Switching On and Off the Full Capacity of an $M/M/\infty$ Queue & Dynamic Pricing and Power Generation Risk Management in Smart Grids

A Dissertation Presented

by

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to

The Graduate School

in Partial Fulfillment of the Requirements

for the Degree of

Doctor of Philosophy

in

Applied Mathematics and Statistics

Stony Brook University

December 2010

Stony Brook University

The Graduate School

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Abstract of the Dissertation

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Part 1. We study a problem of optimal switching on and off the $M/M/\infty$ queue with holding, running and switching costs. We show that an optimal policy either always runs the system, or is defined by two thresholds M and N such that the system is switched on upon an arrival epoch if the system size accumulates to N and is switches off upon a departure epoch if the system size decreases to M.

Part 2. We design an optimal incentive mechanism offered to energy customers at multiple network levels, such as distribution and feeder networks, with the aim of determining the lowest cost aggregate energy demand reduction. Our model minimizes a utility's total cost for this mode of virtual demand generation, i.e., demand reduction, to achieve improvements in both its total systemic costs and load reduction over existing mechanisms. Our scheme assumes that the utility can predict with rebates by observing and learning from their past behavior. Within a single period formulation, we further propose a heuristic

policy that segments the customers according to their likelihood of reducing load. Within a multi-period formulation, we observe that customers who are more willing to reduce their aggregate demand over the entire horizon, rather than simply shifting their load to off-peak periods, tend to receive higher incentives, and vice versa. We further consider integrating the demand response and renewable resources into traditional thermal power generation management. A single-period optimal dispatching problem is considered for a network of energy utilities connected via multiple transmission lines, where we seek to find the lowest operational-cost dispatching of various energy sources to satisfy demand. Our model includes traditional thermal resources and renewable energy resources , together with corresponding power transmission constraints, as available generation capabilities within the grid. A key novel addition is the consideration of demand reduction as a *virtual generation* source that can be dispatched quickly to hedge against the risk of unforeseen shortfall in supply. Demand reduction is dispatched in response to incentive signals sent to consumers. The control options of our optimization model consist of the dispatching order and dispatching amount of the thermal generators together with the rebate signals sent to end-users at each node of the network under a simple demand response policy. Numerical experiments based on our analysis of representative data are presented to illustrate the effectiveness of demand response as a hedging option.

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Chapter 1

Switching On and Off of the Full Capacity of an $M/M/\infty$ Queue

1.1 Introduction

We study an $M/M/\infty$ queue with the controls that turn off the running system or turn on the system to full capacity. System costs include linear holding cost and constant serving cost, as well as switching costs each time the system is switched on or off. The system process is reviewed at either a customer arrival epoch or a departure epoch. The objective is to find a stationary optimal policy to minimize the total discounted cost over an infinite horizon and average cost. We prove that a Blackwell optimal policy is either a full service policy or an (M, N)-policy, where a full service policy is such that never turns off the running server; and an (M, N)-policy is such that there exist two integers M and N, with $0 \leq M < N < \infty$ and for any state $x = (i, \delta) \in S$, a stationary optimal policy is such that

$$\varphi(x) = \begin{cases} 0, & \text{if } i \leq M, \\ 1, & \text{if } i \geq N, \\ \delta, & \text{if } M < i < N. \end{cases}$$

When $M = 0$, the (M, N)-policy is the classic N-policy studied by [5, 16, 33, 55, 48]. Another type of optimal policies for queues with a removable server is called D-policy [29, 28].

The paper is organized as follows. In Section 1.2, we present our model formulation, notations and definitions. Some properties of $M/M/\infty$ are presented in Section 1.3. In Section 1.4, we show the structure of discount-optimal polices when the continuous discount factors are small. The average optimality is shown in Section 1.5. Computation results and numerical experiments are given in section 1.6.

1.2 Problem Formulation

Consider an $M/M/\infty$ queue. Customers arrive according to a Poisson with parameter λ. Service times are exponential with parameter μ and independent. The number of servers is unlimited. The system can be switched on and off. All busy servers operate when the system is on, and all the servers are off when the system is off.

The controller can switch the system from on to off and vice versa any time. The costs include the linear holding cost of h for a unit of time a customer spends in the system, the start-up cost s_1, the shut-down cost s_0, and the running costs c_1 per unit time when the system is on and c_0 per unit time when the server is off, where $c_1 > c_0 \geq 0$. Let $c = c_1 - c_0$. It is clear that without loss of generality we may assume that $c_1 = c$ and $c_0 = 0$. So, throughout the paper we consider the system running time equal to c and the zero idling costs. We also assume that $h > 0$, s_0 and s_1 are nonnegative, but at least one of them is positive.

To simplify the initial analysis, we assume that the server can be turned on and off only at time 0, customer arrival times, and customer departure times. These times are jump epochs for the process $X(t)$ of the number of customers in the system at time t. Let $t_0, t_1, \ldots$ be the sequence of jump epochs. We initially consider the servers can be switched on and off only at jump epochs. Switching takes place only at these times is not restrictive, and the optimal policies described in the paper are also optimal when the system can be turned on and off any time.

Now we define a Continuous-Time Markov Decision Process (CTMDP) for our problem. For this process, decision are chosen only at jump epochs. The state space is $S = \mathbb{N} \times \{0, 1\}$, where $\mathbb{N} = \{0, 1, \ldots\}$. If the state of the system at the decision epoch n is $x_n = (X_n, \delta_n) \in S$, this means that the number of customers in the system is X_n and the state of the system is δ_n, with $\delta_n = 1$ if the system is on and $\delta_n = 0$ if the system is off. The action set is $A = \{0, 1\}$. If the action a_n is selected at a decision epoch t_n, when the system is at a state (X_n, δ_n), the system is switched immediately if $a_n \neq \delta_n$ and its status (on or off) remains unchanged, if $a_n = \delta_n$. In particular, if the system is off, that is $\delta_n = 0$, the decision $a_n = 1$ turns the system on, and, if the system is on, that is $\delta_n = 1$, the decision $a_n = 0$ turns it off.

If the system is off or $X_n = 0$, the time until the next jump epoch, which is an arrival, has an exponential distribution with the intensity λ. If $X_n = i > 0$ and the system is on, the time until the next jump epoch has an exponential distribution with the intensity $\Lambda_i = \lambda + i\mu$, and this jump is an arrival with the probability λ/Λ_i and a departure with the probability $i\mu/\Lambda_i$.

A history of the process up to nth jump, $n = 0, 1, \ldots$, is the sequence t_0, x_0, a_0, $\ldots$, $t_{n-1}, x_{n-1}, a_{n-1}, t_n, x_n$. Let H_n be the set of all histories up to nth decision epoch. Then $H = \cup_{0 \le n < \infty} H_n$ is the set of all histories that contain a finite number of decision epochs. A (possibly randomized) policy π is defined as a transition probability from H to A such that $\pi(A|h_n) = 1$ for each $h_n \in H, n = 0, 1, \ldots$. A *stationary* policy is defined by a mapping $\pi : S \to A$ such that $\pi(x) \in A, x \in S$.

For each initial state of the system $x_0 = (i, \delta)$, and for any policy π, the policy π defines a stochastic sequence $\{x_n, a_n, t_n, n = 0, 1, \ldots\}$, where $t_0 = 0$ and $t_{n+1} \ge t_n$. We denote by $E^{\pi}_{x_0}$ the expectation of this process.

Now we define the cost function. If $x_n = \{i, \delta\}$, and an action a is selected then the cumulative cost during the interval $[t_n, t_{n+1}]$, where $0 \le t_n \le t_{n+1}$ is

$$c(i, \delta, a, t_n, t_{n+1}) = \int_{t_n}^{t_{n+1}} hidt + cI\{\delta = 1\}dt + s_a I\{\delta = a\},$$

where I is the indicator function. The cumulative cost over the interval t is

$$C(t) = \sum_{n=0}^{N(t)} c(X_n, \delta_n, a_n, t_n, t_{n+1}) + c(X_{N(t)}, \delta_{N(t)}, a_{N(t)}, t_{N(t)}, t),$$

where $N(t)+1$ is the number of jump epochs up to time t. Thus, $N(t)$ does not count the jump at $t_0 = 0$.

Let $N_1(t)$ be the number of arrivals and $N_2(t)$ be the number of departures by time t. Since $N_1(t)$ is a Poisson process then with probability 1 $N_1(t) < \infty$ for $t < \infty$ and $N_1(t) \to \infty$ as $t \to \infty$. Since $N(t) = N_1(t) + N_2(t)$ and $N_2(t) \le N_1(t) + X_0$, we have that $N_1(t) \le N(t) \le 2N_1(t) + X_0$. This implies that with probability 1 $N_1(t) < \infty$ for $t < \infty$ and $N_1(t) \to \infty$ as $t \to \infty$. Thus with probability 1 all the epochs t_n are finite and $t_n \to \infty$ as $n \to \infty$.

We observe that $C(t) = \infty$ with probability 1 when $N(t) = \infty$. Now we define the state of the server at time t as $\delta(t) = \delta_n$ for $t_n \le t \le t_{n+1}$, and the number of customers at time t as $X(t) = X_n$ for $t_n \le t \le t_{n+1}$. Using these notations, we can rewrite

$$C(t) = \int_0^t (hX(t) + c\delta(t))dt + \sum_{n=0}^{N(t)} s_{a_n}|a_n - \delta_n|, \tag{1.1}$$

where we use that $|a - \delta| = 0$, if $a = \delta$, and $|a - \delta| = 1$, if $a \neq \delta$. Observe that $C(t)$ is a nondecreasing nonnegative function.

For any initial state of the system $x_0 = (i, \delta)$, and for any policy π, the expected total discounted cost over the infinite horizon is

$$\begin{aligned} V_\alpha^\pi(i,\delta) &= E_{(i,\delta)}^\pi \int_0^\infty e^{-\alpha t} dC(t) \\ &= E_{(i,\delta)}^\pi [\int_0^\infty e^{-\alpha t}(hX(t) + \delta(t)c)dt + \sum_{n=0}^{\infty} e^{-\alpha t_n}|a_n - \delta_n| s_{a_n}]. \end{aligned} \tag{1.2}$$

The average cost per unit time is defined as

$$v^\pi(i,\delta) = \limsup_{t\to\infty} t^{-1} E_{x_0}^\pi C(t). \tag{1.3}$$

Let

$$V_\alpha(i,\delta) = \inf_\pi V_\alpha^\pi(i,\delta), \tag{1.4}$$

$$v = \inf_\pi v^\pi(i,\delta). \tag{1.5}$$

A policy φ is called discount-optimal if $V_\alpha^\varphi(i,\delta) = V_\alpha(i,\delta)$ for any policy π and for any $x_0 = (i,\delta)$. A policy φ is called average optimal if $v^\varphi(i,\delta) = v$ for any policy π and for any $x_0 = (i,\delta)$.

1.3 Properties of $M/M/\infty$ Queues

In this section, let $\Lambda_i = \lambda + i\mu$, $\beta_i = \dfrac{\Lambda_i}{\alpha + \Lambda_i}$, $p_i = \lambda/\Lambda_i$, and $q_i = i\mu/\Lambda_i$, $i = 0, 1, \ldots$. Let T_i be the time until the first departure when the system is on at time 0. Let $N_i(t)$ represent the number of arrivals during time t starting with i customers in the system, thus

$$E[T_i] = E\left[E\left[T_i|N_i(t)\right]\right] = q_i\frac{1}{\Lambda_i} + \sum_{k=1}^{\infty} p_i \dots p_{i+k-1}q_{i+k}\sum_{j=0}^{k}\frac{1}{\Lambda_{i+j}} \tag{1.6}$$

By rearranging the order of summation in (1.6),

$$\begin{aligned}
E[T_i] =& \frac{q_i}{\Lambda_i} + \sum_{j=1}^{\infty}\frac{1}{\Lambda_{i+j}}\sum_{k=j}^{\infty} p_i \dots p_{i+k-1}q_{i+k} \\
=& \lambda^{-1}\left(\frac{\lambda}{\Lambda_i}q_i + \sum_{j=1}^{\infty}\frac{\lambda}{\Lambda_{i+j}}p_i \dots p_{i+j-1}\sum_{k=0}^{\infty} p_{i+j}\dots p_{i+j+k-1}q_{i+k}\right) \\
=& \lambda^{-1}\left(p_iq_i + \sum_{j=1}^{\infty} p_i \dots p_{i+j}\sum_{k=1}^{\infty} P\left(N_{i+j}(t) = k\right)\right) \\
=& \lambda^{-1}\sum_{j=0}^{\infty} p_i \dots p_{i+j}\sum_{k=0}^{\infty} P\left(N_{i+j}(t) = k\right) \\
=& \lambda^{-1}\sum_{j=0}^{\infty} p_i \dots p_{i+j},
\end{aligned} \tag{1.7}$$

where the last equality is by the fact $\sum_{k=0}^{\infty} P\left(N_{i+j}(t) = k\right) = 1$ for all i, j. It is obvious from (1.7) that $E[T_i]$ is decreasing in i. Let $B_{i,\alpha}$ be the expected discount factor for T_i, we have

$$B_{i,\alpha} = E\left[e^{-\alpha T_i}\right] = E\left[E\left[e^{-\alpha T_i}|N_i(t)\right]\right] = \sum_{k=0}^{\infty} P\left(N_i(t) = k\right)\beta_{N_i(t)=k}, \tag{1.8}$$

where $\beta_{N_i(t)=k} = E\left[e^{-\alpha T_i}|N_i(t) = k\right] = \beta_i \dots \beta_{i+k}$. Let $B_{i,\alpha}^k = P(N_i(t) = k)\beta_{N_i(t)=k} = p_i \dots p_{i+k-1}q_{i+k}\beta_{N_i(t)=k}$, where $\beta_i = \dfrac{\Lambda_i}{\alpha + \Lambda_i}$, and we have $B_{i,\alpha} = \sum_{k=0}^{\infty} B_{i,\alpha}^k$. It is obvious that $B_{i,\alpha}$ is decreasing in α and increasing in i. Define the total discounted serving cost and holding cost during T_i as $C_{i,\alpha}$ and $H_{i,\alpha}$ respectively. Thus

$$\begin{aligned}
E\left[C_{i,\alpha}\right] =& E\left[\int_0^{T_i} e^{-\alpha t} c dt\right] = c\left[\int_0^{T_i} e^{-\alpha t} dt\right] \\
=& \frac{c}{\alpha}\left(1 - E\left[e^{-\alpha T_i}\right]\right) = \frac{c}{\alpha}(1 - B_{i,\alpha}), \qquad (1.9) \\
E\left[H_{i,\alpha}\right] =& E\left[\int_0^{T_i} e^{-\alpha t} h(i + N(t)) dt\right] = E\left[E\left[\int_0^{T_i} e^{-\alpha t} h(i+k) dt | N_i(t) = k\right]\right] \\
=& \sum_{k=0}^{\infty} P\left(N_i(t) = k\right) \sum_{j=0}^{k} \beta_{N_i(t)=j} \frac{h(i+j)}{\Lambda_{i+j}} = \frac{h}{\mu} \sum_{j=0}^{\infty} \beta_{N_i(t)=j} q_{i+j} \sum_{k=j}^{\infty} P\left(N_i(t) = k\right) \\
=& \frac{h}{\mu}\left(\beta_{N_i(t)=0} q_i + \sum_{j=1}^{\infty} \beta_{N_i(t)=j} q_{i+j}\left(1 - \sum_{k=0}^{j-1} P\left(N_i(t) = k\right)\right)\right) \\
=& \frac{h}{\mu}\left(\beta_{N_i(t)=0} q_i + \sum_{j=1}^{\infty} \beta_{N_i(t)=j} q_{i+j}\left(p_i \dots p_{i+j-1}\right)\right) \\
=& \frac{h}{\mu} \sum_{j=0}^{\infty} \beta_{N_i(t)=j} P\left(N_i(t) = j\right) = \frac{h}{\mu} B_{i,\alpha}. \qquad (1.10)
\end{aligned}$$

From (1.9), $E\left[C_{i,\alpha}\right]$ is decreasing in i; from (1.10), $E\left[H_{i,\alpha}\right]$ is increasing in i.

Lemma 3.1 *When $\alpha < \frac{c\mu}{h}$, $E\left[H_{i,\alpha}\right] + E[C_{i,\alpha}]$ is decreasing in i.*

Proof By (1.9) and (1.10), we have

$$E\left[C_{i,\alpha} + H_{i,\alpha}\right] = \frac{c}{\alpha}(1 - B_{i,\alpha}) + \frac{h}{\mu} B_{i,\alpha} = \left(\frac{h}{\mu} - \frac{c}{\alpha}\right) B_{i,\alpha} + \frac{c}{\alpha}.$$

Since $B_{i,\alpha} = E\left[e^{-\alpha T_i}\right]$ is increasing in i, thus when $\frac{h}{\mu} - \frac{c}{\alpha} < 0$, i.e., $\alpha < \frac{c\mu}{h}$, $E\left[C_{i,\alpha} + H_{i,\alpha}\right]$ is decreasing in i. $\square$

1.4 Discounted Cost Criterion

In this section we study the expected total cost criterion.

1.4.1 Reduction to Discrete Time and Existence of Stationary Discount-Optimal Policies

In this subsection, we formulate the optimality equation, prove the existence of stationary discount-optimality equations. This is done by reduction of our problem to discrete time.

When the system is on and there are i customers in it, the time until the next jump has an exponential distributions with intensity $\Lambda_i \to \infty$ as $i \to \infty$. Since the jump rates are unlimited, it is impossible to present the problem as a discounted MDP with the discount rate smaller than 1. Thus, we shall present our problem as a negative MDP.

A discrete time MDP is called negative [51, 6, 43, 27], if the costs are nonnegative and the goal is to minimize the expected total rewards. Similarly to discounted MDPs, the value function for a negative MDP satisfies the optimality equation. In addition, if the action sets are finite, there exists a stationary optimal policy. Furthermore, a stationary policy is optimal if and only if it satisfies an optimality equation. This means that for an MDP with a countable state set X, action sets $A(x)$, transition probabilities $p(y|x,a)$, and nonnegative one-step rewards $c(x,a)$, a stationary a stationary policy ϕ is optimal if and only if for all $x \in X$ it satisfies

$$V(x) = c(x,\phi(x)) + \sum_{y\in X} p(y|x,\phi(x))V(y), \tag{1.11}$$

where $V(x)$ is the infimum of the expected total costs starting from state x. In addition, the value function $V(x)$ satisfies the optimality equation

$$V(x) = \min_{a\in A(x)} \{c(x,a) + \sum_{y\in X} p(y|x,a)V(y)\}, \qquad x \in X. \tag{1.12}$$

For our queueing control problem, define the following values:

$$\beta(i,a) = \begin{cases} \dfrac{\lambda}{\lambda+\alpha}, & \text{if } a = 0, \\ \dfrac{\Lambda_i}{\Lambda_i+\alpha}, & \text{if } a = 1, \end{cases} \tag{1.13}$$

$$
p(j|i,a) = \begin{cases} 1, & \text{if } a=0,\ j=i+1, \\ \dfrac{\lambda}{\Lambda_i}, & \text{if } a=1,\ j=i+1, \\ \dfrac{i\mu}{\Lambda_i}, & \text{if } i>0,\ a=1\ j=i-1, \\ 0, & \text{otherwise,} \end{cases} \tag{1.14}
$$

and $c((i,0),0) = \dfrac{hi}{\lambda+\alpha}$, $c((i,1),0) = s_0 + c((i,0),0)$, $c((i,1),1) = \dfrac{hi+c}{\Lambda_i+\alpha}$, and $c((i,0),1) = s_1 + c((i,1),1)$. Let

$$
p_\alpha(j|i,a) = \beta(i,a)p(j|i,a)).
$$

We follow the conventions that $p_\alpha(-1|i,a) = 0$, $V_\alpha(-1,\delta) = 0$, $\sum_{\emptyset} = 0$, and $\prod_{\emptyset} = 1$.

The following theorem is the main result of this subsection.

Theorem 1 *For any $\alpha > 0$ the following statements hold:*

(i) For all $i = 0, 1, \ldots$

$$
V_\alpha(i,\delta) \le (1-\delta)s_1 + \frac{hi}{\mu+\alpha} + \frac{h\lambda}{\alpha(\mu+\alpha)} + \frac{c}{\alpha} < \infty; \tag{1.15}
$$

(ii) For all $i = 0, 1, \ldots$ and all $\delta = 0, 1$ the value function $V_\alpha(i,\delta)$ satisfies the optimality equation

$$
\begin{aligned}
&V_\alpha(i,\delta) \\
&= \min_{a\in\{0,1\}} \{c((i,\delta),a) + p_\alpha(i-1|i,a)V_\alpha(i-1,a) + p_\alpha(i+1|i,a)V_\alpha(i+1,a)\} \\
&= \min\Bigg\{(1-\delta)s_1 + \frac{hi+c}{\alpha+\Lambda_i} + \frac{\lambda}{\alpha+\Lambda_i}V_\alpha(i+1,1) + \frac{i\mu}{\alpha+\Lambda_i}V_\alpha(i-1,1), \\
&\quad \delta s_0 + \frac{hi}{\alpha+\lambda} + \frac{\lambda}{\alpha+\lambda}V_\alpha(i+1,0)\Bigg\};
\end{aligned} \tag{1.16}
$$

(iii) There exists a stationary discount-optimal policy, and a stationary policy ϕ is

discount-optimal if and only if for all $i = 0, 1, \ldots$ and for all $\delta = 0, 1$

$$V_\alpha(i,\delta) = \min_{\phi(i,\delta)\in[0,1]} \{c((i,\delta),\phi(i,\delta)) + p_\alpha(i-1|i,\phi(i,\delta))V_\alpha(i-1,\phi(i,\delta)) \\ + p_\alpha(i+1|i,\phi(i,\delta))V_\alpha(i+1,\phi(i,\delta))\}. \tag{1.17}$$

Proof (i) By definition, $V_\alpha(i,\delta) \le V_\alpha^\pi(i,\delta)$ for any policy π. Consider the policy π that never turns the servers off and immediately turns them on at time 0 if the system is off at time 0. Then $V_\alpha^\pi(i,0) = s_1 + V_\alpha^\pi(i,1)$ or, equivalently, $V_\alpha^\pi(i,\delta) = (1-\delta)s_1 + V_\alpha^\pi(i,\delta)$. Observe that

$$V_\alpha^\pi(0,1) = E\left[\int_0^\infty e^{-\alpha t}\left(hX_0(t)+c\right)dt\right] = hE\left[\int_0^\infty e^{-\alpha t}X_0(t)dt\right] + \frac{c}{\alpha} = \frac{h\lambda}{\alpha(\mu+\alpha)} + \frac{c}{\alpha}, \tag{1.18}$$

where $X_0(t)$ is the number of busy servers at time t if at time 0 the system is empty. The last equality in (1.18) holds because, according to [46, Page 70], $X_0(t)$ has a Poisson distribution with the mean $\lambda\int_0^t e^{-\mu t}dt = \frac{\lambda}{\mu}(1-e^{-\mu t})$. Thus,

$$E\left[\int_0^\infty e^{-\alpha t}X_0(t)dt\right] = \int_0^\infty e^{-\alpha t}\frac{\lambda}{\mu}(1-e^{-\mu t})\,dt = \frac{\lambda}{\alpha(\mu+\alpha)}. \tag{1.19}$$

Also observe that

$$V_\alpha^\pi(i,1) = H_\alpha(i) + V_\alpha^\pi(0,1) = iH_\alpha(1) + V_\alpha^\pi(0,1),$$

where $H_\alpha(i)$ is the expected total discounted holding cost to serve i customers that are in the system at time 0. Since the service times are exponential, $H_\alpha(1) = E\left[\int_0^\xi e^{-\alpha t}hdt\right] = \frac{1}{\mu+\alpha}$, where $\xi \sim \exp(\mu)$. Thus,

$$V_\alpha^\pi(i,\delta) = (1-\delta)s_1 + \frac{hi}{\mu+\alpha} + \frac{h\lambda}{\alpha(\mu+\alpha)} + \frac{c}{\alpha}. \tag{1.20}$$

(ii)We rewrite the objective function V_α^π in (1.2) to exclude the values of t_n and replace $X(t)$ and $\delta(t)$ with X_n and δ_n respectively. Define $\xi_n = t_{n+1} - t_n$, $n = 0, 1, \ldots$. Conditioning on any sequence $\tilde{h} = X_0, \delta_0, a_0, X_1, \delta_1, a_1, \ldots$, the random variables $\xi_0, \xi_1, \ldots$ are independent and either $\xi_n \sim \exp(\lambda)$ when $a_n = 0$ or $\xi_n \sim \exp(\Lambda_{X_n})$

when $a_n = 1$. In the either case,

$$E[e^{-\alpha\xi_n}|\tilde{h}] = \beta(X_n, a_n) \qquad (a.s.). \tag{1.21}$$

In addition, $E\left[\int_0^{\xi_n} e^{-\alpha t} dt|\tilde{h}\right] = E\left[\frac{1-e^{-\alpha\xi_n}}{\alpha}|\tilde{h}\right] = \frac{1-\beta(X_n,a_n)}{\alpha}$ $(a.s.)$. This implies

$$E\left[|a_n - \delta_n|s_{a_n} + \int_0^{\xi_n} (e^{-\alpha t} hX_n + a_n c)dt|\tilde{h}\right] = c((X_n, \delta_n)a_n) \qquad (a.s.). \tag{1.22}$$

Since $t_n = \sum_{i=0}^{n-1} \xi_i$, $t_n \to \infty$ as $n \to \infty$, and ξ_i are independent given the sequence $x_0, a_0, x_1, a_1, \ldots$, we have

$$\begin{aligned} V_\alpha^\pi(i,\delta) &= E_{(i,\delta)}^\pi\left[\sum_{n=0}^\infty \int_{t_n}^{t_{n+1}} e^{-\alpha t}\,(hX(t) + \delta(t)c)\,dt + \sum_{n=0}^\infty e^{-\alpha t_n}|a_n - \delta_n|s_{a_n}\right] \\ &= E_{(i,\delta)}^\pi\left[\sum_{n=0}^\infty \left(e^{-\alpha\sum_{k=0}^{n-1}\xi_k}\int_0^{\xi_n}(hX_n + a_n c)dt + |a_n - \delta_n|s_{a_n}\right)\right] \\ &= E_{(i,\delta)}^\pi\left[\sum_{n=0}^\infty \left(\prod_{i=0}^{n-1}\beta(X_i,a_i)\right)c((X_n,\delta_n),a_n)\right], \end{aligned} \tag{1.23}$$

where the first equality is (1.2), and the last one follows from (1.21) and (1.22).

Since neither the objective function (1.23) nor the transition probabilities depend on times t_n, we have reduced our model to the discrete-time MDP with the same state and action sets as in the original model, the one-step costs $c(x, a)$ and transition probabilities $p_\alpha^*(X'|x, a) = |a - \delta'|p_\alpha(X'|X, a)$, where $x = (X, \delta)$ and $x' = (X', \delta')$. The objective criterion is the expected total discounted reward with the discount factor $\beta(x, a)$ that depends on the state and actions. By repacing the transition probability p_α^* with the transition probability $\tilde{p}_\alpha(x'|x, a) = \beta(x, a)p_\alpha^*$ we reduce this MDP to the standard total-reward MDP with nonnegative costs C. Such MDPs are called negative and their values satisfy the optimality equation (1.11) with $p = \tilde{p}_\alpha$ and $c = C$, that can be rewritten for our problem as (1.16).

(iii) For a negative MDP with finite action sets, an optimal stationary policy always exists [51, 6, 43, 27]. In particular, a stationary policy is optimal if and only if it satisfies (1.12). This implies statement (iii). □ Because of (iii) of Theorem 1, we consider only stationary policies in the remaining part of the paper.

Define $V_\alpha^1(i,\delta)$ and $V_\alpha^0(i,\delta)$ as follows,

$$\begin{cases} V_\alpha^1(i,\delta) = (1-\delta)s_1 + \dfrac{hi+c}{\alpha+\Lambda_i} + \dfrac{\lambda}{\alpha+\Lambda_i}V_\alpha(i+1,1) + \dfrac{i\mu}{\alpha+\Lambda_i}V_\alpha(i-1,1), \\ V_\alpha^0(i,\delta) = \delta s_0 + \dfrac{hi}{\alpha+\lambda} + \dfrac{\lambda}{\alpha+\lambda}V_\alpha(i+1,0). \end{cases}$$

The following lemma follows from optimality equation (1.16).

Lemma 4.2 *The following properties hold for the function $V_\alpha(i,\delta)$.*

(a) If $V_\alpha(i,0) = V_\alpha^1(i,0)$, then $V_\alpha(i,1) = V_\alpha^1(i,1)$.

(b) If $V_\alpha(i,1) = V_\alpha^0(i,1)$, then $V_\alpha(i,0) = V_\alpha^0(i,0)$.

(c) $-s_1 \leq V_\alpha(i,1) - V_\alpha(i,0) \leq s_0$.

Proof

(a) If $V_\alpha(i,0) = V_\alpha^1(i,0)$, then $V_\alpha^1(i,0) \leq V_\alpha^0(i,0)$. Hence $V_\alpha^1(i,1) = V_\alpha^1(i,0) - s_1 \leq V_\alpha^0(i,0) + s_0 = V_\alpha^0(i,1) \Rightarrow V_\alpha(i,1) = V_\alpha^1(i,1)$.

(b) If $V_\alpha(i,1) = V_\alpha^0(i,1)$, then $V_\alpha^0(i,1) \leq V_\alpha^1(i,1)$, hence $V_\alpha^0(i,0) = V_\alpha^0(i,1) - s_0 \leq V_\alpha^1(i,1) + s_1 = V_\alpha^1(i,0) \Rightarrow V_\alpha(i,0) = V_\alpha^0(i,0)$.

(c) From (a), we have $V_\alpha(i,0) = \min\left\{s_1 + V_\alpha(i,1), V_\alpha^0(i,0)\right\} \leq s_1 + V_\alpha(i,1) \Rightarrow V_\alpha(i,0) \leq s_1 + V_\alpha(i,1)$. Similarly from (b), we have

$$\begin{aligned} &V_\alpha(i,1) = \min\left\{V_\alpha^1(i,1), s_0 + V_\alpha(i,0)\right\} \leq s_0 + V_\alpha(i,0) \\ \Rightarrow &V_\alpha(i,1) \leq s_0 + V_\alpha(i,0). \end{aligned}$$

□

1.4.2 Full Service Policy

In subsection 1.4.1, we defined the MDP with X, A, $A(i,\delta) = A$. The class of the policies that never turns the running server off is the class of all policies in the MDP with X, A, $A(i,0) = A$ and $A(i,1) = \{1\}$. This is a sub-model of our original model.

Define by (1.4) $U_\alpha(i,\delta)$ as the optimal total discount cost for this new MDP. From (1.15) we have

$$V_\alpha(i,\delta) \le U_\alpha(i,\delta) \le (1-\delta)s_1 + \frac{hi}{\mu+\alpha} + \frac{h\lambda}{\alpha(\mu+\alpha)} + \frac{c}{\alpha}. \tag{1.24}$$

Theorem 2 *For any $\alpha > 0$ the following statements hold:*

(i) For all $i = 0, 1, \dots$

$$U_\alpha(i,1) = \frac{hi}{\mu+\alpha} + \frac{h\lambda}{\alpha(\mu+\alpha)} + \frac{c}{\alpha}. \tag{1.25}$$

(ii) For all $i = 0, 1, \dots$, the value function $U_\alpha(i,0)$ satisfies the optimality equation

$$U_\alpha(i,0) = \min\left\{ s_1 + \frac{hi+c}{\alpha+\Lambda_i} + \frac{\lambda}{\alpha+\Lambda_i}U_\alpha(i+1,1) + \frac{i\mu}{\alpha+\Lambda_i}U_\alpha(i-1,1), \right.$$
$$\left. \frac{hi}{\alpha+\lambda} + \frac{\lambda}{\alpha+\lambda}U_\alpha(i+1,0) \right\}. \tag{1.26}$$

Proof

(i) Let π be the policy that never turns the running system off. $U_\alpha(i,1) = V_\alpha^\pi(i,1) = \dfrac{hi}{\mu+\alpha} + \dfrac{h\lambda}{\alpha(\mu+\alpha)} + \dfrac{c}{\alpha}$, where the second equality is from (1.20).

(ii) Since $U_\alpha(i,0)$ is the optimal discount cost for the sub-model of the original MDP, it satisfies the optimality equation of the original MDP. Thus, (1.26) follows from (1.16). □

Definition 1 *For an integer $n \ge 0$, a policy is called n-full service if it never turns the running sever off and turns the inactive server on if and only if there are n or more customers in the system. In particular, the 0-full service policy turns on the server at time 0, if it is off, and always keeps it on. A policy is called full service if and only if it is n-full service for some $n \ge 0$.*

The following theorem implies that an n-full service policy is discount-optimal within the class of policies that never turn the running system off.

Theorem 3 *A policy ϕ is discount optimal within the class of the policies that never turns off the server if and only if for all $i = 0, 1, \ldots$,*

$$\phi(i,0) = \begin{cases} 1, & \text{if } i > A(\alpha), \\ 0, & \text{if } i < A(\alpha), \end{cases}$$

where

$$A(\alpha) = \frac{(\mu+\alpha)(c+\alpha s_1)}{h\mu}. \tag{1.27}$$

Before proving Theorem 3, we introduce the definition of passive policies and some lemmas.

Definition 2 *The policy φ with $\varphi(i,\delta) = \delta$ for all $i = 0, 1, \ldots$ and δ is called passive.*

Lemma 4.3 *For any $\alpha > 0$, the passive policy φ is not optimal within the class of policies that never turn off the running system. Furthermore, $V_\alpha^\varphi(i,0) > U_\alpha(i,0)$ for all $i = 0, 1, \ldots$.*

Proof For the passive policy φ

$$V_\alpha^\varphi(i,0) = \sum_{k=0}^{\infty} \left(\frac{\lambda}{\lambda+\alpha}\right)^k \frac{h(i+k)}{\lambda+\alpha} = \frac{hi}{\alpha} + \frac{h\lambda}{\alpha^2}. \tag{1.28}$$

For the policy ϕ that always runs the server

$$V_\alpha^\phi(i,0) = s_1 + \frac{hN}{\mu+\alpha} + \frac{h\lambda}{\alpha(\lambda+\alpha)} + \frac{c}{\alpha}$$

Thus,

$$\begin{aligned} V_\alpha^\varphi(i,0) - V_\alpha^\phi(i,0) &= \left(\frac{hi}{\alpha} + \frac{h\lambda}{\alpha^2}\right) - \left(s_1 + \frac{hi}{\mu+\alpha} + \frac{h\lambda}{\alpha(\lambda+\alpha)} + \frac{c}{\alpha}\right) \\ &= \frac{hi\mu}{\alpha(\lambda+\alpha)} + \frac{h\lambda\mu}{\alpha^2(\mu+\alpha)} - s_1 - \frac{c}{\alpha} > 0, \end{aligned}$$

when $i \geq i^*$, where the value of i^* can be easily computed. Let the initial state be $(i,0)$ when $i < i^*$. Consider a policy π that keeps the server off in states $(j,0)$, $j < i^*$ and switches to an optimal policy when the number of customers in the system reaches i^*. Then $V_\alpha^\varphi(i,0) > V_\alpha^\pi(i,0) \geq U_\alpha(i,0)$, where the first inequality holds because the process hits the state $(i^*, 0)$ with a positive probability. In fact, the process hits this state with probability 1. □

Lemma 4.4 *Let ψ be the policy that turns the system on at time 0 and keeps it on forever, and π be the policy that waits for one arrival and then turns the system on and keeps it on forever. Then*

$$\begin{cases} V_\alpha^\pi(i,0) > V_\alpha^\psi(i,0), & \text{if } i > A(\alpha), \\ V_\alpha^\pi(i,0) < V_\alpha^\psi(i,0), & \text{if } i < A(\alpha), \\ V_\alpha^\pi(i,0) = V_\alpha^\psi(i,0), & \text{if } i = A(\alpha); \end{cases}$$

where $A(\alpha)$ is as in (1.27).

Proof

$$\begin{aligned}
&V_\alpha^\pi(i,0) - V_\alpha^\psi(i,0) \\
=&\left(\frac{hi}{\lambda+\alpha} + \frac{\lambda}{\lambda+\alpha}(s_1 + U_\alpha(i+1,1))\right) - (s_1 + U_\alpha(i,1)) \\
=&\left[\frac{hi}{\lambda+\alpha} + \frac{\lambda}{\lambda+\alpha}\left(s_1 + \frac{h(i+1)}{\mu+\alpha} + \frac{h\lambda}{\alpha(\mu+\alpha)} + \frac{c}{\alpha}\right)\right] - \left[s_1 + \frac{hi}{\mu+\alpha} + \frac{h\lambda}{\alpha(\mu+\alpha)} + \frac{c}{\alpha}\right] \\
=&\frac{hi}{\lambda+\alpha}\frac{\mu}{\mu+\alpha} - \frac{\alpha}{\lambda+\alpha}\left(s_1 + \frac{c}{\alpha}\right) = \frac{h\mu}{(\lambda+\alpha)(\mu+\alpha)}(i - A(\alpha)),
\end{aligned} \tag{1.29}$$

where the first equality holds by (1.26), the second equality holds by (1.25). □

Proof Let ϕ be a stationary optimal policy within the class of the policies that never turn off the running system. Let ψ be the policy that turns the system on at time 0 and keeps it on forever, and π be the policy that waits for one arrival and then turns the system on and keeps it on forever. By (1.26),

$$V_\alpha^\phi(i,0) = \min\left\{s_1 + U_\alpha(i,1), \frac{hi}{\lambda+\alpha} + \frac{\lambda}{\lambda+\alpha}U_\alpha(i+1,0)\right\}. \tag{1.30}$$

We show that if $i > A(\alpha)$, then $\phi(i,0) = 1$. Indeed, let $\phi(i,0) = 0$ for some $i > A(\alpha)$. By Lemma 4.3, $\phi(j,0) = 1$ for some $j > i$. Thus, there exists an $i^* \geq i$ such that $\phi(i^*,0) = 0$ and $\phi(i^*+1,0) = 1$. This implies that $V_\alpha^\psi(i^*,0) \geq V_\alpha^\pi(i^*,0)$, where $i^* > A(\alpha)$. By Lemma 4.4, this is a contradiction. Thus $\phi(i,0) = 1$ for all $i > A(\alpha)$.

If $i < A(\alpha)$, then Lemma 4.4 implies $V_\alpha^\pi(i,0) < V_\alpha^\psi(i,0)$. Thus $\phi(i,0) = 0$ for all $i < A(\alpha)$.

Let $A(\alpha)$ be an integer and $i = A(\alpha)$. In this case, Lemma 4.4 implies $V_\alpha^\psi(i,0) = V_\alpha^\pi(i,0)$. From (1.26), $V_\alpha^\psi(i,0) = V_\alpha^\pi(i,0) = U_\alpha(i,0) = \min\left\{U_\alpha^\psi(i,0), U_\alpha^\pi(i,1)\right\}$. Thus $\phi(i,0) = 1$ or $\phi(i,0) = 0$. □

Corollary 1 *Let*

$$n_\alpha = \lceil A(\alpha) \rceil, \tag{1.31}$$

where $A(\alpha)$ is as in (1.27). Then

$$U_\alpha(i,0) = \begin{cases} \sum_{k=0}^{n_\alpha - i - 1} \left(\frac{\lambda}{\lambda+\alpha}\right)^k \frac{h(i+k)}{\lambda+\alpha} + \left(\frac{\lambda}{\lambda+\alpha}\right)^{n_\alpha - i} \left(s_1 + \frac{hn_\alpha}{\mu+\alpha} + \frac{h\lambda}{\alpha(\mu+\alpha)} + \frac{c}{\alpha}\right), & \text{if } i < n_\alpha, \\ s_1 + \frac{hi}{\mu+\alpha} + \frac{h\lambda}{\alpha(\mu+\alpha)} + \frac{c}{\alpha}, & \text{if } i \geq n_\alpha, \end{cases} \tag{1.32}$$

Proof Theorem 3 implies that n_α-full service policy is discount-optimal within the class of policies that never turn off the running system, where n_α is as in (1.31). □

1.4.3 Reduction to Finite State Space and Existence of Blackwell Optimal Policies

In this section, we explore the existence of Blackwell optimal [36, Chapter 8, Page 233] policy. Define

$$N_\alpha^* = \min\{i \geq 0 : V_\alpha^1(j,0) \leq V_\alpha^0(j,0), \text{ for all } j \geq i\}. \tag{1.33}$$

The following lemma implies that N_α^* is well defined.

Lemma 4.5 $N_\alpha^* \leq n_\alpha$ *for all* $\alpha > 0$.

Proof Let φ be the policy that turns on the system at time 0 and keeps it run forever. Then,

$$V_\alpha^\varphi(n_\alpha,0) = s_1 + V_\alpha^\varphi(n_\alpha,1) \leq s_1 + U_\alpha(n_\alpha,1).$$

Let π be a stationary optimal policy and τ be the time that the system first hits state $(n_\alpha,1)$ under policy π, given the initial state is $(n_\alpha,0)$. Then,

$$V_\alpha^\pi(n_\alpha,0) = R^\pi + E[e^{\alpha\tau}]V_\alpha^\pi(n_\alpha,1),$$

where R^π is the total expected cost during $[0,\tau]$. We have

$$s_1 + U_\alpha(n_\alpha,1) \leq R^\pi + E[e^{\alpha\tau}]U_\alpha(n_\alpha,1) \Rightarrow (1 - E[e^{\alpha\tau}])\,U_\alpha(n_\alpha,1) \leq R^\pi - s_1.$$

Since $V_\alpha^\pi(n_\alpha,1) \leq U_\alpha(n_\alpha,1)$, we have

$$\begin{aligned}
&(1 - E[e^{\alpha\tau}])\,V_\alpha^\pi(n_\alpha,1) \leq R^\pi - s_1 \Rightarrow s_1 + V_\alpha^\pi(n_\alpha,1) \leq R^\pi + E[e^{\alpha\tau}]V_\alpha^\pi(n_\alpha,1)\\
\Rightarrow &V_\alpha^1(n_\alpha,0) = s_1 + V_\alpha^\pi(n_\alpha,1) \leq V_\alpha^\pi(n_\alpha,0).
\end{aligned}$$

Since we also have $V_\alpha^\pi(n_\alpha,0) = \min\left\{V_\alpha^0(n_\alpha,0), V_\alpha^1(n_\alpha,0)\right\}$, thus

$$V_\alpha^1(n_\alpha,0) \leq V_\alpha^0(n_\alpha,0) \Rightarrow N_\alpha^* \leq n_\alpha.$$

□

From Lemma 4.5, N_α^* is bounded from above by n_α for each α. Define an SMDP with finite state space $S' = \{0,1,\ldots,n_\alpha\} \times \{0,1\}$.The state of this SMDP at the decision epoch n is $x_n = (X_n,\delta_n) \in S'$. The action set $A = \{0,1\}$ is the same as the original CTMDP. The time until the next decision epoch is the same as the original CTMDP for $X_n = 0,1,\ldots,n_\alpha - 1$ and $\delta_n = 0,1$. When at state $(n_\alpha,1)$, let τ a random variable that represents the first time the system returns to $(n_\alpha,1)$ before transition to $(n_\alpha - 1,1)$. The transition probabilities $\tilde{p}(j|i,a) = p(j|i,a)$ from (1.14) for $i,j = 0,1,\ldots,n_\alpha - 1$ and $a = 0,1$, except that $\tilde{p}(n_\alpha|n_\alpha,a) = 1$. The one step cost $\tilde{c}((i,\delta),a) = c((i,\delta),a)$ for $i = 0,1,\ldots,n_\alpha - 1$, $\delta = 0,1$ and $a = 0,1$, except $\tilde{c}((n_\alpha,1),1) = E\left[\int_0^\tau (c + hX(t))dt\right]$, and $\tilde{c}((n_\alpha,0),1) = s_1 + \tilde{c}((n_\alpha,1),1)$, where $X(t)$ is the system size at time t. Denote by $\tilde{V}_\alpha(i,\delta)$ as the optimal total discounted cost for this SMDP. Define T_i' as the time for the number of customers in the system becomes $i-1$ if at time 0 it is $i = 1, 2, \ldots$ if the system is running all the time. Let $C'_{i,\alpha}$ be the total holding and serving costs during T'_i, i.e.

$$C'_{i,\alpha} = \int_0^{T'_i} (c + hX(t))dt. \tag{1.34}$$

We show next that this SMDP is equivalent to the original CTMDP, i.e., $\tilde{V}_\alpha(i,\delta) = V_\alpha(i,\delta)$ for all $i = 0, 1, \ldots$ and $\delta = 0, 1$.

Lemma 4.6 *If $V_\alpha^\varphi(i,\delta) = V_\alpha(i,\delta)$ for $i = 0, 1, \ldots, n_\alpha$ and $\varphi(i,\delta) = 1$ for all $i > n_\alpha$, then $V_\alpha^\varphi(i,\delta) = V_\alpha(i,\delta)$ for $i = n_\alpha + 1, n_\alpha + 2, \ldots$.*

Proof For an $\alpha^* > 0$, let φ be an optimal stationary policy for the SMDP defined above. Define φ^* for the original CTMPD as

$$\varphi^* = \begin{cases} \varphi(i,\delta), & \text{if } i \le n_{\alpha^*}, \\ 1, & \text{otherwise.} \end{cases} \tag{1.35}$$

We show that for $\alpha \in (0, \alpha^*]$, $\tilde{V}_\alpha(i,\delta) = V_\alpha(i,\delta)$, for all $i = 0, 1, \ldots$ and $\delta = 0, 1$. Indeed,

$$\tilde{V}_\alpha(i,\delta) = \tilde{V}_\alpha^\varphi(i,\delta) = V_\alpha^{\varphi^*}(i,\delta) \le V_\alpha(i,\delta).$$

On the other hand, since $V_\alpha(i,\delta) = V_\alpha^1(i,\delta)$ for all $i \ge N_\alpha$, then

$$V_\alpha(i,\delta) \le V_\alpha^{\varphi^*}(i,\delta) = \tilde{V}_\alpha^\varphi(i,\delta) = \tilde{V}_\alpha(i,\delta).$$

Thus each optimal stationary policy for the reduced SMDP is also optimal for the original CTMDP. □

Theorem 4 *There exist a Blackwell optimal policy for the original model.*

Proof Consider the SMDP defined before Lemma 4.6. Since $\tilde{V}_\alpha^\phi(i,\delta) > 0$ for all $\alpha > 0$ and any for any policy π, then $\alpha = 0$ is the isolated singularity of every function $\alpha \tilde{V}_\alpha^\pi(i,\delta), \alpha > 0$. According to [17, Theorem 3], this implies that the reduced SMDP has a Blackwell optimal policy φ. Because of Lemma 4.6, the policy φ^* defined in (1.35) is Blackwell optimal for the original problem. □

1.4.4 Structure of Blackwell Optimal Policies

Definition 3 *A policy is called (M, N)-policy if there exists two integers M and N, with $0 \leq M < N < \infty$, such that at state $(i, 0)$, leave the system off if $i < N$ and turn on the system if $i \geq N$; at state $(i, 1)$, leave the system on if $i > M$ and turn off the system if $i \leq M$.*

The main result of this section is Theorem 5.

Theorem 5 *Let $n = \lim_{\alpha \to 0} n_\alpha = \lfloor c/h + 1 \rfloor$, where n_α is as in (1.31).*

(i) When $c < \frac{\lambda}{n}(s_0 + s_1) + \frac{h(n-1)}{2}$, the n-full service policy is Blackwell optimal;

(ii) When $c > \frac{\lambda}{n}(s_0 + s_1) + \frac{h(n-1)}{2}$, there exist two integers M and N, with $0 \leq M < N < \infty$, such that the (M, N)-policy is Blackwell optimal.

Consider an $\alpha^* > 0$ such that a Blackwell optimal policy is discount optimal for all $\alpha \in (0, \alpha^*]$. By Definition 3, M is the threshold that we switch off the system upon a departure, i.e.

$$M = \max\{i \geq 0 : V_\alpha^0(i, 1) \leq V_\alpha^1(i, 1)\}, \qquad \alpha \in (0, \alpha*]. \tag{1.36}$$

In view of Theorem 5 (ii), M is well defined. Let N be the threshold that we turn on the system upon an arrival,, i.e.

$$N = \min\{i \geq 0 : V_\alpha^1(i, 0) < V_\alpha^0(i, 0)\}, \qquad \alpha \in (0, \alpha*]. \tag{1.37}$$

$0 \leq N \leq N_\alpha^*$ for $\alpha \in (0, \alpha^*]$, where N does not depend on α. We first provide some lemmas before proving Theorem 5.

Lemma 4.7 *There exists an $\alpha^* > 0$ such that for all $\alpha \in (0, \alpha^*]$, $E[C'_{i,\alpha}] = \frac{h}{\mu + \alpha} + vE[T'_i] + O(\alpha)$,*

where v is as in (1.5).

Proof Let π be the policy that always run the system. Thus

$$V_\alpha^\pi(i, 1) = E[C'_{i,\alpha}] + E[e^{-\alpha T'_i}]V_\alpha^\pi(i-1, 1).$$

By (1.25) and $\alpha \in (0, \alpha^*]$,

$$\begin{aligned}
E[C'_{i,\alpha}] =& V^{\pi}_{\alpha}(i,1) - V^{\pi}_{\alpha}(i-1,1) + \left(1 - E[e^{-\alpha T'_i}]\right) V^{\pi}_{\alpha}(i-1,1) \\
=& U_{\alpha}(i,1) - U_{\alpha}(i-1,1) + \left(1 - (1 - \alpha E[T'_i] + o(\alpha))\right) V^{\pi}_{\alpha}(i-1,1) \\
=& \frac{h}{\mu+\alpha} + vE[T'_i] + O(\alpha),
\end{aligned}$$

where the second equality holds because $V^{\pi}_{\alpha}(i,1) = U_{\alpha}(i,1)$ for all $i = 0, 1, \ldots,$ and $E[e^{-\alpha T'_i}] = 1 - \alpha E[T'_i] + o(\alpha^2)$ by Taylor expansion, and the last equality holds because the average cost $\alpha V^{\pi}_{\alpha}(i-1,1) = v + O(\alpha)$ for $\alpha \in (0, \alpha^*]$ by Tauberian Theorem [53], where v is the optimal average cost as in (1.5). □

The next lemma implies the monotonicity property of $E[T'_i]$. The stochastic monotonicity for stationary recurrence times in queueing control model with a removable server is considered in [18].

Lemma 4.8 *$E[T'_i] - E[T'_{i+1}] > 0$ and is decreasing in i for $i = 1, 2, \ldots$.*

Proof Since $E[T'_i] = \dfrac{1}{i\mu - \lambda}$ for $M/M/i$ queue [52, Page 292], thus for $M/M/\infty$, $\dfrac{1}{(i+1)\mu - \lambda} < E[T'_i] < \dfrac{1}{i\mu - \lambda}$, thus $E[T'_i] - E[T'_{i+1}]$ is getting smaller as i increases. □

Lemma 4.9 *There exists some $\alpha^* > 0$ such that for $\alpha \in (0, \alpha^*]$, if $V_{\alpha}(i,1) = V^1_{\alpha}(i,1)$, then $V_{\alpha}(i+1,1) = V^1_{\alpha}(i+1,1)$.*

Proof Let i be the smallest integer such that $V_{\alpha}(i,1) = V^1_{\alpha}(i,1)$. Let φ be a stationary optimal policy with $\varphi(i,1) = 1$. Assume $\varphi(i+1,1) = 0$. By Lemma 4.5, there exists a $z \geq i+2$ such that $\varphi(y,1) = 1$ for all $y \geq z$ and $\varphi(z-1,1) = 0$. We have

$$
\begin{aligned}
V_\alpha(z-1,1) = {} & s_0 + \frac{h(z-1)}{\lambda+\alpha} + \frac{\lambda}{\lambda+\alpha}\left(s_1 + E[C'_{z,\alpha}] + E[e^{-\alpha T'_z}]V_\alpha(z-1,1)\right) \\
= & s_0 + \frac{\lambda}{\lambda+\alpha}s_1 + \frac{h(z-1)}{\lambda+\alpha} + \frac{\lambda}{\lambda+\alpha}E[C'_{z,\alpha}] \\
& + E[e^{-\alpha T'_z}]\left(\frac{\lambda}{\lambda+\alpha}s_0 + \frac{\lambda}{\lambda+\alpha}V_\alpha(z-1,0)\right) \\
\geq & s_0 + \frac{\lambda}{\lambda+\alpha}\left(s_1 + E[e^{-\alpha T'_z}]s_0\right) + \frac{h(z-1)}{\lambda+\alpha} + \frac{\lambda}{\lambda+\alpha}E[C'_{z,\alpha}] \\
& + E[e^{-\alpha T'_z}]\left(V_\alpha(z-2,0) - \frac{h(z-2)}{\lambda+\alpha}\right) \\
\geq & s_0 + \frac{\lambda}{\lambda+\alpha}\left(s_1 + E[e^{-\alpha T'_z}]s_0\right) + \frac{h(z-1)}{\lambda+\alpha} + \frac{\lambda}{\lambda+\alpha}E[C'_{z,\alpha}] \\
& + E[e^{-\alpha T'_z}]\left(V_\alpha(z-2,1) - s_0 - \frac{h(z-2)}{\lambda+\alpha}\right) \\
= & \left[E[C'_{z-1,\alpha}] + E[e^{-\alpha T'_z}]V_\alpha(z-2,1)\right] + \left[\frac{h(z-1)}{\lambda+\alpha} + \frac{\lambda}{\lambda+\alpha}(E[C'_{z,\alpha}])\right. \\
& \left. + \left(1 - \frac{\alpha}{\lambda+\alpha}E[e^{-\alpha T'_z}]\right)s_0 + \frac{\lambda}{\lambda+\alpha}s_1\right] - \left[E[C'_{z-1,\alpha}] + E[e^{-\alpha T'_z}]\frac{h(z-2)}{\lambda+\alpha}\right] \\
> & V^1_\alpha(z-1,1),
\end{aligned}
$$

where the first equality holds because $\varphi(z-1,1) = 0$ and $\varphi(z,1) = 1$, the second equality holds because of $V_\alpha(z-1,1) = s_0 + V_\alpha(z-1,0)$, the fist inequality holds because

$$
\begin{aligned}
& V_\alpha(z-2,0) \leq \frac{h(z-2)}{\lambda+\alpha} + \frac{\lambda}{\lambda+\alpha}V_\alpha(z-1,0) \\
\Rightarrow & \frac{\lambda}{\lambda+\alpha}V_\alpha(z-1,0) \geq V_\alpha(z-2,0) - \frac{h(z-2)}{\lambda+\alpha},
\end{aligned}
$$

the second inequality holds by Lemma 2.11 (c), the third equality holds by rearranging the terms such that the first parenthesis equals $V^1_\alpha(z-1,1)$, the last inequality holds because

$$
\begin{aligned}
& \frac{h(z-1)}{\lambda+\alpha} + \frac{\lambda}{\lambda+\alpha}(E[C'_{z,\alpha}]) + \left(1 - \frac{\alpha}{\lambda+\alpha}E[e^{-\alpha T'_z}]\right)s_0 + \frac{\lambda}{\lambda+\alpha}s_1 \\
> & E[C'_{z-1,\alpha}] + E[e^{-\alpha T'_z}]\frac{h(z-2)}{\lambda+\alpha}. \qquad (1.38)
\end{aligned}
$$

Inequality (1.38) holds because $\varphi(i,1) = 1$ and $\varphi(i-1,1) = \varphi(i-1,0) = 1$, we have

$$
\begin{aligned}
&V_\alpha(i,1) = V_\alpha^1(i,1) = E[C'_{i,\alpha}] + E[e^{-\alpha T'_i}]V_\alpha(i-1,1)\\
=&E[C'_{i,\alpha}] + E[e^{-\alpha T'_i}]\left(s_0 + \frac{h(i-1)}{\lambda+\alpha} + \frac{\lambda}{\lambda+\alpha}V_\alpha(i,0)\right)\\
\le&V_\alpha^0(i,1) = s_0 + \frac{hi}{\lambda+\alpha} + \frac{\lambda}{\lambda+\alpha}V_\alpha(i+1,0)\\
\le&s_0 + \frac{hi}{\lambda+\alpha} + \frac{\lambda}{\lambda+\alpha}\left(s_1 + E[C'_{i+1,\alpha}] + E[e^{-\alpha T'_{i+1}}]\left(s_0 + V_\alpha(i,0)\right)\right),\\
\Rightarrow&\frac{hi}{\lambda+\alpha} + \frac{\lambda}{\lambda+\alpha}E[C'_{i+1,\alpha}] + s_0\left(1 - \frac{\alpha}{\lambda+\alpha}E[e^{T'_{i+1}}]\right) + \frac{\lambda}{\lambda+\alpha}s_1\\
\ge&E[C'_{i,\alpha}] + E[e^{-\alpha T'_i}]\frac{h(i-1)}{\lambda+\alpha}\\
\Rightarrow&\frac{hi}{\lambda+\alpha} + \frac{h}{\mu+\alpha} + \nu E[T'_{i+1}] + s_0 + s_1 + O(\alpha) \ge \frac{h}{\mu+\alpha} + \nu E[T'_i] + \frac{h(i-1)}{\lambda+\alpha} + O(\alpha)\\
\Rightarrow&\frac{h}{\lambda+\alpha} + s_0 + s_1 + O(\alpha) \ge \nu\left(E[T'_i] - E[T'_{i+1}]\right).
\end{aligned}
$$

Since $E[T'_i] - E[T'_{i+1}] > 0$ and is decreasing in i by Lemma 4.8,

$$\frac{h}{\lambda+\alpha} + s_0 + s_1 + O(\alpha) > \nu\left(E[T'_{z-1}] - E[T'_z]\right),$$

and hence (1.38) holds. Thus this is a contradiction that $V_\alpha(z-1,1) > V_\alpha^1(z-1,1)$. □

Corollary 2 *$V_\alpha(i,1) = V_\alpha^1(i,1)$ for all $i > M$ and $V_\alpha(i,1) = V_\alpha^0(i,1)$ for all $i \le M$, where M is as in (1.36).*

Proof By definition of M, $V_\alpha(M,1) = V_\alpha^0(M,1)$ and $V_\alpha(i,1) = V_\alpha^1(i,0)$ for all $i > M$. Assume that there exists an $0 \le i \le M$ such that $V_\alpha(i,1) = V_\alpha^1(i,1)$. By Lemma 4.9, $V_\alpha(i+1,1) = V_\alpha^1(i+1,1)$ and by induction we have $V_\alpha(M,1) = V_\alpha^1(M,1)$. This is a contradiction. □

Lemma 4.10 *There exists some $\alpha^* > 0$ such that for $\alpha \in (0,\alpha^*]$, if $V_\alpha(i,0) = V_\alpha^1(i,0)$, then $V_\alpha(i+1,0) = V_\alpha^1(i+1,0)$.*

Proof Let φ be a stationary optimal policy with $\varphi(i,0) = 1$. From Lemma 2.11 (a), $\varphi(i,1) = 1$. Assume $\varphi(i+1,0) = 0$. By Lemma 4.5, there exists a $z \ge i+2$ such that $\varphi(y,0) = 1$ for all $y \geq z$ and $\varphi(z-1,0) = 0$. By Lemma 2.11 (a) and Lemma 4.9, $\varphi(j,1) = 1$ for all $j \geq i$. We have

$$\begin{aligned}
&V_\alpha(z-1,0) = V_\alpha^0(z-1,0) = \frac{h(z-1)}{\lambda+\alpha} + \frac{\lambda}{\lambda+\alpha}\left(s_1 + E[C'_{z,\alpha}] + E[e^{-\alpha T'_z}]V_\alpha(z-1,1)\right) \\
=&\frac{h(z-1)}{\lambda+\alpha} + \frac{\lambda}{\lambda+\alpha}\left[s_1 + E[C'_{z,\alpha}] + E[e^{-\alpha T'_z}]\left(E[C'_{z-1,\alpha}] + E[e^{-\alpha T'_{z-1}}]V_\alpha(z-2,1)\right)\right] \\
=&\left(s_1 + E[C'_{z-1,\alpha}] + E[e^{-\alpha T'_{z-1}}]V_\alpha(z-2,1)\right) + \frac{h(z-1)}{\lambda+\alpha} + \frac{\lambda}{\lambda+\alpha}E[C'_{z,\alpha}] \\
&- \frac{\alpha}{\lambda+\alpha}s_1 - (E[e^{-\alpha T'_{z-1}}] - \frac{\lambda}{\lambda+\alpha}E[e^{-\alpha(T'_{z-1}+T'_z)}])V_\alpha(z-2,1) \qquad (1.39) \\
>&V_\alpha^1(z-1,0),
\end{aligned}$$

where the second equality holds because $\varphi(z-1,0) = 0$ and $\varphi(z,0) = 1$, the second equality holds by rearranging the terms such that the first parenthesis equals $V_\alpha^1(z-1,0)$, the last inequality holds because at $(i,0)$, we have $V_\alpha^0(i,0) \geq V_\alpha^1(i,0)$, thus the policy π that turn on the system at $(i,0)$ and keeps it running forever is superior than the policy ϕ that waits for one more arrival and turns it on at $(i+1,0)$ and keeps it running forever. Thus by similar expansion of $V_\alpha^0(z-1,0)$ as in (1.39) and $V_\alpha^\phi(i,0) \geq V_\alpha^\pi(i,0)$, we have

$$\begin{aligned}
&\frac{hi}{\lambda+\alpha} + \frac{\lambda}{\lambda+\alpha}E[C'_{i+1,\alpha}] - \frac{\alpha}{\lambda+\alpha}s_1 - (E[e^{-\alpha T'_{z-1}}] \\
&- \frac{\lambda}{\lambda+\alpha}E[e^{-\alpha(T'_i+T'_{i+1})}])V_\alpha(i-1,1) \geq 0. \qquad (1.40)
\end{aligned}$$

By Lemma 4.7 and Taylor expansion for $E[e^{-\alpha T'_i}]$ and $E[e^{-\alpha T'_{i+1}}]$ at α, (1.40) implies

$$\begin{aligned}
&\frac{hi}{\lambda+\alpha} + (1 - \frac{\alpha}{\lambda+\alpha})\left(\frac{h}{\mu+\alpha} + vE[T'_{i+1}] + O(\alpha)\right) \\
&- (1 - \alpha E[T'_i] + o(\alpha))\left(1 - \frac{\lambda}{\lambda+\alpha}(1 - \alpha E[T'_{i+1}] + o(\alpha))\right)V_\alpha(i-1,1) \\
=&\frac{hi}{\lambda+\alpha} + (1 - O(\alpha))\left(\frac{h}{\mu+\alpha} + vE[T'_{i+1}] + O(\alpha)\right) \\
&- (1 - \alpha E[T'_i] + o(\alpha))\left(\frac{1 + \lambda E[T'_{i+1}]}{\lambda+\alpha}\right)(v + O(\alpha)) \\
=&\frac{hi - v}{\lambda+\alpha} + \frac{h}{\mu+\alpha} + O(\alpha) \geq 0.
\end{aligned}$$

Since $z-1>i$, thus

$$\frac{h(z-1)}{\lambda+\alpha}+\frac{\lambda}{\lambda+\alpha}E[C'_{z,\alpha}]-\frac{\alpha}{\lambda+\alpha}s_1-(E[e^{-\alpha T'_{z-1}}]-\frac{\lambda}{\lambda+\alpha}E[e^{-\alpha(T'_{z-1}+T'_z)}])V_\alpha(z-2,1)$$
$$=\frac{h(z-1)-v}{\lambda+\alpha}+\frac{h}{\mu+\alpha}+O(\alpha)>0.$$

This is a contradiction that $V_\alpha(z-1,0)>V_\alpha^1(z-1,0)$. □

Corollary 3 *There exists some $\alpha^*>0$ such that for $\alpha\in(0,\alpha^*]$, $V_\alpha(i,0)=V_\alpha^0(i,0)$ for all $i<N$, and $V_\alpha(i,0)=V_\alpha^1(i,0)$ for all $i\geq N$, where N is as in (1.37).*

Proof By definition of N, $V_\alpha(N,0)=V_\alpha^1(N,0)$ and $V_\alpha(i,0)=V_\alpha^0(i,0)$ for all $i<N$. From Lemma 4.10, $V_\alpha(i,0)=V_\alpha^1(i,0)$ for all $i>N$. □

Corollary 4 *There exists $\alpha^*>0$ such that for all $\alpha\in(0,\alpha^*]$, $N=N_\alpha^*$,where N_α^* is as in (1.33).*

Proof By Corollary 3, $V_\alpha(i,0)=V_\alpha^1(i,0)$ for all $i\geq N$, thus $N\geq N^*$. On the other hand, $V_\alpha(N^*,0)=V_\alpha^1(N^*,0)$, thus $N^*\geq N$. □

Corollary 5 *There exists some α^*, such that for all $\alpha\in(0,\alpha^*]$, $M<N$*

Proof Assume that $N\leq M$. Let i be such that $N\leq i\leq M$. By Corollary 3, $V_\alpha(i,1)=V_\alpha^1(i,1)$. However, by Corollary 2, $V_\alpha(i,1)=V_\alpha^0(i,1)$. This is a contradiction. □

Now we give the proof for Theorem 5.

Proof

(i) We use β_0 to denote $\dfrac{\lambda}{\lambda+\alpha}$ throughout the proof. Let ϕ be the n_α-full service policy, where n_α is as in (1.31). Let ψ be the policy that switch off the server at time 0 and then always run the server after the system size accumulated to n_α. ϕ is non-optimal if there exists some $i=0,1,\ldots,n_\alpha-1$, such that $V_\alpha^\psi(i,1)-V_\alpha^\phi(i,1)<0$. First we prove that $V_\alpha^\psi(i,1)-V_\alpha^\phi(i,1)$ is increasing in i, thus the necessary and sufficient condition such that (M, N)-policy is Blackwell optimal can be found when $V_\alpha^\psi(0,1) - V_\alpha^\phi(0,1) < 0$. By the definition of ψ and $V_\alpha^\phi(i,0) = U_\alpha(i,0)$ as in (1.32),

$$\begin{aligned}
&V_\alpha^\psi(i,1) = s_0 + V_\alpha^\phi(i,0) \\
=&s_0 + \sum_{k=0}^{n_\alpha-i-1} \beta_0^k \frac{h(i+k)}{\lambda+\alpha} + \beta_0^{n_\alpha-i}(s_1 + \frac{hn_\alpha}{\mu+\alpha} + \frac{h\lambda}{\alpha(\mu+\alpha)} + \frac{c}{\alpha}).
\end{aligned}$$

Since $V_\alpha^\phi(i,1) = U_\alpha(i,1)$ as in (1.25), thus,

$$\begin{aligned}
&\left(V_\alpha^\psi(i+1,1) - V_\alpha^\phi(i+1,1)\right) - \left(V_\alpha^\psi(i,1) - V_\alpha^\phi(i,1)\right) \\
=&\frac{h}{\alpha} - \frac{h}{\mu+\alpha} - \beta_0^{n_\alpha-i}\frac{h(n_\alpha+1)}{\lambda+\alpha} + \beta_0^{n_\alpha-i}\left(\frac{h\beta_0}{\mu+\alpha} + \frac{c+\alpha s_1}{\lambda+\alpha} + \frac{\alpha h n_\alpha}{(\lambda+\alpha)(\mu+\alpha)}\right) \\
>&\frac{h}{\alpha} - \frac{h}{\mu+\alpha} - \beta_0^{n_\alpha-i}\frac{\alpha^2 s_1 + c\alpha + h\mu}{\mu(\lambda+\alpha)} > 0,
\end{aligned}$$

where the first inequality is by substituting n_α from (1.31), and the last inequality is for $\alpha \in (0, \alpha^*]$. To compare $V_\alpha^\psi(0,1)$ and $V_\alpha^\phi(0,1)$, we have

$$\begin{aligned}
V_\alpha^\psi(0,1) &= s_0 + \sum_{k=0}^{n_\alpha-1} \beta_0^k \frac{hk}{\lambda+\alpha} + \beta_0^{n_\alpha}\left(s_1 + \frac{hn_\alpha}{\mu+\alpha} + \frac{h\lambda}{\alpha(\mu+\alpha)} + \frac{c}{\alpha}\right) \\
&= s_0 + \frac{h}{\alpha}\frac{\beta - n_\alpha\beta_0^{n_\alpha} + (n_\alpha-1)\beta_0^{n_\alpha+1}}{1-\beta_0} + \beta_0^{n_\alpha}\left(s_1 + \frac{hn_\alpha}{\mu+\alpha} + \frac{h\lambda}{\alpha(\mu+\alpha)} + \frac{c}{\alpha}\right) \\
&= s_0 + \frac{h\lambda}{\alpha^2}(1-\beta_0^{n_\alpha}) + \beta_0^{n_\alpha}\left(-\frac{hn_\alpha}{\alpha} + s_1 + \frac{hn_\alpha}{\mu+\alpha}\right) + \beta_0^{n_\alpha}\left(\frac{h\lambda}{\alpha(\mu+\alpha)} + \frac{c}{\alpha}\right) \\
&= s_0 + \frac{h\lambda}{\alpha^2} + \beta_0^{n_\alpha}\left(s_1 + \frac{c}{\alpha} - \frac{h\lambda\mu}{\alpha^2(\mu+\alpha)} - \frac{hn_\alpha\mu}{\alpha(\mu+\alpha)}\right).
\end{aligned}$$

On the other hand, from (1.25) we have

$$V_\alpha^\phi(0,1) = \frac{h\lambda}{\alpha(\mu+\alpha)} + \frac{c}{\alpha}.$$

Thus

$$\begin{aligned}
&V_\alpha^\psi(0,1) - V_\alpha^\phi(0,1) \\
=&s_0 + \beta_0^{n_\alpha} s_1 + (\frac{h\lambda}{\alpha^2} - \frac{h\lambda}{\alpha(\mu+\alpha)}) + \beta_0^{n_\alpha}\left(-\frac{h\lambda\mu}{\alpha^2(\mu+\alpha)} - \frac{hn_\alpha\mu}{\alpha(\mu+\alpha)}\right) \\
=&s_0 + \beta_0^{n_\alpha} s_1 + h\frac{1-\beta_0^{n_\alpha}}{\alpha}\frac{\mu}{\mu+\alpha}\left(\frac{\lambda}{\alpha} - \frac{n_\alpha\beta_0^{n_\alpha}}{1-\beta_0^{n_\alpha}}\right) - \frac{c(1-\beta_0^{n_\alpha})}{\alpha}.
\end{aligned} \tag{1.41}$$

Let $n = n = \lim_{\alpha\to 0} n_\alpha = \lfloor c/h + 1 \rfloor$, then

$$\lim_{\alpha\to 0}\left(V_\alpha^\psi(0,1) - V_\alpha^\phi(0,1)\right) = s_0 + s_1 + \frac{hn(n-1)}{2\lambda} - \frac{cn}{\lambda}, \tag{1.42}$$

which holds because $\lim_{\alpha\to 0}\beta_0^n = 1$, and

$$\lim_{\alpha\to 0}\frac{1-\beta_0^n}{\alpha} = \lim_{\beta_0\to 1}\frac{(1-\beta_0)\sum_{i=0}^{n-1}\beta_0^i}{\lambda\dfrac{1-\beta_0}{\beta_0}} = \lim_{\beta_0\to 1}\frac{\sum_{i=0}^{n-1}\beta_0^i}{\lambda/\beta_0} = \frac{n}{\lambda},$$

and by L'Hôpital's Rule,

$$\begin{aligned}
&\lim_{\alpha\to 0}\frac{\lambda}{\alpha} - \frac{n\beta_0^n}{1-\beta_0^n} = \lim_{\beta_0\to 1}\left(\frac{\beta_0}{1-\beta_0} - \frac{n\beta_0^n}{(1-\beta_0)\sum_{i=0}^{n-1}\beta_0^i}\right) = \lim_{\beta_0\to 1}\frac{\beta_0 - \dfrac{n\beta_0^n}{\sum_{i=0}^{n-1}\beta_0^i}}{1-\beta_0} \\
&= \lim_{\beta_0\to 1}\frac{1 - \dfrac{n^2\beta_0^{n-1}\sum_{i=0}^{n-1}\beta_0^i - n\beta_0^n\sum_{i=1}^{n-1}i\beta_0^{i-1}}{(\sum_{i=0}^{n-1}\beta_0^i)^2}}{-1} = \frac{n^3 - n\dfrac{n(n-1)}{2}}{n^2} - 1 = \frac{n-1}{2}.
\end{aligned}$$

Thus when $c < \frac{\lambda}{n}(s_0+s_1) + \frac{h(n-1)}{2}$, (1.42)$\geq 0 \Rightarrow V_\alpha^\phi < V_\alpha^\psi$ for all $\alpha \in (0,\alpha^*]$, and the n-full service policy is Blackwell optimal for the reduced finite SMDP, thus is optimal for the original CTMDP by Theorem 4.

(ii) When $c > \frac{\lambda}{n}(s_0+s_1) + \frac{h(n-1)}{2}$, (1.42)$< 0 \Rightarrow V_\alpha^\psi < V_\alpha^\phi$ for all $\alpha \in (0,\alpha^*]$. Thus the policy that turn off the running server is Blackwell optimal, i.e. $M \geq 0$. In addition, from Corollary 4 and Corollary 5, we know that $N < \infty$ exists and $M < N$. It suffices to show that $V_\alpha(i,1) = V_\alpha^1(i,1)$ when $i > M$, and $V_\alpha(i,0) = V_\alpha^1(i,0)$ when $i \geq N$, which are followed from Corollary 3 and Corollary 2. This completes the proof.

$\square$

1.5 Average Cost Criterion

The following corollary following directly from Theorem 5 provides the average optimal policy structure.

Corollary 6 *For average costs per unit time, let $n = \lfloor c/h + 1 \rfloor$. Then*

(i) When $c \le \frac{\lambda}{n}(s_0 + s_1) + \frac{h(n-1)}{2}$, any i-full service policy is average-optimal, where $i = 0, 1, \ldots$.

(ii) When $c > \frac{\lambda}{n}(s_0 + s_1) + \frac{h(n-1)}{2}$, there exist two integers M and N, with $0 \le M < N < \infty$, such that the (M, N)-policy is average-optimal.

Proof Because a Blackwell optimal policy is average optimal, thus the claims follows from Theorem 5 except the case $c = \frac{\lambda}{n}(s_0 + s_1) + \frac{h(n-1)}{2}$. Let the equality holds, and let π be any stationary policy. Let ψ_i be any i-full service policy, where $i = 0, 1, \ldots, n$. Let $s = s_0 + s_1$. We have $\frac{c}{h} \le \frac{\lambda}{n}(1+\epsilon)(s_0 + s_1) + \frac{h(n-1)}{2} = \frac{\lambda}{n}s + \frac{h(n-1)}{2}$. Since $\frac{c}{h}$ keeps the same, thus n-full service policy is optimal. Let L^π denote the average queue length under any stationary policy π. Since the average system size under any full service policy is the same, we have $v^{\psi_i} = c + hL^{\psi_i} = c + hL^{\psi_n} = v^{\psi_n} = c + h\lambda/\mu$. Thus any i-full service policy is optimal. We use $v[s]$ to denote v with total switching cost s. Then,

$$v^{\psi_i}[s] \le v^{\psi_i}[(1+\epsilon)s] \le v^\pi[(1+\epsilon)s] \le (1+\epsilon)v^\pi[s] \Rightarrow v^{\psi_i}[s] \le v^\pi[s], \text{ for any } \epsilon,$$

where the second inequality holds because of the optimality of ψ_i, and the third inequality holds because $(1+\epsilon)v^\pi[s]$ is the average cost with holding rate $(1+\epsilon)h$, serving rate $(1+\epsilon)c$ and switching costs $(1+\epsilon)s$. Thus ψ_i is average optimal when $c = \frac{\lambda}{n}(s_0 + s_1) + \frac{h(n-1)}{2}$. $\square$

1.6 Numerical Results and A Sufficient Condition for $M > 0$

We analyze the underlying Markov chain induced by (M, N)-policy. The states (i, δ) where $i < M$, if exists, are transient and the states (i, δ) where $i \geq M$ are recurrent. Let $P_{i,\delta}$ be the stationary probability for state (i, δ). Since $P_{M,1} = 0$, we use P_M instead of $P_{M,0}$ without confusion. $P_{i,\delta}$ satisfies the following equations:

$$
\begin{aligned}
&\lambda P_M = (M+1)\mu P_{M+1,1}, \\
&\lambda P_{M+i,0} = \lambda P_{M+1+i,0}, \ i = 0, 1, \ldots, N-M-2, \\
&(\lambda + (M+1)\mu) P_{M+1,1} = (M+2)\mu P_{M+2,1}, \\
&(\lambda + (M+i)\mu) P_{M+i,1} = \lambda P_{M-1+i,1} + (M+1+i)\mu P_{M+1+i,1}, \ i = 2, \ldots, N-M-1 \\
&(\lambda + N\mu) P_N = \lambda (P_{N-1,0} + P_{N-1,1}) + (N+1)\mu P_{N+1}, \\
&(\lambda + (N+i)\mu) P_{N+i} = \lambda P_{N-1+i} + (N+1+i)\mu P_{N+1+i}, \ i = 1, 2, \ldots. \\
\Rightarrow &P_M = P_{M+1,0} = P_{M+2,0} = \ldots = P_{N-1,0} \\
&P_{M+i,1} = \sum_{j=1}^{i} \frac{\Gamma(M+i-j)\rho^j}{\Gamma(M+i)} P_M, \ \text{for } i = 1, \ldots, N-M, \\
&P_{N+i,1} = \frac{\Gamma(N)\rho^i}{\Gamma(N+i)} P_{N,1} = \sum_{j=1}^{N-M} \frac{\Gamma(N-j)\rho^{i+j}}{\Gamma(N+i)} P_M, \ \text{for } i = 1, 2, \ldots.
\end{aligned}
$$

Since $\sum_{i=1}^{\infty} (P_{i,1} + P_{i,0}) = 1$, we have

$$
(N-M)P_M + \sum_{i=1}^{N-M} \sum_{j=1}^{i} \frac{\Gamma(M+i-j)\rho^j}{\Gamma(M+i)} P_M + \sum_{i=1}^{\infty} \sum_{j=1}^{N-M} \frac{\Gamma(N-j)\rho^{i+j}}{\Gamma(N+i)} P_M = 1. \quad (1.43)
$$

Remember that $M + 1$ is the smallest queue length that the server is kept on when it has been on, and N is the smallest queue length that the server is switched on when it has been off. We can see that when $N = M + 1$, the above equation becomes

$$
\sum_{i=0}^{\infty} \frac{\Gamma(N-1)\rho^i}{\Gamma(N-1+i)} P_M = 1,
$$

where $\rho = \lambda/\mu$. This is the equation for the birth-and-death process truncated at M. Let L be the average queue length under (M, N)-policy, thus

$$L = P_M \sum_{i=M}^{N-1} i + \sum_{i=0}^{\infty} P_{M+i,1}(M + i). \tag{1.44}$$

The busy period starting with i customers in the system for $M/M/\infty$ [31] is

$$\begin{aligned} t_i =& \frac{1}{\lambda}(e^{\rho} - 1)(1 + \frac{1}{\rho} + \frac{2!}{\rho^2} + \ldots + \frac{(i-1)!}{\rho^{i-1}}) - \frac{1}{1!\lambda}(1 + \frac{2!}{\rho} + \ldots + \frac{(i-1)!}{\rho^{i-2}}) \\ & - \frac{1}{2!\lambda}(2! + \frac{3!}{\rho} + \ldots + \frac{(i-1)!}{\rho^{i-3}}) - \ldots - \frac{1}{(i-1)!\lambda}(i-1)!, \end{aligned} \tag{1.45}$$

which can be further reduced to

$$t_i = \frac{1}{\lambda}(e^{\rho} - 1 + \sum_{k=1}^{i-1} \sum_{j=k+1}^{\infty} \frac{\rho^{j-k}}{(k+1)(k+2)\ldots j}). \tag{1.46}$$

Since for $M/M/\infty$ queue, $L^{\phi} = \rho$, thus the long-run average cost per unit time under full service policy ϕ is

$$v^{\phi} = c + hL^{\phi} = c + h\rho. \tag{1.47}$$

This is consistent with our result that $v^{\phi} = \lim_{\alpha \to 0} \alpha V_{\alpha}^{\phi} = \lim_{\alpha \to 0} \alpha U_{\alpha} = c + h\rho$. The long-run average cost per unit time under $(0, N)$-policy π_N can be calculated as

$$v^{\pi_N} = \frac{ct_N + s_0 + s_1}{N/\lambda + t_N} + hL^{\pi_N}, \tag{1.48}$$

where L^{π_N} indicates the dependence of L on π_N. To calculate the average cost for (M, N)-policy when $M > 0$, we first need to derive the busy period denoted by t_{MN}. Recall that t_{MN} starts with N customers in the system and ends with leaving M customers behind. Applying the memoryless property of exponential distribution,

$$t_N = t_{MN} + t_M \Rightarrow t_{MN} = t_N - t_M. \tag{1.49}$$

The average cost under (M, N)-policy π_N is

$$v^{\pi_N} = \frac{ct_{NM} + s_0 + s_1}{(N - M)/\lambda + t_{NM}} + hL^{\pi_N}. \tag{1.50}$$

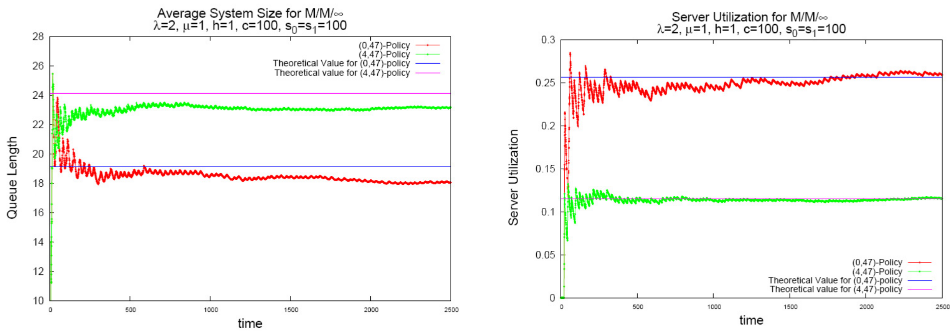


Figure 1.1: Comparing System Size and Server Utilization

We compare t_N and t_{NM} in Figure 1.2. The policy with shorter busy period, which in our case is (M, N^*)-policy, is superior when the serving cost is comparatively large.

The statistics collected from simulation are consistent with theoretical results. From Figure (1.3), we can see that the average system size is larger under (M, N^*)-policy when $M > 0$, while the server utilization is smaller, comparing with $(0, N^*)$-policy. The extra holding cost is traded off by the savings in service cost because of shorter busy period when $M > 0$.

To find the average optimal policy, we need to compare v^ϕ with v^{π_N}. One sufficient condition such that $M > 0$ is superior than $M = 0$ is when

$$v^{\pi_{MN}} < v^{\pi_N} < v^\phi. \tag{1.51}$$

Note that v^{π_N} is convex in N, so a sufficient condition for (1.51) is $v^{\pi_1} < v^{\pi_N}$. We can calculate from (1.43) and (1.45) that $t_1 = e^\rho - 1$, thus

$$\begin{aligned} v^{\pi_1} &= \frac{ct_1 + s_0 + s_1}{1/\lambda + t_1} + h\rho = c(1 - e^{-\rho}) + e^{-\rho}\lambda(s_0 + s_1) + h\rho < v^\phi = c + h\rho \\ \Rightarrow & c > \lambda(s_0 + s_1). \end{aligned} \tag{1.52}$$

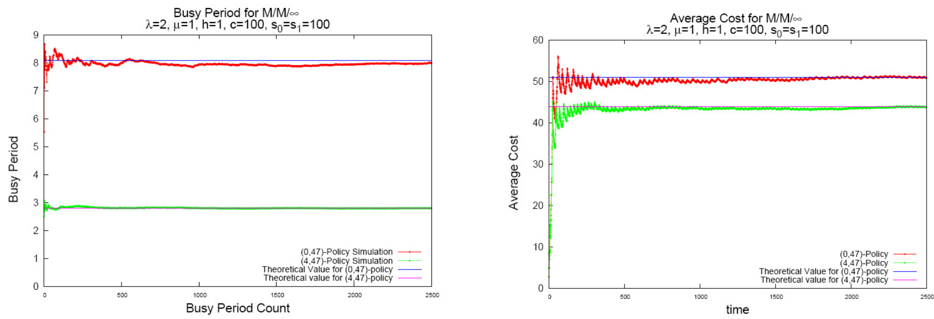


Figure 1.2: Comparing Busy Period and Average Cost

A sufficient condition for $M > 0$ is

$$
\begin{aligned}
&\frac{ct_{NM} + s_0 + s_1}{(N-M)/\lambda + t_{NM}} + hL^{\pi_N} < \frac{ct_N + s_0 + s_1}{N/\lambda + t_N} + hL^{\pi_N} \\
\Leftarrow & h\left(L^{\pi_N} - L^{\pi_N}\right) < \frac{ct_N}{N/\lambda + t_N} - \frac{c(t_N - t_M)}{(N-M)/\lambda + t_N - t_M} \\
\Leftarrow & h\left(P_M - P_0\right)\sum_{i=M}^{N} i < \frac{ct_N}{N/\lambda + t_N} - \frac{c(t_N - t_M)}{(N-M)/\lambda + t_N - t_M} \\
\Leftarrow & \frac{1}{2}h\left(P_M - P_0\right)(N+M)(N-M+1) < \frac{c\lambda\left(Nt_M - Mt_N\right)}{\left(N + \lambda t_N\right)\left(N + \lambda t_N - M - \lambda M\right)} \\
\Leftarrow & \frac{h}{c} < \frac{2\lambda\left(Nt_M - Mt_N\right)}{\left(P_M - P_0\right)\left[(N+\lambda t_N)^2 - (N+\lambda t_N)\left(M + \lambda t_M\right)\right]\left[(N+M)(N-M+1)\right]} \\
\Leftarrow & \frac{h}{c} < \frac{2\left[(Ne^{\rho} - 1) - \lambda t_N\right]}{N(N+1)\left(P_1 - P_0\right)\left[(N+\lambda t_N)^2 - e^{\rho}\left(N + \lambda t_N\right)\right]},
\end{aligned}
\tag{1.53}
$$

where the second inequality is because $(N-M)/\lambda + t_{NM} < N/\lambda + t_N$, the third inequality is because $(N-M)P_M$ is increasing in M so that

$$L^{\pi_N} - L^{\pi_N} < h\left(P_M \sum_{i=M}^{N} i - P_0 \sum_{i=0}^{N} i\right) < h\left(P_M - P_0\right)\sum_{i=M}^{N} i,$$

the last inequality is by applying (1.45),and substitution with $M = 1$, which requires tightest condition for all $M > 0$. Condition (1.53) is consistent with the intuition that the lower the linear holding rate and the higher the constant serving cost, the more likely that the system shut down and leave positive queue size behind due to the increasing service time and cost. As for the numerical example above, the right

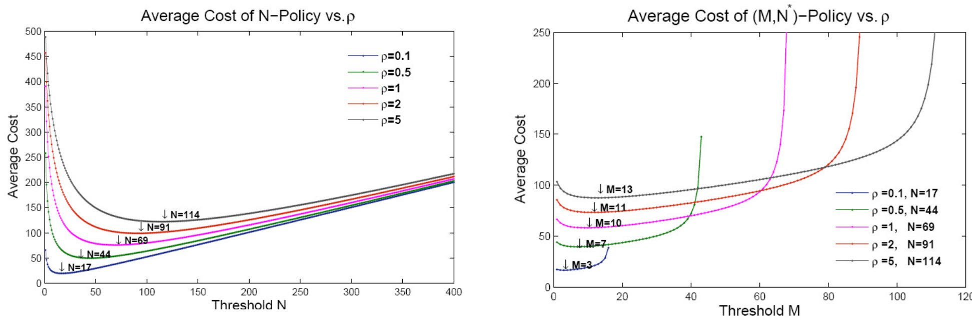


Figure 1.3: Average Cost vs N/M

hand side of (1.53) gives threshold value 0.039, and $h/c = 0.01 < 0.039$ satisfies this condition.

Chapter 2

Incentive Design for Lowest Cost Aggregate Energy Demand Reduction

2.1 Introduction

The advent of Smart Grid technologies such as digital communication devices and advanced metering infrastructures (AMI) has facilitated a better environment for sharing information and data more readily between customers and utilities in a timely fashion. This has focused attention on distributed customer demand response mechanisms, such as dynamic pricing or incentive schemes, as an effective control signal that improves the efficiency of energy usage.

Energy markets exhibit several key characteristics: demand is highly variable over both the time-axis and the price-axis, while supply/generation capacity is relatively inflexible over short horizons. Energy retailing utilities or generating companies may suffer a shortfall in committed supply during peak periods of usage, and this imbalance currently leads to high operating costs due to procurement from secondary spot market sources. In such situations, Smart Grids are envisioned to have the ability to utilize the demand flexibility of customers as a source of *virtual* generation. In effect, customers such as large commercial users, retail operations or consumer homes would be influenced to shift their demand in response to incentive signals. This method of virtual-demand generation, or demand reduction, is emerging as a key advantage for utilities on the Smart Grid: With new technology introduced in the Smart Grid, this option can be instantly initiated via smart metering signals during peak periods at almost zero start-up cost. It can also serve to hedge the utility's financial risk exposures by allowing fine control on the utility's demand-side flexibility.

Dynamic pricing offers customers time-varying electricity prices on a day-ahead or real-time basis, including critical peak pricing (CPP) programs, real-time pricing (RTP) programs, and peak time rebates (PTR) [44]. Dynamic pricing is a mature research field: According to [15], peak-load pricing dates back to Boiteux [7, 8, 9], Houthakker [37], Steiner [50], and Hirshleifer [34]. Dynamic pricing research has been extended in all three areas (CPP, RTP, PTR) in recent years. For instance, the Pacific Northwest National Laboratory conducted an extensive field demonstration in 2006 [12] on the Olympic Peninsula to show the merits of this approach in managing transmission and distribution congestion. Faruqui and Alvarado [23] explores the economic benefits of dynamic pricing compared to traditional time-of-use (TOU) rating mechanisms. In [24], they focus on California's Statewide Pricing Pilot, while Faruqui, Hledik and Tsoukalis [25] show the benefits of reducing peak load through CPP programs. Holland [35] analyzes the short-run efficiency, distributional and environmental impact of RTP, while Borenstein [10] focuses on the long-run efficiency gains from adopting RTP in a competitive electricity market. Barbose [4] reviews the revolution of RTP and provides many case studies of existing RTP programs; Smith [47] considers a linear programming model to determine RTP that minimizes the sum of supply costs and customer curtailment costs. A review of recent dynamic pricing models can be found in [26].

The incentive design problem determines rebates provided to end-users over a fixed tariff to induce a reduction in energy usage. Demand response is modeled as a version of utility or benefit functions, and aggregate demand reduction results from each customer maximizing their utility function. Faruqui and Alvarado [21, 22] apply a quadratic benefit function to design a group of incentive contracts from which customers can voluntarily choose. In this paper, we design an optimal rebate plan for the utility to realize load reduction when the need arises. Rather than providing all customers with the same group of rebate/price contracts, we design a customized, time-varying rebate plan for each customer. The energy utility aggregates the negative demand from these virtual generators and dispatches them based on their unique characteristics. This rebate rate mechanism is optimal in the sense that the utility can achieve the minimal total operating cost, which includes both rebates paid to all the customers and the cost paid on the spot market in case of shortfalls. We contrast the benefits of our model vis-a-vis the equal-rebate mechanism.

The ability of each customer to shift or reduce demand is governed by various factors such as price-demand elasticity, demand variability and flexibility over time. Aalami [1] proposes demand reduction programs for Independent System Operators (ISOs) based on a customer's price elasticity of demand and a quadratic benefit function, and also models a penalty imposed on customers who do not commit to their obligation. Instead of using a quadratic benefit function, we utilize a *load reduction* function that maps a customer's load reduction amount as a (noisy) function of the rebate rate offered. Both linear and nonlinear load reduction functions are considered. We model the case where end-users have been customers of the utility long enough for the utility to possess a reasonable forecast of each end-user's elasticity. Our model further assumes customer responses are mutually independent.

We formulate the corresponding dynamic pricing problem as a stochastic optimization problem whose solution minimizes the total virtual generation cost to an energy utility. Assuming individual customer demands are independent and normally distributed, we formulate both single-period and multi-period problem instances. This normal distributional assumption is motivated by typical market conditions under which the utility operates. Firstly, some customers may represent a conglomeration of multiple real users, e.g., large apartment buildings or commercial offices. Secondly, the utility's user group may consist of a large pool of customers with similar usage profiles, e.g., large neighbourhoods of individual households. In either case, the Central Limit Theorem implies that even if the individual users follow a non-normal energy usage distribution, the aggregate demand from such large sets of end-users approximates the form of the normal distribution. In practice, we expect the trends observed from our results to hold under more general distributional assumptions.

A gradient descent algorithm is derived to solve different formulations of the stochastic optimization problem. We derive various structural results on the optimal rebate scheme and identify a threshold that segments customers for whom no dynamic pricing adjustments should be given. These results motivate a heuristic policy for the single-period problem that segments the customers according to their willingness and likelihood to reduce load. In a multi-period instance of the problem, our results show that customers with higher load flexibility over time receive the larger dynamic pricing adjustments, and vice versa. Moreover, for the same supply shortfall, incentives offered after peak periods are higher than those before peak periods. The smart-grid demand response framework considered in this study can provide significant benefits to energy customers and utilities as well as to higher levels of the energy distribution hierarchy. In addition, the results of our demand response optimization can be used as input to or in conjunction with other smart-grid applications.

The remainder of the paper is organized as follows. Section 2.2 presents our formulation, numerical experiments and theoretical results for the single-period problem; Section 2.3 presents the same for our multi-period problem. Section 2.4 provides concluding remarks.

2.2 Single Period Problem

2.2.1 Problem Formulation

Our single-period formulation variables include (with subscripts $i = 1, \ldots, K$ representing various end-users):

K Total number of end-users;

G Total generation capacity;

d_i Demand level for user i before rebate;

D Total demand before rebate, where

$$D = \sum_{i=1}^{K} d_i; \tag{2.1}$$

$\tilde{d}_i$ Forecast demand level before rebate (represents level below which user i's usage qualifies for the rebate);

d_i^* Demand level after rebate;

r_i Rebate per unit of demand reduction;

$f_i(a_i, r_i)$ General demand reduction function;

a_i Dimensionless quantity measuring end-user's rebate-demand elasticity;

c Spot market price.

We seek to optimize the following objective:

$$\min E\left(\sum_{i=1}^{K} r_i(\tilde{d}_i - d_i^*)^+ + c[D - G - \sum_{i=1}^{K}(d_i - d_i^*)]^+\right) \tag{2.2}$$

subject to

$$d_i^* = d_i - f_i(a_i, r_i), \qquad i = 1, \ldots, K. \tag{2.3}$$

The first term in (3.17) sums up the total rebate amount that the utility pays to each end-user for load reduction from the (pre-announced) forecast level $\tilde{d}_i$, and the second part of (3.17) is the total purchasing cost from the spot market in case of load shortage. Let Obj denote the objective function (3.17).

2.2.2 Algorithm

Assume $\{d_i\}$ has a normal distribution $N(\mu_i, \sigma_i)$, where μ_i and σ_i are known to the utility from historical data. In practice, the standard deviation σ_i is a small fraction of the mean demand μ_i. Moreover, the rebate design problem is usually solved under conditions where the value of the mean demands μ_i are high. Thus, the likelihood of customer demand being negative as a result of this normal-distribution assumption is negligible. We solve the formulation (3.17) using a steepest descent method. For general f_i, the derivative has the form

$$\begin{aligned}&\frac{\partial Obj}{\partial r_i}\\ =&\Phi(\alpha_i)[f_i + r_i\frac{\partial f_i}{\partial r_i} + \tilde{d}_i - \mu_i] + \phi(\alpha_i)\sigma_i - c\Phi(\beta)\frac{\partial f_i}{\partial r_i},\end{aligned} \tag{2.4}$$

where

$$\alpha_i = \frac{\tilde{d}_i + f_i - \mu_i}{\sigma_i}, \qquad \beta = \frac{\sum_{i=1}^{K}\mu_i - \sum_{i=1}^{K} f_i - G}{\sqrt{\sum_{i=1}^{K}\sigma_i^2}},$$

$$\phi(x) = \frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}, \qquad \Phi(x) = Prob[X \le x], \quad X \sim N(0, 1).$$

We shall require that the load reduction functions f_i satisfy the condition

$$\frac{\partial f_i(a_i, r_i)}{\partial r_i}|_{r_i=0} = a_i\mu_i, i = 1, \ldots, K. \tag{2.5}$$

Two cases of $f_i(a_i, r_i)$ are considered where condition (2.5) is satisfied. In one, the load reduction linearly increases in the rebate rate r_i until it reaches an upper bound d_{max}, e.g.,

$$f_i(a_i, r_i) = \min\{\ a_i\mu_i\ \ r_i,\ \ d_{max}\}, \quad i = 1, \ldots, K. \tag{2.6}$$

The other case is where $f_i(a_i, r_i)$ is nonlinear and converges to d_{max} as

$$f_i(a_i, r_i) = d_{max_i}\left\{1 - \frac{1}{r_i a_i \dfrac{\mu_i}{d_{max_i}} + 1}\right\}, \tag{2.7}$$

for $i = 1, \ldots, K$. In practical applications, we expect the load reduction to always have an increasing marginal cost, and this is why we consider concave load reduction functions in our model. Due to this increasing marginal cost structure for both of the $f_i(a_i, r_i)$ functions considered, the total cost for the utility should be convex. This justifies the use of a gradient-descent based algorithm to obtain solutions to the optimization problem (3.17). We use the steepest descent method with the gradient updated by (2.4).

2.2.3 Numerical Examples

Numerical experiments were performed for a wide variety of parameter settings. Tables 3.1 and 3.2 provide a representative sample of the various parameter combinations for (G, σ, c, d_{max}) in our experiments. Symbol G represents the generation capacity, which was varied from 90%D (or a 10% shortfall in generation), 100%D and 110%D (or a 10% surplus in generation). All end-users' demand reduction elasticity were sampled from the same distribution $U[0, 1]$, and thus all end-users statistically exhibit the same mean demand reduction behaviour. The symbol σ_1 indicates that the volatilities σ have been generated from a unimodal distribution, while σ_2 indicates the use of a bi-modal distribution for σ. A spot market cost multiplier c of 20 was chosen, which seems to be fairly typical of peak load conditions especially during summer months. Finally, the following two cases for d_{max} are

Table 2.1: Linear Bounded Load Reduction Function

G, σ, c, d_{max}	Obj	Obj_{eq}	LR	LR_{eq}
90%D,σ_1,20,unif	48.919	56.502	78.844	71.007
100%D,σ_1,20,unif	15.751	16.808	41.820	41.365
110%D,σ_1,20,unif	1.753	2.278	8.287	6.680
90%D,σ_1,20,half	46.244	47.131	80.359	80.400
100%D,σ_1,20,half	15.675	16.635	41.966	41.967
110%D,σ_1,20,half	1.753	2.278	8.287	6.680
90%D,σ_2,20,unif	43.773	49.734	76.376	69.133
100%D,σ_2,20,unif	13.103	13.446	38.779	38.609
110%D,σ_2,20,unif	0.956	1.239	6.112	4.762
90%D,σ_2,20,half	41.769	41.820	77.580	77.602
100%D,σ_2,20,half	13.057	13.338	38.875	39.045
110%D,σ_2,20,half	0.956	1.239	6.112	4.762

considered:

$$d_{max_i} = \frac{\mu_i}{2}, \quad i = 1, \ldots, K, \tag{2.8}$$

$$d_{max_i} = U[0.1, \quad 0.6]\mu_i, \quad i = 1, \ldots, K. \tag{2.9}$$

We use $half$ and $unif$ to indicate whether the d_{max} is generated by (2.8) or (2.9), respectively.

Numerical experiments are conducted for both cases of $f_i(a_i, r_i)$ functions. The value Obj denotes the optimal objective value under the discriminatory rebate plan proposed in this paper, while Obj_{eq} denotes the optimal value obtained for the equal-rebate plan. A quick perusal of Tables 3.1 and 3.2 shows that in all cases using an incentive scheme to drive demand reduction is by itself very valuable in comparison to paying spot market prices to close any generation shortfalls (e.g., the $G = 90\%D$ cases). In addition, the discriminatory rebate scheme is able to wring out about $15 - 20\%$ more cost reduction under this shortfall condition. Recall that these results are under the assumption that all end-users are statistically similar. The benefits of discriminatory incentives are even more significant under conditions where there are statistically distinct classes of customers, as one might expect in practice; such results are omitted due to space limitations.

Table 2.2: Nonlinear Load Reduction Function

G, σ, c, d_{max}	Obj	Obj_{eq}	LR	LR_{eq}
90%D,σ_1,20,unif	89.968	92.156	68.813	68.359
100%D,σ_1,20,unif	21.975	22.744	36.860	36.867
110%D,σ_1,20,unif	1.861	2.347	7.339	5.952
90%D,σ_1,20,half	74.351	75.191	72.067	71.967
100%D,σ_1,20,half	19.727	20.495	38.448	38.509
110%D,σ_1,20,half	1.824	2.321	7.654	6.213
90%D,σ_2,20,unif	79.052	80.970	67.501	67.010
100%D,σ_2,20,unif	17.714	17.865	34.842	34.839
110%D,σ_2,20,unif	0.990	1.263	5.588	4.357
90%D,σ_2,20,half	65.670	66.200	70.415	70.215
100%D,σ_2,20,half	16.068	16.211	36.084	36.196
110%D,σ_2,20,half	0.977	1.254	5.778	4.503

2.2.4 Theoretical Analysis & Observations

$\frac{\sigma}{a\mu}$-Truncation Policy

Fig. 2.1 plots the provided rebates r versus $\sigma/(a\mu)$ for linear f in (2.6), where we can see that $r = 0$ when $\frac{\sigma}{a\mu}$ is large, and vice versa. This observation suggests that there exists a threshold for $\frac{\sigma}{a\mu}$ to "truncate" those end-users who exceed this threshold from being paid. Note that the quantity σ/μ is the (dimensionless) coefficient of variation of the end-user's demand. We call such a policy the $\frac{\sigma}{a\mu}$-*truncation* policy. Before showing how to calculate a good threshold value under this policy, we first consider some properties of the objective function.

Lemma 2.11 *If the load reduction function f is concave, then the objective function* (2.4) *is convex.*

Lemma 2.11 follows from the fact that both our chosen forms for f are concave, together with the form of the objective function Obj and its derivative in (2.4). We now provide a result on the threshold value for $\frac{\sigma}{a\mu}$ such that $r_i = 0$.

Theorem 6 *After the algorithm converges, $r_i = 0$ iff $\frac{\sigma_i}{a_i\mu_i} > c\sqrt{2\pi}\Phi(\beta)$.*

Proof Notice that when the algorithm converges, either $\frac{\partial Obj}{\partial r_i} = 0$ and $r_i > 0$, or $\frac{\partial Obj}{\partial r_i} > 0$ and $r_i = 0$. Thus, on one hand when $r_i = 0$, then $\frac{\partial Obj}{\partial r_i} > 0$. By (2.4), we have

$$\begin{aligned}
\frac{\partial Obj}{\partial r_i} &= (f_i + r_i\frac{\partial f_i}{\partial r_i})\Phi(\frac{f_i}{\sigma_i}) + \phi(\frac{f_i}{\sigma_i})\sigma_i - c\Phi(\beta)\frac{\partial f_i}{\partial r_i} > 0 \\
&\iff \phi(0)\sigma_i - c\Phi(\beta)a_i\mu_i > 0 \\
&\iff \frac{\sigma_i}{a_i\mu_i} > c\sqrt{2\pi}\Phi(\beta),
\end{aligned}$$

where the second to last inequality follows from $f_i = 0$ when $r_i = 0$ and definition (2.5) which yields $\frac{\partial f_i}{\partial r_i}|_{r_i=0} = a_i\mu_i$.

On the other hand, when $\frac{\sigma_i}{a_i\mu_i} > c\sqrt{2\pi}\Phi(\beta)$, assume for contradiction that $r_i > 0$ at convergence. Since from Lemma 2.11 the objective function is convex, then

$$\frac{\partial Obj}{\partial r_i}|_{r_i>0} > \frac{\partial Obj}{\partial r_i}|_{r_i=0} > 0,$$

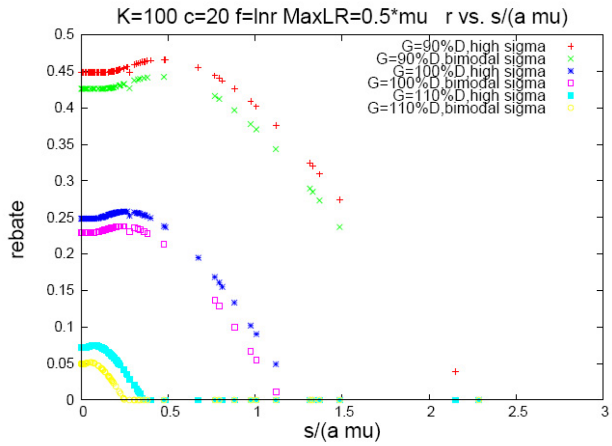
which is a contradiction to the assumption that r_i is at convergence. Hence, $\frac{\sigma_i}{a_i} > c\sqrt{2\pi}\Phi(\beta)$ is also a sufficient condition for r_i to be 0. □

Remark 1 *We can use the value of* $\frac{\sigma_i}{a_i\mu_i}$, $i = 1, \ldots, K$, *to segment end-users, and then pay no rebates to those whose* $\frac{\sigma_i}{a_i\mu_i}$ *value exceeds the threshold* $c\sqrt{2\pi}\Phi(\beta)$. *Thus the utility can exclude those end-users that are of no interest from the perspective of helping to reduce the load.*

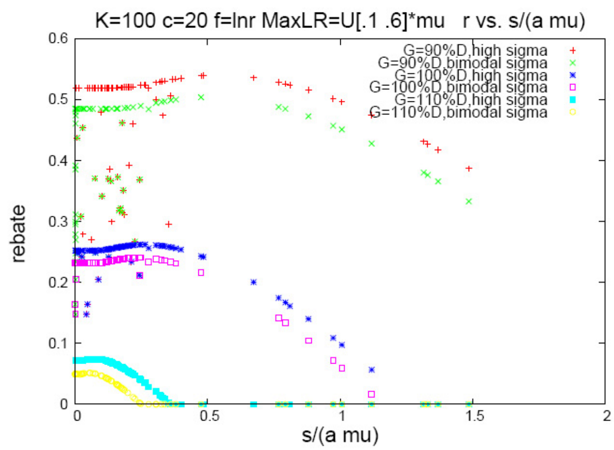
Remark 2 *From Tables 3.1 and 3.2, we can observe that load reduction for* d_{max} *uniformly distributed between* $[0.10.6]\mu$ *is usually smaller than that when* $d_{max} = 0.5\mu$. *The reason for this is that smaller* d_{max} *is a tighter bound when* d_{max} *is uniformly distributed than the fixed half-*μ *case, and thus the utility has has less total reduced load and a higher probability* ($\Phi(\beta)$) *of buying from the spot market.*

Equal-Rebate Plan Vs. Optimal-Rebate Plan

Numerical results show that our gradient based Optimal-Rebate Plan (*ORP*) performs better than the Equal Rebate Plan (*ERP*) when the maximum amount of load



(a) d_{max} half



(b) d_{max} unif

Figure 2.1: Rebate as function of $\sigma/(a\mu)$ for linear f, $K = 100$, $c = 20$.

reduction allowable d_{max} is uniformly distributed rather than d_{max} equaling half of the mean. This is because the uniform-d_{max} case produces a more heterogenous user population, which in turn implies that the utility has a higher opportunity under the *ORP*.

ORP also performs better than *ERP* when there is a larger load shortage to cover. This is due to the fact that *ORP* can find and induce more total load reduction, resulting in smaller penalty costs. When the spot market price is more expensive, *ORP* again performs better than *ERP*. When the penalty cost is cheaper, the utility has a better choice to buy the load elsewhere rather than paying up through the maximum rebate level to obtain d_{max} from each end-user. As previously noted, *ORP* provides even greater benefits over *ERP* when there are statistically distinct classes of customers, which may often arise in practice.

Linear Vs. Nonlinear Load Reduction Function

The case of the linear load reduction function always has a larger improvement in total reduced load and total cost improvement than the case of the nonlinear function. This is because for the same amount of rebate, the linear case always induces more load reduction than the nonlinear case. However, the two cases behave almost the same when the demand for load reduction is not significant. This is because the nonlinear load reduction is approximately equal to the linear one when the rebate amount is small. We believe that the strictly concave non-linear load reduction function is of more practical interest.

2.3 Multi-Period Problem

2.3.1 Multi-Period Problem Formulation

The variables and parameters of the multi-period formulation are essentially the same as those defined for our single-period formulation, together with an additional time (or period) index and the following important additions. Recall that $i = 1, \ldots, K$ indexes end-users, and let the new index $t = 1, \ldots, T$ represent various time-periods.

T Time horizon, e.g., 24 (hours);

K Total number of end-users;

G_t Total generation capacity at time t;

$d_{i,t}$ Demand level before rebate at time t if no load reduction occurs at times $1, \ldots, t-1$;

$d'_{i,t}$ *Actual* demand level before rebate at time t with positive load reduction at time

$1, \ldots, t-1$, where

$$d'_{i,t} = d_{i,t} + \sum_{j=0}^{t-1} \delta \nu^j f_{i,t}(a_{i,t-1-j}, r_{i,t-1-j}) \tag{2.10}$$

(demand reduction function $f_{i,t}$ is defined below);

δ, ν Factors that determine the amount of load shifted from one period to subsequent periods, satisfying the stability condition:

$$\frac{\delta}{1-\nu} < 1;$$

D_t Total *actual* demand before rebate at time t, where

$$D_t = \sum_{i=1}^{K} d'_{i,t}; \tag{2.11}$$

$\tilde{d}_{i,t}$ Forecast for $d'_{i,t}$, the demand level before rebate;

$d^*_{i,t}$ Demand level after rebate (where we assume that $d^*_{i,t}$ are independent over i, t);

$d^{ref}_{i,t}$ Reference demand level below which load reduction by i qualifies for rebate at time t;

$r_{i,t}$ Rebate per unit of demand reduction at time t;

$a_{i,t}$ End-user's "rebate elasticity", or willingness to reduce load at time t;

$f_{i,t}(a_{i,t}, r_{i,t})$ Demand reduction function for the user with rebate elasticity $a_{i,t}$ and offered rebate rate $r_{i,t}$;

c_t Spot market price at time t.

This formulation defines an additional set of variables $d'_{i,t}$ to capture the flexibility of end-users towards sustaining their demand reduction over time, and used to model the shifting of load from one period to subsequent periods in an effort towards responding positively to the utility's rebate signals. The objective of the multi-period formulations is given by

$$\min E\left(\frac{1}{T}\Big[\sum_{t=1}^{T}\sum_{i=1}^{K} r_{i,t}(d_{i,t}^{ref} - d_{i,t}^{*})^{+} + \sum_{t=1}^{T} c_t[D_t - G_t - \sum_{i=1}^{K}(d'_{i,t} - d_{i,t}^{*})]^{+}\Big]\right), \tag{2.12}$$

subject to

$$d_{i,t}^{*} = d'_{i,t} - f_{i,t}(a_{i,t}, r_{i,t}), i = 1, \ldots, K, t = 1, \ldots, T. \tag{2.13}$$

The first term of the objective function (2.12) represents the total rebate amount that the utility pays to all the customers during the period of time $[0, T]$ for the amount of load reduced from their reference levels. Note that the utility accounts for a customer shifting load to available rebates in previous periods by setting the reference level appropriately, so that the rebate pricing is a reasonable indication of whether the load reduction by an end-user during peak hours is valuable. The second part of (2.12) is the utility's total cost in the spot market when there is still a shortage of load after rebates are offered.

2.3.2 Algorithm for Multi-Period Problem

Let OBJ denote the objective function (2.12). Assume $\{d_{i,t}\}$ follows a normal distribution $N(\mu_{i,t}, \sigma_{i,t})$, where $\mu_{i,t}$ and $\sigma_{i,t}$ are inferred by the utility from historical data. With this assumption OBJ can be further reduced, like the single period problem, to

$$
\begin{aligned}
OBJ = \frac{1}{T} \Bigg(& \sum_{t=1}^{T} \sum_{i=1}^{K} r_{i,t} [d_{i,t}^{ref} - \mu_{i,t} \\
& - \sum_{j=0}^{t-2} \delta \nu^j f_{i,t}(a_{i,t-1-j}, r_{i,t-1-j}) \\
& + f_{i,t}(a_{i,t}, r_{i,t})] \Phi(\alpha_{i,t}) + \sum_{t=1}^{T} \sum_{i=1}^{K} r_{i,t} \sigma_{i,t} \phi(\alpha_{i,t}) \\
& + \sum_{t=1}^{T} c_t [\sum_{i=1}^{K} (\mu_{i,t} + \sum_{j=0}^{t-2} \delta \nu^j f_{i,t}(a_{i,t-1-j}, r_{i,t-1-j}) \\
& - f_{i,t}(a_{i,t}, r_{i,t})) - G_t] \Phi(\beta_t) + \sum_{t=1}^{T} c_t \sqrt{\sum_{i=1}^{K} \sigma_{i,t}^2} \phi(\beta_t) \Bigg)
\end{aligned} \tag{2.14}
$$

where

$$
\begin{aligned}
\alpha_{i,t} = & \sigma_{i,t}^{-1} (d_{i,t}^{ref} - \mu_{i,t} - \sum_{j=0}^{t-2} \delta \nu^j f_{i,t}(a_{i,t-1-j}, r_{i,t-1-j}) \\
& + f_{i,t}(a_{i,t}, r_{i,t})),
\end{aligned} \tag{2.15}
$$

$$
\begin{aligned}
\beta_t = & (\sum_{i=1}^{K} \sigma_{i,t}^2)^{-\frac{1}{2}} (\sum_{i=1}^{K} (\mu_{i,t} + \sum_{j=0}^{t-2} \delta \nu^j f_{i,t}(a_{i,t-1-j}, r_{i,t-1-j}) \\
& - f_{i,t}(a_{i,t}, r_{i,t})) - G_t),
\end{aligned} \tag{2.16}
$$

Let OBJ_s be the objective value from period s with $OBJ = \frac{1}{T} \sum_{s=1}^{T} OBJ_s$. Thus, if we choose $d_{i,t}^{ref} = \tilde{d_{i,t}}$, we have

$$
\begin{aligned}
\frac{\partial OBJ}{\partial r_{i,t}} = & \frac{1}{T} \Bigg([\alpha_{i,t} \sigma_{i,t} + r_{i,t} \frac{\partial f_{i,t}(a_{i,t}, r_{i,t})}{\partial r_{i,t}}] \Phi(\alpha_{i,t}) \\
& + \sigma_{i,t} \phi(\alpha_{i,t}) - c_t \frac{\partial f_{i,t}(a_{i,t}, r_{i,t})}{\partial r_{i,t}} \Phi_{\beta_t} \Bigg) \\
& + \frac{1}{T} \sum_{s=t+1}^{T} \Big\{ - r_{i,s} \delta \nu^{s-t-1} \frac{\partial f_{i,t}(a_{i,t}, r_{i,t})}{\partial r_{i,t}} \Phi(\alpha_{i,s}) \\
& + c_s \delta \nu^{s-t-1} \frac{\partial f_{i,t}(a_{i,t}, r_{i,t})}{\partial r_{i,t}} \Phi(\beta_s) \Big\}.
\end{aligned} \tag{2.17}
$$

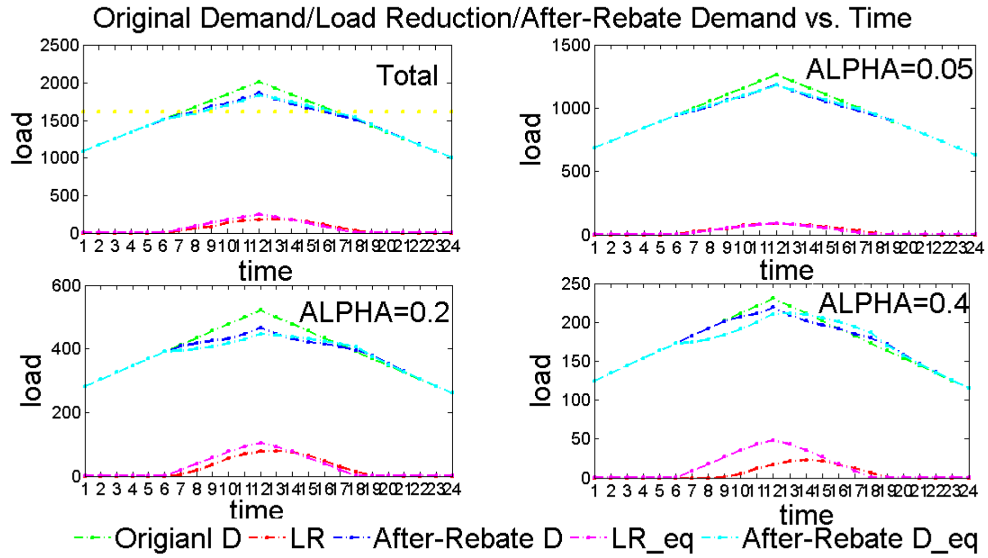


Figure 2.2: Original demand, after-rebate demand and load reduction as functions of time for $T = 24$ hours, $K = 200$, $c = 1$.

With a choice of $d_{i,t}^{ref} = d'_{i,t}$, this renders

$$\begin{aligned}\frac{\partial OBJ}{\partial r_{i,t}} &= \frac{1}{T}\Bigg([\alpha_{i,t}\sigma_{i,t} + r_{i,t}\frac{\partial f_{i,t}(a_{i,t}, r_{i,t})}{\partial r_{i,t}}]\Phi(\alpha_{i,t}) \\ &+\sigma_{i,t}\phi(\alpha_{i,t}) - c_t\frac{\partial f_{i,t}(a_{i,t}, r_{i,t})}{\partial r_{i,t}}\Phi_{\beta_t}\Bigg) \\ &+\frac{1}{T}\sum_{s=t+1}^{T} c_s\delta\nu^{s-t-1}\frac{\partial f_{i,t}(a_{i,t}, r_{i,t})}{\partial r_{i,t}}\Phi(\beta_s). \end{aligned} \tag{2.18}$$

Making the same assumptions for the functional form of the load reduction function $f_{i,t}(a_{i,t}, r_{i,t})$ as in the single-period formulation, we obtain a convex optimization problem in (2.12). Steepest descent is then employed with the gradient update obtained from the appropriate form of (2.17) or (2.18).

2.3.3 Numerical Experiments & Analysis

Fig. 2.2 plots the original demand versus after-rebate demand level as well as the load reduction amount under both the optimal rebate plan (*ORP*) and equal-rebate plan (*ERP*). We observe from these results that customers with smaller load shifting factors (δ and ν) receive higher rebates, and vice versa. In addition, for the same supply shortfall, rebates after peak periods are higher than those before peak periods, in order to reduce load shifting into peak periods.

2.4 Conclusion

We proposed a method for determining lowest cost aggregate demand reduction at multiple network levels such as distribution companies and feeder networks. This is a method of virtual-demand generation (or demand reduction) that is emerging as an option of choice for utilities on the Smart Grid. This method provides several key advantages to the utility: With the new technology in the Smart Grid, this option can be instantly initiated via smart metering signals during peak periods at almost zero start-up cost. It can also serve to hedge the utility's financial risk exposures by allowing fine control on the utility's demand-side flexibility.

We modeled the problem for both single- and multiple-period formulations, each with the objective of minimizing a utility's total operating costs. This includes incentive compensation to end-users for load reduction as well as spot market prices paid to purchase additional units to cover any remaining load shortage. Numerical experiments showed that for each single period, customers with higher rebate-demand elasticity and lower variance should be provided with higher incentive rates; and along multiple periods, customers with smaller likelihood of shifting their load and greater inclination to consume less over the entire horizon should be given higher rebates.

Chapter 3

Integration of Demand Response and Renewable Resources for Power Generation Management

3.1 Introduction

With the advent of the Smart Grid, the infrastructure for energy supply generation and transmission is experiencing a transition from the current centralized system to a decentralized one. The ability to access real-time information on supply availability and prices supported by the demand, as well as the new capability of using dynamic incentive signals to influence demand, offers unique opportunities to improve the overall efficiency of the grid in terms of both long-term supply-demand management as well as near-term dispatching of diverse generation facilities to meet current demand. The responsiveness and flexibility envisioned for the Smart Grid provides additional advantages in facing the significant new challenges [14, 40] of integrating distributed and intermittent generation capability, such as small generators and renewable energy sources (wind, solar, etc.), at a scale that current grid technology is finding hard to achieve. This is becoming more critical as renewable energy technologies are playing an increasingly important role in the portfolio mix of electricity generation.

We consider a single-period optimal dispatching problem of balancing energy transmission for a network of energy utilities across multiple buses interconnected via transmission lines. The grid's generation capability consists of traditional thermal and hydro generators that have been scheduled by the longer horizon (e.g. day-ahead) unit-commitment decision process, augmented by additional smaller capacity "peaker" thermal generators connected to a subset of the buss-es, and the goal is to find the lowest operational-cost dispatching of the peakers to satisfy demand shortfalls in the near-term, in the order of 30 minutes to an hour, subject to power transmission constraints between all pairs of connected buses. This will be familiar to the reader as a version of the well-studied Optimal Power Flow (OPF) problem (see e.g. [20] for a review) adapted for covering short-term generation shortfalls. In addition, we introduce a renewable resource of energy to the grid. We shall consider wind generation as the main renewable resource in this paper. Wind generation has negligible operational costs (in the hourly time-scale) and thus should be the first generator to be dispatched. Indeed, regulations in multiple US states require the use of wind power if it is being generated. However, the intermittent nature of output from wind turbines due to weather conditions is often seen as a potential obstacle to dispatching wind power in the classical sense. Hence, we consider the wind power as a non-dispatchable, variable generation source that is connected in an always-on state to the nearest bus.

Forecasting near-term wind availability and velocity is an imperfect science with significant variability between the forecast and the realized generation. In [19], the authors consider integrating wind power production into existing dispatch models, and analyze the uncertainty of forecast errors for wind power production and its impact on incremental reserve requirements and imbalance costs. The agency charged with controlling the smooth operation of the grid will require that this uncertainty associated with utilizing non-renewable sources be hedged against. This problem is often addressed by balancing energy provided by non-dispatchable sources (such as wind and photovoltaic units) with quickly dispatchable sources (such as small hydro and micro turbine units). [54, 39, 38, 56, 13] study this problem in differing levels of sophistication starting from individual end-users up to local utilities. In particular, a balancing approach to achieve overall dispatchability in a distributed generation network is presented in [54], which consequently converts a group of small distributed generations into a large logical generation station.

Another stream of research focuses on hedging the uncertainty in setting or adjusting transmission parameters by introducing a set of constraints that have the effect of dispatching additional capacity to hedge against the risk of a large unforeseen shortfall in total supply. The study in [11] considers an economic environmental dispatching model where wind and solar energy are both included but constrained to be no more than 30% of the total dispatched capacity. This is recognized in [32] where the authors propose a stochastic programming framework to determine the optimal procurement of interruptible load in order to minimize the risk of a shortfall over multiple periods. In [30], a set of risk constraints, in the form of *chance-constraints*, is imposed to balance risk of shortfalls due to uncertain generation against cost of provisioning corrective generation sources such a peakers. Risk-constrained OPF models often assume Normal distributions as representative of the generation uncertainty as this allows a two-parameter control on the model. (The Normal distribution is fully specified with the first two moments.) [41] generalize this with a point-estimate scheme to solve stochastic OPF problems.

Our model is broadly a "security-constrained" OPF where we consider an alternate approach of accessing the intermittent generator. Our approach differs from standard techniques in two aspects: first, the a risk constraint we impose is based on the *conditional value-at-risk* (CVaR) function, which has certain advantageous properties (refer Section 3.2). Second, in our model an alternative strategy to manage generation shortfall is by incentivizing end-users to reduce their demand in accordance with the shortfall. In [49], we provide a detailed analysis of such *demand response* techniques, and illustrate how a local utility operating at any single bus/node of the network can extract "virtual supply" (i.e., reduce demand) by providing incentives customized to the observed behaviour of each end-user in response to such incentives. Demand response enjoys the advantage of suffering from very little lag in being accessed. In the present article, we shall utilize a simpler demand response policy of providing a single rebate value for every unit of demand reduced by an end-user. The expected (stochastic) response of the end-user is modeled using functional assumptions in common with [49].

The complete set of control options modeled in our optimization problem consists of the dispatching order and the dispatching amount of thermal generators as well as the non-discriminate (equal) rebate signals sent to energy end-users at each node of the network. Further, the transmission limitations of the grid network necessitates additional decisions that maximize the utilization of the flexibility of the network. Any unmatched shortfall must be ultimately fulfilled from the spot market at a high premium (i.e., penalty). On the other hand, any excessive supply over the local demand is assumed to be sold into the spot market, thus acting as an incentive to the grid management entity to access demand response by providing in turn incentives to the end-users.

The remainder of the paper is organized as follows. Section 3.2 introduces both our model notation and formulation. Section 3.3 presents numerical experiments and analysis, followed by some concluding remarks.

3.2 Model Formulation

3.2.1 Model

An electric utility controls the dispatching of committed (regular and peaker) generation units (thermal sources) over a network of multiple local buses interconnected via transmission lines. Define:

N_g number of traditional generator buses;

N_d number of load buses;

N total number of buses.

We assume that power demand ${P^d}_i$ at each bus is normally distributed, ${P^d}_i \sim N(\mu_i, \sigma_i^2)$. This assumption is commonly used in stochastic OPF models, and in particular fits in well with the demand response model we use (refer [49] for further details). The N_g generators are *peak generators*, namely active generators that are currently unused but spinning and thus can be activated within a short time span. Let $P^{g_{min}}_i$ and $P^{g_{max}}_i$ represent the lower and upper bounds on the power outputs from generators i, $i = 1, \ldots, N_g$. Let θ_i^{min} and θ_i^{max} represent the lower and upper bounds on the voltage phase angles of bus i, $i = 1, \ldots, N$.

We assume renewable generation is connected to a single bus such that the total power generated lies within P_{RN}^{min} and P_{RN}^{max}, where the forecasts are $P_{RN} \sim$ (truncated) $N(\mu_{RN}, \sigma_{RN}^2)$. Other renewable resources share similar intermittent and volatile properties with wind and can be modeled in this manner. We assume that the utility also integrates a demand-response policy as a source of virtual generation, where M of the buses offer rebates to customers for reduced energy load. This can be implemented by allowing the utility to interact with different household electric appliances (laundry, temperature control, etc.) through price signals sent to smart meters. The customer response to these incentives is a reduced demand $P_{DR_j}(r)$ with a mean and variance depending on the rebate value r, $P_{DR_j}(r) \sim N(\mu_{DR_j}(r), \sigma^2_{DR_j}(r))$, $j = 1, \ldots, M$. In our numerical experiments we shall assume the mean value of the demand response to be linear in the rebate value:

$$\mu_{DR_j} = b_{1j} r, \qquad j = 1, \ldots, M. \tag{3.1}$$

Finally, any excess (shortfall) of supply is sold to (bought from) the spot market at a unit price c.

In what follows, we shall use the subscripts $_{RN}$ and $_{DR}$ to indicate connections with wind generation resource and demand response, respectively. We also assume that the total demand, demand response and wind power generation are mutually independent random processes.

3.2.2 Decision variables

The optimization problem is formulated with the following decision variables:

- z_i 0/1 indicator of whether traditional generator i is on or off, $i = 1, \ldots, N_g$;
- $P^g{}_i$ total real power output extracted from generator bus i, $i = 1, \ldots, N_g$;
- θ_i voltage phase angle for each bus i, $i = 1, \ldots, N$;
- r non-discriminate unit rebate price offered to all customers on the M participating buses.

Note that $\{z_i\}$ are discrete control variables, while all the other decision variables are continuous. As for the continuous control variable r, the utility is assumed to have the forecasting profiles of each type of load reduction. We formulate the problem as a single-period nonlinear mixed integer program.

3.2.3 Constraints

Bounding Constraints

The real power generation $P^g{}_i$ of each generator bus, the load demand $P^d{}_i$ of each load bus, and the voltage phase angles θ_i of each bus should lie within their minimum and maximum limits. In addition, the total generation load plus the reduced load should balance the total demand. We therefore have the bounding constraints:

$$z_i P^{gmin}_i \le P^g{}_i \le z_i P^{gmax}_i, \quad i = 1, \dots, N_g; \tag{3.2}$$

$$\theta^{min}_i \le \theta_i \le \theta^{max}_i, \quad i = 1, \dots, N; \tag{3.3}$$

$$0 \le r < c; \tag{3.4}$$

$$0 \le P_{DR} \le P^d_{DR}; \tag{3.5}$$

where the last constraint restricts the total demand-response load to be no more than the load of the hosting bus.

Network Balance Constraints

The formulation has to impose power flow balance constraints on the decision variables. For this economic dispatching problem, we use the DC power flow constraints, commonly used in optimal power flow and economic dispatch problems [42],

$$P_i(\theta) = P^g{}_i - P^d{}_i = \sum_{j=1}^{N} \frac{1}{x_{ij}}(\theta_i - \theta_j), \tag{3.6}$$

where x_{ij} is the reactance of the transmission line between bus i and bus j. The generator bus injected with wind power has a real load output of $P^g{}_i + P_{RN}$. The generator bus with demand response options has a real load output of $P^g{}_i + P_{DR}$, and the generator bus with access to the spot market has a real load output of $P^g{}_i$ plus (minus) the amount of load bought from (sold to) the market.

Power balance also has to be achieved at the system level:

$$\sum_{i=1}^{N_g} P^g{}_i + P_{RN} + P_{DR} + S = \sum_{i=1}^{N} P^d{}_i, \tag{3.7}$$

where S represents the load position in the spot market. Thus, we have $S = \sum_{i=1}^{N} P^d{}_i - \sum_{i=1}^{N_g} P^g{}_i - P_{RN} - P_{DR}$, which, if positive, represents a shortfall in supply and, if negative, represents an excess of supply sold to the spot market.

Risk Control

For a loss value $l(u; v)$, which is a cost function of decision variables u and random variables v, the β-Value-at-Risk is the β-quantile of the distribution induced on the loss function. The risk measure β- Conditional Value-at-Risk with confidence level β is defined as the expected value of the cost-based loss function $l(u; v)$ conditional on loss being beyond the $\beta-$th quantile. We denote the β-VaR and β-CVaR values as $\alpha_\beta(u)$ and $\phi_\beta(u)$, where

$$\alpha_\beta(u) \equiv \min\{L_0 \in \mathbb{R} : Prob\{l(u; v) \le L_0\} \ge \beta\}, \tag{3.8}$$

$$\phi_\beta(u) \equiv E[l(u; v) \mid l(u; v) \ge \alpha_\beta(u)]. \tag{3.9}$$

It has been noted that CVaR is a coherent risk measure [3]; moreover, it is a convex function of the decision variables. In our model, we consider CVaR as the risk measure for the power generation portfolio. Let L be a pre-specified risk (loss) level. Then the risk control constraint is formulated as $\phi_\beta(u) \le L$, which can be transformed to the following equivalent formulation under the assumption that v is normally distributed [45]:

$$\mu(u) + (\sqrt{2\pi}(1-\beta)e^{[erf^{-1}(2\beta-1)]^2})^{-1}\sigma(u) \le L, \tag{3.10}$$

$$\mu(u) = E[l(u; v)], \qquad \sigma(u) = Var(l(u; v)),$$

$$erf(z) = \frac{2}{\sqrt{\pi}} \int_0^z e^{-t^2} dt. \tag{3.11}$$

When v has a general distribution, the CVaR constraint can be approximated by sampling a collection of K paths, with $v_1, \dots, v_K$ generated from the probability distribution of v:

$$L_0 + \frac{1}{J(1-\beta)} \sum_{k=1}^{K} (l(u; v) - L_0)^+ \le L. \tag{3.12}$$

Define the cost function $l(z, P^g, r, \theta; P^d, P_{DR}(r), P_{RN})$ as a function of decision variables $z \in \mathbb{R}^{N_g}$, $P^g \in \mathbb{R}^{N_g}$ and r, and random variables $P^d \in \mathbb{R}$, $P_{DR}(r) \in \mathbb{R}^M$ and $P_{RN} \in \mathbb{R}$. In our model, the loss function (total cost) equals the sum of all the generation costs minus the total revenue from the load position exposed to the spot market (positive when selling to the market and negative when purchasing from the market). We assume the production cost f_i, $i = 1, \dots, N_g$, for those operating units to be a quadratic function of the amount dispatched [2], namely

$$f_i = a_{2i}{P^g}_i^2 + a_{1i}{P^g}_i + a_{0i}. \tag{3.13}$$

In addition, suppose the start-up cost for load reduction is zero and that this virtual generator only incurs a "generation" cost linear in the rebate r. In general there is no operational cost for wind renewables. Thus, we have the following form for the loss function in terms of the utility's total cost:

$$\begin{aligned}
& l(z, P^g, r, \theta; P^d, P_{DR}(r), P_{RN}) \\
&= \sum_{i=1}^{N_g} z_i(a_{2i}{P^g}_i^2 + a_{1i}{P^g}_i + a_{0i}) + r \sum_{j=1}^{M} P_{DR_j}(r) \\
&\quad - c(\sum_{i=1}^{N_g} z_i {P^g}_i + \sum_{j=1}^{M} P_{DR_j}(r) + P_{RN} - \sum_{i=1}^{N} Pd_i). \qquad (3.14)
\end{aligned}$$

The mean and variance of the loss induced by the random variables P^d, $P_{DR}(r)$ and P_{RN} are functions of z, P^g and r. More precisely, we have

$$\begin{aligned}
& E[l(z, P^g, r, \theta; P^d, P_{DR}(r), P_{RN})] = \mu(z, P^g, r, \theta) \\
&= \sum_{i=1}^{N_g} z_i(a_{2i}{P^g}_i^2 + a_{1i}{P^g}_i + a_{0i}) + r \sum_{j=1}^{M} \mu_{DR_j}(r) \\
&-c(\sum_{i=1}^{N_g} z_i {P^g}_i + \sum_{j=1}^{M} \mu_{DR_j}(r) + \mu_{RN} - \sum_{i=1}^{N} \mu_i); \qquad (3.15) \\
& Var[l(z, P^g, r, \theta; P^d, P_{DR}(r), P_{RN})] = \sigma^2(z, P^g, r, \theta) \\
&=(r^2 + c^2) \sum_{j=1}^{M} \sigma^2_{DR_j}(r) + c^2(\sum_{i=1}^{N} \sigma_i^2 + \sigma^2_{RN}). \qquad (3.16)
\end{aligned}$$

3.2.4 Complete Model Formulation

The objective is to minimize the total cost (loss function), which yields our final formulation:

$$\begin{aligned}
\min_{z, P^g, r, \theta} \quad & E[l(z, P^g, r, \theta; P^d, P_{DR}(r), P_{RN})], \qquad (3.17) \\
\text{s.t.} \quad & (3.2) - (3.7) \text{ and } (3.10), \\
& z_i = 0 \text{ or } 1, \qquad i = 1, \ldots, N_g. \qquad (3.18)
\end{aligned}$$

3.2.5 Algorithm

Our primary goal in this article is to illustrate the impact that the unpredictable nature of renewable generation has on the network dispatching problem, and to further show that demand response is an effective means to control this stochasticity. Towards this end, we propose to solve instances of the above nonlinear mixed integer program through a two-stage procedure: the outer stage is a listing of possible joint values of the binary variables z_i ($z \in \mathbb{R}^{2^{N_g} \times N_g}$) that indicate whether generator i is dispatched, and the inner stage where each z_i has been fixed and the formulation simplifies to a non-linear formulation that can be efficiently solved using a customized gradient descent algorithm given the expressions for the non-linear constraint (3.10).

3.3 Numerical Example & Analysis

We next consider a representative set of our numerical experiments based on the foregoing analysis, which includes the IEEE 30-bus 6-generator system topology depicted in Fig. 3.1. The parameters for the quadratic cost function a_0, a_1, a_2 and the upper/lower bounds for generator buses are provided in Table 3.1. Voltage phase bounds θ^{min} and θ^{max} for all load buses are 0° and 30°, respectively.

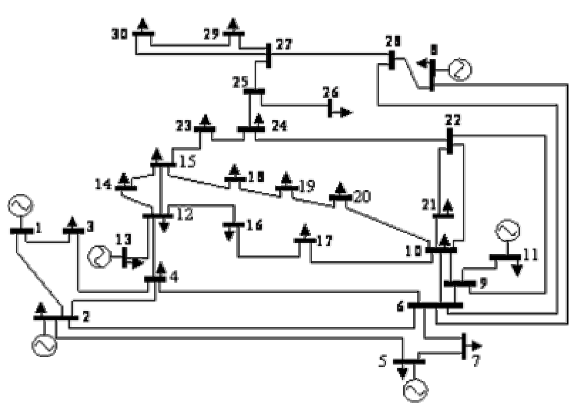


Figure 3.1: IEEE 30-Bus 6-Generator System

Fig. 3.2 shows that the total expected cost when allowing demand response (DR) is lower than the total expected cost without DR for varying spot market price assumptions. In addition, under relatively high spot market prices, the utility can

Table 3.1: Generator Unit Parameters for IEEE 30-Bus System

Bus	a_0	a_1	a_2	P^{gmax}	P^{gmin}
1	0	2.00	0.00375	200	50
2	0	1.75	0.00175	80	20
5	0	1.00	0.00625	50	15
8	0	3.25	0.0083	35	10
11	0	3.00	0.025	30	10
13	0	3.00	0.025	40	12

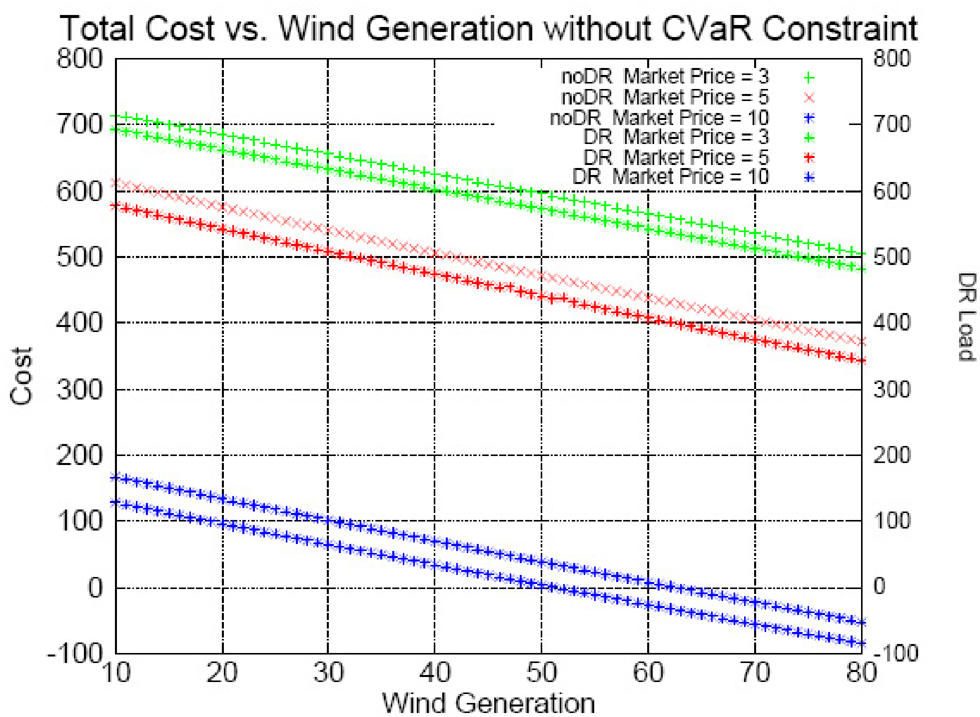


Figure 3.2: Compare Mean Loss vs. Wind with/without DR

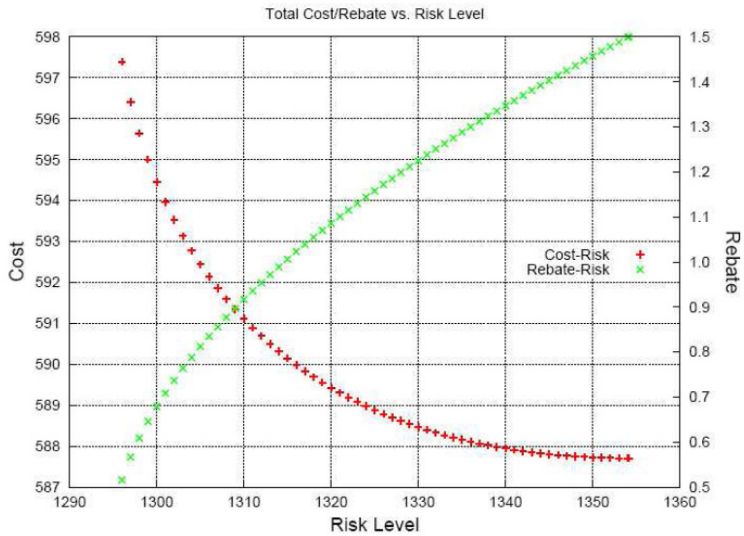
even realize additional profits through the use of DR, which in turn can lead to more efficient overall management of the energy grid. Table 3.2 compares the optimal dispatching policy, rebate price and spot market load position S (positive to buy and negative to sell), for different spot market price and wind generation assumptions, both with and without DR. From Table 3.2 we can see that when the market price is relatively low, wind injection into the system will not change the dispatching load nor the rebate price, and will only reduce the amount of load bought from the spot market. The optimal rebate price seems to be around $c/2$. When the spot market price is relatively high, the utility dispatches more generation to realize a profit from this excess load. Interestingly, more wind injection under such conditions will reduce the dispatched thermal power and rebate price, which seems to result from the power flow constraints limiting the ability of the utility to transfer the excess generated load to the spot-market entry node. The optimal rebate price in this case will be less than $c/2$. Under every condition, the total dispatched thermal generation is lower when DR is allowed, as expected.

Table 3.2: Optimal Dispatching Without CVaR Constraint

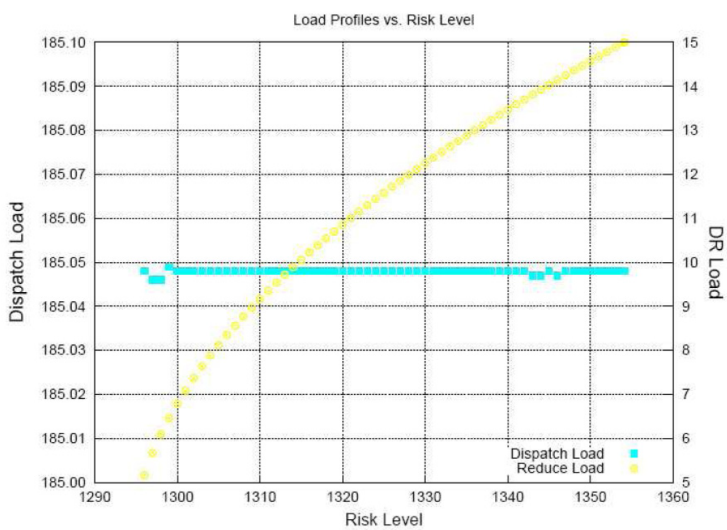
c	Wind	$P^g{}_1$	$P^g{}_2$	$P^g{}_3$	$P^g{}_4$	$P^g{}_5$	$P^g{}_6$	r	S	Cost
3	15	133.3	35.7	16.0	0	0	0	off	83.4	700.2
3	15	133.3	35.7	16.0	0	0	0	1.5	68.4	677.7
3	45	133.3	35.7	16.0	0	0	0	off	53.3	610.2
3	45	133.3	35.7	16.0	0	0	0	1.5	38.4	587.7
5	15	200	58.9	23.4	35	17.8	19.4	off	-86.1	593.8
5	15	196.4	53.5	22	33.4	14.3	16.2	1.81	-85.5	559.7
5	45	188.3	51.9	21.6	30.6	13.3	15.2	off	-82.5	488.7
5	45	178.9	50.1	21.1	27.2	12.1	14.1	1.75	-82.8	457.5
5	80	167.6	47.9	20.6	23.2	10.7	12.8	off	-82.5	372.0
5	80	158.2	46.1	20.1	19.9	10	12	1.68	-79.7	343.2
10	80	126.4	53.9	25.8	35	21.4	30.0	off	-88.9	-52.5
10	80	115.0	51.7	25.2	35	19.9	28.6	1.78	-89.8	-84.8

Fig. 3.3 plots the mean cost, rebate price and load profiles as the tolerated risk levels vary and the spot market price is kept relatively low. The dispatching policy and total dispatch load remains low for different CVaR constraints, due to the relatively low spot market price. Furthermore, the utility prefers to increase load reduction incentive prices to purchase less from the spot market instead of dispatching more generation load due to the higher cost for this extra dispatching.

Fig. 3.3 shows the optimal rebate and the minimum cost as the tolerated risk levels vary, thus providing the efficient frontier for the cost-based loss function. Fig. 3.4 plots the optimal total dispatching amount and demand response load re-



(a) Cost and Rebate against Risk Level



(b) Load Profile vs Risk Level

Figure 3.3: At Market Price=3 Wind=45

duction with respect to these risk tolerances. We see that with relatively high spot market prices, the utility generates (either directly from thermal sources or virtually via load reduction) and sells extra load to the market to make more profit. With a relatively loose CVaR constraint, the utility tends to pay more load reduction incen-

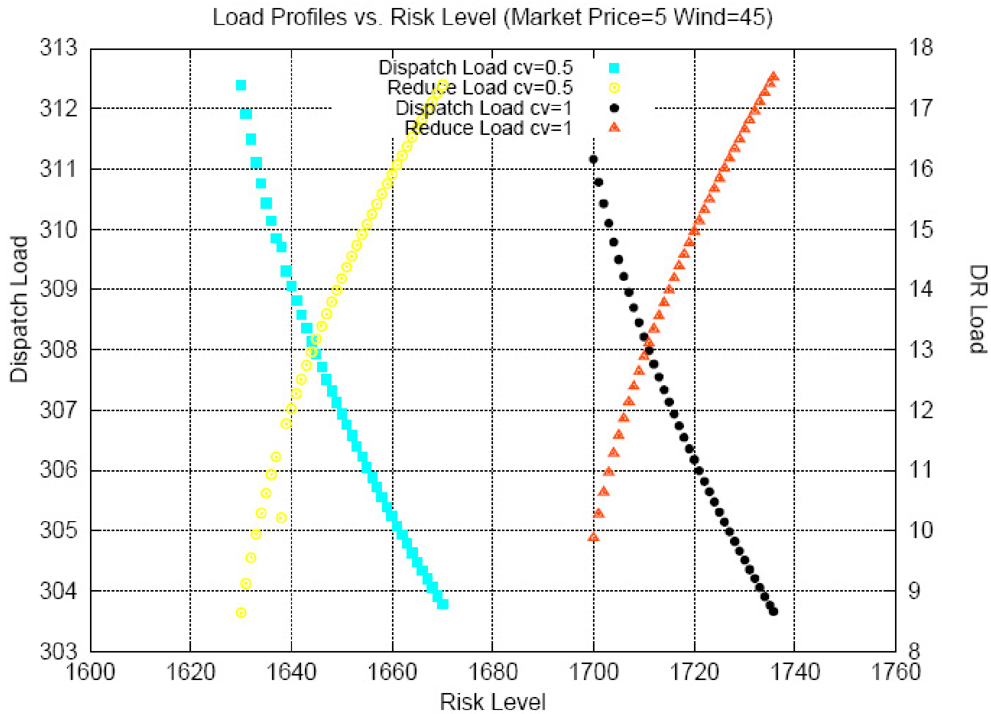


Figure 3.4: Load Profiles vs. Risk Level

tives rather than generate more energy from thermal sources.

Figs. 3.5 and 3.6 show that as the coefficient of variation (CV) for wind generation increases, the mean loss increases while the rebate price decreases under the same risk constraint. This is because with the same risk constraint but a higher proportion of risk being due to wind generation, the most efficient way to keep CVaR within its prescribed limit is to reduce the risk from demand response, which in turn is achieved with a decreasing rebate. Moreover, the changes in cost and rebate are more significant with a tighter risk constraint, which follows from the risk measure in our model being quadratic in r.

Fig. 3.7 describes the optimal cost under changing market price relations with respect to different choices for the buses that interconnect wind, DR or spot market load injections. The optimal costs tend to increase with the spot market price at first, upon reaching a peak, and then tend to decrease because the utility can realize more profits by selling load to the market when the marginal generation cost is lower than the spot market price. In addition, a lower cost is incurred when the phase angle constraints are not active for the bus accessing the spot market or demand reduc-

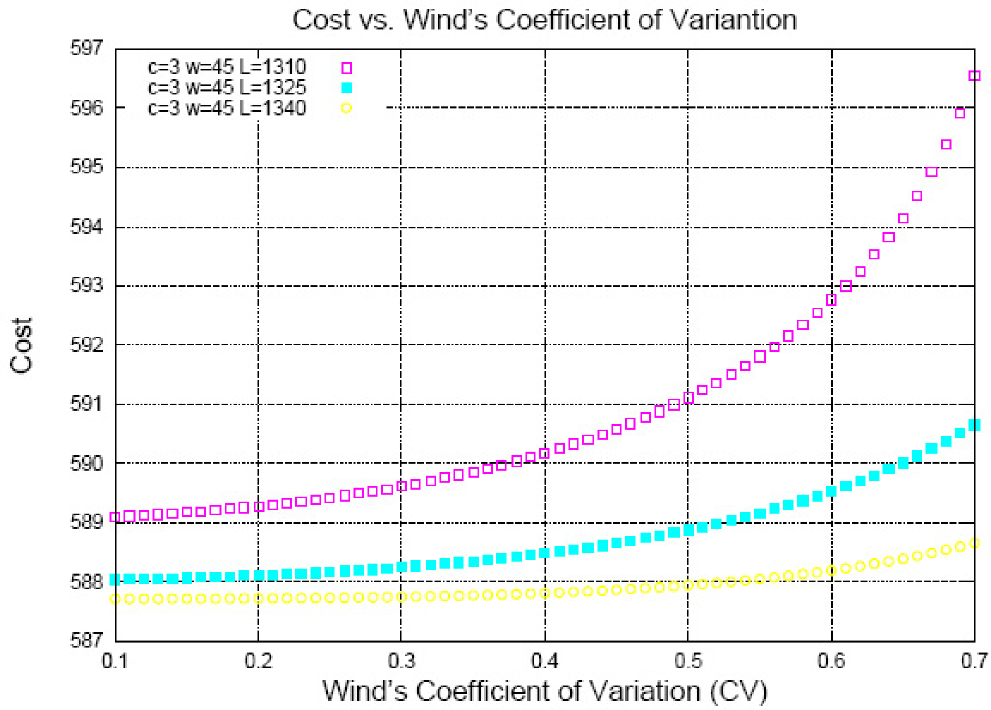


Figure 3.5: Loss vs. Wind's Coefficient of Variation

tion. Fig. 3.8 compares the optimal cost versus risk relations under different wind generation injection bus locations. Lower costs are incurred when the power flow constraints are not active, or when the bus that has wind injection is also attached to a lower marginal cost generator. These results serve to illustrate and quantify the influence that the bus connections for renewable generation and spot market access can have on the resulting optimal dispatch.

We implemented our model with a IEEE 300 bus system. The expected behaviour of the optimal policy dispatching the generation capacity according the increasing order of marginal cost is observed in this case too, as is the fact that the total expected cost is lower when integrated with the demand response program. In addition, the advantage of demand response as a virtual generation source over traditional peakers becomes more stark for this 300 bus case, where the quadratic cost of the thermal generators has a higher penalizing effect while the cost of demand response does not increase with the size of the system load. This independence from

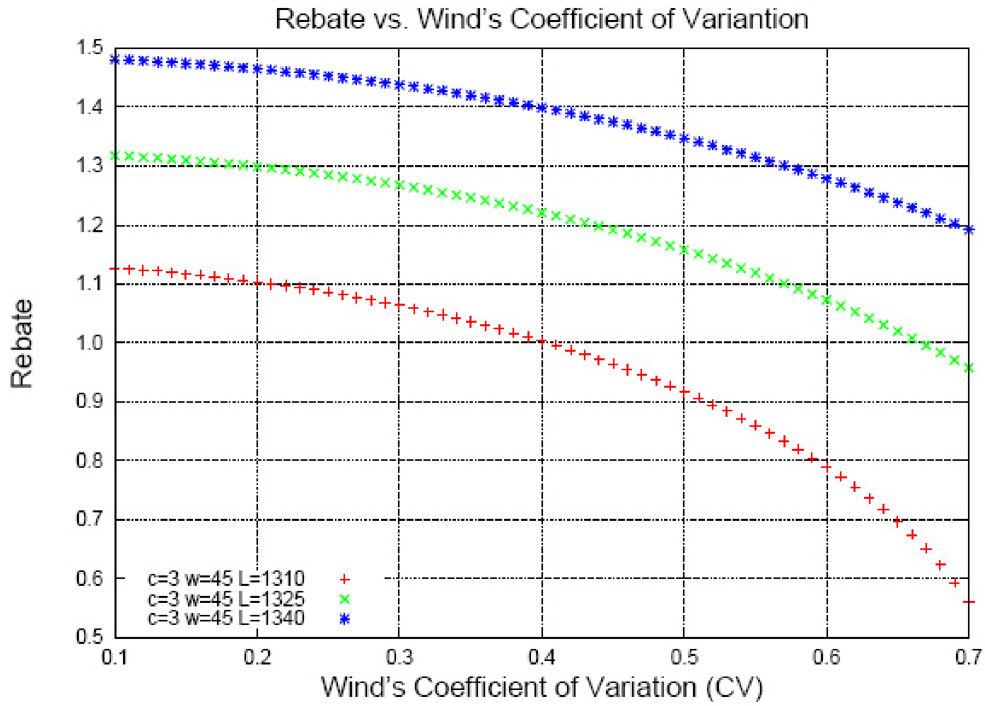


Figure 3.6: Rebate vs. Wind's Coefficient of Variation

system size as well as the almost instantaneous start-up provides strong evidence of the efficacy of demand response as an alternate shortfall reduction method.

Finally, there is an important issue regarding the feasibility of our optimization problem given the various important constraints described in Section 3.2. To elucidate the exposition, our previously described solution approach assumes the existence of a feasible solution, which was easily addressed as part of the above numerical experiments. More generally, however, our approach ensures feasibility by first considering a version of the model in Section 3.2 within the context of a robust optimization framework. Upon solving this problem, the corresponding robust optimal solution provides a lower bound on the potential benefits achievable by the utility. Of course, the utility can decide to directly employ this solution. On the other hand, we can instead exploit this solution to properly define a feasibility region for our original optimization problem and then apply a variant of our original solution approach described above taken over this feasible region. By employing this solution, the utility can further improve upon the benefits realized under our

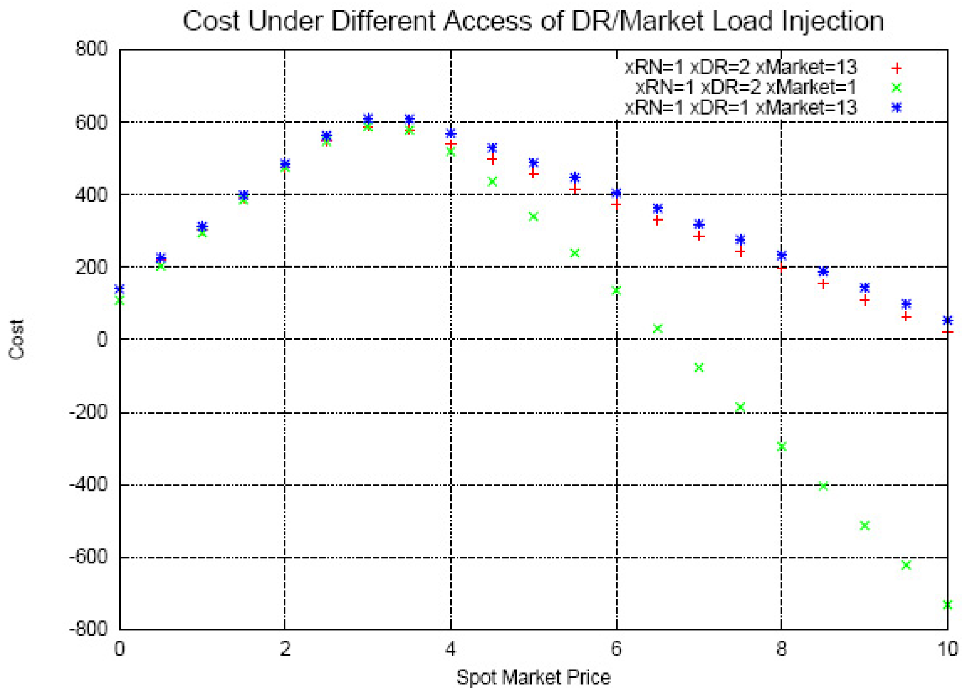


Figure 3.7: Comparing Cost Under Different Bus Connection Settings

optimal solution. These technical details and other more efficient refinements of our optimal solution approach are omitted due to space restrictions.

3.4 Conclusion

We demonstrate the importance of properly capturing in the dispatching problem the effects of any renewable generation sources introduced at various nodes in the distribution network. Although they generate energy at essentially no operational cost, renewable sources can create issues arising from supply forecast uncertainty, where a mismatch in the planned supply and actual demand due to lower than expected renewable generation results in the utility having to purchase expensive units from the spot market. Our approach hedges this uncertainty by imposing a risk constraint on the objective function (total cost of meeting demand) of the dispatching problem, where a risk metric based on the Conditional Value-at-Risk of the cost function was considered. This risk constraint has the effect of slightly increasing

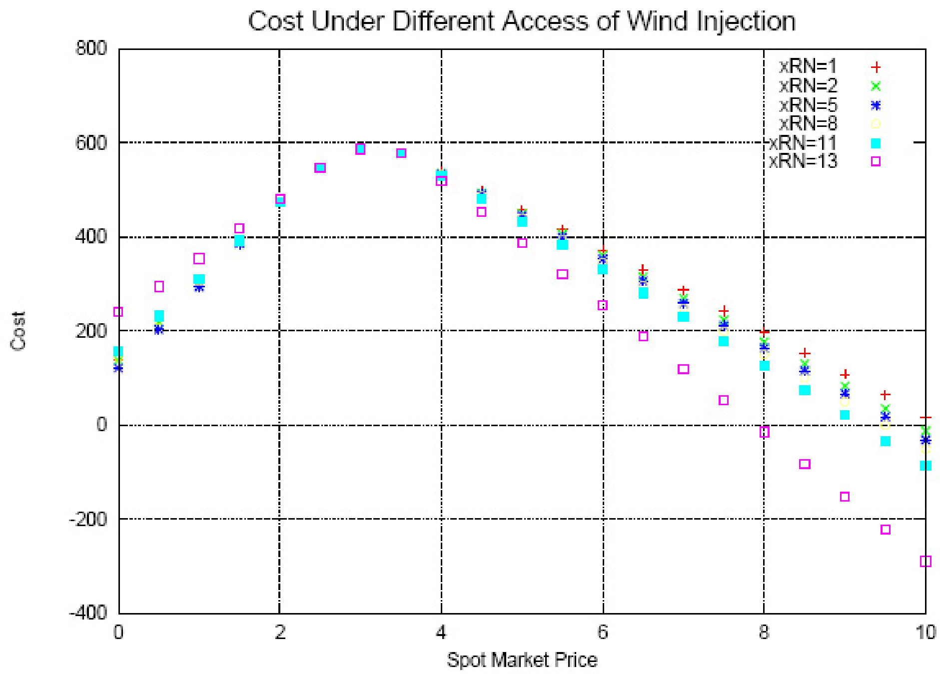


Figure 3.8: Comparing Cost Under Different Wind Connection Settings

the total dispatched generation capacity to hedge the renewable generation uncertainty. An alternative strategy is to incentivize end-users to reduce their load, which can be effectuated almost instantly (as compared to increasing thermal generation) but at the expense of such incentives. Our formulation also considers this strategy.

Results from numerical experiments show that the introduction of renewable sources can have a significant impact on the dispatching policy, as can the location of this introduction within the transmission network. Additionally, demand response load reduction can be an effective tool in controlling shortfalls created by renewable generation forecast mismatches.

Ongoing extensions of the present work include a variant of our formulation based on the AC power flow equations, which yields a highly non-linear stochastic optimization problem. Another important extension we are actively pursuing is to look at a multi-period formulation for the demand response enhanced OPF model. As we point out in [49], demand reduction via incentivization is not a memoryless process, in that reductions in demand in one time period can have the effect of increasing demand in subsequent periods as customers adjust by shifting load rather than reducing their overall consumption.

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