}$ be convergent sequences of pointed metric spaces and let $X$ and $Y$ be their corresponding limits. One says that a sequence of continuous functions $\{f_i\} : \{X_i\} \to \{Y_i\}$ *converges* to a function $f : X \to Y$ if there exists a metric on $X \sqcup \bigsqcup_i X_i$ for which the subspaces $X_i$ converge in the Hausdorff sense to $X$ and such that for any sequence $\{x_i \in X_i\}$ that converges to a point $x \in X$, the following holds.

$$f(x) = \lim_{i \to \infty} f_i(x_i) \tag{1.19}$$

*1.37 Remark.* The limit function $f$ is unique if it exists; i.e. it is independent of the choice of metric on $X \sqcup \bigsqcup_i X_i$.

The following is Gromov's way to produce a notion of convergence for the non-compact case. For technical reasons, the assumption that the spaces be proper (i.e. that the distance function from a point is proper, thus yielding that closed metric balls are compact) is required [24].

**1.38 Definition.** A sequence $\{(X_i, x_i)\}$ of pointed proper metric spaces is said to converge to $(X, x)$ in the pointed Gromov-Hausdorff sense if the following holds: For all $R > 0$ and for all $\varepsilon > 0$ there exists $N$ such that for all $i > N$ there exists an $\varepsilon$-isometry

$$f_i : B_R(x_i) \to B_R(x),$$

with $f_i(x_i) = x$, where the balls are endowed with restricted (not induced) metrics.

In the previous definition, it is enough to verify the convergence on a sequence of balls around $\{x_i\}$ such that their radii $\{R_j\}$ go to infinity. Furthermore, the limit is necessarily also proper as noted by Gromov [23].

Given a pointed space $(X,x)$, Gromov [24] studies the relation between precompactness and the function that assigns to each choice of $R > 0$ and $\varepsilon > 0$ the maximum number $N = N(\varepsilon, R, X)$ of disjoint balls of radius $\varepsilon$ that fit within the ball of radius $R$ centered at an $x \in X$. Furthermore he proves the following result.

**1.39 Theorem** (Gromov's Compactness Theorem [24], Prop.5.2)**.** *Consider a family $(X_i, x_i)$ of pointed path metric spaces, it is pre-compact with respect to the pointed Gromov-Hausdorff convergence if and only if each function $N(\varepsilon, R, \cdot)$ is bounded on $\{X_i\}$. In this case, the family is relatively compact, i.e., each sequence in the $X_i$ admits a subsequence that converges in the pointed Gromov-Hausdorff sense to a complete, proper path metric space.*

*1.40 Remark.* Providing a bound for $N$ is equivalent to providing a bound to the minimum number of balls of radius $2\varepsilon$ required to cover the ball of radius $R$ (see [37]). This will be used instead in the sequel and will also be denoted by $N$.

In the non-compact setting, in order to consider the convergence of sequences of points $\{p_i \in X_i\}$, as in Remark 1.34, the only technicality is the following: In order for a sequence $\{p_i\}$ to be convergent, it has to be bounded. Therefore, there must exist a large enough $R > 0$ such that for all $i$, $p_i \in B_R(x_i)$, where the $x_i \in X_i$ are the distinguished points. Because of this, a sequence $\{p_i \in X_i\}$ is convergent if there exists $R > 0$ for which the sequence $\{p_i \in \overline{B_R(x_i)} \subseteq X_i\}$ is convergent as in Remark 1.34.

To analyze the behavior of sequences of functions defined on convergent sequences of spaces, Gromov [23], as well as Grove and Petersen [25], has given a generalization to the classical Arzelà-Ascoli Theorem. Their setting is that of compact spaces. To analyze the non-compact setting, a further generalization is required. In the proof of the compact case, families of countable dense sets $A_i \subseteq X_i$, $A \subset X$ are considered (since continuous functions are determined by their values on dense sets, and as a basis for the standard diagonalization argument). Also, the codomains satisfy that for every sequence there exist a convergent subsequence. To retain these properties, the assumption of separability for the domains and the requirement of totally bounded metric balls for the codomains are added; these are both controlled by the assumption that all the spaces being considered be proper. Also, by virtue of the Hopf-Rinow Theorem in Riemannian geometry, when the metric spaces considered are Riemannian manifolds these conditions follow from completeness.

**1.41 Theorem** (Arzelà-Ascoli Theorem). *Let $\{(X_i, x_i)\}$ and $\{(Y_i, y_i)\}$ be two convergent sequences of complete proper pointed metric spaces. Let $(X, x)$ and $(Y, y)$ be their corresponding limits. Suppose further that there is an equicontinuous sequence of continuous maps $\{f_i\}$,*

$$f_i : X_i \to Y_i, \tag{1.20}$$

*such that $f_i(x_i) = y_i$, for all $i$. Then there exist a continuous function $f : X \to Y$, with $f(x) = y$, and a subsequence of $\{f_i\}$ that converges to $f$.*

*Proof.* In the compact case, this is the content of the generalization of the Arzelà-Ascoli Theorem given by Grove and Petersen [25]. For the non-compact case, Gromov [23] already proved this for isometries under the assumption of properness. Again, a diagonalization argument is required. Namely, because the sequence $\{f_i\}$ is equicontinuous, for every $\varepsilon > 0$ consider the largest $\delta = \delta(\varepsilon)$ that satisfies the definition of equicontinuity. It follows that $\delta$ is an increasing function of $\varepsilon$ that goes to infinity as $\varepsilon$ does; it may happen that $\delta(\varepsilon) = \infty$ for a finite $\varepsilon$. This implies that for any $R > 0$ there exists $\tilde{R} > 0$,

$$f_i(B_R(x_i)) \subseteq B_{\tilde{R}}(y_i), \tag{1.21}$$

by essentially considering the inverse of $\delta$ as a function of $\varepsilon$ (if $\delta$ is infinite, then the existence of $\tilde{R}$ is clearly also satisfied.). This means that one can now repeat the proof of the compact case for the restrictions $\{f_i|_{B_R(x_i)}\}$ (since $B_R(x_i)$ is separable and $B_{\tilde{R}}(y_i)$ compact). Therefore, consider a sequence of radii $R_j \to \infty$ and apply the standard diagonalization argument to the successive restrictions of the convergent subsequences of $f_i$'s (and subsequences thereof) to $B_{R_j}(x_i) \to B_{\tilde{R}_i}(y_i)$. By uniqueness of the limit, one obtains further and further extensions to a single continuous function $f : X \to Y$ as promised. □

In particular this implies that given a convergent sequence of metric spaces and a sequence of curves, one on each space of the sequence, if their lengths are uniformly bounded, then there is a curve in the limit and a subsequence of curves that converges to it. In the case of sequences of curves within a single metric space, it is well known that the length function is lower semi-continuous, that is that

$$\liminf_{i\to\infty} \ell(\alpha_i) \geq \ell(\lim_{i\to\infty} \alpha_i).$$

This is also true in the case of limits of proper metric spaces, essentially since the condition that the curves be bounded restricts the entire sequence to within a sequence of compact balls, as per Remark 1.34.

**1.42 Corollary.** *Let* $\{(X_i, x_i)\}$ *be a convergent sequence of proper length spaces and let* $(X, x)$ *be their limit. Consider a sequence of curves* $\alpha_i : I \to X_i$ *whose length is uniformly bounded and such that* $\alpha_i(0) = x_i$. *Then there exists a curve* $\alpha : I \to X$, *a limit for a subsequence of* $\{\alpha_i\}$, *such that*

$$\liminf_{i\to\infty} \ell(\alpha_i) \geq \ell(\alpha). \tag{1.22}$$

This will be thus freely used in the sequel.

# Chapter 2

# A case study: Holonomic spaces

Die Einführung von Zahlkoordinaten. . . is eine Vergewaltigung.

*Philosophie der Mathematik und Naturwissenschaft.*
HERMANN WEYL

CERTAINLY, the fact that infinitesimal structures are inherently linear has been very advantageous. In this chapter, the relation between these infinitesimal structures and the global nature of holonomy is abstracted. Actually, the fact that the spaces considered in this chapter be those obtained on the infinitesimal structures over Riemannian manifolds (or in the more general setting of vector bundles with metric connections) will not be discussed. The justification will appear in Chapter 3 as Theorem 3.56.

In the meantime, please consider the following analysis as a classification of twisted metrics, in the sense of Proposition 1.19, with the further assumption that they be locally flat. Three ingredients are to be considered: A normed vector space, a subgroup of the group of norm-preserving linear maps, and a group-norm on said subgroup. *A priori*, this three components need not be related. However, a certain convexity law will be assumed, which will also become natural once the language of the Sasaki metric be introduced in Chapter 3.

A remark on the name: to the experienced reader it should hint a relation with the holonomy action on individual fibers. This is exactly so. The pre-compactness assumption on holonomy is a necessary one. The group-norm is obtained by remembering the definition of holonomy as parallel translation along loops, unwillingly prone to having their lengths measured—by unsympathetic metric geometers with nothing better to do.

## 2.1 Constituents and law.

The notion of a *holonomic space* is introduced in this section. It will be seen in the sequel how these spaces occur as fibers of Euclidean vector bundles with suitable conditions imposed. Several properties of holonomic spaces are also analyzed here.

**2.1 Definition.** Let $(V, \|\cdot\|)$ be a normed vector space, $H \leq Aut(V)$ a subgroup of norm preserving linear isomorphisms, and $L : H \to \mathbb{R}$ a group-norm on $H$. The triplet $(V, H, L)$ will be called a *holonomic space* if it further satisfies the following convexity property:

(P) For all $u \in V$ there exists $r = r_u > 0$ such that for all $v, w \in V$ with $\|v - u\| < r$, $\|w - u\| < r$, and for all $A \in H$,

$$\|v - w\|^2 - \|v - Aw\|^2 \leq L^2(A). \tag{2.1}$$

**2.2 Definition.** Let $(V, H, L)$ be a holonomic space. The *holonomy radius* of a point $u \in V$ is the supremum of the radii $r >$ satisfying the convexity property (P) given by (2.1). It will be denoted by $\mathrm{HolRad}(u)$. It may be infinite.

**2.3 Lemma.** *Given a holonomic space $(V, H, L)$ as above, there exists $r > 0$ such that for $u \in V$, $|u| < r$, and for any $B \in H$,*

$$\|u - Bu\| \leq L(B). \tag{2.2}$$

*Proof.* Simply choose $r = r_0$ as in 2.1, $v = Bu$ and $A = B^{-1}$. □

**2.4 Definition.** Given a holonomic space $(V, H, L)$, the largest radius of a ball satisfying Lemma 2.3 is the *convexity radius of a holonomic space*.

Please take a moment to notice that these radii can in fact be infinite.

*2.5 Remark.* The convexity radius is in general larger than the holonomy radius at the origin, as can be seen in Example 2.21

Recall that the group norm $L$ on $H$ induces a topological group structure on $H$, the $L$-topology (see Proposition 1.10).

**2.6 Lemma.** *Given a holonomic space $(V, H, L)$, the action $H \times V \to V$ is continuous with respect to the $L$-topology on $H$. Furthermore, the bound depends only on the maximum norm when restricted to bounded domains.*

*Proof.* Let $r_0$ be the convexity radius. Let $(a,u) \in H \times V$, with $\|u\| \geq r_0$. Fix $\lambda > 0$ such that $\lambda\|u\| < r$, and let $\varepsilon > 0$. Let $K = \sqrt{1 + \frac{1}{\lambda^2}}$. Note that for any positive real numbers $x, y \in \mathbb{R}$,

$$x + \frac{1}{\lambda}y \leq K\sqrt{x^2 + y^2}.$$

Now, if $\delta = \min\{\frac{\varepsilon}{K}, \frac{r-\lambda\|u\|}{\lambda}\}$ and $\sqrt{L^2(a^{-1}b) + \|u-v\|^2} < \delta$. Notice that

$$\|\lambda v\| \leq \lambda\|u - v\| + \|\lambda u\| < \lambda\delta + \lambda\|u\| \leq r_0.$$

Thus,

$$\begin{aligned}
\|au - bv\| = \|u - a^{-1}bv\| &= \frac{1}{\lambda}\|\lambda u - a^{-1}b\lambda v\| \\
&\leq \frac{1}{\lambda}\left(\|\lambda u - \lambda v\| + \|\lambda v - a^{-1}b\lambda v\|\right) \\
&\leq \|u - v\| + \frac{1}{\lambda}L(a^{-1}b) \\
&\leq K\sqrt{L^2(a^{-1}b) + \|u - v\|^2} \\
&= K\sqrt{L^2(a^{-1}b) + \|u - v\|^2} < K\delta \leq \varepsilon.
\end{aligned}$$

□

Notice that this implies that the $L$-topology is necessarily finer than the subgroup topology induced from $O(V)$. The fact that they be comparable is already somewhat restrictive on what $L$ is allowed to be. This will be even more surprising once the concept of holonomic space be related to its geometric roots and seen that, with the induced topology, $L$ will in general not be continuous.

**2.7 Theorem.** *Let $(V, H, L)$ be a holonomic space.*

$$d_L(u,v) = \inf_{a \in H}\left\{\sqrt{L^2(a) + \|u - av\|^2}\right\}, \tag{2.3}$$

*is a metric on $V$.*

*Proof.* By Lemma 2.6 one sees that the action $H \times V \to V$ is continuous with respect to the $L$-topology on $H$. Letting $G = H$ and $X = V$ in Proposition 1.19 it follows that the $V$ is homeomorphic to $H\backslash(V \times H)$ and that the pullback metric on $V$ is given by (2.3). □

**2.8 Definition.** Given a holonomic space $(V, H, L)$. The metric given by (2.3) will be called *associated holonomic metric* and $V$ together with this metric will sometimes be denoted by $V_L$.

**2.9 Theorem.** *A triplet $(V,H,L)$ is a holonomic space if and only if $id : V \to V_L$ is a locally isometry.*

*Proof.* By property (P), given any point $u \in V$ there exists a radius $r > 0$ such that for all $v, w \in V$, with $\|v-u\| < r$ and $\|w-u\| < r$, and for all $A \in H$,

$$\|v-w\| \leq \sqrt{L^2(A) + \|v - Aw\|^2}.$$

Hence, considering the infimum of the right-hand side, it follows that

$$\|v-w\| \leq d_L(v,w) \leq \sqrt{L^2(id_V) + \|v-w\|^2} = \|v-w\|.$$

Conversely, if the identity is a local isometry, property (P) in Definition 2.1 is also satisfied: Let $B$ be a ball around $u \in V$ on which the identity map $id_V|B$ is an isometry. Therefore, for any $A \in H$ and any pair of points $v, w \in B$,

$$\|v-w\| = \inf_{a \in H}\left\{\sqrt{L^2(a) + \|v-aw\|^2}\right\} \leq \sqrt{L^2(A) + \|v-Aw\|^2}.$$

□

*2.10 Remark.* The holonomy radius is also the radius of the largest ball so that the restricted $d_L$-metric is Euclidean.

**2.11 Proposition.** *Let $(V,H,L)$ be a holonomic space. The original norm on $V$ is recovered by the equation*

$$\|v\| = d_L(v,0) \tag{2.4}$$

*Proof.* Because $H$ acts by isometries on $V$,

$$d_L(v,0) = \inf_{a \in H}\left\{\sqrt{L^2(a) + \|v\|^2}\right\}.$$

The conclusion now follows by letting $a = id_V$. □

**2.12 Corollary.** *Given a holonomic space $(V,H,L)$ the rays emanating from the origin are geodesic rays with respect to $d_L$.*

## 2.2 Consequences.

Several properties will be derived from the definitions. In particular, those concerning the holonomic and convexity radii. These properties will be of two types: one giving conditions for them to be finite; and the other giving a more precise control on the relation between metrics given to $H$. In particular, the standard operator norm on $H$ is seen to be Lipschitz with respect to the constituent group-norm.

**2.13 Proposition.** *Let $(V,H,L)$ be a holonomic space. Then $H = \{id_V\}$ if and only if there exists $u \in V$ for which the holonomy radius is not finite.*

*Proof.* If $H$ is trivial, then $L \equiv 0$, and so $V$ is globally isometric to $V$, hence for any $u \in V$ the holonomy radius is infinite. Conversely, if there exists $u \in V$ with $\mathrm{HolRad}(u) = \infty$, and there is $a \in H$ with $L(a) > 0$ (i.e. $a \neq id_V$), then for any $v \in V$, $\|v - av\| \leq L(a)$ should hold. This is a contradiction since $v \mapsto \|v - av\|$ is clearly not bounded unless $a = id_V$. $\square$

**2.14 Corollary.** *Let $(V,H,L)$ be a holonomic space. Then the function $u \mapsto \mathrm{HolRad}(u)$ is positive. Furthermore, it is finite provided $H$ is nontrivial.*

**2.15 Proposition.** *Let $(V,H,L)$ be a holonomic space. The function $u \mapsto \mathrm{HolRad}(u)$ is continuous.*

*Proof.* By Proposition 2.13 one can assume, with no loss of generality, that $H \neq \{id_V\}$. Let $u \in V$ and let $\rho(u)$ be the holonomy radius at $u$. Let $v \in V$ with $\|v - u\| < \rho(u)$, i.e. $v \in B_{\rho(u)}(u)$, then by maximality of $\rho(v)$, it has to be at least as large as the radius of the largest ball around $v$ completely contained in $B_{\rho(u)}(u)$,

$$\rho(v) \geq \rho(u) - \|u - v\|.$$

Also, by maximality of $\rho(u)$, if follows that $\rho(v)$ cannot be strictly larger than the smallest ball around $v$ that contains $B_{\rho(u)}(u)$,

$$\rho(v) \leq \rho(u) + \|u - v\|.$$

Therefore, at any given point $u \in V$ and any $\varepsilon > 0$, there exists $\delta = \min\{\rho(u), \varepsilon\}$ such that for any $v \in V$, $\|u - v\| \leq \delta$ it follows that

$$|\rho(u) - \rho(v)| \leq \|u - v\| \leq \varepsilon. \qquad \square$$

For many applications, having an exact formula for the convexity radius, which in turn is bounded below by the holonomy radius at the origin, is desirable. In fact, with the further assumption on the normed vector space to be an inner product space, the a formula for the holonomy radius at zero will also be given.

**2.16 Theorem.** *Let $(V, H, L)$ be a holonomic space. Then the convexity radius is given by*

$$\mathrm{CvxRad} = \inf_{a \in H} \frac{L(a)}{\|id_V - a\|}, \tag{2.5}$$

*where for any $T : V \to V$, $\|T\|$ denotes its operator norm.*

*Proof.* Let $u \in V$ with $\|u\| \le \frac{L(A)}{\|id_V - A\|}$, then

$$\|Au - u\| \le \|A - id_v\| \|u\| \le L(A)$$

which proves that

$$\mathrm{CvxRad} \ge \inf_{a \in H} \frac{L(a)}{\|id_V - a\|}.$$

Now, let $\rho > \frac{L(A)}{\|id_V - A\|}$ and let $\varepsilon > 0$ be such that

$$\varepsilon < \|A - id_V\|\rho - L(A) = \|A - id_V\| \left( \rho - \frac{L(A)}{\|A - id_V\|} \right) > 0. \tag{2.6}$$

Then, by the definition of operator norm, there exists $u \in V$ with $\|u\| = \rho$ such that

$$\|A - id_V\|\rho \ge \|Au - u\| > \|A - id_V\|\rho - \varepsilon.$$

The second inequality, together with (2.6), yields that

$$\|A - id_V\|\rho - \varepsilon > L(A).$$

This proves that CvxRad cannot be strictly larger than $\frac{L(A)}{\|id_V - A\|}$ for any $A$, and thus for all. □

Recall that by Examples 1.8 and 1.9 and by Proposition 1.10, the operator norm and any composition of it with a non decreasing subadditive function is a group-norm; and that given a group-norm $N$, a left-invariant metric is obtained by

$$d_N(g, h) = N(g^{-1}h).$$

With this, the group norm in the definition of a holonomic space, the usual operator norm and the convexity radius are related in the following Lipschitz condition.

**2.17 Corollary.** *Given a holonomic space* $(V, H, L)$ *then*

$$\|a-b\| \leq \frac{1}{\mathrm{CvxRad}} L(a^{-1}b),$$

*for all* $a, b \in H$.

**2.18 Theorem.** *Let* $(V, H, L)$ *be a holonomic space and suppose further that* $V$ *is an inner product space and that the norm is given by* $\|\cdot\|^2 = \langle \cdot, \cdot \rangle$. *Then the holonomy radius at the origin is given by*

$$\mathrm{HolRad}(0) = \inf_{a \in H} \frac{L(a)}{\sqrt{2\|id_V - a\|}}, \tag{2.7}$$

*where for any* $T : V \to V$, $\|T\|$ *denotes its operator norm.*

*Proof.* Using the inner product, and the fact the symmetry of the group-norm $L$, $L(A^{-1}) = L(A)$, and that $H$ acts by isometries, (2.1) is equivalent to

$$\|v-w\|^2 - \|Av-w\|^2 \leq L^2(A),$$

which when expanded out yields,

$$\|v\|^2 + \|w\|^2 - 2\langle v, w\rangle - \|v\|^2 - \|w\|^2 + 2\langle Av, w\rangle \leq L^2(A),$$

and thus

$$2\langle Av - v, w\rangle \leq L^2(A).$$

Thus if $\|v\|, \|w\| \leq \frac{L(A)}{\sqrt{2\|id_V - A\|}}$ then

$$2\langle Av - v, w\rangle \leq 2\|A - id_v\|\|v\|\|w\| \leq L^2(A).$$

Since the inequality has to hold for any $A$, it follows that

$$\mathrm{HolRad}(0) \geq \inf_{a \in H} \frac{L(a)}{\sqrt{2\|id_V - a\|}}.$$

Furthermore, for $\rho > \frac{L(A)}{\sqrt{2\|id_V - A\|}}$, let $\varepsilon > 0$ such that

$$\varepsilon < \|id_V - A\|\rho - \frac{L^2(A)}{2\rho} = \frac{\|id_V - A\|}{\rho}\left(\rho^2 - \frac{L^2(A)}{2\|id_V - A\|}\right) > 0.$$

By the definition of operator norm, there exists $v \in V$, with $\|v\| = \rho$ and

$$\|id_V - A\|\rho \geq \|Av - v\| > \|id_V - A\|\rho - \varepsilon.$$

Set $w = \frac{\rho}{\|Av-v\|}(Av - v)$. It now follows that

$$2\langle Av - v, w\rangle = 2\rho\|Av - v\| > 2\rho(\|id_V - A\|\rho - \varepsilon) > L^2(A),$$

by the previous second inequality.

Thus, (2.1) cannot hold for $\rho > \frac{L(A)}{\sqrt{2\|id_V - A\|}}$ and the claim follows. $\square$

**2.19 Corollary** (cf. Corollary 2.17)**.** *Given a holonomic space $(V, H, L)$, with $V$ an inner product space, then*

$$\sqrt{2\|a-b\|} \leq \frac{1}{\mathrm{HolRad}(0)} L(a^{-1}b),$$

*for all $a, b \in H$.*

For most of the applications, the convexity radius, it's formula, and the fact that it is bounded below by the holonomy radius, will be used more than the formula (2.7). In fact, the following result will be used in the sequel.

**2.20 Corollary.** *Given a holonomic space $(V, H, L)$ then*

$$\|a - b\| \leq \frac{1}{\mathrm{HolRad}(0)} L(a^{-1}b), \tag{2.8}$$

*for all $a, b \in H$.*

After these concepts are reinterpreted in terms of holonomy and lengths of loops, (2.8) states that one can control the holonomy by controlling the length of a loop generating. And viceversa.

## 2.3 Examples

Here are two examples. The first one shows that indeed the convexity and holonomy radii are different. The second, seemingly trivial, will play a significant rôle in the study of the occurrence of isolated 2-dimensional singularities, as seen in Chapter 5

**2.21 Example.** The existence of an $r > 0$ satisfying (2.2) (guaranteed for holonomic spaces by 2.3) is not equivalent to the existence of an $r' > 0$ satisfying (2.1). This follows from

(2.7) by considering the following action: Let $V = \mathbb{C}^2$, $H = \mathbb{R}$,

$$t \cdot (z,w) = (e^{it}z, e^{\sqrt{2}it}w),$$

and $L(t) = |t|$. Indeed, by Theorem 2.18,

$$\mathrm{HolRad}(0) = \inf_{a \in H} \frac{L(a)}{\sqrt{2\|id_V - a\|}} = \lim_{t \to 0^+} \frac{|t|}{\sqrt{2\sqrt{2 - 2\cos(\sqrt{2}t)}}} = 0,$$

whereas, any positive $r \leq \frac{1}{\sqrt{2}}$ will make (2.2) hold. Hence, by Theorem 2.16,

$$\mathrm{CvxRad} \geq \frac{\sqrt{2}}{2}.$$

Finally, consider the following example.

**2.22 Example.** Let $r > 0$ and let $H$ be the group generated by a rotation by $0 < \alpha < \pi$. Let $L_r$ be the group-norm given by

$$L(a) = \begin{cases} 2r & a \neq e, \\ 0 & otherwise. \end{cases} \tag{2.9}$$

Consider $V$ to be $\mathbb{R}^2$ with the standard inner product. Then $(V, H, L_r)$ is a holonomic space. This can be seen directly will also follow from Theorem 3.56 when considering the flat metric

$$ds^2 = dr^2 + \left(\frac{\alpha r}{2\pi}\right)^2 d\theta^2$$

on $\mathbb{R}^2 \setminus \{0\}$ (see Chapter 5).

# Chapter 3

# Shaping directions after Sasaki.

> Has lineas deſcribere Geometria non docet ſed poſtulat. Poſtulat enim ut Tyro eaſdem accurate deſcribere prius didicerit quam limen attingat Geometriæ; dein, quomodo per has operationes Problemata ſolvantur, docet.
>
> *Philoſophiæ Naturalis Principia Mathematica*
> Isaac Newton

It is a matter of choice to make the tangent bundle (or arbitrary vector bundles) into a metric space. First one begins by assigning a Riemannian metric to the base manifold and then, in the case of the tangent bundle, magic occurs and there is a natural most obvious pick: the Sasaki metric. In the more general setting of vector bundles, this metric occurs when one further picks a bundle metric (tautologically a priori given for the tangent bundle) and a compatible connection (once more already granted as the Levi-Civita connection).

A Riemannian metric (or more generally a bundle metric) is an inner product on each tangent space. One reason why inner products are beautiful on their own is because they yield a way to measure distances between tangent vectors, notwithstanding the fact that when you think of the tangent space as the space of directions, it doesn't seem very natural to think of distances between directions. Only after one makes an obvious identification (denoted in the sequel by $\mathfrak{I}$ after Gromoll [21]) does it become natural, at least in the case of the Euclidean Geometry.

Once this *infinitesimal* notion of distance (which is called a *Riemannian manifold* or *Riemannian metric*) is given (or chosen), the actual notion of distance comes from the age-old practice of integration.

In general, the starting point for studying the metric geometric properties of bundles over Riemannian manifolds is to consider their total spaces as Riemannian manifolds such that the projection is a Riemannian submersion. Existence and naturality of such metrics has been addressed and studied from a purely differential geometric viewpoint (see [30] or [48] for the tangent bundle).

One procedure to view a vector bundle as a Riemannian submersion is to endow the base with a Riemannian metric and to require that the bundle be equipped with a bundle metric and any compatible bundle connection. These two ingredients provide a plethora of metrics on the total space of the bundle (see [1]), perhaps the simplest of which is the Sasaki-type metric, introduced for the tangent bundle by Sasaki [42]. These are just a particular case of the general construction over locally fibered maps as in [28].

**To the experienced reader:** In this chapter, the notions of connection, affine connection, bundle metric, and parallel translation are reviewed. Also, the differential and the length structures of the Sasaki-type metrics are analyzed, yielding a complete description of the fibers as holonomic spaces; showing a way to recover the Riemannian structure on $M$ via the Sasaki-type metric and the knowledge of the norm; and providing a condition on the arrows of a category whose objects are Riemannian manifolds so that the Sasaki-type metric renders the tangent bundle construction a functor from said category to itself.

## 3.1 Connections and the canonical splitting.

In the sequel, the discussion will be focussed on general vector bundles over Riemannian manifolds. The basic example being the tangent bundle (consider already in Chapter 1).

**3.1 Definition.** A vector bundle is a triple $(E,\pi M)$ where $E,M$ are differential manifolds and $\pi : E \to M$ is a surjective submersion (i.e. a surjective smooth map whose derivative $\pi_*$ is surjective when restricted to any tangent space), such that for any point $p \in M$ $\pi^{-1}(p)$ is a vector space, and there exists a neighborhood $U \ni p$ such that there exists a diffeomorphism

$$\psi_U : \pi^{-1}(u) \to U \times \pi^{-1}(p),$$

such that for any two such diffeomorphisms $\psi_U, \psi_V$, the map $\psi_U \circ \psi_V^{-1}$ is linear on the second factor, whenever $U \cap V$ is nonempty.

A bit of nomenclature: Let $M$ denote a differential manifold. A vector bundle over $M$ will be denoted by $(E,\pi)$ where $E$ is its total space and $\pi : E \to M$ is the projection map .

Notice that the tangent bundle is in fact a vector bundle, since its differential structure was locally modeled as a product (cf. Chapter 1). Also, every diagram is assumed to be commutative unless otherwise stated.

**3.2 Definition.** Let $(E, \pi_1)$ and $(F, \pi_2)$ be vector bundles over a manifold $M$. Define their *Whitney sum* as a vector bundle over $M$ with total space denoted by $E \oplus F$ and projection map $\pi_1 \circ pr_1 = \pi_2 \circ pr_2$ fitting into the universal diagram for the pullback. Namely,

$$\begin{tikzcd}
\forall \tilde{E} \arrow[dr, dotted, "\exists! \psi"] \arrow[drr, bend left, "q_2"] \arrow[ddr, bend right, "q_1"'] & & \\
& E \oplus F \arrow[r, "pr_2"'] \arrow[d, "pr_1"] & F \arrow[d, "\pi_1"'] \\
& E \arrow[r, "\pi_2"] & M
\end{tikzcd} \tag{3.1}$$

*3.3 Remark.* The projections from the Whitney sum to its factors are also vector bundles regarding, e.g., the pullback $\pi_1^* = E\oplus$ as a functor from the category of bundles over $M$ to the category of bundles over $E$.

**3.4 Proposition.** *Given a vector bundle, $(E, \pi)$, there are two vector bundle structures with total space $TE$, namely the standard projection, $(TE, \pi_E)$,*

$$\pi_E : TE \to E,$$

*and the* secondary structure, $(TE, \pi_*)$,

$$\pi_* : TE \to TM.$$

*Let $TE \oplus_2 TE$ denote the Whitney sum using the secondary structure; that is $(X, Y) \in TE \oplus_2 TE$ satisfy $\pi_* X = \pi_* Y$:*

$$\begin{tikzcd}
TE \oplus_2 TE \arrow[r, "pr_2"] \arrow[d, "pr_1"'] & TE \arrow[d, "\pi_*"'] \\
TE \arrow[r, "\pi_*"] & TM
\end{tikzcd} \tag{3.2}$$

**3.5 Definition.** Following the notation in [21], given a (normed) vector space $V$, there is a canonical isomorphism between $V \times V$ and $TV$, given by

$$\mathfrak{I}_v(w) f = \mathfrak{I}(v, w) f = \left.\frac{d}{dt}\right|_{t=0} f(v + tw). \tag{3.3}$$

That is, $\mathfrak{I}_v w$ is the directional derivative at $v$ in the direction $w$.

This construction already shows us the following statement (cf. [28]).

**3.6 Proposition.** *Given any vector bundle $(E,\pi)$, (3.3) yields a bundle isomorphism between $\oplus^2 E := E \oplus E$ and the vertical distribution $\mathcal{V} = \ker \pi_* \subseteq TE$, in a natural way; that is, there is a natural transformation (also denoted by $\mathfrak{I}$) from the functor $\oplus^2$ to the functor $T$.*

*Proof.* The naturality: Let $(E,\pi_1)$ and $(F,\pi_2)$ be vector bundles over $M$, and let $\varphi : E \to F$ be a morphism between them. Then,

$$\begin{tikzcd} \oplus^2 E \arrow[r, "\oplus^2\varphi"] \arrow[d, "\mathfrak{I}"'] & \oplus^2 F \arrow[d, "\mathfrak{I}"'] \\ TE \arrow[r, "\varphi_*"] & TF \end{tikzcd} \qquad \begin{tikzcd} (e,\tilde{e}) \arrow[r, mapsto] \arrow[d, mapsto] & (\varphi e, \varphi\tilde{e}) \arrow[d, mapsto] \\ {[e+t\tilde{e}]_e} \arrow[r, mapsto] & \varphi_*([e+t\tilde{e}]_e) = [\varphi e + t\varphi\tilde{e}]_{\varphi_* e} \end{tikzcd} \tag{3.4}$$

with $[\alpha(t)]_{\alpha(0)} = \dot{\alpha}(0)$, where $\alpha$ is a curve. The fact that it maps into the vertical distribution follows from

$$\pi_{1*}[e+t\tilde{e}]_e = [\pi_1(e+t\tilde{e})]_{\pi e} = [\pi_1 e]_{\pi e} = 0, \tag{3.5}$$

since by assumption $\pi_1 e = \pi_1 \tilde{e} = \pi_1(e+t\tilde{e})$. Surjectivity can also be verified. $\square$

**3.7 Corollary.** *Let $f : M \to N$ be a smooth map between smooth manifolds. Then*

$$f_{**} \circ \mathfrak{I} = \mathfrak{I} \circ (\oplus^2 f_*). \tag{3.6}$$

**3.8 Definition** (Dieudonné [17])**.** A connection on a vector bundle $(E,\pi)$ is a bundle morphism $C : E \oplus TM \to TE$ with respect to both bundle structures on $TE$:

$$\begin{tikzcd} E \oplus TM \arrow[rr, "C"] \arrow[dr, "pr_1"'] & & TE \arrow[dl, "\pi_E"] \\ & E & \end{tikzcd} \qquad \begin{tikzcd} E \oplus TM \arrow[rr, "C"] \arrow[dr, "pr_2"'] & & TE \arrow[dl, "\pi_*"] \\ & TM & \end{tikzcd} \tag{3.7}$$

*3.9 Remark.* One should read $C(e,u)$ as the "horizontal lift of $u$ at $e$", since given a connection $C$ one can define the horizontal space as $\mathcal{H}_e = C(\{e\} \times T_{\pi(e)}M)$ as well as a projection onto the vertical space $\mathcal{V}$, also denoted by $\mathcal{V}$. Parallel translation along curves in $M$ of vectors in $E$ is defined as horizontal lifts to $E$.

**3.10 Proposition.** *Given a connection $C$ on $(E,\pi)$ there exists a bundle isomorphism $\Xi = \Xi_C : E \oplus TM \oplus E \to TE$ as bundles over $E$.*

*Proof.* Define $\Xi$ by

$$\Xi(e,u,f) = C(e,u) + \mathfrak{I}(e,f). \tag{3.8}$$

This map is a bundle map in view of the following diagram.

$$\begin{tikzcd} E\oplus TM\oplus E \arrow[rr, "\Xi"] \arrow[dr, "pr_1"'] & & TE \arrow[dl, "\pi_E"] \\ & E & \end{tikzcd} \qquad \begin{tikzcd} (e,u,f) \arrow[rr, mapsto] \arrow[dr, mapsto] & & C(e,u)+\mathfrak{I}(e,f) \arrow[dl, mapsto] \\ & e & \end{tikzcd}$$

In order to prove that it is an isomorphism, an inverse can be produced:

$$\Xi^{-1}(X) = (\pi_E X, \pi_* X, \mathfrak{I}^{-1}_{\pi_E X}\mathcal{V}X) \tag{3.9}$$

with $\mathcal{V}$ as in 3.9. □

**3.11 Definition.** A metric on a vector bundle $(E,\pi)$ is a function $g : \oplus^2 E \to \mathbb{R}$ such that when restricted to the fibers it is a non-degenerate positive definite inner product. Given $(E,\pi)$ and a vector bundle with metric $(F,\tilde{\pi},h)$ there is a natural metric on $\pi^* F = E \oplus F$ as a bundle over $E$ given by the pullback metric

$$\pi^* h = h \circ (\oplus^2 pr_2). \tag{3.10}$$

Notice that when $E = TM$ the this is the precise formulation of a Riemannian metric on $M$.

*3.12 Remark.* Given two bundles with metrics $(E,\pi_1,g)$, $(F,\pi_2,h)$ over $M$, there is a natural metric on their Whitney sum as bundles over $M$:

$$g \oplus h = g \circ (\oplus^2 pr_1) + h \circ (\oplus^2 pr_2). \tag{3.11}$$

**3.13 Definition** (Fisher and Laquer [19])**.** Let $h : \oplus^2 E \to \mathbb{R}$ be a metric on a vector bundle $(E,\pi)$. There is an associated map $Th : TE \oplus_2 TE \to \mathbb{R}$ given by

$$Th(X,Y) = \left.\frac{d}{dt}\right|_{t=0} h(u(t),v(t)), \tag{3.12}$$

where $u$ and $v$ are curves in $E$ with $\pi \circ u = \pi \circ v$ such that $\dot{u}(0) = X$ and $\dot{v}(0) = Y$ (Recall $\pi_* X = \pi_* Y$ from 3.4). As the notation suggests $Th$ is essentially $h_* : TE \oplus_2 TE \to T\mathbb{R}$ (in fact it is exactly $pr_2 \circ \mathfrak{I}^{-1} \circ h_*$).

## 3.2 Parallelism.

As discussed in the introduction, the notion of parallelism has been central to geometry. It happen rather quickly —about two thousand years after the infamous fifth postule was introduced by Euclid— that mathematicians realized that there is not a global well defined notion for objects in space (in a generalized space) to be parallel in a consistent way. However, at the infinitesimal level, parallelism can be described as integral submanifolds of the horizontal distribution (not necessarily of maximal dimension). In particular for curves, it is seen to always exist given certain initial conditions. This notion was first introduced by Levi-Civita [31] after the introduction of the notion of absolute calculus by Ricci and Levi-Civita [38].

The following technical statements will be needed in the sequel. As known, many structures on vectors bundles are transfered automatically by universality. In particular, given a vector bundle with connection $(E, \pi, C)$ over a manifold $M$, parallel translation along a curve $\alpha : I \to M$ is the trivialization of $\alpha^*E$ such that the vertical projection coincides with the projection onto the linear factor:

**3.14 Proposition.** *Let $(E, \pi, C)$ be a vector bundle with connection over a manifold $M$, and let $\alpha : I \to M$ be a smooth curve. The pullback becomes a bundle with connection $(\alpha^*E, \alpha^*\pi, \alpha^*C)$. Moreover, since $I$ is contractible, $\alpha^*E$ is trivial. $\alpha^*C$ yields a trivialization, called* parallel translation, *by considering the flow $P = P^\alpha$ of the vector field*

$$e \mapsto [\alpha^*C]((s,e), \frac{d}{ds}) = (s, C(e, \dot{\alpha})), \tag{3.13}$$

*under the usual presentation of pullbacks a subsets of cartesian products. Furthermore, $P$ satisfies the following properties.*

$$P_* \frac{\partial}{\partial t} = [\alpha^*C](P, \frac{d}{ds}) \tag{3.14}$$

$$P_t \circ P_\tau = P_{t+\tau} \tag{3.15}$$

$$(\alpha^*\pi) \circ P(t, (s,e)) = s + t \tag{3.16}$$

$$P_t(e + \lambda f) = P_t e + \lambda P_t f \tag{3.17}$$

$$P_{t*}[\alpha^*C](e, v) = [\alpha^*C](P_t(e), v) \tag{3.18}$$

$$P_{t*}\mathfrak{I}(e, f) = \mathfrak{I}(P_t e, P_t f) \tag{3.19}$$

*so that if* $I = [a,b]$*,* $\alpha^*E \cong I \times [\alpha^*E]_a$ *by*

$$(s,e) \overset{\mathcal{P}}{\longmapsto} (s, P_{a-s}e) \tag{3.20}$$

*Proof.* The first equation is the definition of a flow, and the second is the usual local 1-parameter group property. The third equation follows from the fact that this vector field is $(\alpha^*\pi)$-related to $\frac{d}{ds}$. Linearity in the fourth is a direct consequence of a connection $C$ being a bundle map with respect to the secondary bundle structure of $TE$ over $TM$. The fifth equation is an application of the fact that for any manifold $N$ and for any vector field $Y \in \mathfrak{X}(N)$ with flow $\Phi$, $Y$ is $\Phi_t$-related to itself. The last one follows from linearity of $P_t$.

Finally, $\mathcal{P}$ is an bundle morphism that is a linear isomorphism on the fibers and hence a diffeomorphism:

$$\mathcal{P}^{-1}(s,e) = (s, P_{s-a}e)$$

□

3.15 **Proposition** (Fisher and Laquer [19]). *Given a vector bundle with metric and connection* $(E,h,C,\nabla)$*, the following are equivalent.*

1. *Parallel translation is by isometries.*

2. *For all* $\xi \in TM$ *and all sections* $\sigma, \tau \in \Gamma(E,\pi)$*,*

$$Th(C(\sigma,\xi), C(\tau,\xi)) = 0. \tag{3.21}$$

*3.16 Remark.* One says that the connection is compatible with the metric if these properties hold.

Notice that a connection on $(E,\pi,M)$ can be interpreted as a splitting $C$ of the following short exact sequence of bundles over the the total space $E$.

$$\begin{tikzcd} 0 \arrow[r] & \pi^*E \arrow[r, "\mathfrak{I}"] & TE \arrow[r, "\psi"] & \pi^*TM \arrow[r] \arrow[l, bend left, "C"] & 0 \end{tikzcd} \tag{3.22}$$

where $\psi = (\pi_E, \pi_*)$, by regarding $C(e,u)$ as the horizontal lift of the vector $x \in M_{\pi e}$ to $e$ as in the following definition. In particular, for the tangent bundle, by considering the Levi-Civita connection one gets a canonical splitting of $TTM$.

With this splitting in mind, one defines vertical and horizontal lifts as follows(c.f. [27]).

**3.17 Definition.** Given elements $e, f \in E$, $u \in TM$ such that $\pi(e) = \pi(f) = \pi_M(u) = p$ the *horizontal lift* of $u$ over $e$ is given by

$$u^h(e) = C(e, u).$$

The *vertical lift* of $f$ over $e$ is

$$f^v(e) = \mathfrak{I}_e(f).$$

Lastly, for the case of loops based at a point on $M$, one can consider the corresponding parallel translates. The extent to which this parallel translations depend on the particular choice, and thus measuring the failure of having a global way of telling when two vectors (e.g. two directions in the case of the tangent bundle) are parallel, is given by the holonomy groups.

**3.18 Definition.** Given a bundle with metric and connection, parallel translation yields a map from the space $\Omega_p$ of piecewise smooth loops at a point $p \in M$ to the group $GL(E_p)$ by

$$\alpha \in \Omega_p \mapsto H(\alpha) = P_1^\alpha. \tag{3.23}$$

The holonomy group $Hol_p$ at the point $p$ on the base manifold is then defined as the continuous image of the map $H$.

It can be shown that these groups are all isomorphic and that they admit a Lie group structure. Holonomy groups have proved to be useful in detecting special types of geometries. In fact, the holonomy groups of simply connected irreducible Riemannian manifolds (i.e. the holonomy groups of the Levi-Civita connection) have been classified by Berger [7] (see [8]).

## 3.3 Connections as *bona fide* derivatives

It is also quite standard to think of connections on vector bundles as covariant derivatives. This is equivalent. In this section, all the previous definitions are interpreted using covariant derivatives instead.

**3.19 Definition** (Connections and metrics on vector bundles)**.** As before, given a vector bundle $\pi : E \to M$ over a smooth manifold $M$, a bundle metric is a choice of inner products on each fiber, $E_p$, that depends smoothly on the base space. Namely, it is a smooth section $h \in \Gamma(Sym^2(E^*))$ satisfying the non-degeneracy assumption.

A connection on a bundle $\pi : E \to M$ is a map $\nabla$ ,

$$\nabla : \Gamma(E) \to \Gamma(T^*M \otimes E), \tag{3.24}$$

that satisfies the Leibniz rule $\nabla(f\sigma) = df \otimes \sigma + f\nabla(\sigma)$ for any section $\sigma$ and any smooth function $f$ on $M$. Given a bundle metric, the connection is said to be *metric* if it further satisfies that $\nabla(h) = 0$, where $\nabla$ the induced connection on $\mathrm{Sym}^2 E^*$. More explicitly, a connection is metric if and only if for any sections $\sigma, \tau \in \Gamma(E)$ and any vector field $X \in \mathfrak{X}(M)$,

$$X(h(\sigma,\tau)) = h(\nabla_X\sigma,\tau) + h(\sigma,\nabla_X\tau). \tag{3.25}$$

*3.20 Remark.* The standard (Kozul) covariant derivative definition of a connection is equivalent and is recovered by the following equation. Let $Y : M \to E$ be a section of the bundle $(E,\pi)$, let $x \in TM$; then

$$\mathfrak{I}(Y,\nabla_x Y) = Y_* x - C(Y,x). \tag{3.26}$$

**3.21 Definition** (Parallel translation)**.** Given a vector bundle $\pi : E \to M$ with connection $\nabla$, a section $\sigma \in \Gamma(E)$ is *parallel* if $\nabla\sigma \equiv 0$. A section $\sigma$ along a curve $\alpha : [0,1] \to M$ is *parallel* if $\nabla_{\dot\alpha}\sigma \equiv 0$. Given any curve $\alpha$ and a vector $u \in E$ with $\pi(u) = \alpha(0)$ there exists a unique parallel section $t \mapsto P_t^\alpha(u)$ along $\alpha$ with $P_0^\alpha(u) = u$.

It follows that the transformation $u \mapsto P_t^\alpha(u)$ is linear with respect to $u$ for any $t$. Furthermore, if the connection is metric then $P_t^\alpha(u)$ is an isometry with respect to the bundle metric. $P_t^\alpha(u)$ is frequently called *parallel translation* of $u$ along $\alpha$ at time $t$.

**3.22 Definition** (Holonomy Groups)**.** Given a vector bundle $\pi : E \to M$ with connection $\nabla$, and given any $p \in M$, the *holonomy group* of $\nabla$ is the collection, denoted by $Hol_p(\nabla)$, of $P_1^\alpha : E_p \to E_p$ where $\alpha : [0,1] \to M$ is a loop at $p$; i.e. $\alpha(0) = \alpha(1) = p$. If $M$ is connected it follows that, for all point $p,q \in M$, $Hol_p(\nabla)$ is isomorphic to $Hol_q(\nabla)$, but this isomorphism is not canonical. Furthermore, if $\nabla$ is metric with respect to $h$, then for all $p \in P$, $Hol_p(\nabla)$ is a subgroup of the orthogonal group $O(E_p)$ with respect to $h_p$.

In order to define metrics of Sasaki-type, the following construction is required.

**3.23 Definition** (Vertical lifts)**.** Given a vector bundle $\pi : E \to M$, consider a vector $u \in E_p$, the *vertical lift* of $u$ is the map $u^v : E_p \to T(E_p) \subseteq TE$ given by

$$v \mapsto \dot\gamma(0), \tag{3.27}$$

where $\gamma$ is the curve in $E$ given by $\gamma(t) = v + tu$. It follows that $\pi_*(u^v) \equiv 0$.

In fact, if one starts with a section $\sigma \in \Gamma(E)$, in this fashion one produces a vector field $\sigma^v \in \mathfrak{X}(E)$, the vertical lift of $\sigma$, that satisfies that

$$\pi_*(\sigma^v) \equiv 0. \tag{3.28}$$

3.24 **Definition** (Horizontal lifts)**.** Given a vector bundle $\pi : E \to M$ with connection $\nabla$, consider a tangent vector $x \in T_pM$, the *horizontal lift* of $x$ is the map $x^h : E_p \to TE$ given by

$$v \mapsto \dot{\sigma}(0), \tag{3.29}$$

where $\sigma(t) = P_t^\alpha(v)$ and $\alpha$ is any curve on $M$ such that $\alpha(0) = p$ and $\dot{\alpha}(0) = x$; i.e. $x^h(v)$ is the derivative of the parallel translation of $v$ in the direction of $x$. It follows that $\pi_*(x^h) \equiv x$.

In fact, if one starts with a vector field $X \in \mathfrak{X}(M)$, in this fashion one produces a vector field $X^h \in \mathfrak{X}(E)$, the horizontal lift of $\sigma$, that satisfies that

$$\pi_*(X^h) \equiv X. \tag{3.30}$$

3.25 *Remark.* Given a vector bundle $\pi : E \to M$ with metric connection $\nabla$ and bundle metric $h$, consider any vector $\xi \in T_uE$, with $p = \pi(u)$, then $\xi$ can be expressed as

$$\xi = \sigma^v(u) + x^h(u) \tag{3.31}$$

for some uniquely determined $x \in T_pM$ and $\sigma \in E_p$.

## 3.4 Global shape.

Having reviewed all the needed concepts, it is now appropriate to introduce the Sasaki-type metrics on the total spaces of vector bundles over Riemannian manifolds.

Let $(E, \pi)$ be a vector bundle over $M$ and suppose it has a covariant derivative $\nabla^E$ and a compatible metric $h$. Consider a trivialization over a coördinate neighborhood of $M$ given by coördinates functions $x^i$ on $M$ and by sections $e_j$ on $E$. Consider the coördinates $r^j$ such that for any $e \in E$ $e = r^j(e)e_j$.

Before that, and because it will be useful for certain later results, an local expression of the vertical and horizontal lifts will be given in terms of a choice of trivialization of the bundle over a coördinate chart of the base manifold.

**3.26 Proposition.** *Given $e, f \in E$, with $\pi e = \pi f$, the vertical and horizontal lifts are given by*

$$f^v(e) = \mathfrak{I}_e(f) = r^j(f)e_j^v \tag{3.32}$$

$$\left(\frac{\partial}{\partial x^i}\right)^h(e) = C_e\left(\frac{\partial}{\partial x^i}\right) = \frac{\partial}{\partial x^i} - r^j\Gamma_j^k\left(\frac{\partial}{\partial x^i}\right)e_k^v \tag{3.33}$$

*where $e_k^v = \frac{\partial}{\partial r^k}$ and $\Gamma$ is given by*

$$\nabla^E e_i = \Gamma_i^k e_k. \tag{3.34}$$

*Proof.* Both are easy consequences of the definitions using (3.26) and (3.3). □

**3.27 Proposition.** *Given $\sigma, \tau \in \Gamma(E)$, $X, Y \in \mathfrak{X}(M)$. Then the Lie bracket at a point $e \in E$ is given by the following formulæ.*

$$[\sigma^v, \tau^v] = 0 \tag{3.35}$$

$$[X^h, \sigma^v] = (\nabla_X^E \sigma)^v \tag{3.36}$$

$$[X^h, Y^h] = [X, Y]^h - (R_{X,Y}^E e)^v \tag{3.37}$$

*where $R^E$ is the curvature of $\nabla^E$.*

*Proof.* This a straight forward computation which follows directly from (3.32) and (3.33). □

### 3.4.1 Sasaki metrics

This subsection reviews the definitions of Sasaki-type metrics on general vector bundles, states several assorted properties thereof.

In view of the splitting of $TE$ given in 3.10 and 3.12 there is a very natural way to define a complete Riemannian metric on the total space $E$, the Sasaki-type metric. Benyounes, Loubeau, and Wood [6] have introduced a larger class of such metrics of which the Sasaki-type is a particular case.

Recall the definition of vertical and horizontal lifts from Definition 3.17.

**3.28 Definition** ([42])**.** Given a vector bundle with metric and compatible connection $(E, \pi, h, \nabla^E)$ over a Riemannian manifold $(M, g)$, the *Sasaki-type metric* $\textsc{g} = \textsc{g}(g, h, \nabla^E)$ is defined as follows

$$\mathrm{G}(e^v, f^v) = h(e, f) \tag{3.38}$$

$$\mathrm{G}(e^v, x^h) = 0 \tag{3.39}$$

$$\mathrm{G}(x^h, y^h) = g(x, y), \tag{3.40}$$

*3.29 Remark.* An equivalent phrasing of G can be given in terms of the connection map [27], $\kappa : TE \to E$, uniquely determined by requiring that

$$\kappa(\sigma_* x) = \nabla^E_x \sigma; \tag{3.41}$$

so that G becomes

$$\mathrm{G}(\xi, \eta) = g(\pi_* \xi, \pi_* \eta) + h(\kappa \xi, \kappa \eta), \tag{3.42}$$

for vectors $\xi, \eta \in T_e TE$.

3.30 **Proposition.** *Given a curve $\alpha : I \to M$ (parametrized by arc length), the (trivial) pullback bundle $\alpha^* E$ (as in 3.14) is further isometric to $I \times \mathbb{R}^k$ where $k$ is the rank of $E$.*

*Proof.* In view of 3.14 and 3.15, by parallel translation one gets that

$$\alpha^* \mathrm{G} = \ell(\alpha)^2 dt^2 + \alpha^* h_p,$$

where $p = \alpha(0)$, and $\ell$ denotes the length of $\alpha$. □

3.31 **Proposition.** *The length distance on $(E, \mathrm{G})$ is expressed as follows. Let $u, v \in E$, then*

$$d_E(u, v) = \inf\left\{ \sqrt{\ell(\alpha)^2 + \|P^\alpha_1 u - v\|^2} \;\middle|\; \alpha : [0,1] \to M, \alpha(0) = \pi u, \alpha(1) = \pi v \right\}. \tag{3.43}$$

*Furthermore, if $\pi u = \pi v$ then*

$$d_E(u, v) = \inf\{\sqrt{L(a)^2 + \|au - v\|^2} : a \in Hol_p\}, \tag{3.44}$$

*with $L$ being the infimum of lengths of loops yielding a given holonomy element.*

*Proof.* The first expression is the definition of distances by 3.30. And (3.44) follows by dividing the set of curves $\alpha$ according to the holonomy element they yield. □

3.32 **Proposition.** *Let $(E, \pi, h, \nabla^E)$ be a vector bundle with metric and compatible connection over $(M, g)$ and consider its corresponding Sasaki metric G. The Levi-Civita covariant deriva-*

*tive $\nabla$ of $\mathrm{G}$ is the given at a point $e \in E$ by*

$$\nabla_{\sigma^v} \tau^v = 0 \tag{3.45}$$

$$\nabla_{\sigma^v} Y^h = (\mathfrak{F}_e(Y,\sigma))^h \tag{3.46}$$

$$\nabla_{X^h} \tau^v = (\mathfrak{F}_e(X,\tau))^h + (\nabla^E_X \tau)^v \tag{3.47}$$

$$\nabla_{X^h} Y^h = (\nabla_X Y)^h - \frac{1}{2}(R^E_{X,Y} e)^v, \tag{3.48}$$

*for vector fields $X, Y \in \mathfrak{X}(M)$ and sections $\sigma, \tau \in \Gamma(E)$. Where $\mathfrak{F}$ is given by the equation*

$$2g(\mathfrak{F}_e(x,f),y) = h(R^E_{x,y} e, f). \tag{3.49}$$

*Proof.* All of these are similar; for instance, the case $\nabla_{\sigma^v} Y^h$ goes as follows.

$$\begin{aligned} 2\mathrm{G}(\nabla_{\sigma^v} Y^h, \tau^v) &= \sigma^v(\mathrm{G}(Y^h,\tau^v)) + Y^h(\mathrm{G}(\tau^v,\sigma^v)) - \tau^v(\mathrm{G}(\sigma^v, Y^h)) \\ &\quad + \mathrm{G}(\tau^v,[\sigma^v,Y^h]) + \mathrm{G}(Y^h,[\tau^v,\sigma^v]) - \mathrm{G}(\sigma^v,[Y^h,\tau^v]) \\ &= 0 + Y(h(\tau,\sigma)) - 0 \\ &\quad + h(\tau, -\nabla^E_Y \sigma) + 0 - h(\sigma, \nabla^E_Y \tau) = 0, \end{aligned}$$

$$\begin{aligned} 2\mathrm{G}(\nabla_{\sigma^v} Y^h, Z^h) &= \sigma^v(\mathrm{G}(Y^h,Z^h)) + Y^h(\mathrm{G}(Z^h,\sigma^v)) - Z^h(\mathrm{G}(\sigma^v, Y^h)) \\ &\quad + \mathrm{G}(Z^h,[\sigma^v,Y^h]) + \mathrm{G}(Y^h,[Z^h,\sigma^v]) - \mathrm{G}(\sigma^v,[Y^h,Z^h]) \\ &= 0 + 0 - 0 \\ &\quad + 0 + 0 + h(\sigma, R^E_{Y,Z} e) \\ &= 2g(\mathfrak{F}_e(Y,\sigma),Z). \end{aligned}$$

□

3.33 **Corollary.** *In the case when the vector bundle is $(TM, g, \nabla)$, one can recover the formulæ obtained by Kowalski [29], that is*

$$\mathfrak{F}_u(x,v) = \frac{1}{2} R(u,v)x. \tag{3.50}$$

*Proof.* In this case, $h = g$, $R^{TM} = R$ and thus

$$2g(\mathfrak{F}_u(x,v),y) = h(R^{TM}_{x,y} u, v) = g(R(x,y)u,v) = g(R(u,v)x,y).$$

□

### 3.4.2 Metric properties

The setting is as follows. Given a Riemannian manifold $(M,g)$ (i.e. a metric space) and a vector bundle over it with certain additional structure, one produces another Riemannian manifold $(E,\textsc{g})$. It has been thoroughly investigated in the case when $E$ is the tangent bundle. In that case, it was noted by Musso and Tricerri [35] that $\textsc{g}$ is flat if and only if $g$ is also. In general, one also has the requirement that $E$ admit a flat metric. Even without those assumptions, certain flatness is still present.

Certainly, by construction, $\textsc{g}$ renders $\pi$ a Riemannian submersion; but additional properties occur.

**3.34 Proposition.** *Let $(E,\textsc{g})$ be a vector bundle with a Sasaki metric over a Riemannian manifold $(M,g)$. The following properties hold:*

1. *$\pi : E \to M$ is a Riemannian submersion.*

2. *$\varsigma : M \to E$, the zero section, is a totally convex isometric embedding.*

3. *For any $p \in M$ the fiber $F_p = \pi^{-1}(p)$ is totally geodesic, flat, and equidistant.*

*Proof.* The fact that $\pi : E \to M$ is a Riemannian submersion follows from the description of $\textsc{g}$ in 3.28.

To show that the metric induced on it coincides with $g$ (i.e. $g = \varsigma^*\textsc{g}$), and that the second fundamental form vanishes, it suffices to observe that the tangent spaces to $\Sigma_0$ coincide with the horizontal distribution. But given the fact that the connection $C$ is bilinear (cf. 3.8) yields

$$\varsigma_*\frac{\partial}{\partial x^i} = \frac{\partial}{\partial x^i} = \frac{\partial}{\partial x^i} - r^j(0)\Gamma_{ij}^k e_k^v = \left(\frac{\partial}{\partial x^i}\right)^h \in \mathcal{H}$$

and

$$\nabla_{X^h} Y^h = (\nabla_X Y)^h - (R^E_{XY}0)^v \in \mathcal{H},$$

thus proving the claim. To prove that the image $\Sigma_0$ of $\varsigma : M \to E$ is a locally convex submanifold, recall that

$$L(c) = \int \textsc{g}(\dot c, \dot c)^{\frac{1}{2}} \geq \int \textsc{g}(\mathcal{H}\dot c, \mathcal{H}\dot c)^{\frac{1}{2}} = L(\pi \circ c), \tag{3.51}$$

so convexity follows since for any curve joining $\varsigma(p)$ and $\varsigma(q)$ within $\Sigma_0$ is necessarily smaller than any other curve having non vanishing vertical component.

The fibers are totally geodesic in view of (3.45). Flatness is yet another application of (3.45), since the curvature is tensorial. □

**3.35 Lemma.** *Let $e \in E$ and let $\Sigma_0$ be the zero section. Then*

$$d(e,\Sigma_0) = \sqrt{\mu(e)} = |e|. \tag{3.52}$$

*Proof.* Let $e \in E$ and let $p \in M$ such that $d(e,\varsigma(p)) = d(e,\Sigma_0)$. Then, since the tangent vector at $\varsigma(p)$ to any minimizing geodesic to $e$ is perpendicular to $\Sigma_0$, it is also vertical, and, by 3.34, will remain in $E_p$. Therefore, $\pi u = p$ and there exists a unique such $p \in M$. □

**3.36 Proposition.** *Let $e \in E$, then $t \mapsto te$ is a minimizing geodesic for all time; i.e. a ray.*

*Proof.* This follows from the lemma and the fact that any minimizing geodesic between two points is also minimizing between any other two points along its trace.

□

*3.37 Remark.* These properties can also be derived from (3.43) or (3.44) directly.

As an immediate consequence,

**3.38 Corollary.** *$M$ can be thought of as a submanifold $\Sigma_0 \subseteq E$ and $\pi$ can be constructed intrisically as the retraction*

$$\rho = \varsigma \circ \pi, \tag{3.53}$$

*which can be interpreted geometrically as follows. Let $e \in E$, then*

$$\rho(e) = \textit{unique point in } \Sigma_0 \textit{ at distance } \sqrt{\langle e,e\rangle} \textit{ from } e. \tag{3.54}$$

With this in mind, the next classical result now follows.

**3.39 Corollary.** *$E$ is isomorphic to the normal bundle of $\varsigma$ and the isomorphism is given by*

$$\beta(e) = \mathfrak{I}(\rho(e),e) = \dot{\gamma}_e(0) \tag{3.55}$$

*where $\gamma_e$ is the unique geodesic from $\rho(e)$ to $e$ with speed $d(e,\Sigma_0)$.*

*3.40 Remark.* This map is nothing but the inverse of the exponential map at $\rho(e)$ restricted to $\mathcal{V}$.

Another important fact is that of metric completeness.

**3.41 Proposition.** *The Sasaki metric is complete if and only if $(M,g)$ is complete.*

*Proof.* In view of the classical theorem of Hopf-Rinow, the equivalence follows from viewing $M$ as the zero section $\Sigma_0$ and from considering the following sets, in lieu of metric balls,

$$C_p(R) = \{v \in E | \pi(v) \in B_p(r), \mu(v) = r^2\},$$

which are sequentially compact —and thus compact—, together with the fact that

$$B_v(r) \subseteq C_{\pi(v)}(r + \|v\|).$$

□

Summarizing, the differential and metric geometric properties of these metrics:

1. The Sasaki-type metric G is complete if and only if $g$ is also complete.
2. The projection $\pi : E \to M$ is a Riemannian submersion.
3. The fibers $E_p$ are totally geodesic and flat.
4. The zero section $\varsigma : M \to E$, $\varsigma(p) = 0_p \in E_p$ is an isometric embedding (i.e. it is distance preserving).
5. The rays $t \mapsto tu$ are geodesic rays for $t \geq 0$ and are unique in joining $u$ to the closest point to the zero section.

## 3.5 Means of identification: Norms and their derivatives.

A somewhat natural question is whether one can recover the base manifold from the Riemannian structure on the total space of its tangent bundle. Topological considerations aside, this question will be partially addressed here from a metric and differential geometric viewpoint.

It will be seen that there are certain geometric objects that essential recover the structure, but a definite answer is yet to be found. The first section establishes how to recover said structure through the knowledge of certain gradient vector field, or more precisely through a particular function, namely $\mu(e) = \|e\|$. The second section, reviews the notion of almost tangent structure and recalls a result of Thompson and Schwardmann [48].

### 3.5.1 Canonical constructs

Given any vector bundle with metric $(E, \pi, h)$ over $(M, g)$ endowed with a Riemannian metric of Sasaki-type certain amount of information about $g$ can be recover once one has information about the following vector field.

**3.42 Definition.** The canonical vector field of $(E, \pi)$ denoted by $\mathfrak{X}$ is given by

$$\mathfrak{X}(e) = \mathfrak{J}(e, e). \tag{3.56}$$

It is smooth, being given as the pullback of the identity map. Notice that the vanishing set of $\mathfrak{X}$ is exactly the zero section. Therefore, to recover the structure, in this case, should mean to exhibit $\pi$.

**3.43 Proposition.** *Let $(E, \pi, h, \nabla^E, \mathrm{G}, \triangledown)$ be a vector bundle with metric connection with a Riemannian metric of Sasaki-type. Let $\mathfrak{X}$ be its canonical vector field, let $\Phi$ be its flow, and let $\Sigma = \{e|\mathfrak{X}(e) = 0\}$. Then $\mathfrak{X}$ is complete. Furthermore, by virtue of Lemma 3.35, $\pi$ may be regarded as the map*

$$\pi(e) = \lim_{t \to -\infty^+} \Phi_t(e) \tag{3.57}$$

*Proof.* By the definition of the canonical vector field, one sees that it is essentially the position vector field when restricted to the fibers. Because of this, this is enough to see that the flow is given by

$$\Phi_t(e) = \exp(t)e,$$

which proves both claims by thinking of $\pi$ as the retraction $\rho$ onto $\Sigma$ given in Lemma 3.35. ☐

To further recover the structure, one sees that the covariant derivative of the canonical vector field behaves as expected, as stated in the following.

**3.44 Proposition.** *Let $(E, \pi, h, \nabla^E, \mathrm{G}, \triangledown)$ be as before.*

1. *The canonical vector field $\mathfrak{X}$ is vertical:*

$$\pi_* \mathfrak{X} = 0. \tag{3.58}$$

2. *The covariant derivative of the canonical vector field $\mathfrak{X}$ at any point $e \in E$ coincides with the vertical projection:*

$$\triangledown \mathfrak{X} = \mathcal{V}, \tag{3.59}$$

*where $\mathcal{V}$ is the projection onto the vertical subbundle.*

*Proof.* The first part is an immediate consequence of the definition, since $\mathfrak{I}$ maps onto the vertical bundle. For the second part, consider a local frame $e_j \in \Gamma(E)$ as at the beginning of the section. Then $\mathcal{X}$ can be written as follows.

$$\mathcal{X} = r^j e_j^v, \tag{3.60}$$

so that for a vertical lift $f^v$,

$$\triangledown_{f^v} \mathcal{X} = f^v(r^j)e_j = (r^i(f)e_i)^v(r_j) = r^i(f)\delta_i^j e_j^v = f^v,$$

since $e_i^v = \frac{\partial}{\partial r^i}$. Now, for a horizontal lift $x^h$,

$$(\triangledown_{x^h}\mathcal{X})_e = x^h(r^j)e_j^v(e) + r^j(e)\triangledown_{x^h} e_j^v(e) = r^j(e)(\mathfrak{F}_e(x, e_j))^h = (\mathfrak{F}_e(x,e))^h = 0$$

where the second equality follows from (3.47) since

$$x^h(r^j)e_j = [x(r^j) - r^k\Gamma_k^\ell(x)e_\ell^v(r^j)]e_j = -r^k\Gamma_k^j(x)e_j = -r^k\nabla_x^E e_k,$$

and the last equality follows from the fact that $h(R_{x,y}^E\sigma,\tau) = -h(R_{x,y}^E\tau,\sigma)$:

$$2g(\mathfrak{F}_e(x,e),y) = h(R_{x,y}^E e, e) = 0.$$

$\square$

**3.45 Corollary.** *The canonical splitting of $TE$ can be recovered from the knowledge of $\mathrm{G}$ and $\mathcal{X}$.*

Analogously, if one is stead given the norm —a seemingly weaker assumption— by virtue of the following considerations, the canonical vector field can be recovered and thus so will the projection map.

**3.46 Definition.** The canonical function of $(E,\pi,h)$ denoted by $\mu = \mu_E$ is given by

$$\mu(e) = \sqrt{h(e,e)} = \sqrt{\mathrm{G}(\mathcal{X},\mathcal{X})}. \tag{3.61}$$

**3.47 Proposition.** *Let $\mathrm{G}$ be a Sasaki metric, then the gradient and the Hessian of $\mu^2$ are given by*

$$\nabla\mu^2 = 2\mathcal{X} \tag{3.62}$$

$$\triangledown\nabla\mu^2 = 2\mathcal{V} \tag{3.63}$$

*Proof.* The second equation follows from the first in view of 3.44. To establish the first, one observes that

$$\mathrm{G}(\nabla\mu^2,\xi) = \xi h(e,e) = \xi\mathrm{G}(\mathfrak{X},\mathfrak{X}) = \mathrm{G}(2\mathfrak{X},\nabla_\xi\mathfrak{X}) = \mathrm{G}(2\mathfrak{X},\mathcal{V}\xi)$$

so that for a vertical $f^v$,

$$\mathrm{G}(\nabla\mu^2,f^v) = \mathrm{G}(2\mathfrak{X},f^v),$$

and for a horizontal $x^h$,

$$\mathrm{G}(\nabla\mu^2,x^h) = 0 = \mathrm{G}(2\mathfrak{X},x^h).$$

□

3.48 **Corollary.** *The level sets $\Sigma_r := \mu^{-1}(r)$ are submanifolds of $E$ for all $r \geq 0$ and in particular $\Sigma_0 = \varsigma(M)$ where $\varsigma \in \Gamma(E)$ with $\varsigma(p) = 0$; i.e. $\varsigma$ is the zero section.*

### 3.5.2 Almost tangent structures

The notion of tensor structure was introduced by Clark and Bruckheimer [16] in the context of the extensively studied $G$-structures. Examples of this are the structures given by metric tensors (of any signature), orientations, almost complex structures, etc.

It is common knowledge that any paracompact manifolds admit metric tensors, but certain other structures impose, by nature, conditions on dimension; as well as some integrability notions and conditions.

The total space of a tangent bundle, being even dimensional and orientable, has a plethora of these structures. A particularly pertinent example is that of an almost tangent structure.

3.49 **Definition.** Let $N$ be a smooth manifold. An almost tangent structure is a bundle endomorphism $S$ on $TN$ satisfying $S^2 = 0$ and such that $\mathrm{Im}S = \mathrm{Ker}S$.

The existence of an almost tangent structure requires the dimension of $N$ to be even and in the case of the total space of a tangent bundle, this structure is quite canonical, thus the suggestive name.

3.50 **Proposition.** *The map $S = \mathfrak{I}\circ(\pi_M \oplus \pi_*)$ is an almost tangent structure.*

*Proof.* The fact that $S^2 = 0$ is immediate from the definition and both $\pi_*$ and $\mathfrak{I}$ being onto yields that

$$\mathcal{V} = \mathrm{Im}S = \ker S.$$

□

Thompson and Schwardmann [48] give a comprehensive review of the theory of tangent manifolds; i.e. those with an integrable almost tangent structure. One such structure is said to be integrable if its Nijenhuis tensor vanishes; or equivalently, if the distribution $\mathrm{Im}S = \ker S$ is integrable. In particular, the following is true.

3.51 **Proposition** ([48]). *Let $(N, \pi, M, S)$ denote Riemannian manifolds $N$ and $M$, a Riemannian submersion $\pi$ and an almost tangent structure on $N$ such that* $\mathrm{Im}S = \ker \pi_*$. *Then $\pi$ is a fibre bundle with fibers diffeomorphic to a product* $(\mathbb{S}^1)^k \times \mathbb{R}^\ell$. *So that if the fibers are simply connected, then $N$ is affinely diffeomorphic to $TM$.*

In view of this last result, one is necessarily left pondering the following.

3.52 **Question.** *What is the simplest additional geometric information needed to recover $S$ from the knowledge of* $\mathrm{G}$*?*

Even though this question remains open, in the sequel, using the techniques from Gromov's theory of convergence, several structures are analyzed. Many of these questions hint that the interplay between the metric properties on the total space of the tangent bundle and that of the base is indeed quite rich.

## 3.6 Holonomy: from global to local.

Aside from introducing the specific terminology, exhibiting quite explicitly the length metric structure of Sasaki-type Riemannian metrics as seen in Theorem 3.28, the main observation in this chapter comes from the realization that the fibers, even though they were seen to be dull from the viewpoint of differential geometry, carry a lot of information in their lack of convexity: The way these vector spaces sit inside the total space is a manifestation of the latter's global geometry. And this is the content of this section.

### 3.6.1 Fibers as holonomic spaces

Recall from Definition 2.1 that a holonomic space has three constituents: a normed vector space, a subgroup of the norm-preserving linear isomorphisms, and a group-norm on said subgroup. Furthermore, the law they have to abide by is the convexity property (P). In this section, the fibers are seen to satisfy this, together with the holonomy group at their base-point an the following group-norm.

3.53 **Theorem** (Solórzano [45]). *Let $Hol_p$ be the holonomy group over a point $p \in M$ of a bundle with metric and connection and suppose that $M$ is Riemannian. Then the function* $L_p : Hol_p \to \mathbb{R}$,

$$L_p(A) = \inf\{\ell(\alpha) | \alpha \in \Omega_p, P_1^\alpha = A\}, \tag{3.64}$$

*is a group-norm for $Hol_p$*

*Proof.* Positivity is immediate from the fact that it is defined as an infimum of positive numbers. To prove non-degeneracy suppose that an element $A \neq I$ has zero length. There exists $u \in E_p$ such that $Au \neq u$; thus, by (3.44), choosing $a = A$ yields $d(u, Au) = 0$. A contradiction.

The length of the inverse of any holonomy element is the same because the infimum is taken essentially over the same set (the same curves but traversed in the opposite direction). Finally, to establish the triangle inequality, note that the set of loops that generate $AB$ contains the concatenation of loops generating $A \in Hol_p$ with loops generating $B \in Hol_p$. □

3.54 **Definition.** The function $L_p$, defined by (3.64) will be called *length-norm* of the holonomy group induced by the Riemannian metric at $p$.

*3.55 Remark.* The fact that connection be metric is used twice in proof that $L_p$ is indeed non-degenerate. This is because once can then produce a Riemannian metric on the total space of the bundle, namely that of Sasaki type.

3.56 **Theorem** (Holonomic fiber theorem. Solórzano [45]). *Let $E_p$ be the fiber of a vector bundle with metric and connection $E$ over a Riemannian manifold $M$ at a point $p$. Let $Hol_p$ denote the associated holonomy group at $p$ and let $L_p$ be the group-norm given by* (3.64). *Then $(E_p, Hol_p, L_p)$ is a holonomic space. Moreover, if $E$ is endowed with the corresponding Sasaki-type metric, the associated holonomic distance coincides with the restricted metric on $E_p$ from $E$.*

*Proof.* According to the definition given in 2.1, the only remaining condition is given by (2.1). To see this, one needs only to note that the fiber $E_p$ is a totally geodesic submanifold of $E$. With this, given any point $u \in E_p$, let $r = \mathrm{CvxRad}_p(E) > 0$, the convexity radius; thus, for any pair of points $v, w \in B_r^E(p) \cap E_p$ there exists a unique geodesic from $v$ to $w$. This geodesic is necessarily $t \mapsto u - t(v - u) \in E_p$, and thus the distance

$$d(u, v) = \|u - v\|,$$

proving that the metric is locally Euclidean and by Theorem 2.9 the claim follows. □

This fact, as innocent as it seems, will be proved to be rather powerful in Chapter 5.

### 3.6.2 Holonomy Radius of a Riemannian Manifold

Given a Riemannian manifold $(M,g)$, in view of the fundamental theorem of Riemannian Geometry, one immediately obtains a vector bundle, a connection and a bundle metric compatible with the connection; i.e. the tangent bundle, the Levi-Civita connection and the metric itself. With this at hand, together with Theorem 3.56, there is no further need to motivate the following definition.

3.57 **Definition.** Let $(M,g)$ be a Riemannian manifold and let $p \in M$. The *holonomy radius* of $M$ at $P$ and denoted by $\mathrm{HolRad}_M(p)$ is defined to be the supremum of $r > 0$ such that for all $u, v \in M_p$ with $\|u\|, \|v\| \leq r$ and for all $a \in Hol_p$

$$\|u - v\|^2 - \|au - v\|^2 \leq L_p^2(a), \tag{3.65}$$

where $L_p$ is the associated length norm on $Hol_p$.

*3.58 Remark.* This is simply the holonomy radius at the origin of the holonomic space $(T_pM, Hol_p, L_p)$ (see Definition 2.2).

3.59 **Theorem.** *Given a Riemannian manifold M. The function that assigns to each point its holonomy radius is strictly positive.*

*Proof.* This is a direct consequence of 2.14 and the fact that the tangent spaces are holonomic by the holonomic fiber theorem3.56. □

*3.60 Remark.* This fact also follows directly from geometric considerations given that $0 < \mathrm{CvxRad}_{TM}(0_p) \leq \mathrm{HolRad}_M(p)$, where $\mathrm{CvxRad}_{TM}$ is the convexity radius of $TM$ with its Sasaki metric.

The kind reader might raise a natural question at this point:

3.61 **Question.** *Is the function*

$$\mathrm{HolRad} : M \to \mathbb{R}$$

*continuous?*

A partial answer is given in Chapter 5: the function is at least upper semi-contiuous. Of course, the only relevant case is when the function is finite. For otherwise flatness occurs, as per the next result.

3.62 **Proposition.** *If there exists a point p in a Riemanian manifold M for which the holonomy radius is not finite, then M is flat.*

*Proof.* by Proposition 2.13, the existence of such point is equivalent to the group being trivial. In particular, the restricted holonomy group is trivial, which in turn is equivalent to flatness. □

*3.63 Remark.* The converse is certainly not true. Consider for example a cone metric on $\mathbb{R}^2 \setminus \{0\}$, or the infinite Möbius strip, or the Klein bottle, all of which are flat but have non-trivial holonomy.

Summarizing one gets the following statement.

**3.64 Corollary.** *Let $M$ be a simply connected Riemannian manifold. If there is a point on $M$ with infinite holonomy radius, then $M$ is isometric to a Euclidean space.*

## 3.7 Twofold examples

In the case when $(M, g)$ is a two-fold more can be said from the Gauß-Bonnet Theorem. Furthermore, in the particular case of the $\mathbb{S}^2$ or $\mathbb{H}^2$, $L$ can be computed by virtue of the isoperimetric inequality.

Recall the following classical result.

**3.65 Lemma.** *Let $(M^2, g)$ be a 2-dimensional Riemannian manifold and let $\gamma : [0, \ell] \subseteq \mathbb{R} \to M$ be any curve parametrized by arc length. Let $k$ be a signed geodesic curvature of $\gamma$ with respect to an orientation of $\gamma^* TM$. Let $\theta(t)$ be the angle between $\dot{\gamma}$ and its parallel translate at time $t$. Then*

$$2\pi - \theta(t) = \int_0^t k \tag{3.66}$$

*Assume further that $\gamma$ is a loop. Then, possibly up to a reversal in orientation, the holonomy action of $\gamma$ at $p = \gamma(0)$ is the rotation by $2\pi - \int_0^\ell k$.*

*Proof.* Consider a compatible parallel almost complex structure on $\gamma^* TM$, $J$. With respect to the orthonormal frame given by $\{\dot{\gamma}, J(\dot{\gamma})\}$, $\nabla_{\dot{\gamma}} \dot{\gamma} = kJ(\dot{\gamma})$, and thus the equations for any parallel vector field $P = a\dot{\gamma} + bJ(\dot{\gamma})$ along $\gamma$ are given by

$$\begin{aligned} \dot{a} &= kb \\ \dot{b} &= -ka \end{aligned}$$

which integrates to a rotation by $-\int k$ as claimed. □

**3.66 Theorem.** *Let $M^2$ be a complete simply connected two-dimensional non-flat space-form with curvature K. Let $L : \mathbb{S}^1 \to \mathbb{R}$ be the associated length-norm on the holonomy group. Then*

$$L(\theta) = \frac{\sqrt{4\pi|\theta| \pm \theta^2}}{\sqrt{|K|}}, \tag{3.67}$$

*for $-\pi \leq \theta \leq \pi$, where the sign is opposite to the sign of the curvature.*

*Proof.* By the Gauß-Bonnet Theorem, $\theta = 2\pi - \int k = KA$, where $A$ is the area of the region enclosed by any loop $\gamma$, so that

$$A = \left|\frac{\theta}{K}\right|. \tag{3.68}$$

Now, the isoperimetric inequality in this case (see [36]) is given by

$$\ell^2 \geq 4\pi A - KA^2, \tag{3.69}$$

where the equality is achieved when $\gamma$ is metric circle. So, by direct substitution of (3.68) into (3.69) the claim follows. $\square$

**3.67 Corollary.** *Let $M^2$ be a simply connected two-dimensional non-flat space-form with curvature K. The holonomy radius at any point $p \in M$ is given by*

$$\inf_{-\pi \leq \theta \leq \pi} \sqrt{\frac{4\pi|\theta| \pm \theta^2}{2|K|\sqrt{2 - 2cos(\theta)}}}. \tag{3.70}$$

*Proof.* In view of (2.7), the only remain part is to compute $\|a - id\|$ for any holonomy element $a$. Since all of them are rotations by some angle $\theta$, if follows that $\|au - u\| = \|a - id\|\|u\|$ for any given $u \in T_pM$. Hence a direct application of the law of cosines yields that

$$\|a - id_V\| = \sqrt{2 - 2\cos\theta} \tag{3.71}$$

and hence the result. $\square$

## 3.8 Categorical concerns.

This section is devoted to determining to what extent does the construction of the Sasaki metric in the case of tangent bundles produces a functor at the level of Riemannian manifolds. The issue, of course, is to determine a reasonable class of maps.

The concept of natural bundles and of metrics has been studied by Terng [47] and Kowalski and Sekizawa [30] respectively, among others. In essence, the idea is to understand what constructions are functorial. In the case of bundles, the question is the following.

**3.68 Question.** *What bundle constructions are well defined up to diffeomorphism?*

Examples of these are tangent bundles, cotangent bundles, their products, etc. In fact, Terng [47] proves that natural bundles are in one-to-one correspondence to isomorphism classes of modules of jet groups.

For metrics, the question can be posed as follows.

**3.69 Question.** *What Riemannian metrics on tangent bundles are preserved under (local) isometries of the base?*

In the category of Riemannian manifolds and local isometries, the notion of *natural metric* for the tangent bundle (one for which the total differential as a map of tangent bundles is also a local isometry) has been extensively studied by Sasaki [42]; Kowalski and Sekizawa [30]; Kolář, Michor, and Slovák [28]; among others . A full classification was obtained by Abbassi and Sarih [2], yielding the following

**3.70 Theorem** (Abbassi and Sarih [2])**.** *Any natural metric on the tangent bundle rendering the projection Riemannian and preserving the natural splitting of the second tangent bundle can be written as follows:*

$$\langle X^h, Y^h\rangle_u = \langle X, Y\rangle_p, \tag{3.72}$$

$$\langle X^h, Y^v\rangle_u = 0, \tag{3.73}$$

$$\langle X^v, Y^v\rangle_u = \alpha(\|u\|^2)\langle X, Y\rangle_p + \beta(\|u\|^2)\langle u, X\rangle_p\langle u, Y\rangle_p, \tag{3.74}$$

*where $u, X, Y$ are tangent vectors at $p$; the superscripts denote the usual horizontal and vertical lifts; and $\alpha, \beta : [0,\infty) \to \mathbb{R}$ satisfy the following.*

$$\alpha(t) > 0,\ \alpha(t) + t\beta(t) > 0. \tag{3.75}$$

In particular, for $\alpha \equiv 1, \beta \equiv 0$, one gets the metric introduced by Sasaki [42].

Trying to extend the class of maps from local isometries to a larger class that includes isometric immersions and Riemannian submersions leads naturally to the following definition.

**3.71 Definition** (Fischer [18]). A Riemannian map $f$ is a smooth map between Riemannian manifolds such that it satisfies that, for every $p$,

$$f_{*p}\Big|(\ker(f_{*p})^{\perp} \longrightarrow \mathrm{Im}(f_{*p}) \tag{3.76}$$

is a linear isometry.

In particular, Fischer [18] observes that these maps are locally the composition of Riemannian submersions and isometric immersions, and have constant rank. In particular, from the classical Constant Rank Theorem (see [41]), it follows that the fibers and the image of a Riemannian map are smooth manifolds.

**3.72 Question.** *Does the Sasaki construction of a metric render the tangent bundle into a functor from the category of Riemannian manifolds with Riemannian maps to itself?*

Unfortunately, the answer is negative. In order for the differential of a map to be Riemannian, the original map needs to be Riemannian and satisfy an extra condition. Namely that the fibers and the image be totally geodesic.

**3.73 Definition.** Let $\varphi : (M, g) \to (\overline{M}, \overline{g})$ be a Riemannian map. A vector field $x \in \mathfrak{X}(M)$ is called *basic* if

1. for any point $p \in M$, $x(p) \in (\ker(\varphi_{*p}))^{\perp}$; and
2. there exists a vector field $X \in \mathfrak{X}(\overline{M})$, $\varphi$–related to $x$, i.e. such that

$$X \circ \varphi = \varphi_* \circ x. \tag{3.77}$$

Just as in the particular case of Riemannian submersions (or of isometric immersions), it is possible to produce basic vector fields in a neighborhood of a point $p \in M$ such that $x(u)$ is prescribed, by the standard procedure: By the first assumption, consider $X(\varphi(p)) = \varphi_* x(p)$, smoothly extend it to a vector field on $U \subseteq \varphi(M)$ and on $\overline{M}$ and then consider, for any point $q \in \varphi^{-1}(U)$, $x(q)$ to be the unique tangent vector, perpendicular to the kernel of $\varphi_*$, whose projection is $X(\varphi(q))$.

**3.74 Definition.** Let $\varphi : (M, g) \to (\overline{M}, \overline{g})$ be a Riemannian map. The *second fundamental form $B$ of $\varphi$* is a bilinear bundle map $B : \oplus^2(\ker(\varphi_*))^{\perp} \to T\overline{M}$ over $\varphi$ given by

$$B(u, x) = \overline{\nabla}_{\varphi_* u} X - \varphi_* \nabla_u x. \tag{3.78}$$

for basic vector fields.

The proof that this is tensorial, bilinear and symmetric product can be given by the same argument as for isometric immersions. It follows that $B$ vanishes identically if and only if the image $\varphi(M)$ is a totally geodesic submanifold of $\overline{M}$.

For notational purposes, for any $x \in T_pM$, denote by $x_\top \in \ker \varphi_{*p}$ and $x_\perp \in (\ker(\varphi_*))^\perp$ the unique such vectors such that

$$x = x_\top + x_\perp. \tag{3.79}$$

For brevity, this will be called *the* $(\perp, \top)$ *splitting*. Furthermore, an additional second fundamental form is given for the fibers.

**3.75 Definition.** Let $\varphi : (M, g) \to (\overline{M}, \overline{g})$ be a Riemannian map. The *fiberwise second fundamental form $\alpha$ of $\varphi$* is a bilinear map $\alpha : \oplus^2 \ker(\varphi_*) \to TM$ given by

$$\alpha(t, s) = (\nabla_t s)_\perp. \tag{3.80}$$

for vector fields tangential to the fibers. Associated to it, the *fiber shape operator*, $S$ : $\ker \varphi_* \oplus (\ker \varphi_*)^\perp \to TM$, given by

$$S_t(x) := S(t, x) = -(\nabla_x t)_\perp \tag{3.81}$$

Either of these tensors measures how much the fibers differ from being totally geodesic. In fact both of them vanish identically if and only if this be the case.

**3.76 Lemma.** *Let $\varphi : (M, g) \to (\overline{M}, \overline{g})$ be a Riemannian map, let their tangent bundles be given their corresponding Sasaki metrics $\mathrm{G}(g)$ and $\mathrm{G}(\overline{g})$ respectively, and let $C$ and $\overline{C}$ be their corresponding Levi-Civita connections. Let $u \in \mathfrak{X}(M)$ and $U \in \mathfrak{X}(\overline{M})$ be $\varphi$–related, and let $v \in TM$. Then*

$$\varphi_{**}C(u, v) = \overline{C}(\varphi_* u, \varphi_* v) + \mathfrak{I}(\varphi_* u, \overline{\nabla}_{\varphi_* v} U - \varphi_* \nabla_v u) \tag{3.82}$$

*Proof.* Recall that by (3.26), for any vector field $y \in \mathfrak{X}(M)$,

$$\mathfrak{I}(y, \nabla_x y) = y_* x - C(y, x),$$

and that for any map

$$\varphi_{**}\mathfrak{I}(y, x) = \mathfrak{I}(\varphi_* y, \varphi_* x),$$

by Corollary 3.7, which produces

$$
\begin{aligned}
\varphi_{**}C(u,v) &= \varphi_{**}u_*v - \mathfrak{I}(\varphi_*u, \varphi_*\nabla_v u) && (3.83)\\
&= U_*\varphi_*v - \mathfrak{I}(\varphi_*u, \varphi_*\nabla_v u) && (3.84)\\
&= \overline{C}(U, \varphi_*v) + \mathfrak{I}(\varphi_*u, \overline{\nabla}_{\varphi_*v}U - \varphi_*(\nabla_v u)) && (3.85)\\
&= \overline{C}(\varphi_*u, \varphi_*v) + \mathfrak{I}(\varphi_*u, \overline{\nabla}_{\varphi_*v}U - \varphi_*(\nabla_v u)), && (3.86)
\end{aligned}
$$

as promised. □

**3.77 Proposition.** *Let $\varphi : (M,g) \to (\overline{M},\overline{g})$ be a Riemannian map, let their tangent bundles be given their corresponding Sasaki metrics $\mathrm{G}(g)$ and $\mathrm{G}(\overline{g})$ respectively, and let $C$ and $\overline{C}$ be their corresponding Levi-Civita connections and let $u,v \in TM$. Then*

$$
\varphi_{**}C(u,v) = \overline{C}(\varphi_*u, \varphi_*v) + \mathfrak{I}(\varphi_*u, B(u_\perp, v_\perp) - \phi_*(T(u,v) - \alpha(u_\top, v_\top))) \qquad (3.87)
$$

*where $T$ is given by*

$$
T(u,v) = S_{u_\top}(v_\perp) + S_{v_\top}(u_\perp).
$$

*Equivalently, for $u, X \in T_pM$,*

$$
\varphi_{**}X^h = (\varphi_*X)^h + [B(u_\perp, X_\perp) - \varphi_*(T(u,X) - \alpha(u_\top, X_\top))]^v. \qquad (3.88)
$$

*Proof.* From Lemma 3.76, and because any $u \in T_pM$ can be extended into a projectable vector field, one sees that

$$
\overline{\nabla}_{\varphi_*v}U - \varphi_*(\nabla_v u) = B(u_\perp, v_\perp) - \phi_*(T(u,v) - \alpha(u_\top, v_\top).
$$

Consider first the $(\perp, \top)$ splitting:

$$
(\nabla_u v)_\perp = (\nabla_{v_\perp}(u_\top) + \nabla_{v_\top}(u_\top) + \nabla_{v_\perp}(u_\perp) + \nabla_{v_\top}(u_\perp))_\perp. \qquad (3.89)
$$

Now, recall that $B$ is given by

$$
\begin{aligned}
B(u_\perp, v_\perp) &= \overline{\nabla}_{\varphi_*v_\perp}U - \varphi_*(\nabla_{v_\perp}u_\perp) && (3.90)\\
&= \overline{\nabla}_{\varphi_*v}U - \varphi_*[(\nabla_{v_\perp}u_\perp)_\perp], && (3.91)
\end{aligned}
$$

and that

$$
\alpha(u_\top, v_\top) = (\nabla_{u_\top}v_\top)_\perp. \qquad (3.92)
$$

Lastly, extend $v$ to a vector field. Thus,

$$\begin{aligned}\varphi_*(\nabla_{v_\perp}(u_\top)+\nabla_{v_\top}(u_\perp)) &= \varphi_*(\nabla_{v_\perp}(u_\top)+\nabla_{u_\perp}(v_\top)) && (3.93)\\ &= \varphi_*(T(u,v)), && (3.94)\end{aligned}$$

since $\varphi_*(\nabla_{v_\top}(u_\perp)) = \varphi_*(\nabla_{u_\perp}(v_\top))$. To get (3.88), one has only to remember that $X^v(u) = \mathfrak{I}(u,X)$ and $X^h(u) = C(u,X)$ for elements $u, X \in TM$ on the same fiber. □

3.78 **Corollary.** *Let $\varphi : (M,g) \to (\overline{M},\overline{g})$ be a Riemannian map and let their tangent bundles be given their corresponding Sasaki metrics* $\mathrm{G}(g)$ *and* $\mathrm{G}(\overline{g})$ *respectively. Let $u \in T_pM$ and let $x \in (\ker(\varphi_{*p}))^\perp$, then*

$$\varphi_{**}C(u,x) = \overline{C}(\varphi_*u,\varphi_*x) + \mathfrak{I}(\varphi_*u, B(u_\perp,x_\perp)) - \mathfrak{I}(\varphi_*u,\varphi_*(T(u,x))). \tag{3.95}$$

*Proof.* By Proposition 3.77,

$$\begin{aligned}\varphi_{**}C(u,x) &= \overline{C}(\varphi_*u,\varphi_*x) + \mathfrak{I}(\varphi_*u, B(u_\perp,x_\perp) - \varphi_*(T(u,x) - \alpha(u_\top,x_\top)))\\ &= \overline{C}(\varphi_*u,\varphi_*x) + \mathfrak{I}(\varphi_*u, B(u_\perp,x_\perp) - \varphi_*(T(u,x)))\\ &= \overline{C}(\varphi_*u,\varphi_*x) + \mathfrak{I}(\varphi_*u, B(u_\perp,x_\perp)) - \mathfrak{I}(\varphi_*u,\varphi_*(T(u,x))).\end{aligned}$$

□

3.79 **Corollary.** *Let $\varphi : (M,g) \to (\overline{M},\overline{g})$ be a Riemannian map, let their tangent bundles be given their corresponding Sasaki metrics* $\mathrm{G}(g)$ *and* $\mathrm{G}(\overline{g})$ *respectively, and let $C$ and $\overline{C}$ be their corresponding Levi-Civita connections. The map $\varphi$ has totally geodesic fibers and totally geodesic image if and only if for all $X \in TM$,*

$$\begin{aligned}\varphi_{**}X^h &= (\varphi_*X)^h && (3.96)\\ \varphi_{**}X^v &= (\varphi_*X)^v. && (3.97)\end{aligned}$$

*Equivalently, $\varphi_{**}$ commutes with vertical and horizontal projections.*

*Proof.* The assumption of total geodesy is equivalent to the vanishing of $B, \alpha, T$ and $S$ and, thus, by (3.88) and by Corollary 3.7, the claim follows. □

One can now characterize the kernel of $\varphi_{**}$

3.80 **Proposition.** *Let $\varphi : (M,g) \to (\overline{M},\overline{g})$ be a Riemannian map, let their tangent bundles be given their corresponding Sasaki metrics* $\mathrm{G}(g)$ *and* $\mathrm{G}(\overline{g})$ *respectively, and let $C$ and $\overline{C}$ be their* *corresponding Levi-Civita connections. A vector $X^h + Y^v \in TTM$ is in the kernel of $\varphi_{**}$ if and only if $X$ is in the kernel of $\varphi_*$, and*

$$Y_\perp = T(u,X) + \alpha(u_\top, X_\top) \tag{3.98}$$

*Proof.* From Proposition 3.77 and Corollary 3.7,

$$\varphi_{**}(X^h + Y^v) = (\varphi_* X)^h + (\varphi_* Y + B(u_\perp, X_\perp) - \varphi_*(T(u,X) - \alpha(u_\top, X_\top)))^v.$$

This already implies that $\varphi_* X = 0$. Now, this also implies that

$$\varphi_* Y + B(u_\perp, X_\perp) - \varphi_*(T(u,X) - \alpha(u_\top, X_\top)) = 0.$$

Since $B$ is perpendicular to the image of $\varphi$, it follows that

$$B(u_\perp, X_\perp) = 0 \tag{3.99}$$

and

$$\varphi_*(Y - T(u,X) - \alpha(u_\top, X_\top)) = 0. \tag{3.100}$$

However, both $T$ and $\alpha$ are defined by taking their $\perp$ –component, hence one is left only with

$$Y_\perp = T(u,X) + \alpha(u_\top, X_\top),$$

as claimed. □

**3.81 Corollary.** *Let $\varphi : (M,g) \to (\overline{M}, \overline{g})$ be a Riemannian map, let their tangent bundles be given their corresponding Sasaki metrics $\mathrm{G}(g)$ and $\mathrm{G}(\overline{g})$ respectively, and let $C$ and $\overline{C}$ be their corresponding Levi-Civita connections. Suppose that $\varphi$ has totally geodesic fibers and totally geodesic image. Then vector $X^h + Y^v \in TTM$ is in the kernel of $\varphi_{**}$ if and only if $X$ and $Y$ are in the kernel of $\varphi_*$, and in particular, a vector $X^h + Y^v \in TTM$ is perpendicular to the kernel of $\varphi_{**}$ if and only if $X$ and $Y$ are perpendicular to the kernel of $\varphi_*$.*

*Proof.* By Proposition 3.80, and Corollary 3.79, (3.98) reduces to $Y_\perp = 0$, which is equivalent to $\varphi_* Y = 0$ as claimed. The conclusion about $(\ker \varphi_*)^\perp$ now follows since the Sasaki metric renders vertical lifts and horizontal lifts perpendicular to each other and preserves orthogonality:

$$G(g)(X_\top^h + Y_\top^v, Z^h + W^v) = g(X_\top, Z) + g(Y_\top, W),$$

which implies that $g(X_\top, Z) = 0 = g(Y_\top, W)$, thus completing the proof. □

**3.82 Lemma.** *Let $\varphi : (M, g) \to (\overline{M}, \overline{g})$ be a Riemannian map and let their tangent bundles be given their corresponding Sasaki metrics $\mathrm{G}(g)$ and $\mathrm{G}(\overline{g})$ respectively. Let $u \in T_pM$ and let $x, y \in (\ker(\varphi_{*p}))^{\perp}$, then*

$$[(\varphi_*)^* \mathrm{G}(\overline{g})]\,(\mathfrak{I}_u(x), \mathfrak{I}_u(y)) = \mathrm{G}(g)(\mathfrak{I}_u(x), \mathfrak{I}_u(y)) \tag{3.101}$$

*Proof.* Recall that by Corollary 3.7,

$$\varphi_{**}\mathfrak{I}_u(x) = \mathfrak{I}_{\varphi_* u}(\varphi_* x)$$

From this it follows that

$$\begin{aligned}
[(\varphi_*)^* \mathrm{G}(\overline{g})](\mathfrak{I}_u(x), \mathfrak{I}_u(y)) &= \mathrm{G}(\overline{g})(\varphi_{**}\mathfrak{I}_u(x), \varphi_{**}\mathfrak{I}_u(y)) \\
&= \mathrm{G}(\overline{g})(\mathfrak{I}_{\varphi_* u}(\varphi_* x), \mathfrak{I}_{\varphi_* u}(\varphi_* y)) \\
&= \overline{g}_{\varphi(p)}(\varphi_* x, \varphi_* y) \\
&= g_p(x, y) \\
&= \mathrm{G}(g)(\mathfrak{I}_u(x), \mathfrak{I}_u(y)),
\end{aligned}$$

since by assumption $\varphi^* \overline{g} = g$ when restricted to $(\ker(\varphi_{*p}))^{\perp}$. □

**3.83 Lemma.** *Let $\varphi : (M, g) \to (\overline{M}, \overline{g})$ be a Riemannian map and let their tangent bundles be given their corresponding Sasaki metrics $\mathrm{G}(g)$ and $\mathrm{G}(\overline{g})$ respectively. Let $u \in T_pM$ and let $x, y \in (\ker(\varphi_{*p}))^{\perp}$, then*

$$\begin{aligned}
[(\varphi_*)^* \mathrm{G}(\overline{g})](C(u, x), C(u, y)) = \mathrm{G}(g)&(C(u, x), C(u, y)) \\
&+ \overline{g}(B(u_{\perp}, x), B(u_{\perp}, y)) \\
&+ g(T(u, x), T(u, y)).
\end{aligned} \tag{3.102}$$

*Proof.* By Corollary 3.78,

$$\varphi_{**}C(u, x) = \overline{C}(\varphi_* u, \varphi_* x) + \mathfrak{I}(\varphi_* u, B(u_{\perp}, x_{\perp})) - \mathfrak{I}(\varphi_* u, \varphi_*(T(u, x))) \tag{3.103}$$

Notice now that this is an orthogonal decomposition of $\varphi_{**}C(u, x)$ and therefore the claim follows. □

**3.84 Lemma.** *Let $\varphi : (M, g) \to (\overline{M}, \overline{g})$ be a Riemannian map and let their tangent bundles be given their corresponding Sasaki metrics $\mathrm{G}(g)$ and $\mathrm{G}(\overline{g})$ respectively. Let $u \in T_pM$ and let*

$x, y \in (\ker(\varphi_{*p}))^{\perp}$, *then*

$$[(\varphi_*)^* \mathrm{G}(\overline{g})](C(u,x), \mathfrak{I}(u,y)) = g(T(u,x), y) \tag{3.104}$$

*Proof.* By Corollaries 3.78 and 3.7,

$$\varphi_{**}C(u,x) = \overline{C}(\varphi_* u, \varphi_* x) + \mathfrak{I}(\varphi_* u, B(u_\perp, x_\perp)) - \mathfrak{I}(\varphi_* u, \varphi_*(T(u,x)))$$

$$\varphi_{**}\mathfrak{I}(u,y) = \mathfrak{I}(\varphi_* u, \varphi_* y)$$

Now, again, since these are orthogonal splittings, the only remaining term is

$$\begin{aligned} \mathrm{G}(\overline{g})(\mathfrak{I}(\varphi_* u, \varphi_*(T(u,x))), \mathfrak{I}(\varphi_* u, \varphi_* y)) &= \overline{g}(\varphi_* T(u,x), \varphi_* y) \\ &= g(T(u,x), y). \end{aligned}$$

□

**3.85 Theorem.** *Let $\varphi : (M, g) \to (\overline{M}, \overline{g})$ be a Riemannian map and let their tangent bundles be given their corresponding Sasaki metrics $\mathrm{G}(g)$ and $\mathrm{G}(\overline{g})$ respectively. Suppose further that the fibers and the image of $\varphi$ are totally geodesic. Let $u \in T_pM$ and let $X, Y \in (\ker \varphi_{*p})^{\perp}$. Then, the pullback metric on the orthogonal complement to $\ker \varphi_{**u}$ is given by*

$$(\varphi_*)^* \mathrm{G}(\overline{g})(X^v, Y^v) = \mathrm{G}(g)(X^v, Y^v). \tag{3.105}$$

$$(\varphi_*)^* \mathrm{G}(\overline{g})(X^h, Y^v) = 0 \tag{3.106}$$

$$(\varphi_*)^* \mathrm{G}(\overline{g})(X^h, Y^h) = \mathrm{G}(g)(X^h, Y^h) \tag{3.107}$$

*and thus, $\varphi_*$ is also a Riemannian map.*

*Proof.* Because of the totally geodesic assumption, by Corollary 3.81 it follows that $X^v, X^h$, $Y^v, Y^h$, are in the kernel of $\varphi_{**}$. Now, the equations are simply a restatement of the content of Lemmata 3.83,3.84, and 3.82. □

For the particular case when $\varphi$ is a isometric immersion, the formula for the induced metric is given in the next statement.

**3.86 Theorem.** *Let $\iota : (M, g) \to (\overline{M}, \overline{g})$ is an isometric immersion and let their tangent bundles be given their corresponding Sasaki metrics $\mathrm{G}(g)$ and $\mathrm{G}(\overline{g})$ respectively. Then, at a point $u \in TM$,*

$$(\iota_*)^* \mathrm{G}(\overline{g})(X^v, Y^v) = \mathrm{G}(g)(X^v, Y^v). \tag{3.108}$$

$$(\iota_*)^* \mathrm{G}(\overline{g})(X^h, Y^v) = 0 \tag{3.109}$$

$$(\iota_*)^* \mathrm{G}(\overline{g})(X^h, Y^h) = \mathrm{G}(g)(X^h, Y^h) + \overline{g}(B(u,X), B(u,Y)). \tag{3.110}$$

*Proof.* This follows from the previous lemmata. The only observation is that since the map has zero dimensional fibers, $T, S$ and $\alpha$ necessarily vanish. □

And thus, a stronger result is obtained for isometric immersions.

**3.87 Corollary.** *Let $\iota : (M, g) \to (\overline{M}, \overline{g})$ is an isometric immersion and let their tangent bundles be given their corresponding Sasaki metrics $\mathrm{G}(g)$ and $\mathrm{G}(\overline{g})$ respectively. Then the induced metric coincides with the Sasaki metric iff the embedding is totally geodesic.*

In view of Proposition 3.32, one gets, seemingly for free, the next result.

**3.88 Proposition.** *Let $\iota : (M, g) \to (\overline{M}, \overline{g})$ is an isometric immersion and let their tangent bundles be given their corresponding Sasaki metrics $\mathrm{G}(g)$ and $\mathrm{G}(\overline{g})$ respectively. Assume further that the immersion is totally geodesic. Then $\iota_*$ is also totally geodesic.*

*Proof.* Since $\iota$ is totally geodesic it follows that

$$\iota_* R(x,y)z = \overline{R}(\iota_* x, \iota_* y)(\iota_* z),$$

where $R$ and $\overline{R}$ are the corresponding Riemann curvature tensors. With this at hand, it remains to show that the push forward of the covariant derivative coincides with the covariant derivative "of the push forward". Let $y \in \mathfrak{X}(M)$ and consider a $\iota$–related field $Y \in \mathfrak{X}(\overline{M})$. By Proposition 3.32, at a point $u \in TM$,

$$\begin{aligned}
\overline{\nabla}_{(\iota_* x)^h} Y^h &= (\overline{\nabla}_{\iota_* x} Y)^h - \frac{1}{2}(\overline{R}(\iota_* x, Y)(\iota_* u))^v \\
&= (\iota_* \nabla_x y)^h - \frac{1}{2}(\iota_* R(x,y)u))^v \\
&= \iota_{**}[(\nabla_x y)^h - \frac{1}{2}(R(x,y)u)^v] \\
&= \iota_{**}(\nabla_{x^h} y^h),
\end{aligned}$$

$$\begin{aligned}
\overline{\nabla}_{(\iota_* x)^v} Y^h &= \frac{1}{2}(\overline{R}(\iota_* u, Y)(\iota_* x))^v \\
&= (\iota_* \frac{1}{2} R(u,y)x))^v \\
&= \iota_{**}[\frac{1}{2}(R(u,y)x)^v] \\
&= \iota_{**}(\nabla_{x^v} y^h),
\end{aligned}$$

The case $\nabla_{x^h} Y^v$ follows from the fact that the Lie bracket of $\iota$–related fields is again $\iota$–related, and the last case follows from the fact that $\nabla_{x^v} y^v = 0$ as well as $\overline{\nabla}_{(\iota_* x)^v} Y^v = 0$. $\square$

**3.89 Lemma.** *Let $\varphi : (M, g) \to (\overline{M}, \overline{g})$ be a Riemannian map and let their tangent bundles be given their corresponding Sasaki metrics $\mathrm{G}(g)$ and $\mathrm{G}(\overline{g})$ respectively. Suppose further that the fibers are totally geodesic. Then, the fibers of $\varphi_*$ are totally geodesic.*

*Proof.* By Corollary 3.81, the kernel of $\varphi_{**}$ is given by horizontal lifts and vertical lifts of elements in the kernel of $\varphi_*$, by the assumption that the fibers are totally geodesic. Also by this assumption, for any $X, Y, x \in \ker \varphi_{*p}$ and $u \in (\ker \varphi_{*p})^\perp$,

$$R(X, Y)u, R(u, Y)X \in \ker \varphi_{*p}.$$

To see this, extend them to projectable fields and thus, since the fibers are totally geodesic, $\nabla_X u$, $\nabla_u(\nabla_Y X)$ and $[u, Y]$ are all tangential to the fiber and thus in the kernel of $\varphi_{*p}$. From this, and again by Proposition 3.32, the claim follows. $\square$

In light of these results, the following definition should require no further motivation.

**3.90 Definition.** A smooth map between Riemannian manifolds is a *geodesic Riemannian map (GR)* if it is a Riemannian map with totally geodesic fibers and totally geodesic image.

Notice that the composition of Riemannian maps needs not be Riemannian, nor does the composition of GR maps as can be seen by considering the following example.

**3.91 Example.** Let $\varphi : \mathbb{R} \to \mathbb{R}^2$ be given by $t \mapsto \frac{1}{\sqrt{2}}(t, t)$ and $\rho : \mathbb{R}^2 \to \mathbb{R}$ by $(x, y) \mapsto x$. Both these maps are geodesic Riemannian, yet their composition $\rho \circ \varphi(t) = \frac{t}{\sqrt{2}}$ is not.

**3.92 Theorem** (Geodesic category theorem)**.** *The Sasaki metric construction renders the tangent bundle a functor from the category of geodesic Riemannian maps (and compositions thereof) to itself. Furthermore, the canonical projection remains a natural transformation.*

*Proof.* The fact that the projection is, by the construction of the Sasaki metric, a Riemannian submersion with totally geodesic fibers, and hence GR (see Proposition 3.34). By Proposition 3.88, if the image of a Riemannian map $\varphi$ is totally geodesic the so is its tangent bundle, which is the image of $\varphi_*$. Lastly, by Lemma 3.89, the fibers $\varphi_*$ are also totally geodesic. $\square$

Requiring that a map be geodesic Riemannian is still a weak assumption from the metric geometric point of view. As maps between metric spaces, it is not necessarily true that totally geodesic injective isometric immersions are totally convex, i.e distance preserving. On the other hand, Riemannian submersions are always submetries.

# Chapter 4

# Holonomy: A global perspective through norms

C'est véritablement utile puisque c'est joli.

*Le Petit Prince*
Antoine de Saint-Exupéry

As a by-product of the previous considerations, a natural topology can be given to the holonomy groups that doesn't necessarily coincide with the classical Lie group topology.

This new topology arises from the observations in Theorems 3.53 and 3.56, as well as from Proposition 1.10. Namely, looking at the infimum of lengths of loops one produces a metrizable topology for the holonomy groups.

Controlling the length of loops that generate a given holonomy element has many applications, as pointed out by Montgomery [32] , in Control Theory, Quantum Mechanics, or sub-Riemannian geometry (see [33]).

Therefore, considering the infimum $L(a)$ of lengths of loops that generate a given holonomy element $a$ is quite natural and it exhibits the fibers of a vector bundle as holonomic spaces, which in turn shows that the global shape of the space determines how the individual fibers bend within the total space.

Although the function $a \mapsto L(a)$ is in general not even upper-semicontinuous when regarded as a function on the holonomy group with the subspace topology (or even its Lie group topology), as pointed out by Wilkins [50], the following results gives a more positive outcome.

**4.1 Theorem.** *Let $H$ be the holonomy group of a metric connection on a vector bundle $E$ over a Riemannian manifold. There exists a finer metrizable topology on $H$, given by $d(a,b) = L(a^{-1}b)$,* *so that the function $a \mapsto L(a)$ is continuous with respect to this topology and furthermore, the group action $H \times E_p \to E_p$ remains continuous.*

*Proof.* By 2.6 the action map $H \times E_p \to E_p$, is continuous, so by 1.15, the identity map is continuous from the $L$-topology to the Lie topology. Furthermore, by 1.12 $L$ is continuous with respect to the $L$-topology . □

Now, the following fact hints a type of 'wrong way' inheritance.

**4.2 Proposition** ([50, 43]). *Let $\pi : P \to M$ be a smooth principal bundle over a smooth manifold $M$, let a smooth connection on $\pi : P \to M$ be given, and let $H_p$ denote the holonomy group of this connection attached to some element $p$ of $P$. Suppose that $H_p$ is compact. Then there exists a constant $K$ such that every element of $H_p$ can be generated be a loop of length not exceeding $K$.*

So, in the language of the induced length structure the following is true.

**4.3 Theorem.** *Let $E \to M$ be a vector bundle with bundle metric and compatible connection. Let $H$ be the holonomy group of this connection. If $H$ is compact with the standard Lie group topology (in particular bounded with respect to any —invariant— metric), then $H$ with the induced length metric given by* (3.64) *is bounded.*

Tapp [46] introduces a way to measure the size of a holonomy transformation as a supremum over *acceptable* left invariant metrics. A smooth invariant metric $m$ is acceptable if for any $X \in \mathfrak{k} = \mathfrak{g}(\Phi)$, the Lie algebra of $\Phi$,

$$\|X\|_m \leq \sup_{v, \|v\|=1} \|X(v)\|, \tag{4.1}$$

where $X(v)$ means the evaluation of the fundamental vector on $F$ associated with $X$. The size of a holonomy transformation $A$ is then defined as the supremum of its distances to the identity $dist_m(A, Id)$ over acceptable metrics $m$. And the following fact relates this 'size' to the norm defined by (3.64), whenever there are curvature bounds.

**4.4 Proposition** (Tapp [46]. Proposition 7.1). *Let $E \to B$ be a Riemannian vector bundle over a compact simply connected manifold $B$. Let $\nabla$ be a compatible metric connection and let its curvature $R$ be bounded in norm, $|R| \leq C_R$. Fix a point $x \in B$ and let $Hol(\nabla)$ be the corresponding holonomy group at $x$. Then there exists a constant $C(B)$ such that for any loop $\alpha$ in $B$, $|P_\alpha| \leq C \cdot C_R \cdot \ell(\alpha)$, where $P_\alpha \in Hol(\nabla)$ stands for the holonomy transformation induced by $\alpha$.*

**4.5 Theorem.** *With the assumptions as in the previous statement, the norm given by* (3.64) *and Tapp's holonomy size are related by* $|g| \leq C \cdot C_R \cdot L(g)$, *so that the induced length topology is finer than that of Tapp's holonomy size.*

*Proof.* This is immediate from the inequality, since the infimum is taken over loops with the same holonomy transformation associated. □

Finally, with the additional assumption of completeness, the following result gives a converse to Theorem 4.2.

**4.6 Theorem.** *Let* $E \to B$ *be a Riemannian vector bundle with compatible connection over a complete Riemannian manifold* $B$. *The holonomy group is compact if and only if the restricted holonomy group is compact and its associated length norm is bounded.*

*Proof.* The necessity is the content of Theorem 4.2. For the sufficiency, suppose that the holonomy group has infinitely many connected components. Consider a sequence $\{a_i\}$ of inequivalent classes. Let $\gamma_i$ be a loop generating $a_i$ such that $L(a_i) = \ell(\gamma_i)$; these exist by the completeness of the metric and an application of Arzelà-Ascoli Theorem as pointed out by Montgomery [32]. Since the lengths of the $\gamma_i$'s are bounded by assumption, another application of Arzelà-Ascoli Theorem, now to the sequence $\gamma_i$, yields a uniformly convergent subsequence, also denoted by $\{\gamma_i\}$. Therefore, for $i >> 0$, all loops are homotopic. This is a contradiction to the following fact: different connected components of the holonomy group represent different homotopy classes. □

**4.7 Corollary.** *In the case of complete Riemannian manifolds, the holonomy group is compact if and only if the length norm is bounded.*

*Proof.* By virtue of the classification theorem of Berger [7], the restricted holonomy group is always compact. □

*4.8 Remark.* Completeness is really essential as looking again at a cone metric on the punctured plane shows.

# Chapter 5

# Convergence

> You take an obvious concept of a limit, and then, by the power of analysis, you can go to the limit many times, which creates structures that you have not seen before. You think you have not done anything but, amazingly, you have achieved something.
>
> *Notices of the American Mathematical Society, 2010*
> Mikhael Gromov

The structures that become apparent in a weak limiting process, such as the one introduced by Gromov, are necessarily robust. In this chapter, the collection of vector bundles with a metric of Sasaki type with an upper bound on their rank is seen to be pre-compact. Furthermore, their limits retain a surprising amount of information, even when there is no additional conditions imposed on their base spaces (such as curvature bounds of any sort).

Holonomic spaces —in particular the fibers of said bundles— are also seen to converge, and this convergence is compatible with that of their ambient spaces: the fibers converge to fibers. Because of this, and because of the rich structure of the holonomic spaces, their limits are necessarily nice. Topologically, they are the quotient of a Euclidean spaces by a compact Lie subgroup of the orthogonal group, called the *wane group*.

This group, which is produced by waning holonomy elements, will be seen to play a rôle in the degeneration of the notion of parallelism that vector bundles with connection have. This degeneration occurs only at the level of uniqueness, not of existence.

## 5.1 Holonomic spaces revisited.

Because holonomic spaces arise as fibers of vector bundles with metric connections over Riemannian manifolds (by Theorem 3.56), studying their convergence properties becomes natural when trying to understand the behavior of their metrics of Sasaki-type under limits. Also, given the underlying linear nature of the holonomic spaces, a $C^0$-convergence of the metrics to semimetrics is obtained, which implies the pointed Gromov-Hausdorff convergence of the holonomic spaces to precisely described spaces. Metrically, the description of their induced limit metrics is slightly more elusive.

**5.1 Theorem.** *Given a finite dimensional vector space, the collection of all holonomic space metrics $(V, d_L)$ is precompact in the $C^0$ sense. Namely, for any sequence $(V, H_i, L_i)$ there exists a subsequence (denoted without loss of generality with the same index i) for which the metrics $d_{L_i} : V \times V \to \mathbb{R}$ converge uniformly on bounded domains to a semi-metric $\rho : V \times V \to \mathbb{R}$.*

*Proof.* The strategy is the following: first use Arzelà-Ascoli on balls of a fixed radius $r > 0$ around the origin; next argue that these convergences can be made to agree on $V$; and finally, argue that the limit function is a semi-metric.

Let $V$ be a finite dimensional normed vector space. For any $r > 0$ let $\varepsilon > 0$ and consider $\eta = \min\{1, \frac{\varepsilon^2}{(1+2\sqrt{r})^2}\}$, $\delta = \frac{\eta}{\sqrt{2}}$.

Let $u, v, u', v' \in V$ be such that $\|u\|, \|v\|, \|u'\|, \|v'\| < r$ and

$$\sqrt{\|u-u'\|^2 + \|v-v'\|^2} < \delta. \tag{5.1}$$

Then, for any normed preserving linear map $a : V \to V$,

$$\begin{aligned} \Big| \|au - v\| - \|au' - v'\| \Big| &\leq \|au - au'\| + \|v - v'\| && (5.2)\\ &\leq \|u - u'\| + \|v - v'\| && (5.3)\\ &< \sqrt{2}\sqrt{\|u-u'\|^2 + \|v-v'\|^2} < \eta, && (5.4) \end{aligned}$$

by the triangle inequality for $\|\cdot\|$ and because $\|au - au'\| = \|u - u'\|$, $a \in O(V)$, (5.1) and $\delta = \frac{\eta}{\sqrt{2}}$.

In particular,

$$\|au - v\|^2 \leq \|au' - v'\|^2 + \eta^2 + 4r\eta,$$

which follows by direct squaring and by noticing that $\|au - v\| \leq 2r$, by the triangle inequality.

Consider now any group-norm $L : H \to \mathbb{R}$ for any $H \leq O(V)$. By adding $L^2(a)$ to both sides, one sees that,

$$L^2(a) + \|au - v\|^2 \leq L^2(a) + \|au' - v'\|^2 + \eta^2 + 4r\eta$$

now, by taking the square root and applying the triangle inequality,

$$\sqrt{L^2(a) + \|au - v\|^2} \leq \sqrt{L^2(a) + \|au' - v'\|^2} + \eta + 2\sqrt{r\eta}.$$

Now, since $\eta$ is less than one it follows that $\eta \leq \sqrt{\eta}$, so that

$$\sqrt{L^2(a) + \|au - v\|^2} \leq \sqrt{L^2(a) + \|au' - v'\|^2} + \varepsilon$$

holds.

Therefore:

$$\begin{aligned} d_L(u,v) &= \inf_a \sqrt{L^2(a) + \|au - v\|^2} && (5.5)\\ &\leq \inf_a \sqrt{L^2(a) + \|au' - v'\|^2} + \varepsilon && (5.6)\\ &= d_L(u',v') + \varepsilon. && (5.7) \end{aligned}$$

Because of (5.4), and by interchanging $u, v$ with $u', v'$, it now follows that

$$|d_L(u,v) - d_L(u',v')| < \varepsilon, \tag{5.8}$$

thus proving that the family $\{d_L\}$ is equicontinuous on balls of a fixed radius $r > 0$ around the origin in $V$.

To prove uniform boundedness, one needs to observe that for any $L$ the following is true:

$$d_L(u,v) \leq \sqrt{L^2(id_V) + \|id_V u - v\|^2} = \|u - v\| \leq 2r. \tag{5.9}$$

The hypotheses of the classical Arzelà-Ascoli's theorem now apply to get a uniform limit on the ball of radius $r > 0$ (times itself). Consider a countable exhaustion of $V$ by balls of radius $r_i \to \infty$. By a diagonal argument for any sequence of metrics $\{d_{L_i}\}$ one gets a pointwise limit $\rho$ on $V$ that is uniform on compact sets.

Finally, except for nondegeneracy, all the properties of (semi)metrics are well behaved under limits:

$$\rho(u,v) = \lim_{i\to\infty} d_{L_i}(u,v) \geq 0, \tag{5.10}$$

$$\rho(u,v) = \lim_{i\to\infty} d_{L_i}(u,v) = \lim_{i\to\infty} d_{L_i}(v,u) = \rho(v,u), \tag{5.11}$$

$$\rho(u,u) = \lim_{i\to\infty} d_{L_i}(u,u) = 0, \tag{5.12}$$

$$\begin{aligned} \rho(u,v) &= \lim_{i\to\infty} d_{L_i}(u,v) & (5.13)\\ &\leq \lim_{i\to\infty} d_{L_i}(u,w) + \lim_{i\to\infty} d_{L_i}(w,v) & (5.14)\\ &= \rho(u,w) + \rho(w,v). & (5.15) \end{aligned}$$

Therefore for any family of holonomic spaces $\{(V,d_{L_i)}\}$ there exists a subsequence that converges uniformly on compact sets. □

Nowhere in the proof was the fact that the function $d_L : V \times V \to \mathbb{R}$ was nondegenerate used; only properties of semi-metrics were required. However, the restriction to holonomic spaces yields nondegeneracy of $d_L$ and is of interest for the sequel as they occur naturally. In general, the limit $\rho$ will be degenerate unless further assumptions are made (see Theorem 5.10).

**5.2 Corollary.** *Given a finite dimensional normed vector space, the collection of all pointed holonomic space metrics $((V,d_L),0)$ is pre-compact in the pointed Gromov-Hausdorff sense.*

*Proof.* By Theorem 5.1 for any sequence $(V,d_i)$ of holonomic space metrics there exists a subsequence for which the semi-metrics $d_i$ converge uniformly on compact sets to a semi-metric $\rho$ on $V$. The quotient space $Q = V/\sim$, where $u \sim v$ if and only if $\rho(u,v) = 0$ for $u,v \in V$, is naturally a metric space; the metric is given by the distance $d_\infty([u],[v]) = \rho(u,v)$ for any choice of representatives (or as the usual —not Hausdorff— distances between subsets of $V$). Therefore the convergent subsequence of metrics yields a convergent sequence of metric spaces $(V,d_i) \to (Q,d_\infty)$. □

**5.3 Corollary.** *The space of holonomic metrics on inner-product spaces of dimension at most $k$ is precompact in the Gromov-Hausdorff sense. More explicitly, for any family of holonomic spaces $\{(V_i,H_i,L_i)\}$, where $V_i$ has dimension at most $k$ and its norm is induced by an inner product, there exists a subsequence that converges in the pointed Gromov-Hausdorff sense.*

*Proof.* By passing to a subsequence, one can assume that all the vector spaces in the sequence have the same dimension. Now, by Sylvester's Law of Inertia— which in particular states that any two positive definite symmetric bilinear forms on a finite dimensional vector space are isometric—, all the norms can be made to coincide, by way of some isometries $\phi_i : V \to V$. By defining $\tilde{H}_i = \phi_i^{-1} H_i \phi_i$ and $\tilde{L}_i : \tilde{H}_i \to \mathbb{R}$ by $\tilde{L}_i(b) = L_i(\phi_i b \phi_i^{-1})$, the sequence $\{(V, \tilde{H}_i, \tilde{L}_i)\}$ now satisfies the hypotheses of Theorem 5.1. □

More can be said about the limiting metric spaces when there is more information about the underlying subgroups of isometries. Before that, recall that by Lemma 1.16, for any sequence of isometries $\{\varphi_i\}$, if there exists a point $x$ such that $\{\varphi_i(x)\}$ converges, then there exists a subsequence $\{\varphi_{i_k}\}$ that converges to an isometry. Of course, in the case when the norm on a finite dimensional vector space $V$ is given by an inner product, then this is easily seen, since the group $O(V)$ is a compact Lie group.

This fact is essential for producing a subset of $O(V)$ that determines the degeneracy of the limit semi-metric. Later, this subset can be replaced by a group (at the time of writing, it is not clear that the set produced in the next result is not already a group).

**5.4 Theorem.** *Let $V$ be a finite dimensional normed vector space and $\{H_i\}$ be a sequences of subgroups of the group of norm preseving linear maps, denoted here by $O(V)$. Consider a sequence of group-norms $\{L_i : H_i \to \mathbb{R}\}$ such that the semi-metrics $d_i = d_{L_i}$, given by*

$$d_{L_i}(u,v) = \inf_a \sqrt{L_i^2(a) + \|au - v\|^2}, \tag{5.16}$$

*for any $u,v \in V$, converge uniformly on compact sets to a semi-metric $d_\infty$ on $V$. Then, there exists a set $G_0 \subseteq O(V)$, given by:*

$$G_0 = \{g \in O(V) | g = \lim_{i_n \to \infty} a_{i_n}, \lim_{i_n \to \infty} L_{i_n}(a_{i_n}) = 0\}, \tag{5.17}$$

*such that for any $u,v \in V$,*

$$d_\infty(u,v) = 0 \tag{5.18}$$

*if and only if there exists $g \in G_0$ such that $v = gu$.*

*Proof.* Let $u,v \in V$ be such that $d_\infty(u,v) = 0$. This means that for any choice of $\varepsilon > 0$ there exists $N = N_\varepsilon > 0$ such that for any $j > N$,

$$d_j(u,v) < \varepsilon \tag{5.19}$$

In particular by (5.16) there exists $a_j(\varepsilon) \in H$ with

$$\sqrt{L_j^2(a_j) + \|a_j u - v\|^2} < \varepsilon, \tag{5.20}$$

which in turn gives that

$$L_j(a_j), \|a_j u - v\| \le \varepsilon. \tag{5.21}$$

By letting $\varepsilon = \frac{1}{n}$ and recursively choosing $j = j_n = \max\{[N_{\frac{1}{n}}], j_{n-1}\} + 1$, one produces a sequence $\{b_n = a_j(\frac{1}{n})\}$ for which $b_n u \to v$ and by Lemma 1.16, passing to a further subsequence if needed, such that it converges in $O(V)$ to some $g$, with $gu = v$ and $\lim_n L_{j_n}(b_n) = 0$ as required.

Conversely, let $u \in V$ and consider $v = gu$ with $g \in G_0$, with $a_{i_n} \to g$. Then for all $\varepsilon > 0$ there exists $i_n >> 0$ such that

$$d_{i_n}(u, gu) \le \sqrt{L_{i_n}^2 a_{i_n} + \|a_{i_n} u - gu\|^2} \le \varepsilon. \tag{5.22}$$

So the claim now follows by the uniform convergence of $d_i \to d_\infty$. □

As mentioned before, it is not clear at this point whether $G_0$ is a subgroup of $O(V)$, since for two different elements in $G_0$ the subsequences determining them might in principle be disjoint (i.e. have no common subsequence).

Nevertheless, the characterization given by the previous theorem is still quite good as will be seen in the sequel. If one however insists upon having a group action to determine the degeneracy of $d_\infty$, this can be achieved by the following result. The drawback is that this new presentation says nothing about how to explicitly construct said group directly from the knowledge of $L_i$.

**5.5 Theorem.** *Let $V$ be a finite dimensional normed vector space and $H$ be a subgroup of the group of linear norm preserving isomorphisms, $O(V)$. Consider now a sequence of group-norms $\{L_i : H \to \mathbb{R}\}$ such that the semi-metrics $d_i = d_{L_i}$, given by*

$$d_{L_i}(u, v) = \inf_a \sqrt{L_i^2(a) + \|au - v\|^2}, \tag{5.23}$$

*for any $u, v \in V$, converge uniformly on compact sets to a semi-metric $d_\infty$ on $V$. Then, there exists a closed subgroup of $O(V)$, given by*

$$G = \{g \in O(V) | \forall u \in V, d_\infty(u, gu) = 0\}, \tag{5.24}$$

*such that for any $u, v \in V$*

$$d_\infty(u, v) = 0 \tag{5.25}$$

*if and only if there exists $g \in G$ such that $v = gu$.*

*5.6 Remark.* Consider $g \in G_0$, as in Theorem 5.4. Then, for any $u \in V$, $d_\infty(u, gu) = 0$ by Theorem 5.4. Thus

$$G_0 \subseteq G.$$

**5.7 Definition.** The group $G$ will be henceforth called the *wane group* of a convergent sequences of holonomic spaces.

*Proof of Theorem 5.5.* Three statements must be proved: 1) the equivalence between having zero distance and being related by an element in $G$; 2) the fact that $G$ is actually a group; and 3) that this group is closed in $O(V)$.

To prove the equivalence first consider let $v \in V$ with $d_\infty(u, v) = 0$, then by Theorem 5.5 there exists $g \in G_0 \subseteq G$ (by Remark 5.6) with $v = gu$.

Conversely, for any $g \in G$ and for any $u \in V$

$$d_\infty(u, gu) = 0, \tag{5.26}$$

by the definition of $G$.

As the reader might have noticed, this doesn't prove that $G \subseteq G_0$ since this only implies that for any $u \in V$ and for any $g \in G$ there exists $h \in G_0$ such that

$$gu = hu. \tag{5.27}$$

Bear in mind that this equality is attained only at $u \in V$, since in principle $h$ depends on $u$. What was accomplished was the following: For any $u \in V$, $\{hu | h \in G_0\} = \{gu | g \in G\}$, that is that the equivalence classes determined by $G$ and by $G_0$ (in turn determined by the degeneracy of $d_\infty$) in $V$ are the same, as promised.

Secondly, to prove that $G$ is indeed a group, notice that because $d_\infty$ is already known to be a semi-metric,

$$d_\infty(u, u) = 0, \tag{5.28}$$

regardless of $u \in V$. So $id_V \in G$.

Let $g, h \in G$, then by the triangle inequality of $d_\infty$,

$$d_\infty(u, ghu) \leq d_\infty(u, hu) + d_\infty(hu, g(hu)) = 0 + 0 \tag{5.29}$$

and, of course,

$$d_\infty(u, g^{-1}u) = d_\infty(g(g^{-1}u), g^{-1}u) = 0. \tag{5.30}$$

Therefore $G$ is a subgroup of the group of norm preserving linear maps, $O(V)$.

Finally, to see that $G$ is closed, notice that for any $u \in V$ the assignment $\varphi_u : O(V) \to \mathbb{R}$, given by

$$\varphi_u : g \mapsto d_\infty(u, gu), \tag{5.31}$$

is a composition of continuous functions and thus itself continuous. Because of this, $G$ can be represented as the following intersection of closed sets,

$$G = \bigcap_{u \in V} \varphi_u^{-1}(0), \tag{5.32}$$

and thus $G$ is closed. □

*5.8 Remark.* If $V$ is further assumed to be an inner product space, then $O(V)$ is a compact Lie group; furthermore, because by Theorem 5.5, $G$ is also a compact Lie group.

**5.9 Corollary.** *Let $\{V_i\}$ be a collection of finite dimensional inner product vector spaces and consider a sequence of holonomic space metrics $\{d_i\}$ on $\{V_i\}$. In addition, suppose that the sequence of metric spaces $\{(V_i, d_i)\}$ is a convergent sequence in the Gromov-Hausdorff sense. Then there exists a positive integer $k$ and a closed subgroup $G \leq O(k)$ such that*

$$(V_i, d_i) \xrightarrow{pt-GH} \mathbb{R}^k/G, \tag{5.33}$$

*where the metric on the limit is obtained as in Corollary 5.3.*

*Proof.* As in Corollary 5.3, one can pass to a subsequence and assume that $\{V_i\}$ has constant dimension $k$ and such that the norms are the constant. The conclusion now follows from Theorem 5.4. □

Recall that in view of Theorem 2.9 for a holonomic space $(V, H, L)$ the *holonomy radius* at a point $u \in V$, $\mathrm{HolRad}(u)$, is the largest $r > 0$ for which the metric $d_L$ is isometric to the Euclidean metric when restricted to the ball of radius $r$ around $u \in V$. In the special case when $u = 0$, by Corollary 2.20, given a holonomic space $(V, H, L)$,

$$\mathrm{HolRad}(0) \leq \inf_{a \in H} \frac{L(a)}{\|a - id_V\|}. \tag{5.34}$$

Because of this, if one has certain control on the holonomy radii of the sequence, the following holds.

**5.10 Theorem.** *Let $(V, H, L_i)$ be a convergent sequence of holonomic spaces. Suppose further that there exists a constant $c > 0$ such that*

$$c \leq \mathrm{HolRad}_i(0). \tag{5.35}$$

*Then the limit semi-metric $d_\infty$ is nondegenerate. That is that $d_\infty$ is a metric.*

*Proof.* Consider $g \in G_0$, as in 5.4, and let $\{a_{i_n}\}$ be any defining sequence for $g$. That is such that $a_{i_n} \to g$ and $\lim_{i_n \to \infty} L_{i_n}(a_{i_n}) = 0$. By Lemma **??**, for each $a_{i_n}$,

$$c \leq \mathrm{HolRad}_{i_n}(0) \leq \frac{L_{i_n}(a_{i_n})}{\|a_{i_n} - id_V\|}. \tag{5.36}$$

Thus for any $\varepsilon > 0$ and for any $N > 0$ there exists $i_n > N$,

$$\|a_{i_n} - id_V\| \leq \frac{L_{i_n}(a_{i_n})}{c} \leq \frac{\varepsilon}{c}. \tag{5.37}$$

Therefore there is a subsequence of $\{a_i\}$ that converges to the identity map. Because $\{a_{i_n}\}$ converges to $g$, it follows that $g = id_V$ and now Theorem 5.4 yields the claim. □

Finally, as an application of these concepts the upper semi-continuity of the holonomy radius of a connection over a Riemannian manifold can be asserted (recall Definition 2.2). In the case of a holonomic space $(V, H, L)$, by Proposition 2.15, the holonomy radius is a continuous function on $V$.

**5.11 Proposition.** *Given a vector bundle $\pi : E \to M$ with metric connection $\nabla$ and bundle metric $h$ over a Riemannian manifold $(M, g)$, then the holonomy radius* $\mathrm{HolRad} : M \to \mathbb{R}$ *is an upper semicontinuous function.*

*Proof.* Let $p \in M$ and consider a sequence $\{p_i\} \subseteq M$ converging to $p$. By Remark 1.35 and by the fact that the fibers of $\pi$ are equidistant, it follows that the holonomic metrics $\{d_{L_{p_i}}\}$ converge in the $C^0$ sense to the holonomic metric $d_{L_p}$.

Let $\rho = \mathrm{HolRad}(p)$. Now, since the metrics $\{d_{L_{p_i}}\}$ converge uniformly when restricted to the ball of radius $\rho$, and the metric $d_{L_p}$ is Euclidean on that ball, let

$$\tilde{\rho} = \limsup_{i \to \infty} \mathrm{HolRad}(p_i).$$

Let $u, v \in E_p$ with $\|u\|, \|v\| < \tilde{\rho}$. Then, there exist (sub)sequences $\{u_i\} \subseteq E_{p_i}$, $\{v_i\} \subseteq E_{p_i}$ with

$\|u_i\|, \|v_i\| < \tilde{\rho}$, converging to $u$ and $v$ respectively such that

$$d_{L_{p_i}}(u_i, v_i) = \|u_i - v_i\|. \tag{5.38}$$

Therefore,

$$d_{L_p}(u,v) = \lim_{i\to\infty} d_{L_{p_i}}(u_i, v_i) = \lim_{i\to\infty} \|u_i - v_i\| = \|u - v\|, \tag{5.39}$$

thus proving that $\tilde{\rho} \leq \rho$ as promised. □

## 5.2 Structures: emergence and waning.

Since vector bundles (with metric connections) appear naturally as associated objects to Riemannian manifolds, providing the latter with additional structures (as Poisson brackets, control structures, orientations, holomorphic structures, etc.), it is natural to investigate the behavior of these bundles under limits of their bases.

One reason to analyze them from the viewpoint of holonomic spaces is given by Theorem 3.56. Furthermore, the next result gives yet another reason.

**5.12 Theorem.** *Given a vector bundle $\pi : E \to M$ with metric connection $\nabla$ and bundle metric $h$ over a Riemannian manifold $(M,g)$, consider a point $p \in M$ and let $(V,H,L)$ be the holonomic space $(E_p, Hol_p(\nabla), L_p)$. Then the Gromov-Hausdorff distance between $(V, d_L)$ and $\pi^{-1}(B_R(p)) \subseteq E$ (with the restricted metric from $E$) is finite and bounded by $2R$.*

*Proof.* By Theorem 3.56, the inclusion $(E_q, d_{L_q}) \hookrightarrow E$ is an isometric embedding in the sense of metric spaces for any q. Furthermore, because the projection map is a Riemannian submersion, parallel translation along any minimal geodesic in $M$ connecting the points $p,q \in M$ renders the fibers equidistant. Therefore, the distance between the central fiber and any other fiber over a ball of radius $R$ is bounded by $R$. From this, for any $p \in M$ and $R > 0$ the inclusion map of the central fiber

$$E_p \hookrightarrow \pi^{-1}(B_R(p)) \subseteq E \tag{5.40}$$

is an $R$-isometry. By Proposition 1.33, the claim now follows. □

In particular, for the tangent bundle:

**5.13 Corollary.** *Given a Riemannian manifold $(M,g)$ and a point $p \in M$ let $(V, d_L)$ be the holonomic space $(M_p, Hol_p(g), L_p)$ then the Gromov-Hausdorff distance between $(V, d_L)$ and $\pi_M^{-1}(B_R(p)) \subseteq TM$ is bounded by $2R$.*

**5.14 Theorem** (Sasaki-type metric Compactness Theorem)**.** *Given a precompact collection of (pointed) Riemannian manifolds $\mathcal{M}$ and a positive integer $k$, the collection* $\mathrm{BWC}_k(\mathcal{M})$ *of vector bundles with metric connections of rank $\leq k$ endowed with metrics of Sasaki-type is also precompact. The distinguished point for each such bundle is the zero section over the distinguished point of their base.*

*5.15 Remark.* Passing to a subsequence is unavoidable as can be seen in Example 5.40.

*Proof of Theorem 5.14.* Fix $\varepsilon > 0$ and $R > 0$. Following Theorem 1.39 and Remark 1.40, define $C = C(\epsilon, R) > 0$ by

$$C := \max_{i \leq k} N(\varepsilon, R, \mathbb{E}^i), \tag{5.41}$$

where $\mathbb{E}^i$ is the Euclidean space of dimension $i$.

Let $(E, h, \nabla) \xrightarrow{\pi_E} (M, g)$ be any bundle with metric connection and let $N(R, \varepsilon)$ be the uniform bound on the number of balls of radius $\varepsilon$ needed to cover a ball of radius $R$ on $\mathcal{M}$. Consider $p \in M$ and its zero section $0 = \varsigma(p)$.

Since $\pi_E$ is a Riemannian submersion, $\pi_E(B_R(0)) = B_R(p)$. Let $A$ be any $\varepsilon$-net in $B_R(p)$. Since for each $a \in A$, $E_p$ is flat, let $A_a$ be any $\varepsilon$-net in $B_R(\varsigma(a)) \subseteq T_aM$. The cardinality of $A_a$ can be chosen to less than $C(\varepsilon, R)$, because the identity map is a distance non-increasing map between the (induced) Euclidean metric on $E_p$ and the restricted metric.

Let $u \in B_R(0)$, then let $a \in A$ such that $d(a, \pi_E(u)) < \varepsilon$. Let $\gamma$ be any minimal geodesic connecting $\pi_m u$ to $a$ and let $v \in T_aM$ be the parallel image of $u$ along $\gamma$. Finally consider $u_a \in A_a$ to be such that $|u_a - v| < \varepsilon$. Hence,

$$d(u, u_a) \leq d(\pi_M u, a) + d(v, u_a) \leq 2\varepsilon.$$

Therefore, given any $R > 0$ , and for any $\varepsilon > 0$, the following holds.

$$N(2\varepsilon, R, E) \leq N(\varepsilon, R, M) + C(\varepsilon, R). \tag{5.42}$$

So that if the assignment $M \mapsto N(\varepsilon, R, M)$ is bounded on $\mathcal{M}$, then so is $E \mapsto N(\varepsilon, R, E)$ on $\mathrm{BWC}_k(\mathcal{M})$.

Therefore, in view of Gromov's Compactness Theorem 1.39, this finishes the proof. $\square$

**5.16 Proposition.** *For any sequence of Riemannian manifolds $\{(X_i, p_i)\}$ converging to $(X_\infty, x_\infty)$ consider a convergent family of bundles with metric connection $(E_i, h_i, \nabla_i)$ over it converging to $(E_\infty, y_\infty)$. Then there exist continuous maps $\pi_\infty : E_\infty \to X_\infty$, $\varsigma_\infty : X_\infty \to E_\infty$, $\mu_\infty : E_\infty \to \mathbb{R}$, and a subsequence, without loss of generality also indexed by $i$, such that:*

1. *the projection maps $\pi_i : E_i \to X_i$ converge to $\pi_\infty : E_\infty \to X_\infty$, which is also a submetry;*

2. *the zero section maps $\varsigma_i : X_i \to E_i$ converge to $\varsigma_\infty : X_\infty \to E_\infty$, which is also a isometric embedding;*

3. $\pi_\infty \circ \varsigma_\infty = id_{X_\infty}$*; and*

4. *the maps $\mu_i : E_i \to \mathbb{R}_{\geq 0}$, given by*

$$\mu_i(u) = d_{E_i}(u, \varsigma_i \circ \pi_i(u)) = \sqrt{h_i(u,u)},$$

*converge to $\mu_\infty : E_\infty \to \mathbb{R}_{\geq 0}$, also given by*

$$\mu_\infty(y) = d_{E_\infty}(y, \varsigma_\infty \circ \pi_\infty(y)).$$

*Proof.* Since all of these maps preserve the distinguished points (consider $0 \in \mathbb{R}_{\geq 0}$), by the Arzelà-Ascoli Theorem 1.41, one only has to check equicontinuity. But this is immediate from the fact that the $\pi_i$ are submetries [22], $\varsigma_i$ isometric embeddings, and $\mu_i$ both distance functions and submetries. In fact, their limits will share these properties, as noted by Petersen [37, Section 10.1.3].

The equation $\pi_\infty \circ \varsigma_\infty = id_{X_\infty}$ holds since the corresponding equation holds for every $i$. Finally, the equation $\mu_\infty(y) = d_{E_\infty}(y, \varsigma_\infty \circ \pi_\infty(y))$ holds, since for any sequence $\{u_i\}$ converging to $y \in Y$ the geodesics $t \mapsto tu_i = \varsigma_i \circ \pi_i(u_i) + tu_i$ are rays (see Proposition 3.34), hence isometric embeddings, and thus also converge to a minimal geodesic. □

Because of Theorem 5.12, the fiberwise behavior is also controlled.

**5.17 Proposition.** *Let $\pi_i : (E_i, 0_{*_i}) \to (X_i, *_i)$ be a convergent sequence of pointed spaces as before. Let $\pi_\infty : (E_\infty, 0_{*_\infty}) \to (X_\infty, *_\infty)$ be their limit. Then if $q \in X_\infty$ and $\{q_i \in X_i\}$ is any sequence converging to $q$. Then, by passing to a subsequence if needed, for any $\varepsilon > 0$,*

$$\pi_i^{-1}(B_\varepsilon(q_i)) \xrightarrow{pt-GH} \pi_\infty^{-1}(B_\varepsilon(q)). \tag{5.43}$$

*Furthermore,*

$$\pi_i^{-1}(q_i) \xrightarrow{pt-GH} \pi_\infty^{-1}(q). \tag{5.44}$$

*Proof.* Since the pointed sequence converges with distinguished point $*_i$, it also converges with respect to the points $q_i$.

Consider, as in the proof of Proposition 5.16, the minimizing geodesics $\gamma_i$ given by $t \mapsto tu_i$ for any convergent sequence of points $u_i \in \pi_i^{-1}(q_i)$. Then, the sequence converges to a minimizing geodesic and since the sequence of maps $\pi_i \circ \gamma_i \equiv q_i$ also converges, it follows that the limit $Q$ of the fibers (which is is known to exist by Theorem 5.1 or Corollary 5.2 ) is inside the fiber over the limit (see Remark 1.35). More precisely, there exists an isometric embedding

$$Q \hookrightarrow \pi_\infty^{-1}(q). \tag{5.45}$$

To prove that this is indeed surjective, and thus proving (5.44), the statement of (5.43) will be proved first.

For any $\varepsilon > 0$, the sequence $(\pi_i^{-1}(B_\varepsilon(0_{q_i})), 0_{q_i})$ also converges (or a subsequence thereof) by Theorem 5.14. Any sequence of points $u_i \in \pi_i^{-1}(B_\varepsilon(q_i))$ that converges, necessarily converges to a point $y \in \pi_\infty^{-1}(B_\varepsilon(q))$ since there exists $N > 0$ such that for any $i > N$

$$d_i(q_i, \pi_i(u_i)) \le \varepsilon, \tag{5.46}$$

so that, by continuity,

$$d_\infty(q, \pi_\infty(y)) = \lim_{i\to\infty} d_i(q_i, \pi_i(u_i)) \le \varepsilon \tag{5.47}$$

Conversely, consider any $y \in \pi_\infty^{-1}(B_\varepsilon(q))$ and any sequence $\{u_i\}$ converging to $y$. By looking again at $\gamma_i$, the minimizing geodesics from $\varsigma_i \circ \pi_i(u_i)$ to $u_i$, one sees that a subsequence of $\{\varsigma_i \circ \pi_i(u_i)\}$ converges to $\varsigma_\infty \circ \pi_\infty(y)$. By Proposition 5.16, $\pi_\infty$ is an isometry when restricted to the image of $\varsigma_\infty$; therefore there exists a subsequence of $\{\pi_i(u_i)\}$ that converges to $\pi_\infty(y)$. Now, because

$$\pi_\infty(y) \in B_\varepsilon(q), \tag{5.48}$$

it follows that there exists $N > 0$ such that for all $i > N$,

$$\pi_i(u_i) \in B_\varepsilon(q_i). \tag{5.49}$$

Thus, for all $\varepsilon > 0$,

$$\pi_i^{-1}(B_\varepsilon(q_i)) \xrightarrow{pt-GH} \pi_\infty^{-1}(B_\varepsilon(q)). \tag{5.50}$$

Furthermore, this convergence is attained in a compatible way with the convergence of their ambient spaces.

To finish the proof of (5.44) consider any $y \in \pi_\infty^{-1}(q)$ and any sequence $\{u_i\}$ converging to $y$. Since

$$\pi_\infty^{-1}(q) \subseteq \pi_\infty^{-1}(B_\varepsilon(q))$$

for any $\varepsilon > 0$, by (5.43) the sequence can be assumed to satisfied that

$$u_i \in \pi_i^{-1}(B_\varepsilon(q_i)).$$

It is better to denote this sequence by $\{u_i^\varepsilon\}$, since it in fact depends on $\varepsilon$. Now, by Theorem 5.12, for any $\varepsilon > 0$,

$$d_{GH}(\pi_i^{-1}(B_\varepsilon(0_{q_i})), \pi_i^{-1}((q_i))) < 2\varepsilon.$$

One can consider a sequence $\{\widetilde{u_i^\varepsilon} \in \pi_i^{-1}(q_i)\}$ with

$$d_{E_i}(u_i^\varepsilon, \widetilde{u_i^\varepsilon}) < \varepsilon. \tag{5.51}$$

Now, again by a diagonalization argument, consider $\varepsilon = \frac{1}{i}$ and define

$$v_i = \widetilde{u_i^\varepsilon}. \tag{5.52}$$

By definition, $v_i \in \pi_i^{-1}(q_i)$ and for any $\varepsilon > 0$, there exist $N$ for such that for any $i > N$,

$$2/i < \varepsilon,$$

and as such,

$$d_{E_i}(u_i^{1/i}, v_i) < \frac{\varepsilon}{2}, \tag{5.53}$$

and

$$d(u_i^{1/i}, y) < \frac{\varepsilon}{2}. \tag{5.54}$$

Therefore, for any $y$ over $q$, a sequence $\{v_i\}$ over $q_i$ that converges to $y$ was produced. By Remark 1.35, $Q \cong \pi_\infty^{-1}(q)$ and the claim follows. $\square$

**5.18 Proposition.** *For any sequence of Riemannian manifolds $\{(X_i, p_i)\}$ converging to $(X_\infty, x_\infty)$ consider a convergent family of bundles with metric connection $(E_i, h_i, \nabla_i)$ over it converging to $(E_\infty, y_\infty)$ and $\pi_\infty : E_\infty \to X_\infty$ as in Proposition 5.16. Then the fibers of $\pi_\infty$ are equidistant.*

*Proof.* Let $p, q \in X_\infty$ be arbitrary and consider sequences $\{p_i\}, \{q_i\} \subseteq X_i$ converging to $p, q \in X_\infty$ respectively. Because for each $i$ the fibers of $\pi_i$ are equidistant, the distance between the fibers $\pi_i^{-1}(p_i)$ and $\pi_i^{-1}(q_i)$ is equal to the distance between $p_i$ and $q_i$.

Let $u, v \in E_\infty$ and consider sequences $\{u_i \in \pi_i^{-1}(p_i)\}$ and $\{v_i \in \pi_i^{-1}(q_i)\}$ converging to $u, v$ respectively. Then,

$$d_{E_i}(u_i, v_i) \geq d_{X_i}(p_i, q_i); \tag{5.55}$$

which in turn implies that

$$d_{E_\infty}(u,v) \geq d_{X_\infty}(p,q). \tag{5.56}$$

This proves that the distance between the fibers is at least the distance between their base points.

It remains to show that for any $u \in \pi_\infty^{-1}(p)$ there exists $v \in \pi_\infty^{-1}(q)$ with

$$d_{E_\infty}(u,v) = d_{X_\infty}(p,q). \tag{5.57}$$

To see this, consider any sequence $\{u_i \in \pi_i^{-1}(p_i)\}$ converging to $u$. Let $\{v_i \in \pi_i^{-1}(q_i)\}$ be a sequence such that

$$d_{E_i}(u_i,v_i) = d_{X_i}(p_i,q_i). \tag{5.58}$$

Let $\alpha_i$ be minimizing geodesics connecting $u_i$ to $v_i$. By the Arzelà-Ascoli Theorem, there exists a subsequence of $\{\alpha_i\}$ that converges to a minimizing geodesic in $E_\infty$ connecting $u$ to some point $v \in E_\infty$. It follows that the corresponding subsequence of $\{v_i\}$ converges to $v$. From this it follows, since fibers converge to fibers, that for any $p,q \in X_\infty$ and for any $u \in \pi_\infty^{-1}(p)$ there exists

$$v \in \pi_\infty^{-1}(q), \tag{5.59}$$

such that, by continuity,

$$d_{E_\infty}(u,v) = \lim_{i\to\infty} d_{E_i}(u_i,v_i) = \lim_{i\to\infty} d_{X_i}(p_i,q_i) = d_{X_\infty}(p,q). \tag{5.60}$$

□

The fibers of $\pi_\infty$ can be naturally identified with the quotient of any given fiber by a closed subgroup of the orthogonal group in view of Theorem 5.9, as stated in the following result.

**5.19 Theorem.** *Let $\pi_i : E_i \to X_i$ be a convergent sequence of vector bundles with bundle metric and compatible connections $\{(E_i,h_i,\nabla_i)\}$, with limit $\pi : E \to X$. Then there exists a positive integer $k$ such that for any point $p \in X$ there exists a compact Lie group $G \leq O(k)$, called the* wane group *that depends on the point, such that the fiber $\pi^{-1}(p)$ is homeomorphic to $\mathbb{R}^k/G$, i.e. the orbit space under the standard action of $G$ on $\mathbb{R}^k$.*

*5.20 Remark.* Recall that the main feature of $G$ is that its orbits coincide with the "orbits" of the following set $G_0$ (see Theorems 5.4 and 5.5). Let $p \in X$ and let $p_\in X_i$ be a sequence that converges to $p$. Suppose, by passing to a subsequence, that the sequence of holonomy groups is constant, $H \equiv Hol_{p_i}(\nabla_i)$, and the fibers have constant dimension. Then let $G_0$ is given by

$$G_0 = \left\{ g = \lim_{i_n \mapsto \infty} a_{i_n} \middle| a_{i_n} \in H, \lim_{i_n \mapsto \infty} L_{i_n}(a_{i_n}) = 0 \right\}, \tag{5.61}$$

where $L_i(a_i)$ is the infimum of the lengths of loops at $p_i \in X_i$ that generate $a_i$ by parallel translation (as in Definition 3.54).

By virtue of Corollaries 5.3 and 5.9, the limit metric of $\pi^{-1}(p)$ could, in principle, be given more explictly, once the behavior of these lengths is known.

*Proof of Theorem 5.19.* Let $\{p_i\}$ be a sequence converging to $p$. Then, by Proposition 5.17, there exists a subsequence of $\{\pi_i^{-1}(p_i)\}$ that converges to the fiber $\pi^{-1}(p)$. Let $V_i = \pi_i^{-1}(p_i)$, $H_i = Hol_{p_i}(\nabla_i)$, and $L_i : H_i \to \mathbb{R}$ the induced length norm. Then by Corollary 5.9, applied to the sequence $\{(V_i, H_i, L_i)\}$, the conclusion now follows. □

Summarizing, in the case of the collection of Sasaki metrics on the tangent bundles of a convergent sequence of Riemannian manifolds, the following holds.

**5.21 Theorem.** *Let $\{M_i\}$ be a family of Riemannian manifolds with an upper bound on their dimension that converges in the (pointed) Gromov-Hausdorff sense to $X$. Then there exists a subsequence of $\{TM_i\}$, with their Sasaki metrics, that converges to a space $Y$. Furthermore,*

1. *there exists a continuous map $\pi : Y \to X$, that is a submetry with equidistant fibers.*
2. *there exists a positive integer $k$ such that for any $p \in X$ there exists a closed subgroup $G \leq O(k)$ such that $\pi^{-1}(p)$ is homeomorphic to $\mathbb{R}^k/G$.*

*Proof.* Because the sequence $\{M_i\}$ is convergent, by Theorem 5.14, there is a subsequence of $\{TM_i\}$ that converges to a space $Y$. By Propositions 5.16 and 5.18, the promised $\pi : Y \to X$ exists and has the required properties. Finally, by Theorem 5.19 the rest of the claim holds. □

Theorems 5.10 and 5.19 together give a criterion for the fibers of $\pi_\infty$ in Theorem 5.16 to be vector spaces:

**5.22 Theorem.** *Let $\pi_i : E_i \to X_i$ be a convergent sequence of vector bundles with bundle metric and compatible connections $\{(E_i, h_i, \nabla_i)\}$, with limit $\pi : E \to X$. Suppose further that exist a uniform positive lower bound for the holonomy radii of $\pi_i : E_i \to X_i$ as in Definition 2.2. Then the fibers of $\pi_\infty$ are vector spaces.*

*Proof.* Again, by reduction to the case where the rank is constant, the conclusion follows from Theorem 5.10. □

Another piece of information inherited by the limits is that of "scalar multiplication". The standard $\mathbb{R}$ actions converge to an $\mathbb{R}$ action on the limit. This action doesn't need to be such that if for a non zero $u$, $au = bu$ then $a = b$.

**5.23 Theorem.** *For any sequence of Riemannian manifolds $\{(X_i, p_i)\}$ converging to $(X_\infty, x_\infty)$ consider a convergent family of bundles with metric connection $(E_i, h_i, \nabla_i)$ over it converging to $(E_\infty, y_\infty)$. There exists a continuous $\mathbb{R}$-action*

$$\mathbb{R} \times E_\infty \to E_\infty$$

*such that there exists a subsequence of $\{E_i\}$ such that the standard $\mathbb{R}$-actions given by scalar multiplication converge uniformly on compact sets to it.*

*Proof.* The existence of said map follows from an application of the Arzelà-Ascoli theorem by the following reasoning. Regarding $\mathbb{R} \times E_i$ as a metric space with the standard product metric, one sees that by requiring $a, b \in \mathbb{R}$, $u, v \in E_i$ such that

$$\sqrt{|a|^2 + d^2(0_i, u)}, \sqrt{|b|^2 + d^2(0_i, v)} \leq R,$$

for some fixed $R >> 1$. Recall that the distance function on $E_i$ is given as in (3.43) and that therefore the distance between re-scalings of a common vector is bounded above by their linear distance, that is

$$d(au, b, u) \leq \|u\|_i |a - b|. \tag{5.62}$$

Also,

$$\begin{aligned} d(bu, bv) &= \inf_\alpha \sqrt{\ell^2(\alpha) + |b|^2 \|P^\alpha u - v\|^2} \\ &= \max\{1, |b|\} \inf_\alpha \sqrt{\ell^2(\alpha) + \|P^\alpha u - v\|^2} \\ &\leq R d(u, v), \end{aligned}$$

and therefore

$$\begin{aligned} d(au, bv) &\leq d(au, bu) + d(bu, bv) \\ &\leq R|a - b| + R d(u, v) \\ &\leq \sqrt{2} R \sqrt{|a - b|^2 + d^2(u, v)}. \end{aligned}$$

This proves that the family of maps $(a, u) \mapsto au$ is equicontinuous when restricted to balls of a given radius. Thus by the Arzelà-Ascoli theorem, there exists a convergent subsequence and thus the required map exists. Furthermore, since for any $a$ the map $u \mapsto au$ is also a limit of the corresponding re-scaling maps, the defining properties of an $\mathbb{R}$-action are also verified, namely: For all $u \in E_\infty$ and for all $a, b \in \mathbb{R}$

$$1 \cdot u = u, \tag{5.63}$$

$$a \cdot (b \cdot u) = (ab) \cdot u \tag{5.64}$$

□

5.24 *Remark.* As expected, multiplication by zero yields the *zero section* (as defined in Proposition 5.16), namely

$$0 \cdot u = \varsigma_\infty \circ \pi_\infty(u). \tag{5.65}$$

**5.25 Corollary.** *For any sequence of Riemannian manifolds $\{(X_i, p_i)\}$ converging to $(X_\infty, x_\infty)$ consider a convergent family of bundles with metric connection $(E_i, h_i, \nabla_i)$ over it converging to $(E_\infty, y_\infty)$. For any $u \in E_\infty$, the map*

$$t \mapsto tu,$$

*for $t \geq 0$ is a geodesic parametrized proportional to arc-length.*

*Proof.* This again follows from the fact that the corresponding maps for any sequence $\{u_i\}$, with $u_i \in E_i$ are geodesic rays (cf. Proposition 3.34) and an application of the Arzelà-Ascoli theorem. □

**5.26 Corollary.** *For any sequence of Riemannian manifolds $\{(X_i, p_i)\}$ converging to $(X_\infty, x_\infty)$ consider a convergent family of bundles with metric connection $(E_i, h_i, \nabla_i)$ over it converging to $(E_\infty, y_\infty)$, and let $\mu_\infty : E_\infty \to \mathbb{R}$ as in Proposition 5.16.*

*Then for any $u \in E_\infty$ and for any $a \in \mathbb{R}$,*

$$\mu_\infty(au) = |a| \mu_\infty(u) \tag{5.66}$$

*Proof.* This can be verified in two ways: 1) Since $\mu_\infty$ is the limit of the norms and since scalar multiplication satisfies said equation at the level of norms, then so will the limit satisfy it; 2) In view of the previous corollary, since $\mu_\infty(v)$ is also the distance between $v$ and $0 \cdot v = \varsigma_\infty \pi_\infty(v)$ and the map $t \mapsto tau$, being part of a geodesic ray, is a minimal geodesic. □

**5.27 Corollary.** *For any sequence of Riemannian manifolds $\{(X_i, p_i)\}$ converging to $(X_\infty, x_\infty)$ consider a convergent family of bundles with metric connection $(E_i, h_i, \nabla_i)$ over it converging to $(E_\infty, y_\infty)$. For any $u \in E_\infty$ there exists a sequence $\{u_i\}$, with $u_i \in E_i$ such that $\|u_i\| = \mu_\infty(u)$ for all $i$.*

*Proof.* There are two cases given by whether $\mu_\infty(u) = 0$ or no. If it is then for any sequence of points $\{x_i\}$ converging to $\pi_\infty(u)$ it follows that $u_i = \varsigma_\infty(x_i)$ converges to $u$ as required. If $\mu_\infty(u) \neq 0$ then, without loss of generality, one can consider a sequence $\tilde{u}_i$ converging to $u$ such that for all $i$, $\|\tilde{u}_i\| \neq 0$. Then by letting $u_i = (\mu_\infty(u)/\|\tilde{u}_i\|)\tilde{u}_i$, the conclusion also follows since $\|\cdot\|$ converges to $\mu_\infty$. □

## 5.3 Isolated degenerations.

Consider a sequence of twofolds converging in the pointed Gromov-Hausdorff sense to a cone in such a way that away from the point the convergence is smooth. Then the sequence of tangent bundles converges to the tangent bundle with an appropriate fibre added above the tip. This fibre is determined by the cone angle since it determines the holonomy group of the cone: the fibre is homeomorphic to the quotient of any tangent space by the holonomy action and its restricted metric is the canonical one. This is the content of Theorem 5.32 below.

In order to prove this fact, as well as to give a more precise statement, the metric structure of the cone, as a Riemannian manifold, will be analyzed. The standard way to produce a metric space structure on an abstract connected Riemannian manifold is by constructing the length structure on curves and then by defining the infimum over possible paths connecting two given points. By the Hopf-Rinow Theorem, the metric completeness guaranties the existence of minimizing geodesics.

Suppose now that a closed submanifold $\Sigma$ is removed from a complete Riemannian manifold $N$.

**5.28 Proposition.** *Given a complete Riemannian manifold $N$ and a codimension 2 submanifold $\Sigma$, the metric completion of $N \setminus \Sigma$ is uniquely isometric to $N$.*

*Proof.* As a Riemannian manifold itself, the complement $M = N \setminus \Sigma$ can still be given a metric structure. If the codimension of said submanifold is at least 2, then by standard transversality theory, any minimizing geodesic between two points in $N$ can be arbitrarily approximated by curves that miss $\Sigma$.

This proves that the restricted metric on $M \subseteq N$ coincides with its induced metric; indeed, the induced distance on $M$ can only increase, but by the described approximation it is seen to be equal.

The metric completion of $M$ is in a sense the smallest metric space $X$ containing $M$ as a dense set: if $Y$ is any complete metric space and $f$ is any uniformly continuous function from $M$ to $Y$, then there exists a unique uniformly continuous function $F$ from

$X$ to $Y$, which extends $f$. In particular, for any such $f$ and any point $p \in \Sigma$, there exists a neighborhood (within a coördinate chart) $U \subseteq N$ of $p$ such that

$$\overline{U \cap M} = \overline{U}, \tag{5.67}$$

and as such, $F$ can be uniquely defined at $p$. □

The codimension assumption is necessarily optimal —not only because of potential disconnections— as can be seen in elementary examples (e.g. consider a 2-torus with a generating circle removed).

In the Gromov Hausdorff Theory of convergence of metric spaces, one usually restricts one's attention to complete metric spaces. In particular, when a limit exists, it is defined to be complete.

Consider now the case when a sequence of metric spaces converge in the sense that they satisfy the definition without the assumption of completeness (neither for the terms in the sequence nor for the "limit" space).

**5.29 Proposition.** *Let $\{(X_i, x_i)\}$ be a sequence of (not necessarily complete) pointed metric spaces. Suppose that there exists a (not necessarily complete) pointed metric space $(X, x)$ such that together they satisfy the conditions of Definition 1.38. Then their completion $\{\overline{X_i}, x_i\}$ and $\overline{X}, x$ also satisfy the conditions of Definition 1.38, and thus $\overline{X}$ is the pointed Gromov-Hausdorff limit of the sequence of completions.*

*Proof.* Recall that by Definition 1.38 a sequence $\{(X_i, x_i)\}$ of pointed proper metric spaces is said to converge to $(X, x)$ in the pointed Gromov-Hausdorff sense if the following holds: For all $R > 0$ and for all $\varepsilon > 0$ there exists $N$ such that for all $i > N$ there exists an $\varepsilon$-isometry

$$f_i : B_R(x_i) \to B_R(x), \tag{5.68}$$

with $f_i(x_i) = x$, where the balls are endowed with restricted (not induced) metrics.

A space is proper if distance balls are compact. In the case of manifolds the convergence essentially says that given a positive number $r$, the balls of radius $r$ around the distinguished points $x_1$ converge in the Gromov-Hausdorff sense to the ball of radius $r$ around $x$ in such a way that

$$\lim_{i \to \infty} x_i = x. \tag{5.69}$$

The original technical reason for completeness is that for bounded metric spaces the Hausdorff distance between a subspace and its closure is zero, but they don't need to be isometric. However, if a sequence of spaces satisfies properties of the definition of pointed Gromov-Hausdorff convergence, that means that for arbitrary fixed radius, the Gromov-Hausdorff distance between the balls of that radius and the corresponding ball in the expected limit is going to zero, then by a standard triangle inequality argument, the distance between the completions of said balls is also going to zero.

For the latter, this is a statement of complete spaces and thus within the usual Gromov-Hausdorff theory. The only subtlety is observe that the completion of a closed ball of a given radius is the closed ball of the same radius in the completion, but this is immediate.

□

**5.30 Proposition.** *Given a convergent sequence of Riemannian metrics $\{g_i\}$ converging $\mathcal{C}^k$-smoothly to a Riemannian metric $g$, the sequence of Sasaki metrics $\mathrm{G}(g_i)$ converges $\mathcal{C}^{k-1}$-smoothly to the Sasaki metric $\mathrm{G}(g)$.*

*Proof.* In local coördinates, the tangent bundle is described as follows. Let $\{x^i : U \to \mathbb{R}\}$ be a local chart for the base. Then $u^i : \pi^{-1}(U) \to \mathbb{R}$, given by

$$u^i(p,v) = dx^i_p(v). \tag{5.70}$$

In this terms the Sasaki metric $\mathrm{G}$ with respect to a Riemannian metric $g$ is given as follows (cf. Definition 3.28).

$$\mathrm{G}_{n+i,n+j} = g_{ij} \circ \pi \tag{5.71}$$

$$\mathrm{G}_{i,n+j} = u^\ell(\Gamma^k_{j\ell} g_{ik}) \circ \pi \tag{5.72}$$

$$\mathrm{G}_{ij} = g_{ij} \circ \pi + u^k u^\ell (\Gamma^\alpha_{ik} \Gamma^\beta_{\alpha k} g_{j\beta}) \circ \pi \tag{5.73}$$

Because of these expressions, one sees that the Sasaki metric $\mathrm{G}(g)$ as a function of $\{x^i, u^i\}$ is of class $\mathcal{C}^{k-1}$ if $g$ is of class $\mathcal{C}^k$. Furthermore, if a sequence of metrics is converging smoothly, then so is their corresponding sequence of Sasaki metrics. □

Even though all such cones are homeomorphic to the $\mathbb{R}^2$, the geometry of their tangent bundles is very sensitive to the opening angle. As a working definition of a 2-dimensional cone consider the following metric in polar coördinates.

$$g = dr^2 + \left(\frac{\varphi r}{2\pi}\right)^2 d\theta^2 \tag{5.74}$$

As seen before, the geometry of the restricted metric on individual tangent spaces is determined by the length-norm on the holonomy group. Now is time to see this local structure gives back information about the completion of the Sasaki metric.

**5.31 Proposition** (Cone completion)**.** *The completion of the Sasaki metric of a metric 2-dimensional cone is obtained by attaching*

$$\mathbb{R}^2 \big/ \overline{Hol} \tag{5.75}$$

*as the corresponding fibre over the tip. Here $Hol$ denotes the holonomy group of* (5.74)*.*

At this point it is important to notice that even when the fibre over the tip is a metric cone, the restricted metric on this cone does not in general coincide with that of the base cone. This only happens when

$$\varphi = \frac{2\pi}{n} \tag{5.76}$$

for some integer $n$. That is that they coincide only when the space was already an orbifold.

This apparent discontinuity is worsen by the misleading illusion that if two cones have very close opening angles their behavior should be similar. However both the global and infinitesimal analysis shows that they are indeed quite different:

- These cones are actually infinite Gromov-Hausdoff distance apart.
- Their holonomies may be abysmally different.

*Proof of Proposition 5.31.* From (5.74), one sees that if one cuts open the cone along a constant $\theta$ ray, a "fundamental region" for the cone is a hinge of opening angle $\varphi$. Look at the shaded region in Figure 5.1. In that case, an angle $0 < \varphi < \pi$ is considered. The picture is correct although it might be misleading for values of $\varphi$ larger than $2\pi$ (which the fomula (5.74) certainly allows for).

Since the metric is flat, the only way to generate nontrivial holonomy is to go around the tip (Labeled $O$). The holonomy transformation at a point $P$ with respect to a loop $\gamma$ based at $P$ is a rotation by

$$\alpha(\gamma) = w_O(\gamma) \cdot \varphi \mod 2\pi, \tag{5.77}$$

where $w_O$ is the winding number of $\gamma$ around $O$ (for an a priori fixed orientation) and as such the holonomy group is generated by $\varphi \mod 2\pi$.

Recall that the length-norm $L$ is given by infimum of lengths of loops generating any given holonomy. If the angle is larger than or equal to $\pi$ then, regardless of the nontrivial holonomy element, the infimum is achieved by $2 \cdot \overline{PO}$, i.e. twice the distance from the point considered to the tip of the cone; indeed, this is achieved by a sequence of very small loops around the tip.

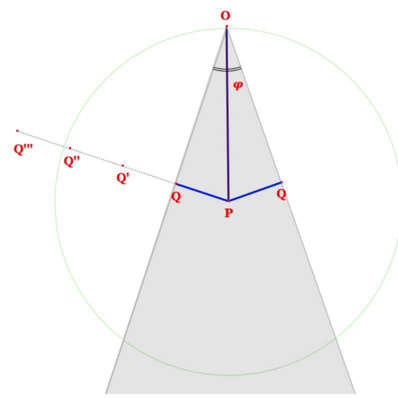


Figure 5.1: *Cut-open cone of angle $\theta$. A circle with center $P$ and radius $\overline{PO}$ only contains a few segments of the form $\overline{PX}$, $X = Q, Q', Q'', Q'''$ showing that it is eventually shorter to go by the tip $O$ instead.*

If the angle is smaller, then shortcuts occur for a little while: Let $Q$ be as in the figure, the perpendicular projection of $P$ onto the slit. Then, there exists a natural number $N$ such that

$$N \cdot \overline{PQ} \leq \overline{PO} < (N+1) \cdot \overline{PQ}, \tag{5.78}$$

and thus the length norm $L(n \cdot \varphi)$ is $2n \cdot \overline{PQ}$ if $0 < |n| \leq N$ and $2 \cdot \overline{PO}$ otherwise. Notice that

$$\overline{PQ} = \sin \frac{\varphi}{2} \cdot \overline{PO} \tag{5.79}$$

By Theorem 3.28, the restricted metric on the tangent space at $P$ is a holonomic space metric

$$d(u,v) = \inf_n \sqrt{L(n \cdot \varphi \mod 2\pi)^2 + \|R_{n\cdot\varphi} u - v\|^2} \tag{5.80}$$

where $R$ stands for the corresponding rotation.

Thus, for any sequence of points $P$ converging to $O$, the holonomic spaces converge to the quotient of $\mathbb{R}^2$ given by the following semimetric:

$$\rho(u,v) = \inf_n \|R_{n\cdot\varphi} u - v\|. \tag{5.81}$$

There are two possible situations. One where the holonomy group $H = \langle\varphi\rangle$ is finite, and hence closed and discrete; or when it is not, and hence dense. This are determined by whether

$$\varphi \in 2\pi\mathbb{Q}. \tag{5.82}$$

In either case, let $G = \overline{H} \subseteq \mathbb{S}^1$ and notice that (5.81) is precisely the metric on the orbit space

$$\mathbb{R}^2/G, \tag{5.83}$$

which is again a (non-degenerate 2-dimensional) cone if $G$ is discrete or a ray $[0,\infty)$ (a degenerate cone) if $G = \mathbb{S}^1$.

Finally, consider any Cauchy sequence on the tangent bundle to the cone with the tip removed. Because by construction the canonical projection is a Riemannian submersion the image of this sequence is still Cauchy. Now, without loss of generality assume that the limit is the tip. Then consider the sequence of corresponding tangent planes (those that contain the starting sequence).

As per the previous discussion, said sequence of tangent spaces converges to $\mathbb{R}^2/G$ in the Gromov-Hausdorff sense. Since this convergence obtained by looking at restricted metrics, the considered Cauchy sequence remains a Cauchy sequence for any metric on the disjoint union of the tangent spaces with the limiting cone that realizes the convergence as a Hausdorff convergence. This gives a correspondence between points on the limiting cone and Cauchy sequences whose projections converge to the tip. □

These facts together yield the following result.

**5.32 Theorem.** *Given a sequence of 2-dimensional metrics converging in the pointed Gromov-Hausdorff sense to a flat cone with opening angle $\varphi$, such that the convergence is smooth away from the tip, the sequence of tangent bundles (with their Sasaki metrics) converges to the metric completion of the tangent bundle of the cone. The fiber over the tip is isometric to the quotient of $\mathbb{R}^2$ by the closure of the holonomy group $H$ of the flat cone (with the standard quotient/cone metric). Since $H$ is generated by a rotation by*

$$\theta = 2\pi \sin(\varphi), \tag{5.84}$$

*the fiber over the tip is in general a different cone (e.g. it can be a ray if $\sin(\varphi)$ is not rational).*

*Proof.* The claim is now justified by the following facts:

1. By assumption the convergence is smooth away from the tip and hence (By Proposition 5.30) the convergence of the Sasaki metrics is also smooth away from the tip;

2. Any given tangent space inside the tangent bundle is of codimension the dimension of the base; that is 2. Thus the removal of a single fibre affects not the metric structure of the tangent bundle as per Proposition 5.28;

3. The completion of the tangent bundle of a cone minus the tip is described as in the claim in view of Proposition 5.31; and

4. By Proposition 5.29, Gromov-Hausdorff limits commute with completions.

□

Since the metric on the cone is flat, this analysis can be extended to 2-folds converging smoothly away from a discrete set to a flat metric with isolated conic singularities; each cone point has a cone angle that determines the topology of the fiber above it. Examples of these spaces are polyhedra —in particular the Platonic solids— with a flat metric on their faces. Since by assumption a polyhedron is a two-dimensional manifold, the edges can be smoothen out so that the only singularities of the metric occur at the vertices. In this case, at each vertex, the 'angle' $\theta$ is given as follows.

5.33 **Definition.** Let $P$ be a flat polyhedron with vertices $\{V_i\}$ and faces $\{F_j\}$. The *angle defect* at a vertex $V_i$ is the difference

$$\theta_i := 2\pi - \sum \theta_j(V_i) \tag{5.85}$$

where $\theta_j(V_i)$ is the angle at $V_i$ of the face $F_j$.

5.34 **Proposition.** *Let $P$ be a flat polyhedron with vertices $\{V_i\}$ and angle defects $\{\theta_i\}$. Then the metric completion of the Sasaki metric on $TP$ is obtained by attaching, over each vertex $V_i$,*

$$R^2/H, \tag{5.86}$$

*where $H$ is the closure of the group of rotations by $\theta_i$ in $O(2)$.*

*Proof.* The argument is identical to that of Proposition 5.31. If desired, this can be seen by looking at the tangent cone at the given vertex. □

5.35 **Corollary.** *Given a sequence of Riemannian metrics converging smoothly away from a discrete set to a flat polyhedron, then their Sasaki metrics converge to the completion of the Sasaki metric on the polyhedron and the singular fibers are homeomorphic to the quotient of $\mathbb{R}^2$ by the closure of the group of rotations by the angle defect at the given vertex.*

## 5.4 Further examples.

Other consequences of Theorems 5.16 and 5.5 are the following.

5.36 **Example.** Given any compact Riemannian manifold $(M^n, g)$ and let $X_i$ be the metric space obtained by rescaling $g$ into $\frac{1}{i^2}g$. Then tangent spaces converge to $\mathbb{R}^n/\overline{Hol(g)}$. Here $\overline{Hol(g)}$ denotes the closure in $O(n)$.

*Proof.* Let $p \in M$ and let $(V, H, L)$ be the associated holonomy space given by Theorem 3.56 at $p$. By Theorem 5.12, because $M$ is compact, it follows that

$$d_{GH}(TM, V) \leq 2\text{diam}(M). \tag{5.87}$$

Thus, by re-scaling, $\text{diam}(X_i) \to 0$ and the limit $Y = \lim_i TX_i$ is equal to the limit of the holonomic metrics at $p$. To analyze these spaces notice that the Sasaki metric re-scales like the base metric does; indeed, by re-scaling, the Levi-Civita connection remains constant and thus the horizontal lifts remain unchanged (cf. Definition **??**). Define

$$(V_i, H_i, L_i)$$

to be the corresponding holonomic spaces at $p \in X_i$. Again, because the connection is unchanged, it follows that

$$H_i \equiv H.$$

Also, since by Definition 3.54 $L$ is an infimum of lengths,

$$L_i = \frac{1}{i} L.$$

Finally, the norm on $V_i$, denoted by $\|\cdot\|_i$, is given by

$$\|\cdot\|_i = \frac{1}{i}\|\cdot\|,$$

where $\|\cdot\|$ is the norm on $V$. Notice that by considering the map $\phi_i : V \to V$, given by

$$\phi_i : u \mapsto iu, \tag{5.88}$$

one gets an isometry of the holonomic spaces

$$\phi_i : (V, H, L_i) \to (V_i, H_i, L_i). \tag{5.89}$$

Therefore, the limit $Y$ is the quotient $\mathbb{R}^n/G$ for some compact Lie group $G$, by Theorem 5.5. Furthermore, consider $G_0$ as in Theorem 5.4. Because for any $a \in H$,

$$\lim_{i \to \infty} L_i(a) = 0,$$

it follows that $G_0 = \overline{H}$; thus proving the claim.

$\square$

5.37 *Remark.* A theorem of Wilking [49] states that any closed subgroup of $O(n)$ can be realized as the closure of a holonomy group of a compact smooth manifold. By Theorem 5.36 one thus recovers all linear metric quotients of $\mathbb{R}^n$.

5.38 **Example.** Let $\{(M_i, g_i)\}$ and $\{(N_i, h_2)\}$ be two convergent sequences of complete Riemannian manifolds, with limits $X$ and $Y$ respectively. Let $\{E_i \to M_i\}$ and $\{F_i \to N_i\}$ be two convergent sequences of vector bundles with connections endowed with their metrics of Sasaki-type. Let $E \to X$ and $F \to Y$. Then, for the product metrics on $M_i \times N_i$ the limit converges to $E \times F \to X \times Y$.

*Proof.* For each $i$ the bundles $E_i \times F_i \to M_i \times N_i$ are endowed with the product connection, the product bundle metric, from which it follows that for any curve $\gamma = (\gamma_1, \gamma_2)$, the parallel translation along $\gamma$ splits in the following way.

$$P^{\gamma} = P^{\gamma_1} \oplus P^{\gamma_2}$$

From this it follows that the product metric of metrics of Sasaki-type coincides with the metric of Sasaki-type on the product. Now, because the spaces are products, the limit of the product is the product of the limits. □

5.39 *Remark.* Because of this, it follows that for any $(p,q) \in M_i \times N_i$,

$$Hol_{(p,q)}(\tilde{g}_i) = Hol_p(E_i) \times Hol_q(F_i). \tag{5.90}$$

Furthermore, the length norm of $(a,b)$ is given by

$$L_{(p,q)}((a,b)) = \sqrt{L_p^2(a) + L_q^2(b)} \tag{5.91}$$

5.40 **Example.** Let $(M_1^{n_1}, g_1)$ and $(M_2^{n_2}, g_2)$ be two complete Riemannian manifolds; suppose further that $M_2$ compact. Consider the metrics

$$\{\tilde{g}_i = g_1 + \frac{1}{i^2} g_2\}$$

on $M_1 \times M_2$, which to converge to $(M_1, g_1)$. If $\pi : Y \to M_1$ is the limit of their corresponding tangent bundles endowed with their metrics of Sasaki-type, then the fibers of $\pi$ are homeomorphic to

$$\mathbb{R}^{n_1} \times \left(\mathbb{R}^{n_2} / \overline{Hol(g_2)}\right). \tag{5.92}$$

However, for the constant sequence $\{(M_1, g_1)\}$ the limit is the canonical projection $TM_1 \to M_1$. This proves that passing to a subsequence in Theorem 5.14 is in general unavoidable.

*Proof.* Because these metrics are product metrics, where the second factor is re-scaled, the limit is the limit of the factors, by Example 5.38.

Now, since only one of the factor is being re-scaled, while the other remains constant, the group-norm becomes degenerate on $\{id\} \times Hol(M_2)$, thus yielding the desired result by Example 5.36. $\square$

# Chapter 6

# A weak notion of parallelism on singular spaces

ε΄. Καὶ ἐὰν εἰς δύο εὐϑείας εὐϑεῖα ἐμπίπτουσα τὰς ἐντὸς καὶ ἐπὶ τὰ αὐτὰ μέρη γωνίας δύο ὀρϑῶν ἐλάσσονας ποιῇ, ἐκβαλλομένας τὰς δύο εὐϑείας ἐπ᾽ ἄπειρον συμπίπτειν, ἐφ᾽ ἃ μέρη εἰσὶν αἱ τῶν δύο ὀρϑῶν ἐλάσσονες.

*Fifth Postulate, Elements.*
Euclid

Perhaps the most general setting for a notion of parallel translation is that of a pair of metric spaces and a surjective submetry between them. Even in this generality one can talk about parallel translation as long as one is willing to loosen it by considering, instead of functions, relations or —equivalently— set-valued functions. A notion of parallelism is this weak sense is the prescription of a class of curves on the domain of the given submetry: horizontal curves.

Once this class is given, two points are "parallel" if they can be joined by a horizontal curve. This is seen to be an equivalence relation. Holonomy groups control the extent to which this notion fails to produce a global notion.

In the setting of limits of vector bundles with connection endowed with their metrics of Sasaki type the following general considerations will be considerably better behaved. Yet, in giving a precise framework the assumptions will be kept to a bare minimum.

Lastly, the wane groups, in controlling the departure of the fibers from vector spaces, also effect the non-uniqueness of parallel translates.

## 6.1 Definitions

In this section, several concepts are recalled and introduced. In particular, a notion of *holonomy monoid* will be given for submetries. Recall that a *submetry* $\pi : Y \to X$ is a map (*a fortiori* surjective and continuous) such that for any radius $r$ the image of any metric ball of radius $r$ is again a ball of the same radius $r$.

**6.1 Definition.** Given a submetry $\pi : Y \to X$ a curve $\gamma : [0,1] \to Y$ is *horizontal* if and only if

$$\ell(\gamma) = \ell(\pi\gamma). \tag{6.1}$$

The set of all such curves will be denoted by $\mathcal{H}(\pi)$.

**6.2 Definition.** Given a curve $\alpha : [0,1] \to X$ and a point $u \in \pi^{-1}\alpha(0)$ a *parallel transport of $u$ along $\alpha$* is a horizontal $\gamma$ such that $\gamma(0) = u$ and $\pi\gamma = \alpha$.

It is easy to produce examples where given $\alpha$ and $u$ there exist no parallel translation as well as examples where there are even infinitely many such lifts. However, in the case of limits of metrics of Sasaki type, there will always exist at least one lift given $u$.

**6.3 Definition.** Given a curve $\alpha : [0,1] \to X$ and a point $u \in \pi^{-1}\alpha(0)$ *the parallel translation of $u$ along $\alpha$* is given as relation $\mathcal{P} \subseteq \pi^{-1}(\alpha(0)) \times \pi^{-1}(\alpha(1))$. This can be regarded as a set-valued function

$$\mathcal{P}^{\alpha} : \pi^{-1}(\alpha(0)) \dashrightarrow \pi^{-1}(\alpha(1)),$$

given by

$$\mathcal{P}^{\alpha}(u) = \{\gamma(1) | \gamma \in \mathcal{H}(\pi), \pi\gamma = \alpha\} \subseteq \pi^{-1}(\alpha(1)). \tag{6.2}$$

In the setting of limits of Sasaki-type metrics, there are examples where uniqueness is not satisfied, and thus such that it is necessary to talk about relations ( as set-valued functions) and not of single-valued functions.

**6.4 Theorem.** *Given a submetry $\pi : Y \to X$, and given two curves $\alpha, \beta : I \to X$ such that $\alpha(1) = \beta(0)$, then*

$$\mathcal{P}^{\beta\cdot\alpha} = \mathcal{P}^{\beta} \circ \mathcal{P}^{\alpha}, \tag{6.3}$$

*where $\beta \cdot \alpha$ stands for the concatenation of $\alpha$ and $\beta$. Also,*

$$\mathcal{P}^{\alpha^-} = (\mathcal{P}^{\alpha})^*, \tag{6.4}$$

*where $\alpha^-$ is the reverse curve.*

*Furthermore, given a fixed $x \in X$, the set*

$$\mathcal{H}_x := \{\mathcal{P}^\alpha | \alpha(0) = \alpha(1) = x\} \tag{6.5}$$

*is a *-semigroup with identity.*

*Proof.* Because parallel translation is defined by horizontal curves, and the concatenation of curves is additive in length, it follows that the the concatenation of horizontal curves is horizontal, thus proving the first claim. The second claim follows by reversing the direction of the horizontal curves.

In particular, for the set of parallel translations along loops, since it is closed under composition and under the involution if follows that it is indeed a monoid. □

**6.5 Definition.** Given a submetry $\pi : Y \to X$ and a point $x \in X$, the monoid with involution

$$\mathcal{H}_x := \{\mathcal{P}^\alpha | \alpha(0) = \alpha(1) = x\} \tag{6.6}$$

will be called *Holonomy monoid* of $\pi$ at $x$.

Notice that this coincides with the usual holonomy group in the case of a metric of Sasaki type, as well as in the case of Riemannian submersions in general ([22]). It will be seen that their departure from being groups is equivalent to the non-uniqueness of loops.

## 6.2 Existence and invariance

When studying limits of spaces, it is important to determine what properties "pass to the limit". Take the example of minimal geodesics. It is not true in general that any minimal geodesic in the limit arises as a limit of minimal geodesics. It is then a natural question to ask whether the same is true for horizontal curves.

The next results states that this is not true, and that every horizontal curve is the limit of horizontal curves.

**6.6 Theorem.** *Horizontal curves are the uniform limits of horizontal curves.*

*Proof.* Let $\pi : E \to X$ be, as before, a limit of vector bundles $\pi_i : E_i \to X_i$ and let $\gamma$ be a horizontal curve between $u \in E$ and $v \in E$ and consider a sequence $\{\gamma_i\}$ of piecewise smooth curves converging uniformly to $\gamma$ such that their lengths $\ell(\gamma_i)$ converge to $\ell(\gamma)$ . Let $\alpha_i = \pi_i\gamma_i$, and let $\widetilde{\gamma_i}$ be the unique horizontal lifts of $\alpha_i$ with $\widetilde{\gamma_i}(0) = \gamma_i(0)$.

Because $\alpha_i = \pi_i \circ \gamma_i$ holds, it follows that $\alpha$ is the limit $\{\alpha_i\}$ and since $\widetilde{\gamma}_i$ have uniformly bounded lengths, one can assume that they converge uniformly. Indeed, a uniform upper bound $C$ on lengths implies that

$$\widetilde{\gamma}_i(t) \in B_C(\gamma_i(0)) \subseteq E_i,$$

and thus the convergence can be regarded as a Hausdorff convergence as in Remark 1.34.

Furthermore, it follows that if $\widetilde{\gamma}$ is the limit of $\widetilde{\gamma}_i$, then $\pi\widetilde{\gamma} = \alpha$.

The claim is that $\widetilde{\gamma} = \gamma$. In fact, for each $i$, since the Riemannian structure on $\pi_i^{-1}\alpha_i$ is flat and Euclidean, the curve

$$\vartheta_i : t \mapsto P_t^{\alpha_i}\left((1-t)\gamma_i(0) + t(P_t^{\alpha_i})^{-1}(\gamma_i(1))\right) \tag{6.7}$$

is shorter than $\gamma_i$ with the same endpoints, and its length is given by

$$\ell(\vartheta_i) = \sqrt{\ell^2(\alpha_i) + \|P^{\alpha}(\gamma_i(0)) - \gamma_i(1)\|^2}. \tag{6.8}$$

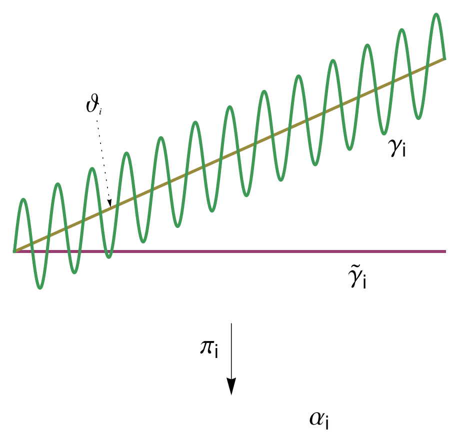


Figure 6.1: The geometry on $\alpha_i^* E_i$

Now by the lower semi-continuity of the length functions, it follows that

$$\ell(\gamma) = \lim_{i\to\infty}(\ell(\gamma_i)) \geq \lim_{i\to\infty}(\ell(\alpha_i)) \geq \ell(\alpha) = \ell(\gamma), \tag{6.9}$$

and that

$$\ell(\gamma) = \lim_{i\to\infty}(\ell(\gamma_i)) \geq \lim_{i\to\infty}(\ell(\widetilde{\gamma}_i)) \geq \ell(\widetilde{\gamma}) \geq \ell(\alpha) = \ell(\gamma). \tag{6.10}$$

Therefore, $\widetilde{\gamma}$ is horizontal. For the curves $\vartheta_i$, since

$$\ell(\gamma_i) \geq \ell(\vartheta_i) \geq \ell(\alpha_i), \tag{6.11}$$

it follows from (6.8) that

$$\|P^{\alpha}(\gamma_i(0)) - \gamma_i(1)\| \xrightarrow[i\to\infty]{} 0, \tag{6.12}$$

which means

$$\|\tilde{\gamma}_i(1) - \gamma_i(1)\| \xrightarrow[i\to\infty]{} 0. \tag{6.13}$$

This now says that $\{\widetilde{\gamma}_i(1)\}$ converges to $\gamma(1)$, as required.

Now, to see that this is true not only at the endpoints, notice that any segment of a horizontal curve is still horizontal, and that one can restrict the $\gamma_i$ and the $\alpha_i$ accordingly. The claim now follows and with it the end of the proof. □

**6.7 Corollary.** *Horizontal curves have constant norm and constant re-scalings of horizontal curves are horizontal.*

*Proof.* This is true since, before passing to the limit, the class of horizontal curves is closed under re-scaling and scalar multiplication is a uniform limit of scalar multiplications. □

**6.8 Theorem** (Parallel translation existence Theorem)**.** *For any sequence of Riemannian manifolds $\{(X_i, p_i)\}$ converging to $(X, p)$ consider a convergent family of bundles with metric connection $(E_i, h_i, \nabla_i)$ over it converging to $(E, \varsigma(p))$ with $\pi : E \to X$ as in Proposition 5.16. Let $\alpha : I \to X$ be any rectifiable curve. Then for any $u \in \pi^{-1}(\alpha(0))$ there exists $v \in \pi^{-1}(\alpha(1))$ such that there exists a horizontal path $\gamma$ from $u$ to $v$. In other words,*

$$\mathcal{P}^{\alpha}(u) \neq \varnothing. \tag{6.14}$$

*Proof.* Let $\alpha_i$ be piecewise smooth curves converging uniformly to $\alpha$. Let $u_i \in \pi_i^{-1}(\alpha_i(0))$ be a sequence of points converging to $u \in \pi^{-1}(\alpha(0))$, Let $\gamma_i$ be the parallel translation along $\alpha_i$ with $\gamma_i(0) = u_i$. Since

$$\ell(\gamma_i) = \ell(\alpha_i)$$

by construction, the convergence in length of $\{\alpha_i\}$ gives a uniform upper bound on the lengths of $\gamma_i$. Thus, by the Arzelà-Ascoli theorem, there exists a subsequence of $\{\gamma_i\}$, without loss of generality labeled again by $\gamma_i$, that converges uniformly to a curve $\gamma$ with $\gamma(0) = u$.

Now, by the lower semi-continuity of the the lengths,

$$\ell(\alpha) = \lim \ell(\alpha_i) = \lim_{i\to\infty} \ell(\gamma_i) \geq \ell(\gamma) \geq \ell(\alpha), \tag{6.15}$$

which implies equality and thus finishes the proof. $\square$

It will now be seen that the limit of the total spaces of a sequence of vector bundles with corresponding Riemannian metrics of Sasaki type contains many flats, i.e. isometric embeddings of $[a,b] \times [0,\infty) \subseteq \mathbb{R}^2$.

**6.9 Definition.** Let $X$ be a (pointed) limit of complete geodesic spaces $X_i$ (e.g. Riemannian manifolds)and $x, y \in X$. A curve $\alpha : [0,1] \to X$ is a *minimal limit geodesic* if

1. $\alpha(0) = x$ and $\alpha(1) = y$;
2. $\ell(\alpha) = d(x,y)$, i.e. if it is a minimal geodesic; and
3. there exist sequences $\{x_i \in X_i\}$, $\{y_i \in X_i\}$ and minimal geodesics $\alpha_i$, with $\alpha_i(0) = x_i$ and $\alpha_i(1) = y_i$ such that $\{\alpha_i\}$ converges to $\alpha$.

**6.10 Proposition.** *For any sequence of Riemannian manifolds $\{(X_i, p_i)\}$ converging to $(X,p)$ consider a convergent family of bundles with metric connection $(E_i, h_i, \nabla_i)$ over it converging to $(E, \varsigma(p))$ with $\pi : E \to X$ as in Proposition 5.16. Let $\alpha : I \to X$, parametrized by arc-length, be a limit geodesic. Then for any $u \in \pi^{-1}(\alpha(0))$ there exists an isometric embedding*

$$\varphi = \varphi_{u,\alpha} : [0, \ell(\alpha)] \times [0,\infty) \to E_\infty \tag{6.16}$$

*with*

$$\pi_\infty \varphi(t,s) = \alpha(t) \tag{6.17}$$

*and such that*

$$u = \varphi(0, \mu_\infty(u)). \tag{6.18}$$

*Furthermore, if $\gamma(t) := \varphi(t, \mu_\infty(u))$, then $\gamma$ is horizontal and*

$$\varphi(t,s) = \frac{s}{\mu_\infty(u)} \gamma(t) \tag{6.19}$$

*Proof.* Let $\alpha_i$ be geodesics (parametrized by arc-length) such that $\alpha = \lim_{i\to\infty} \alpha_i$. Let $u_i \in \pi_i^{-1}(\alpha_i(0))$, with $\|u_i\| = \mu_i(u)$ and such that $\{u_i\}$ converges to $u$. Let $\gamma_i$ be the unique parallel translation along $\alpha_i$ with initial value $u_i$; that is that

$$\gamma_i(t) = P_t^{\alpha_i}(u_i). \tag{6.20}$$

It is only needed to show that the map

$$\varphi_i(t,s) = \frac{s}{\|u_i\|}\gamma_i(t) \tag{6.21}$$

is an isometric embedding for each $i$. Once this is done an application of the Arzelà-Ascoli theorem completes the proof.

Indeed, by (3.43) the distance between two images is given by

$$d(\varphi(t,s),\varphi(t',s')) = \inf_\beta \sqrt{\ell^2(\beta) + \|P^\beta(\varphi(t,s)) - \varphi(t',s')\|^2} \tag{6.22}$$

$$= \inf_\beta \sqrt{\ell^2(\beta) + \frac{1}{\|u_i\|^2}\|(sP^\beta P^\alpha_t - s'P^\alpha_{t'})(u_i)\|^2}, \tag{6.23}$$

$$\tag{6.24}$$

where the infimum is over curves $\beta$ connecting $\pi_i(\varphi(t,s)) = \alpha_i(t)$ to $\pi_i(\varphi(t',s')) = \alpha_i(t')$. Now, given that parallel translation is by linear isometries of the fiber, then

$$\|(sP^\beta P^\alpha_t - s'P^\alpha_{t'})(u_i)\| \geq |s-s'|\|u_i\| \tag{6.25}$$

since the closest points between the spheres of radius $s\|u_i\|$ and $s'\|u_i\|$ with common center is given by the right hand side. Also, for any $\beta$

$$\ell(\beta) \geq |t-t'| \tag{6.26}$$

which is the distance between its endpoints. Thus

$$d(\varphi(t,s),\varphi(t',s')) \geq \sqrt{|t-t'|^2 + |s-s'|^2}. \tag{6.27}$$

However, by chosing $\beta = \alpha|[\min\{t,t'\},\max\{t,t'\}]$ in (6.23), the reverse inequality from (6.27) is obtained, thus yielding the claim and finishing the proof. □

## 6.3 Influence of the wane groups and holonomy

Since one can think of examples of non uniqueness (consider any isolated conic singularity as in Theorem 5.32), it is only natural to wonder what conditions guarantee uniqueness in parallel translation. In principle, there are two ways in which parallel translation can fail to be unique along a curve $\alpha$. One pertaining the relation $\mathcal{P}^\alpha$: whether or not this relation is in fact a function; and another, seemingly more drastic: whether there is more than one horizontal curve with given initial value over $\alpha$. These are in fact the same as per the following result.

**6.11 Proposition.** *Suppose $\pi : E \to X$ is a pointed Gromov-Hausdorff limit of a sequence metrics of Sasaki type. Then for all curves $\alpha : I \to X$, $\mathcal{P}^\alpha$ is a function if and only if for all curves $\beta : I \to X$ there exists a unique horizontal curve over it with given initial point.*

*Proof.* The necessity is immediate since given a curve $\alpha$, if given initial data there is exactly one horizontal curve, then the relation determined is such that there is only one pair $(x, y) \in \mathcal{P}^\alpha$ for any given $x \in \pi^{-1}(\alpha(0))$. Conversely, to see that it is sufficient, if there is a curve such that there are two distinct horizontal curves with common initial point and common endpoint (thus keeping the overall relation a function), then —since they are distinct curves after all— there exists a time for which their endpoints are different. Now, since they are projected onto the same curve, by restricting the curve downstairs, there exists a curve $\beta$ with a parallel translation relation that is not a function. □

In fact, the holonomy monoids defined in Definition 6.5 already determine the non-uniqueness of parallel translation globally, as seen in the next result.

**6.12 Theorem.** *Suppose $\pi : E \to X$ is a pointed Gromov-Hausdorff limit of a sequence metrics of Sasaki type. The holonomy monoids are indeed groups if and only if parallel translation is unique.*

*Proof.* Since, if at all, the inverse is given by $^*$ (by Proposition 1.28), if follows that parallel translations along loops are functions if and only if holonomy monoids are groups. Now, if there is a curve $\alpha$ such that there are two distinct horizontal curves over it with same initial value but different endpoints, then $\alpha^{-1}\alpha$ will be a curve for which the parallel translation relation is not a function. □

The condition that parallel translations be unique already implies some further control on the possible collapses of the fibers, namely the following fact.

**6.13 Theorem.** *Suppose $\pi : E \to X$ is a pointed Gromov-Hausdorff limit of a sequence metrics of Sasaki type. For any $x \in X$ and consider $G_x \leq O(k)$, the subgroup guaranteed by Theorem 5.19. If parallel translations are unique then for all $x, y \in X$, the corresponding wane groups are isomorphic,*

$$G_x \cong G_y, \tag{6.28}$$

*up to conjugation by an element in $O(k)$.*

*Proof.* Now, the contrapositive statement says that if the exist two points with non isomorphic groups then parallel translation is not unique. To prove this let $\{x_i\}, \{y_i\} \subseteq X_i$ be sequences that converge to $x, y \in X$ respectively, such that $G_x$ and $G_y$ are not isomorphic.

Let $u_i \in \pi_i^{-1}(x_i)$ converging to some $u \in \pi^{-1}(x)$. To bring this to the level of holonomic spaces, fix a minimal geodesic $\alpha_i$ from $x_i$ to $y_i$ and isomorphisms

$$\phi_i : \pi_i^{-1}(x_i) \to \mathbb{R}^k, \text{ and} \tag{6.29}$$

$$\varphi_i : \pi_i^{-1}(y_i) \to \mathbb{R}^k, \tag{6.30}$$

such that for all $i, j$,

$$\phi_i(u_i) = \phi_j(u_j), \text{ and} \tag{6.31}$$

$$\varphi_i(P^{\alpha_i} u_i) = \varphi_j(P^{\alpha_j} u_j). \tag{6.32}$$

Let $\tilde{u} = \phi_i(u_i), v = \varphi_i(P^{\alpha_i}(u_i))$, and

$$P_i = \varphi_i \circ P^{\alpha_i} \circ \phi_i^{-1}; \tag{6.33}$$

thus for all $i$,

$$P_i(\tilde{u}) = v \tag{6.34}$$

Again without loss of generality, $\{P_i\}$ converges in $O(k)$ to a map $P$, with

$$P(\tilde{u}) = v. \tag{6.35}$$

By assumption, since $G_x \ncong G_y$, there exists $g \in G_x$ such that for all $h \in G_y$ (or reversely there exists $h \in G_y$ such that for all $g \in G_x$),

$$P(g\tilde{u}) \neq hP(\tilde{u}), \tag{6.36}$$

for otherwise $PG_yP^{-1} = G_x$, a contradiction to them being different. Suppose without loss of generality that it is not the parenthetical case, i.e. that it is $g \in G_x$ that exists. It follows that $g$ is necessarily not the identity map.

Back at the level of fibers, this says that I can find elements $\tilde{u}_i = \phi_i^{-1}(\tilde{u}) \in \pi_i^{-1}(x_i)$ that converge to $u$ but such that their parallel translates along $\alpha_i$ remain away from the parallel translates along $\alpha_i$ of the $u_i$ by a definite amount. Passing to the limit (and taking a further subsequence if needed), the corresponding horizontal curves connecting them converge to distinct horizontal curves with the same starting point $u$. $\square$

**6.14 Corollary.** *Suppose $\pi : E \to X$ is a pointed Gromov-Hausdorff limit of a sequence metrics of Sasaki type. If parallel translations are unique then all fibers of $\pi$ are homeomorphic.*

*Proof.* This again follows from Theorem 5.19, since for each $x \in X$ the topology of the fiber is determined by $G_x$. $\square$

The proof of Theorem 6.13 hints that the not only do the wane groups have to be conjugate to each other, but that the conjugating element has to occur as a limit or parallel translations. It has yet to be seen to what extent is the converse true.

# Chapter 7

# Future directions

> Time was when all the parts of the subject were dissevered, when algebra, geometry, and arithmetic either lived apart or kept up cold relations of acquaintance confined to occasional calls upon one another; but that is now at an end; they are drawn together and are constantly becoming more and more intimately related and connected by a thousand fresh ties, and we may confidently look forward to a time when they shall form but one body with one soul.
>
> *Presidential Address to British Association, 1869.*
> J.J. Sylvester

Several questions have arisen during the research that lead to this report. The original motivation was to investigate the geometry of the Riemannian metric of Sasaki type. This lead to the introduction of the notions of *holonomic space* and of *holonomy radius of a Riemannian manifold*.

Even though the notion of holonomic space can seem artificial, it serves the purpose of displaying the strong geometric interactions —at the level of the fibers— of the holonomy groups with the fibers' metric; and, as a side-effect, it produces a metric structure on the said groups that is geometric in nature. At a philosophical level, this shows that nature doesn't need to be smooth or even continuous, given that the length-norms are not continuous and yet are useful.

In trying to understand the limits of tangent bundles and their relations to the limit of their bases, the notion of holonomy radius (a slight weakening of the convexity radius of the Sasaki-type metric at the zero section) controls the collapsing in the fiberwise direction. This collapse is further described by the *wane groups*; their existence is yet another surprising consequence of the robust algebraic nature of Parallelism in smooth spaces.

## On the wane groups

In order to exhibit the wane group at a point $x \in X$ there are several choices involved. In particular, the realization of the fiber as a limit of holonomic spaces. Changing these choices, only changes the group up to conjugation.

As seen in the case of conical singularities, the isomorphism type seems to be such that the groups can only 'decrease' in a neighborhood; i.e. for any point $p$ in the example there exists a neighborhood $U$ such that for any $q \in U$, there exists an element $g \in O(V)$ such that

$$gG_q g^{-1} \subseteq G_p. \tag{7.1}$$

**7.1 Question.** *Is there a meaningful topology on the conjugacy classes of subgroups of the orthogonal group for which the assignment of wane groups is continuous? Does the 'natural' partial order —given by inclusion— topology work?*

**7.2 Question.** *Given a limit space $X$, what are the defining properties of the wane group map? Is there a sheaf-theoretic description of said map?*

Even if this is not the case, a classification problem still arises:

**7.3 Question.** *How many conjugacy classes of wane groups are there for a particular limit space?*

Again, in the conic example it was seen that only at the tip is the space singular and only at the tip is the wane group nontrivial.

**7.4 Question.** *What is the exact relationship between a non-trivial wane group and the presence of singularities on the base space?*

And, of course, one can hope that in understanding this, a natural stratification, in terms of wane-types, could yield more information about the limiting process.

**7.5 Question.** *Is there a meaningful stratification of the limit space according to wane groups? If so, are the strata smooth in any sense?*

## On holonomy groups, monoids, and lengths.

Given a notion of horizontality, holonomy follows. In the limits, holonomy is not given by groups, or even by isomorphic structures (since the domains of definition change). However, it can be seen that the holonomy monoids at different points $p, q \in X$ are still weakly related: Consider any parallel translation $P$ from $p$ to $q$, then the corresponding holonomy monoids are related as follows. Let $a, b \in \mathcal{H}_p$, then the map to $\mathcal{H}_q$ given by

$$a \mapsto PaP^* \tag{7.2}$$

is satisfies that

$$PabP^* \subseteq PaP^*PbP^*. \tag{7.3}$$

Which is the best one expects without further assumptions (since $P^*$ will in general not be the inverse of $P$).

**7.6 Question.** *How far are the holonomy monoids from groups? Is there a 'big" submonoid that is a group?*

In the smooth case, the holonomy groups come with a length-norm associated to them. In the singular case, a function $Ł = Ł_p : \mathcal{H}_p \to \mathbb{R}$ satisfying the same properties of a group norm and

$$Ł(a^*) = Ł(a) \tag{7.4}$$

is produced in the same way: by looking at the infimum of length of loops generating a given holonomy. In particular it follows that if $a$ is not the identity, then $Ł(a) > 0$. It is not immediate the it induces a metric (or a reasonable topology) on $\mathcal{H}_p$: in the group case, this follows from the existence of inverses.

The fact that the construction of this norm is entirely metric raises several question in the smooth setting.

**7.7 Question.** *To what extent does the length-norm determine the algebraic nature of the holonomy group?*

**7.8 Question.** *What can be said about the length topology on the holonomy group? Heuristically, is it close to the Lie group topology? How can this be measured?*

As examples show, this length can fail to be continuous. However, in the 2-dimensional space forms considered, the function was in fact continuous (yet not smooth).

**7.9 Question.** *When is the length-norm continuous with respect to the standard Lie group topology?*

In the book of Hille [26], the mere assumption of measurability of a subadditive function essentially already yields its boundedness. In the work of Bingham and Ostaszewski [9], the assumption is weakened to being Baire.

**7.10 Question.** *Is the length-norm at least Baire or measurable (with respect to the Lie group Haar measure)?*

## On the limiting metrics and parallelism

The existence of a function Ł and the scaling-invariant notion of parallelism were essential ingredients in giving an explicit expression of the distance function of a Sasaki-type metric in terms of holonomy and the metric on the fibers. However, the linear structure of the fibers was also used in changing any path in the total space for a linear path.

**7.11 Question.** *Is the metric on the limit given by a formula similar to* (3.43) *or* (3.44)*? Namely, is the metric given as follows?*

$$d(u,v) = \inf_{\substack{\mathcal{P}, \\ w \in \mathcal{P}(u)}} \sqrt{\text{Ł}^2(\mathcal{P}) + d_F^2(w,v)} \tag{7.5}$$

*where,* $\mathcal{P}$ *is the parallel translate along any curve from* $\pi(u)$ *to* $\pi(v)$ *and* $d_F$ *stands for the induced metric on the fibers.*

Even if this is not true, one can still ask the following partial question. In the smooth setting, the induced metric on the fibers is flat Euclidean. In view of the possible nontriviality of the wane group, the fiber is not expected to be flat in that sense. However, the fibers, as quotients of Euclidean space are equipped with a natural quotient metric. For simplicity, one says that this is the natural flat metric.

**7.12 Question.** *Are the limit fibers intrinsically flat?*

In Theorem 6.13 it was seen that a necessary condition for uniqueness of parallel translates was that there be a unique wane group up to conjugation.

**7.13 Question.** *Are there examples of limits with unique wane groups and non-unique parallel translates?*

In fact (as seen earlier in this chapter), to exhibit a particular wane group, one has to give an explicit presentation of the fiber as a quotient of holonomic space $V/G$. Given a curve $\alpha : I \to X$, and two such holonomic space presentations, the following subset of $O(k)$ needs to be understood.

**7.14 Definition.** Let $\alpha : I \to X$, and let $G_x$ and $G_y$ be representatives of the wane groups (well-defined up to choice of basis) at the end points $x = \alpha(0)$ and $y = \alpha(1)$, then the set $\Lambda_\alpha \subseteq O(k)$, given by

$$\Lambda_\alpha = \{P \in O(k) | P = \lim_{i\to\infty} P^{\alpha_i}, \alpha = \lim_{i\to\infty} \alpha_i\}, \tag{7.6}$$

is the *set of parallel translation germs* along $\alpha$ with respect to $G_x$ and $G_y$.

**7.15 Conjecture.** *Suppose $\pi : E \to X$ is a pointed Gromov-Hausdorff limit of a sequence metrics of Sasaki type. For any $x \in X$ and consider $G_x \leq O(k)$, the subgroup guaranteed by Theorem 5.19. Let $\alpha : I \to X$, let $G_x$ and $G_y$ be representatives of the wane groups at the end points $x = \alpha(0)$ and $y = \alpha(1)$, then the corresponding set of parallel translation germs $\Lambda$ is invariant under the left and right actions of $G_y$ and $G_x$ given by group multiplication. Furthermore, if parallel translation is unique along $\alpha$ if and only if the second orbit space*

$$G_y \backslash \Lambda / G_x \tag{7.7}$$

*consists of a single element.*

As mentioned in the introduction, Rieffel [39] introduces a Lipschitz seminorm of a very natural space of matrix-valued functions to control distances between vector bundles. In essence, he regards Euclidean vector bundles as a certain type of map into the space of self-adjoint idempotent matrices. In light of this, together with the existence of the limit sets $\Lambda$ of germs of parallel translates, one can imagine taking limits of these maps of matrices.

**7.16 Question.** *Does there exist a 'virtual' vector bundle over the limit space together with a canonical identification that recovers the limit of total spaces? If so, is there a well-defined parallel translation on it, such that the identifications of which give the one described in this report? How is it related to the $\Lambda$'s described in this Chapter?*

It seems that this vector bundle would depart even further from the geometry of the limiting space. However, this departure occurs already at the level of the limit fibers.The cone example shows that the fibers of a limit of tangent bundles need not coincide with the tangent cone. However, there might be a relationship between them since both are in a way compatible with re-scalings.

**7.17 Question.** *Under what conditions are the fibers of a limit of tangent bundles and the tangent cones related?*

## Holonomy radius and tangency

Already in the smooth setting, the analysis of the holonomic-space structure of the fibers of a metric bundle produces the new synthetic notion of a holonomy radius. Because of Corollary 2.20, it follows that

$$\|P^{\alpha} - id\| \leq \frac{1}{\mathrm{HolRad}_p} \ell(\alpha), \tag{7.8}$$

for any loop $\alpha$ at $p$. This suggests that there must be a functional relation between the holonomy radius and curvature. The surprising fact is that it seems to indicate that the relation with curvature is at $p$ itself, not necessarily in a neighborhood.

**7.18 Question.** *What is the exact relationship (if any) between the holonomy radius and the curvature at the point (or in a neighborhood thereof)?*

Also, in the particular case when $\pi : Y \to X$ is a limits of tangent bundles.

**7.19 Question.** *What is the defining property of a vector? More explicitly, of an element of $Y$, when can a curve on $X$ be associated to it?*

## Isoperimetry

In the case of space forms in dimension 2, the problem of finding the minimizers for the length norm is the same as finding minimizers for area. This is the classical isoperimetric problem. In that setting, it is know that solutions have to have constant geodesic —read "mean"— curvature.

**7.20 Question.** *What are the conditions for a curve to be a minimizer?*

In the isoperimetric case, the absolute minimizer path need not contain an a priori given point. If such a condition is imposed, then the solution might not be smooth at the given point (but will remain so elsewhere).

**7.21 Question.** *Are the minimizers of the isoholonomic problem smooth away from the base point?*

## Closing remarks

The list of questions presented here is but a subtle hint of this beautiful and rich field; it will serve as a basis for pursuing further the topics already discussed in this report. The results obtained here required very few additional assumptions; therefore they necessarily give information about robust geometric properties that were not apparent when restricting the attention to smooth spaces.

Further assumptions can and will be made in order to give more precise formulations of the aforementioned questions in hope to find the beautiful answers that Nature has so far hinted.

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