Infinitely primitively renormalizable polynomials

with bounded combinatorics

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Abstract of the Dissertation

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Infinitely renormalizable quadratic polynomials have been heavily studied. In the context of quadratic-like renormalization, one may try to prove the existence of *a priori* bounds, a definite thickness for the annuli corresponding to the renormalizations. In 1997, M. Lyubich showed that *a priori* bounds imply local connectivity of the Julia set and combinatorial rigidity for the corresponding quadratic polynomial [Lyu97]. In a paper from 2006, J. Kahn showed that infinitely renormalizable quadratic polynomials of bounded primitive type admit *a priori* bounds [Kah06]. In 2002, H. Inou generalized some of the polynomial-like renormalization theory to polynomials of higher degree with several critical points [Ino02]. In my thesis, I generalize Kahn's theorem to the context of polynomials of higher degree admitting infinitely many primitive renormalizations of bounded type around each of their critical points. These *a priori* bounds imply local connectivity and rigidity.

Contents

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Chapter 1

Introduction

In recent years, the theory of renormalization, which involves looking at a small piece of a dynamical system and rescaling to obtain a new dynamical system of the same type, has been established as a powerful tool in many different situations. In my thesis, I prove the following theorems about renormalization in the dynamics of complex polynomials:

1. Complex polynomials admitting infinitely many bounded-type, primitive renormalizations around each critical point have *a priori* bounds, which means that all of the renormalizations lie in a compact space. (Actually, we prove the existence of *beau* bounds, a stronger form of *a priori* bounds.)

2. The Julia sets of such polynomials are locally connected, which means that there is a topological model for the dynamics.

3. The dynamical systems arising from such polynomials are rigid: Combinatorial equivalence implies conformal equivalence.

My thesis is among the first steps toward understanding infinitely renormalizable complex polynomials of degree ≥ 3 with more than one critical point. The existence of *a priori* bounds in the context of polynomials having more than one critical point is new, generalizing a result of Jeremy Kahn for polynomials having exactly one critical point [Kah06]. Furthermore, my local connectivity result is the first among infinitely renormalizable complex polynomials that are not real and have more than one critical point, and my approach to combinatorial rigidity furnishes a new proof in the case where there is exactly one critical point.

The existence of more than one critical point creates additional combinatorial difficulties, which are overcome by using certain decompositions of the relevant domains. When there is exactly one critical point, the parameter space is one-dimensional, and there are well-known compactness properties involving wakes, limbs, and ray portraits. When there is more than one critical point, the parameter space is much more complicated, necessitating a new approach to the rigidity problem that does not rely on the one-dimensional properties.

1 Selected historical background

The study of holomorphic dynamics originated with the analysis of P. Fatou and G. Julia in the early 1900s. Afterward, L. Ahlfors and L. Bers applied the theory of quasiconformal maps to the study of Kleinian groups, dynamical systems generated by groups of Möbius transformations of the Riemann sphere. The field of rational dynamics was then dormant until the 1980s, when D. Sullivan rejuvenated the field by using quasiconformal deformations to prove his "no wandering domains" theorem [Sul85]. Around the same time, A. Douady and J. Hubbard offered a systematic treatment of the dynamics of complex quadratic polynomials in their Orsay notes [DH].

Complex quadratic polynomials $f_c(z) = z^2 + c$, with $c \in \mathbb{C}$, provide the easiest examples of holomorphic dynamical systems having critical points, but as of 2016, they are still not completely understood. Understanding their dynamics amounts to understanding the corresponding parameter space, the Mandelbrot set, and the central conjecture in this field is that the Mandelbrot set is locally connected. Local connectivity of the Mandelbrot set at c is equivalent to combinatorial rigidity of f_c. One important tool used to study local connectivity is the puzzle decomposition, which first appeared in the work of Branner and Hubbard in the context of cubic polynomials [BH88, BH92]. By applying puzzle techniques to quadratic polynomials with connected Julia sets, J. C. Yoccoz proved that if f_c does not have neutral periodic points and admits at most finitely many renormalizations, then the corresponding Julia set is locally connected, and the Mandelbrot set is locally connected at c [Hub93, Mil00, Roe00]. The remaining parameters consist of those c such that f_c admits infinitely many renormalizations.

The results obtained by Yoccoz have been generalized in a number of ways. Using the quasi-additivity law, Kahn and Lyubich showed that if a complex polynomial $z \mapsto z^d + c$, with $d \geq 2$, admits at most finitely many renormalizations, and if every periodic point is repelling, then the Julia set is locally connected [KL09a]. Avila, Kahn, Lyubich, and Shen showed that under the same conditions, the polynomial is combinatorially rigid [AKLS09]. Yarrington showed that for a polynomial of degree ≥ 3 with at least two critical points, a certain combinatorial condition implies local connectivity of the Julia set, provided the Julia set is connected, every periodic point is repelling, each fixed ray lands at a distinct fixed point, and no critical point admits a renormalization [Yar95].

To help put the results of my thesis in context, some results from the theory of infinite renormalization, *a priori* bounds, local connectivity, and rigidity are summarized below.

A priori bounds

- Sullivan obtained complex *a priori* bounds for real quadratic polynomials admitting infinitely many renormalizations of bounded type [Sul88].

- Lyubich obtained *a priori* bounds for complex quadratic polynomials that satisfy the secondary limbs condition and have sufficiently big combinatorial depth [Lyu97].

- Lyubich and Yampolsky obtained complex *a priori* bounds for the cases not covered by [Sul88] and [Lyu97], real quadratic polynomials admitting infinitely many renormalizations of "essentially bounded but unbounded" type, completing the study of complex *a priori* bounds for real quadratic polynomials [LY97]. These complex *a priori* bounds were also obtained independently, using essentially different methods, by Graczyk and Świątek and by Levin and van Strien in [GS96, LvS98].

- Kahn obtained *a priori* bounds for quadratic polynomials admitting infinitely many primitive renormalizations of bounded type [Kah06]. While Sullivan's proof in the context of real quadratic polynomials exploits the real symmetry, Kahn's proof for complex quadratic polynomials relies on two very general tools: the canonical weighted arc diagram and the quasi-additivity law. Kahn's proof is valid without any significant changes in the case of complex polynomials $z \mapsto z^d + c$, with $d \geq 2$.

- Kahn and Lyubich obtained *a priori* bounds for quadratic polynomials that admit infinitely many primitive renormalizations and satisfy the decoration condition or the molecule condition [KL08, KL09b]. These conditions allow certain unbounded combinatorics.

Local connectivity

- Hu and Jiang showed that the Feigenbaum quadratic polynomial has a locally connected Julia set [HJ98].

- Jiang showed that if a quadratic polynomial admits infinitely many renormalizations, the existence of unbranched *a priori* bounds implies local connectivity of the Julia set [Jia00]. See also theorem VI in [Lyu97] and proposition 4.14 in [McM96].

- Lyubich showed that if a quadratic polynomial satisfies the secondary limbs condition, the existence of *a priori* bounds implies the existence of unbranched *a priori* bounds [Lyu97]. Consequently, for any quadratic polynomial satisfying the secondary limbs condition, the existence of *a priori* bounds implies local connectivity of the Julia set.

- Levin and van Strien showed that the Julia set of a polynomial $z \mapsto z^{2d} + c$, where d is an integer and c is real, is either locally connected or totally disconnected [LvS98]. Independently, Lyubich and Yampolsky observed that theorem 1.1 in [LY97] remains valid for real polynomials $z \mapsto z^d + c$, which implies that the corresponding Julia sets are locally connected.

Rigidity

- McMullen proved that robust infinitely renormalizable quadratic polynomials do not admit invariant line fields on their Julia sets [McM94].

- Lyubich showed that if a quadratic polynomial satisfies the secondary limbs condition, the existence of *a priori* bounds implies combinatorial rigidity [Lyu97]. In particular, real quadratic polynomials satisfying the secondary limbs condition are combinatorially rigid.

- Inou proved that robust infinitely renormalizable complex polynomials with more than one critical point do not admit invariant line fields on their Julia sets [Ino02].

- Kozlovski, Shen, and van Strien showed that for a real polynomial of degree ≥ 2 with connected Julia set, if every critical point is real and nondegenerate, and if there are no neutral periodic points, then topological conjugacy on $\mathbb{R}$ implies quasiconformal conjugacy on $\mathbb{C}$ [KSvS07].

- Cheraghi showed that if a complex polynomial $z \mapsto z^d + c$, with $d \geq 2$, satisfies the secondary limbs condition, then the existence of *a priori* bounds implies combinatorial rigidity [Che10].

2 The results

A *polynomial-like* map $f : U \to V$ is a holomorphic branched covering map of degree $d \geq 2$, where U and V are topological disks properly contained in $\mathbb{C}$, and $\overline{U}$ is a compact subset of V. The *Julia set* of f is

$$K(f) = \bigcap_{n=0}^{\infty} f^{-n}(\overline{U}).$$

The Julia set is compact, perfect, and full. If a set W is chosen such that $K(f) \subset W$ and $f|W$ is a polynomial-like map of degree d, then $K(f|W) = K(f)$.

Let $f : U \to V$ be a polynomial-like map, and let c be a critical point of f. We say that f is *primitively renormalizable around* c with *period* $p \geq 2$ if there are topological disks U' and V' containing c such that $f^p : U' \to V'$ is a polynomial-like map, $K(f^p|U')$ is connected, and $U', f(U'), \ldots, f^{p-1}(U')$ are pairwise disjoint. We call the polynomial-like map $f^p : U' \to V'$ a *primitive renormalization* of f.

Let $\mathcal{R}(c) = \{p_n\}$ denote the set of periods p_n such that f admits a primitive renormalization $f^{p_n} : U^n \to V^n$ around c. If $|\mathcal{R}(c)| = \infty$, then we say that f is *infinitely* primitively renormalizable around c; in this case, if there is a positive number B such that $p_{n+1}/p_n \leq B$ for each n, then we say that the infinitely many primitive renormalizations around c are of *bounded type* or have *bounded combinatorics*. The main theorem is that such maps enjoy a compactness condition called *a priori* bounds.

Theorem A. *If a polynomial-like map f admits infinitely many primitive renormalizations of bounded type around each of its critical points, then it has a priori bounds.*

This theorem is a generalization of a theorem due to Jeremy Kahn for maps of degree 2, which necessarily have only one critical point [Kah06]. Kahn's proof remains valid for polynomial-like maps of degree ≥ 3 having exactly one critical point, so the heart of A is the case where there is more than one critical point. The *a priori* bounds condition implies that we can choose the domains associated with the renormalizations in such a way that the annuli $V^n \setminus \overline{U^n}$

have modulus bounded away from 0. This allows us to prove the following theorem.

Theorem B. *The Julia set $K(f)$ is locally connected.*

Local connectivity is more than just an arcane topological property. Let $F : \mathbb{C} \to \mathbb{C}$ be a complex polynomial of degree $d \geq 2$. Assume that F is monic and that its polynomial-like restriction to a disk of large radius admits infinitely many primitive renormalizations of bounded type around each of its critical points. The Julia set K_F of this restriction of F coincides with the set of z in $\mathbb{C}$ such that $F^n(z) \not\to \infty$ as $n \to \infty$. Since K_F is connected and full, the Riemann mapping theorem implies that there is a unique conformal isomorphism $\Phi_F : \mathbb{C} \setminus \overline{\mathbb{D}} \to \mathbb{C} \setminus K_F$ normalized such that $\Phi_F(z)/z \to 1$ as $z \to \infty$. By the classical theorem of Carathéodory, local connectivity of K_F implies that Φ_F extends continuously to a map $\mathbb{C} \backslash \mathbb{D} \to \mathbb{C}$. In this case, Douady showed that K_F is homeomorphic to the quotient space obtained from $\overline{\mathbb{D}}$ by collapsing to single points the convex hulls of points in $\partial\mathbb{D}$ mapped to the same point by Φ [Dou93].

Let $G : \mathbb{C} \to \mathbb{C}$ be another complex polynomial of degree d. Assume that G is monic and that G is combinatorially equivalent to F. By combinatorial equivalence, we mean that if $\Phi_G : \mathbb{C} \setminus \overline{\mathbb{D}} \to \mathbb{C} \setminus K_G$ is the unique normalized conformal isomorphism, then $\Phi_G \circ \Phi_F^{-1} : \mathbb{C} \setminus K_F \to \mathbb{C} \setminus K_G$ extends to a homeomorphism $\mathbb{C} \to \mathbb{C}$. The following theorem says that F is combinatorially rigid.

Theorem C. *The homeomorphism $\Phi_G \circ \Phi_F^{-1} : \mathbb{C} \to \mathbb{C}$ is holomorphic.*

Hence, $F = G$.

3 The proofs

The first step in proving theorem A is to implement Jeremy Kahn's "improvement of life" philosophy in the context of polynomial-like maps of degree $d \geq 2$ with more than one critical point. To this end, we need a notion of renormalization that respects geometry, and this requires us to work in a larger space of maps, the space of *pseudo-polynomial-like maps*. These maps restrict to polynomial-like maps.

Let $\mathbf{f} : (\mathbf{V}, \mathbf{K}) \to (\mathbf{V}, \mathbf{K})$ be a pseudo-polynomial-like map, and assume that its polynomial-like restriction admits a primitive renormalization of period p around one of its critical points. Then there is a little Julia set K_0 associated with this renormalization, and there is the associated cycle of *little Julia sets* of $\mathbf{f}$,

$$\mathcal{K} = \bigcup_{j=0}^{p-1} K_j \subset \mathbf{K}.$$

If $\mathbf{f}$ has exactly one critical point, then $\mathcal{K}$ contains the critical value. However, when there is more than one critical point, it can happen that some critical values of $\mathbf{f}$ are outside of $\mathcal{K}$. For now, we will assume that the critical values of $\mathbf{f}$ are contained in $\mathcal{K}$.

For each $j = 0, \ldots, p-1$, we can define the pseudo-polynomial-like *canonical renormalization* $R_j\mathbf{f} : \mathbf{V}_j \to \mathbf{V}_j$ of $\mathbf{f}$ around K_j of *period* p. Let γ_j be the hyperbolic geodesic in $\mathbf{V} \setminus \mathcal{K}$ homotopic to K_j. The special property of $R_j\mathbf{f}$ is that the Julia set of the polynomial-like restriction of $R_j\mathbf{f}$ is canonically identified with K_j, and the hyperbolic length of the core geodesic in the annulus $\mathbf{V}_j \setminus K_j$ is equal to the hyperbolic length of γ_j. We prove the following "improvement of life" theorem.

Theorem D. *There exists a threshold $\mu > 0$, depending only on certain combinatorial data, such that if $\mathbf{f}'$ is a canonical renormalization of a pseudo-polynomial-like map $\mathbf{f}$ (with the specified combinatorics), then* $\operatorname{mod}(\mathbf{f}') < \mu$ *implies* $\operatorname{mod}(\mathbf{f}) < \mu/2$.

Now, assume that the polynomial-like restriction of $\mathbf{f}$ admits infinitely many primitive renormalizations of bounded type around each critical point. A deceptively strong assumption about the critical points of $\mathbf{f}$, which we will not discuss further, allows us to consider the "tree of renormalizations" growing from $\mathbf{f}$, consisting of the canonical renormalizations of $\mathbf{f}$, the canonical renormalizations of the canonical renormalizations of $\mathbf{f}$, and so on. Theorem D implies that the moduli of these pseudo-polynomial-like renormalizations are bounded away from 0.

Proving theorem A now amounts to removing the combinatorial assumptions we made on the critical points of $\mathbf{f}$. This follows from decomposing $\mathbf{f}$ into small "pieces" that do satisfy the assumptions, so we obtain a lower bound for the moduli of all maps in the "forest" of trees of renormalizations for a general pseudo-polynomial-like map whose polynomial-like restriction admits infinitely many primitive renormalizations of bounded type around each of its critical points.

This implies that for a sufficiently deep level of renormalization, there are definitely thick annuli separating the little Julia sets of each deeper level of renormalization, which is the key component in the proof of theorem B. We finish the proof by appealing to the theory of Yoccoz puzzles and compactness theorems for univalent maps.

The proof of the rigidity theorem, theorem C, amounts to building a quasiconformal map $\mathbb{C} \to \mathbb{C}$ that is homotopic, relative the postcritical set of F, to a homeomorphism conjugating F and G. We do this by splicing together quasiconformal maps, defined on finitely connected planar domains with bounded geometry, in such a way that the resulting quasiconformal map is in the desired homotopy class.

To this end, we consider a compact space $\mathcal{X}$ of pseudo-polynomial-like maps with specified combinatorial data and geometric bounds. We associate to any $\mathbf{f} : \mathbf{V} \to \mathbf{V}$ in $\mathcal{X}$ a planar Riemann surface $S(\mathbf{f})$ without cusps that is homeomorphic to $\mathbb{R}^2 \setminus \{1, \ldots, p\}$. Let Γ be the core geodesic in the annulus $\mathbf{V} \setminus \mathbf{K}$. There is a unique cycle of little Julia sets $\mathcal{K} = K_0 \cup \cdots \cup K_{p-1}$ for $\mathbf{f}$. For each $j = 0, \ldots, p-1$, let γ_j be the hyperbolic geodesic in $\mathbf{V} \setminus \mathcal{K}$ homotopic to K_j. The Riemann surface $S(\mathbf{f})$ is the domain bounded by Γ and certain equidistant curves for the geodesics $\bigcup_j \gamma_j$. The renormalization combinatorics of the polynomial-like restriction of $\mathbf{f}$ determine a "marking" of the domain $S(\mathbf{f})$.

Pick a basepoint $\mathbf{f}_*$ in X. Let $\mathrm{Teich}(\mathbf{f}_*)$ denote the reduced Teichmüller space of $S(\mathbf{f}_*)$. The key step in the proof of theorem C is the following theorem.

Theorem E. *The map $\Psi : X \to \mathrm{Teich}(\mathbf{f}_*)$, associating to a map $\mathbf{f}$ in X the marked domain $S(\mathbf{f})$, is continuous.*

Then $\Psi(X)$ is a compact subspace of $\mathrm{Teich}(\mathbf{f}_*)$. The fact that our renormalizations have uniformly bounded combinatorics allows us to decompose the domain of F, our starting polynomial, as the union of the postcritical set of F and countably many domains with bounded geometry. This gives us a quasiconformal map $\mathbb{C} \to \mathbb{C}$ that sends the postcritical set of F to the postcritical set of G, but this map is probably not in the right homotopy class. A construction similar to the one for $S(\mathbf{f})$ allows us to adjust our map so that it is in the correct homotopy class, and the proof is finished.

4 The structure of the chapters

There are three chapters following the introduction and background material.

In chapter 3, we generalize Kahn's improvement of life theorem to the context of pseudo-polynomial-like maps with more than one critical point. First, we use a combinatorial model to study the combinatorics of arcs joining the little Julia sets to each other and to ∞. There is an exponential growth of certain canonical weights, which is encapsulated in the main inequality. We use the canonical arc diagram to relate hyperbolic geometry and combinatorics. The covering lemma of Kahn and Lyubich [KL09c] gives us a threshold allowing us to control degenerating geometry. At the end of the chapter, we prove the existence of *a priori* bounds, theorem A. Actually, we obtain *beau* bounds.

In chapter 4, we describe a decomposition that allows us to apply theorem A to a polynomial admitting infinitely many primitive renormalizations, with bounded combinatorics, around one of its critical points. This amounts to the observation that after throwing away the first few levels of renormalization, the subsequent renormalizations always involve the same set of critical points of the original map. Using this decomposition, we prove *a priori* bounds around the relevant critical points. Then we prove local connectivity of the Julia set, theorem B. At the end of the chapter, we cycle trees to label the little Julia sets according to their nested structure. This simplifies notation for the proof of rigidity.

In chapter 5, we prove the rigidity theorem, theorem C. We begin by describing some geometric objects associated with simple, closed geodesics in a hyperbolic Riemann surface. Then we describe a compact space containing all of the renormalizations of the polynomials under consideration. This allows us to prove that the domains associated with one level of renormalization have bounded geometry. Next, we describe another compact space controlling the geometry of the domains associated with two consecutive levels of renormalization. The proof of rigidity amounts to building a quasiconformal map in the wrong homotopy class and fixing it. Building the wrong map relies on bounded geometry for the domains associated with one level of renormalization. Fixing the homotopy class relies on bounded geometry for the domains associated with two consecutive levels of renormalization. The proof of rigidity also relies on two intuitively obvious topology theorems, which we prove at the end of the chapter.

Chapter 2

Background

5 Notation and terminology

We will use the following basic definitions:

- A *topological disk* is a simply connected domain in a Riemann surface.
- A *topological annulus* is a doubly connected domain, with finite modulus, in a Riemann surface.
- A (non-degenerate) *continuum* is a compact, connected subset of $\mathbb{C}$ containing at least two points. A continuum K is *full* if $\mathbb{C} \setminus K$ is connected.
- A *component* of a topological space means a connected component.

We will use the following notations:

- $\mathrm{Crit}(f)$ denotes the set of critical points of a smooth map f.
- $\mathrm{Dil}(h)$ denotes the quasiconformal dilatation of the quasiconformal map h.
- $\mathrm{diam}_X(Y)$ denotes the diameter of a subset Y of a metric space X.
- $\mathrm{dist}_X(a, b)$ denotes the distance, in a metric space X, between points a and b.
- $\mathbf{f}$ denotes a pseudo-polynomial-like map.
- $f : (V, U) \to (V', U')$ means that $U \subset V$, $U' \subset V'$, $f : V \to V'$, and $f|U : U \to U'$.

- f^n denotes the n-th iterate of f, when $n \geq 0$ is an integer.
- $G = (V, E)$ denotes the graph with vertices V and edges E.
- $\mathrm{mod}(A)$ and $\mathrm{mod}(\mathbf{f})$ denote the modulus of the annulus A and the modulus of the pseudo-polynomial-like map $\mathbf{f}$, respectively.
- $|\gamma|_X$ denotes the length of curve curve with respect to the metric on X.
- $\mathcal{L}(\Gamma)$ denotes the extremal length of the path family Γ.
- $\mathcal{W}(\Gamma)$ denotes the extremal width of the path family Γ.

Let X and Y be topological spaces, and let $f : X \to X$ and $g : Y \to Y$ be continuous maps. We say that f and g are *topologically conjugate*, denoted $f \sim_{\mathrm{top}} g$, if there is a homeomorphism $h : X \to Y$ such that $h \circ f = g \circ h$. Suppose that X and Y are Riemann surfaces and that f and g are holomorphic maps. We say that f and g are *quasiconformally conjugate*, denoted $f \sim_{\mathrm{qc}} g$, if there is a quasiconformal map $h : X \to Y$ such that $h \circ f = g \circ h$. We say that f and g are *conformally conjugate* if there is a conformal isomorphism $h : X \to Y$ such that $h \circ f = g \circ h$.

6 Polynomials

Let $f : \mathbb{C} \to \mathbb{C}$, $f(z) = z^d + a_{d-1}z^{d-1} + \cdots + a_0$, be a polynomial of degree $d \geq 2$. The *filled Julia set* of f is the set $K = K(f) = \{z \in \mathbb{C} : f^n(z) \not\to \infty\}$. We call $\mathbb{C} \setminus K(f)$ the *basin of* ∞.

Let $g : \mathbb{C} \to \mathbb{C}$ be another polynomial. We say that f and g are *hybrid conjugate*, denoted $f \sim_{\mathrm{hyb}} g$, if there is a quasiconformal conjugacy $h : \mathbb{C} \to \mathbb{C}$ with $\mathrm{Dil}(h) = 0$ almost everywhere on K.

Theorem 6.1. *The filled Julia set K is compact and full. It is connected if and only if* $\mathrm{Crit}(f) \subset K$.

We will only consider polynomials with connected Julia sets. In this case, there is a unique conformal isomorphism $B = B_f : \mathbb{C} \setminus K \to \mathbb{C} \setminus \overline{\mathbb{D}}$ such that $B(z)/z \to 1$ as $z \to \infty$. The map B satisfies $B(f(z)) = (f(z))^d$.

Consider the foliations of the domain $\mathbb{C} \setminus \overline{\mathbb{D}}$ by radial line segments $\{re^{2\pi i\theta} : r \in (1, +\infty)\}$, for all $\theta \in \mathbb{R}/\mathbb{Z}$, and by circles $\{z \in \mathbb{C} : |z| = e^r\}$, for all $r \in (1, +\infty)$. We call $R^\theta = R^\theta(f) = B^{-1}(e^{2\pi i\theta}(0, +\infty))$ the *external ray* of angle $\theta \in \mathbb{R}/\mathbb{Z}$. We say that R^θ *lands* on a point $a \in \partial K$ if $\overline{R^\theta} \subset \mathbb{C}$ is equal to $R^\theta \cup \{a\}$. We call $E^r = E^r(f) = B^{-1}(\{z \in \mathbb{C} : |z| = e^r\})$ the *equipotential* of level r.

Theorem 6.2. *(Assume that K is connected.) Let $a \in \partial K$ be a repelling periodic point. Then there is a finite, positive number of external rays landing on a.*

Let $a \in \partial K$ be a repelling p-periodic point. Let $\mathcal{R}(a) = \mathcal{R}_f(a)$ be the union of external rays landing on a. Let $\overline{a} = \{a = a_0, \ldots, a_{p-1}\}$ be the p-cycle containing a. Let $\mathcal{R}(\overline{a}) = \mathcal{R}_f(\overline{a}) = \bigcup_{a \in \overline{a}} R(a)$. Let E^r be an equipotential, and let D be the bounded component of $\mathbb{C} \setminus E^r$. Let Γ be the set $(\partial D) \cup \overline{\mathcal{R}(\overline{a}) \cap D}$. For each integer $n \geq 0$, let $\mathcal{Y}^n$ be the set consisting of $\overline{P} \subset \mathbb{C}$, where $P \subset \mathbb{C}$ is a bounded component of $f^{-n}(\Gamma)$. We call $\mathcal{Y}^n$ the set of Yoccoz *puzzle pieces* of depth n. The construction of these pieces depends on the choice of equipotential E^r and the repelling periodic cycle $\overline{a}$, but our notation will not reflect this.

7 Polynomial-like maps

For a thorough introduction to the theory of polynomial-like map, the reader is invited to consult [DH85]. The following theorem says that polynomial-like maps behave like polynomials.

Theorem 7.1 (Straightening theorem)**.** *Let $f : U \to V$ be a polynomial-like map (with connected K) of degree d. Then f is hybrid conjugate to a polynomial of degree d, unique up to affine conjugacy. If* $\mathrm{mod}(V \setminus \overline{U}) \geq \mu > 0$*, then the dilatation of the quasiconformal map providing the hybrid conjugacy is controlled by μ.*

Proof. See [DH85]. □

Strictly speaking, polynomials are not polynomial-like maps. However, for the following theorem, let us adopt the convention that a polynomial of degree d, with connected Julia set, is also called a polynomial-like map.

Theorem 7.2. *Fix an integer $d \geq 2$ and a real number $\mu > 0$. The space of polynomial-like maps $f : U \to V$ of degree d, with $K(f)$ connected and* $\mathrm{mod}(V \setminus \overline{U}) \geq \mu$*, is compact, up to affine conjugation.*

Proof. See theorem 5.8 in [McM94]. □

8 Pseudo-polynomial-like maps

The theory of pseudo-polynomial-like maps is developed in [Kah06]. We discuss here only the theory necessary for our results.

A *pseudo-polynomial-like map* $\mathbf{f}$ consists of the following objects:

- topological disks $\mathbf{U}'$ and $\mathbf{U}$ properly contained in $\mathbb{C}$,
- a holomorphic branched covering map $f : \mathbf{U}' \to \mathbf{U}$ of degree $d \geq 2$,
- a holomorphic immersion $i : \mathbf{U}' \to \mathbf{U}$, and
- compact, connected, and full sets $\mathbf{K} \subset \mathbf{U}$ and $\mathbf{K}' \subset \mathbf{U}'$.

We require that $\mathbf{K}' = f^{-1}(\mathbf{K}) = i^{-1}(\mathbf{K})$. Abusing notation, we will call $\mathbf{f} : (\mathbf{U}, \mathbf{K}) \to (\mathbf{U}, \mathbf{K})$ a pseudo-polynomial-like map of degree d. We call $\mathbf{K}$ the *filled Julia set* of $\mathbf{f}$. The modulus of $\mathbf{f}$ is defined to be $\mathrm{mod}(\mathbf{f}) = \mathrm{mod}(\mathbf{U} \setminus \mathbf{K})$.

The following theorem says that pseudo-polynomial-like maps behave like polynomial-like maps.

Theorem 8.1. *Let $\mathbf{f} : (\mathbf{U}, \mathbf{K}) \to (\mathbf{U}, \mathbf{K})$ be a pseudo-polynomial-like map of degree d. Then i restricts to an embedding near $\mathbf{K}$. There exist domains $U, V \subset \mathbf{U}$ such that $f \circ i^{-1} : U \to V$ is a polynomial-like map of degree d, and $K(f \circ i^{-1}|U) = \mathbf{K}$. If $\mathrm{mod}(\mathbf{f}) \geq \mu > 0$, then the domains U and V can be chosen so that $\mathrm{mod}(V \setminus \overline{U}) \geq \epsilon$, where $\epsilon = \epsilon(d, \mu) > 0$.*

Proof. See lemma 2.4 in [Kah06]. □

We call $f \circ i^{-1} : U \to V$ a *polynomial-like restriction* of $\mathbf{f}$.

We equip the set of pseudo-polynomial-like maps of degree $d \geq 2$ with a topology: The domains are given the Carathéodory topology, and the maps are given the topology of uniform convergence on compact sets. For the following theorem, let us also call $i, f : \mathbb{C} \to \mathbb{C}$ a pseudo-polynomial-like map of degree d when $i \in \mathrm{Aut}(\mathbb{C})$, f is a polynomial of degree d, and $f \circ i^{-1}$ is a polynomial with connected Julia set.

Theorem 8.2. *Fix an integer $d \geq 2$ and a real number $\mu > 0$. The space of pseudo-polynomial-like maps $\mathbf{f}$ of degree d, with $\mathrm{mod}(\mathbf{f}) \geq \mu$, is compact, up to pre- and post-composition by two independent affine maps.*

For the proof of this theorem, we will need the following estimate.

Lemma 8.3. *Let $A \subset \mathbb{C}$ be an annulus, with core geodesic $\gamma \subset A$ and* $\mathrm{mod}(A) \geq \mu > 0$*. Let $D \subset \mathbb{C}$ denote the bounded component of $\mathbb{C} \setminus A$. There exists $r = r(\mu) > 0$ such that*

$$\mathrm{dist}_{\mathbb{C}}(\gamma, D) > r \cdot \mathrm{diam}_{\mathbb{C}}(\gamma).$$

Proof. See theorem 2.5 in [McM94]. $\square$

Proof of theorem 8.2. Let $\{i_n, f_n : (U_n, K_n') \to (V_n, K_n)\}_n$ be a sequence of pseudo-polynomial-like maps, where each i_n is a holomorphic immersion, and each f_n is a holomorphic branched covering map of degree d. Pre- and post-composing by two affine maps of the form $z \mapsto Az + B$, where $(A, B) \in (\mathbb{C} \setminus \{0\}) \times \mathbb{C}$, we can assume that $\{0, 1\} \subset K_n' \subset \overline{\mathbb{D}}$, $\{0, 1\} \subset K_n \subset \overline{\mathbb{D}}$, and $i_n(0) = 0$. Passing to a subsequence, we can assume that there are compact, connected sets K' and K of Euclidean diameter 1 such that $K_n' \to K'$ and $K_n \to K$ in the Hausdorff topology.

Let $\gamma_n \subset V_n \setminus K_n$ be the core geodesic. Let $A_n \subset V_n \setminus K_n$ denote the hyperbolic collar around γ_n. Since $\mathrm{mod}(V_n \setminus K_n) \geq \mu$, we know that $|\gamma_n|_{V_n \setminus K_n} \leq L$, where $L = L(\mu) > 0$. Then $\mathrm{mod}(A_n) \geq m$, where $m = m(L) > 0$. Let $D_n \subset V_n$ be the topological disk bounded by γ_n. Lemma 8.3 implies that

$$\mathrm{dist}_{\mathbb{C}}(\partial V_n, 0) > \mathrm{dist}_{\mathbb{C}}(\gamma_n, D_n) > r \cdot \mathrm{diam}_{\mathbb{C}}(\gamma) > r,$$

where $r = r(m) > 0$. This shows that each topological disk V_n contains the round disk of radius r around 0. Then by passing to a subsequence, we can assume that there is a topological disk $V \subset \mathbb{C}$ such that $(V_n, 0) \to (V, 0)$ in the Carathéodory topology. Clearly, $\mathrm{mod}(V \setminus K) \geq \mu$.

Since $f_n : U_n \to V_n$ is a branched covering map of degree d with critical values in $K_n \subset V$, we know that $\mathrm{mod}(U_n \setminus K_n') = \mathrm{mod}(V_n \setminus K_n)/d \geq \mu/d$. By an argument similar to the one in the paragraph above, we can assume that there is a topological disk $U \subset \mathbb{C}$ such that $(U_n, 0) \to (U, 0)$ in the Carathéodory topology. Then $\mathrm{mod}(U \setminus K') \geq \mu/d$.

Let $X_n = i_n^{-1}(\{0, 1\})$. Passing again to a subsequence, we can assume that there is a finite set X such that $X_n \to X$ in the Hausdorff topology. Then $\{i_n | U_n \setminus X_n\}_n$ is a normal family, because it consists of functions omitting the values 0 and 1. Then this family is uniformly bounded on compact subsets of $U \setminus X$. Our normalization implies that $\{i_n | U_n \setminus K_n'\}_n$ is uniformly bounded, so in fact, $\{i_n\}_n$ is uniformly bounded on compact subsets of U. Passing again to a subsequence, we can assume that there is a holomorphic function $i : U \to V$ such that $i_n \to i$ uniformly on compact subsets of U. Since i must assume the values 0 and 1, it is non-constant, and as a non-constant limit of holomorphic immersions, Hurwitz' theorem implies that i is a holomorphic immersion. (In the case where $U = \mathbb{C}$, we know that i is an entire immersion, $i^{-1}(K) = K'$, and i is bijective on a neighborhood of K'. It follows that i is affine.)

Now, let us consider the maps $\{f_n\}_n$. Theorem 5.6 in [McM94] asserts that by passing to a subsequence, we can assume that there is a holomorphic branched covering map $f : (U, 0) \to (V, 0)$ of degree $\leq d$ such that $f_n \to f$ uniformly on compact subsets of U. Since $\mathrm{mod}(U_n \backslash K'_n) \geq \mu/d$, we can assume, by passing to a subsequence, that $\mathrm{Crit}(f_n) \to \mathrm{Crit}(f) \subset U$ in the Hausdorff topology. Then $\deg(f) = d$, and we are finished. $\square$

There is a notion of iteration of pseudo-polynomial-like maps. (Iteration is defined as for holomorphic correspondences. See [Kah06].) We will briefly describe the construction of $\mathbf{f}^2 : \mathbf{U} \to \mathbf{U}$, the second iterate of a pseudo-polynomial-like map $\mathbf{f} : \mathbf{U} \to \mathbf{U}$ consisting of the maps $i, f : \mathbf{U}^1 \to \mathbf{U}$. The fibered product

$$\mathbf{U}^2 := \{(x, y) \in \mathbf{U}^1 \times \mathbf{U}^1 : f(x) = i(y)\}$$

is a Riemann surface, because the derivative of i is non-vanishing. Consider the diagram,

$$\begin{tikzcd}
 & & \mathbf{U}^2 \arrow[dl, "\pi_{\text{left}}"'] \arrow[dr, "\pi_{\text{right}}"] & & \\
 & \mathbf{U}^1 \arrow[dl, "i"'] \arrow[dr, "f"] & & \mathbf{U}^1 \arrow[dl, "i"'] \arrow[dr, "f"] & \\
\mathbf{U} & & \mathbf{U} & & \mathbf{U},
\end{tikzcd}$$

where π_{left} and π_{right} denote the projections to the left and right factors. One checks that $\mathbf{U}^2$ is a topological disk, $i_2 := i \circ \pi_{\text{left}}$ is a holomorphic immersion, and $f_2 := f \circ \pi_{\text{right}}$ is a holomorphic branched covering map of degree d^2. Then $\mathbf{f}^2$ consists of the maps $i_2, f_2 : \mathbf{U}^2 \to \mathbf{U}$.

For any integer $n \geq 1$, we have the n-th iterate of $\mathbf{f}$, denoted $\mathbf{f}^n : \mathbf{U} \to \mathbf{U}$, which consists of the following data:

$$\begin{tikzcd}
 & \mathbf{U}^n \arrow[dl, "i_n"'] \arrow[dr, "f_n"] & \\
\mathbf{U} & & \mathbf{U}.
\end{tikzcd}$$

The filled Julia set $\mathbf{K}$ is embedded in each $\mathbf{U}^n$. Abusing notation, we identify $\mathbf{K}$ with its image under the embedding. We can also consider the restriction of $\mathbf{f}$ to $\mathbf{U}^n$, which consists of the following data:

$$\begin{tikzcd} & \mathbf{U}^n \arrow[dl, "i"'] \arrow[dr, "f"] & \\ \mathbf{U}^{n-1} & & \mathbf{U}^{n-1}. \end{tikzcd}$$

9 Renormalization

9.1 Polynomial-like renormalization

Let $f : U \to V$ be a polynomial-like map of degree d, and let $c \in \mathrm{Crit}(f)$. Given an integer $p \geq 2$, we say that f is *primitively renormalizable* around c, with *period* p, if there are domains U' and V' such that

1. $c \in U' \subset U$,
2. $f^p : U' \to V'$ is a polynomial-like map with $K(f^p|U')$ connected, and
3. the topological disks $U'_j = f^j(U')$, for each $j = 0, \ldots, p-1$, are pairwise disjoint.

We say that the polynomial-like map $f^p|U'$ is a *primitive renormalization* of *period* p. For each $j = 0, \ldots, p-1$, let $K_j = f^j(K(f^p|U'))$. Each set K_j is a *little filled Julia set*, and $\mathcal{K} = \bigcup_{j=0}^{p-1} K_j$ is the *cycle of little Julia sets*. It is known that $f(\mathcal{K}) = \mathcal{K}$.

Each little Julia set K_j has a neighborhood on which f^p restricts to a polynomial-like map with filled Julia set K_j. If K_j contains a critical point of f, then any such polynomial-like restriction is a polynomial-like renormalization.

If f' is a polynomial-like renormalization of a quadratic-like map f, then $\deg(f') = 2$. If f' is a polynomial-like renormalization of a polynomial-like map f, with $\deg(f) \geq 3$, then it is possible that $\deg(f') > \deg(f)$.

Lemma 9.1. $\deg(f^p|U') \leq 2^{d-1}$.

To prove this lemma, we will use the following fact:

Lemma 9.2. *Fix an integer $d \geq 2$. If a set positive integers $\{r_j\}_{j=1}^N$ satisfies $\sum r_j = d-1$, then $\prod(1+r_j) \leq 2^{d-1}$.*

Proof. First, we observe that if each $r_j = 1$, then $N = \sum r_j = d - 1$, and $\prod(1 + r_j) = 2^N = 2^{d-1}$. Otherwise, there is an index n such that $r_n \geq 2$. In this case, $N \leq d - 2$, because $2 + (N - 1) \leq r_n + \sum_{j \neq n} r_j = d - 1$. We modify the positive integers r_j, $j = 1, \dots, N$, to obtain positive integers s_j, $j = 1, \dots, N+1$, as follows: $s_n = r_n - 1$, $s_{N+1} = 1$, and $s_j = r_j$, $j \neq n$. Clearly, $\sum s_j = \sum r_j = d - 1$. Since $r_n \geq 2$, we know that $(1 + s_{N+1})(1 + s_n) = 2r_n \geq 1 + r_n$. We compute

$$\begin{aligned}
\prod(1 + r_j) &= (1 + r_n) \left(\prod_{j \neq n} (1 + r_j) \right) = (1 + r_n) \left(\prod_{j \neq n, N+1} (1 + s_j) \right) \\
&\leq (1 + s_{N+1})(1 + s_n) \left(\prod_{j \neq n, N+1} (1 + s_j) \right) = \prod(1 + s_j).
\end{aligned}$$

Iterating this process, we see that the largest product is obtained when each $r_j = 1$. $\square$

Proof of lemma 9.1. We will apply the Riemann-Hurwitz formula to the holomorphic branched covering map $f : U \to V$ of degree d. We obtain $\chi(U) = d \cdot \chi(V) - \sum(d_j - 1)$, where d_j denotes the local degree of f near the critical point c_j, and j indexes all of the critical critical points of f. This reduces to $\sum r_j = d - 1$, where $r_j = d_j - 1$. By the lemma above, we see that $\deg(f^p|U') \leq \prod d_j \leq 2^{d-1}$. $\square$

We will say that the polynomial-like renormalization $f^p|U'$ is *good*, or that f *admits a good renormalization*, if $f(\mathrm{Crit}(f)) \subset \mathcal{K}$. The planar domain $V \setminus \mathcal{K}$ is a hyperbolic Riemann surface. For each $j = 0, \dots, p-1$, there is a unique, peripheral, simple, closed geodesic $\gamma_j \subset V \setminus \mathcal{K}$ going around K_j.

Lemma 9.3. *If $f : U \to V$ admits a good renormalization $f^p|U'$, then $2^{-(d-1)}|\gamma_0|_{V\setminus\mathcal{K}} \leq |\gamma_j|_{V\setminus\mathcal{K}} \leq 2^{d-1}|\gamma_0|_{V\setminus\mathcal{K}}$.*

Proof. Using the fact that $f^p|U'$ is good, we know that $f : U \setminus f^{-1}(\mathcal{K}) \to V \setminus \mathcal{K}$ is a covering map. Consider the index $j + 1$ of γ_{j+1} as an element of $\mathbb{Z}/p\mathbb{Z}$. Let $f^*\gamma_{j+1}$ denote the connected component of $f^{-1}(\gamma_{j+1})$ homotopic to γ_j in $f^{-1}(V \setminus \mathcal{K}) = U \setminus f^{-1}(\mathcal{K})$. By the Schwarz lemma, we know that if d_j is the local degree of f near K_j, then

$$|\gamma_{j+1}|_{V\setminus\mathcal{K}} = \frac{1}{d_j}|f^*\gamma_{j+1}|_{U\setminus f^{-1}(\mathcal{K})}.$$

Since $U \setminus f^{-1}(\mathcal{K})$ is contained in $V \setminus \mathcal{K}$, the Schwarz lemma also tells us that

$$|f^*\gamma_{j+1}|_{U\setminus f^{-1}(\mathcal{K})} \geq |f^*\gamma_{j+1}|_{V\setminus\mathcal{K}}.$$

Since the geodesic γ_j is homotopic to $f^*\gamma_{j+1}$ in $V \setminus \mathcal{K}$, we know that

$$|f^*\gamma_{j+1}|_{V\setminus\mathcal{K}} \geq |\gamma_j|_{V\setminus\mathcal{K}}.$$

Iterating these three inequalities, we see that

$$|\gamma_0|_{V\setminus\mathcal{K}} \geq \frac{1}{d_{p-1}d_{p-2}\cdots d_j}|\gamma_j|_{V\setminus\mathcal{K}} \geq \frac{1}{d_{p-1}\cdots d_0}|\gamma_0|_{V\setminus\mathcal{K}}.$$

To finish the proof, we apply lemma 9.1. □

Remark 9.1. One problem with polynomial-like renormalization is that there is too much freedom in the choice of domains. Even if a renormalizable polynomial-like map has large modulus, we can choose domains such that a polynomial-like renormalization has small modulus. Consequently, we want to define a renormalization that takes advantage of all the modulus available. This leads us to canonical renormalization, defined below.

9.2 Canonical renormalization

Let $\mathbf{f} : \mathbf{U} \to \mathbf{U}$ be a pseudo-polynomial-like map of degree d, consisting of the following data:

$$\begin{tikzcd} & \mathbf{U}' \arrow[dl, "i"'] \arrow[dr, "f"] & \\ \mathbf{U} & & \mathbf{U}. \end{tikzcd}$$

Let $g := f \circ i^{-1} : U \to V$ be a polynomial-like restriction of $\mathbf{f}$. Suppose that g admits a primitive renormalization, with period p, around one of its critical points. Let $\mathcal{K} = \bigcup K_j$ be the corresponding cycle of little Julia sets. Suppose that $g(\mathrm{Crit}(g)) \subset \mathcal{K}$. Under these conditions, we say that $\mathbf{f}$ is *primitively renormalizable* with *period p*.

Choose a little Julia set $K_j \subset \mathcal{K}$. We will define the *canonial renormalization* of $\mathbf{f}$ *around* K_j. The domain $\mathbf{U} \setminus \mathcal{K}$ is a hyperbolic Riemann surface. Let $\gamma \subset \mathbf{U} \setminus \mathcal{K}$ denote the geodesic homotopic to $K_j \subset \mathcal{K}$. Let $D \subset \mathbf{U}$ be the topological disk bounded by γ.

Let $A = A_\gamma(\mathbf{U} \setminus \mathcal{K})$ be the annulus covering space for $\mathbf{U} \setminus \mathcal{K}$ corresponding to γ, and let $\pi : A \to \mathbf{U} \setminus \mathcal{K}$ be a covering map. (See figure 2.1.) Identify γ with the core geodesic in A, and let $A_0 \subset A \setminus \gamma$ be the component on which π is injective. Gluing A and D by the conformal isomorphism $\pi : A_0 \to D \setminus K_j$, we obtain a topological disk $\mathbf{V}$.

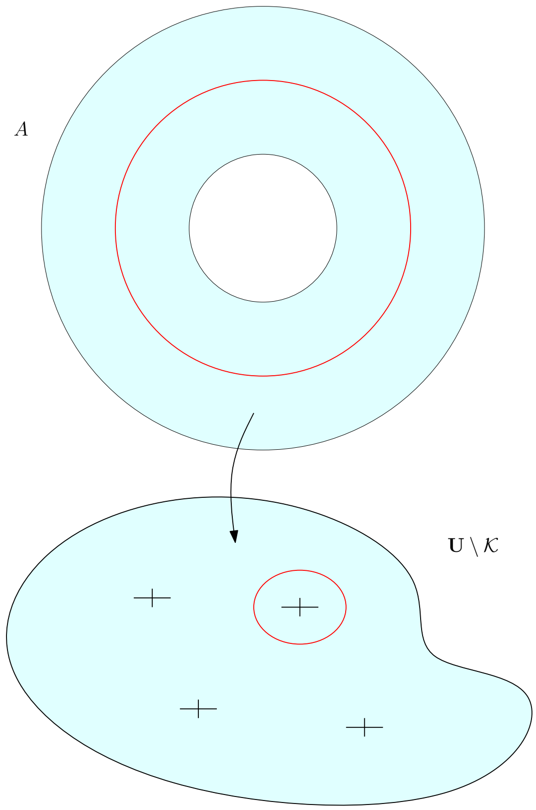

Figure 2.1: The annulus A covers $\mathbf{U} \setminus \mathcal{K}$. The red geodesic in A maps isometrically to the red geodesic in $\mathbf{U} \setminus \mathcal{K}$. The portion of A on the inside of the red geodesic maps bijectively to the portion of $\mathbf{U} \setminus \mathcal{K}$ on the inside of the red geodesic.

Let $\mathbf{f}^p : \mathbf{U}^p \to \mathbf{U}^p$ be the p-th iterate of $\mathbf{f}$. It consists of the following data:

$$\begin{tikzcd} & \mathbf{U}^p \arrow[dl, "i_p"'] \arrow[dr, "f_p"] & \\ \mathbf{U} & & \mathbf{U}. \end{tikzcd}$$

Define $\mathcal{K}^p = f_p^{-1}(\mathcal{K}) \subset \mathbf{U}^p$. Let $\gamma' \subset \mathbf{U}^p \setminus \mathcal{K}^p$ be the geodesic homotopic to K_j. Let $D' \subset \mathbf{U}^p$ be the topological disk bounded by γ.

Let $A' = A_{\gamma'}(\mathbf{U}^p \setminus \mathcal{K}^p)$ be the annulus covering space for $\mathbf{U}^p \setminus \mathcal{K}^p$, and let $\pi' : A' \to \mathbf{U}^p \setminus \mathcal{K}^p$ be a covering map. Identify γ' with the core geodesic in A', and let $A'_0 \subset A' \setminus \gamma'$ be the component on which π' is injective. Gluing A' and D' by the conformal isomorphism $\pi' : A'_0 \to D' \setminus K_j$, we obtain a topological disk $\mathbf{V}'$.

The holomorphic branched covering map $f_p : \mathbf{U}^p \to \mathbf{U}$ restricts to a covering map $f_p : \mathbf{U}^p \setminus \mathcal{K}^p \to \mathbf{U} \setminus \mathcal{K}$ and an isometry $f_p : \gamma' \to \gamma$. The covering map $f_p : \mathbf{U}^p \setminus \mathcal{K}^p \to \mathbf{U} \setminus \mathcal{K}$ lifts to a covering map $f' : A' \to A$. Similarly, the holomorphic immersion $i_p : \mathbf{U}^p \setminus \mathcal{K}^p \to \mathbf{U} \setminus \mathcal{K}$ lifts to an immersion $i' : A' \to A$.

$$\begin{tikzcd} A' \arrow[r, dashed, "i',f'"] \arrow[d, "\pi'"] & A \arrow[d, "\pi"] \\ \mathbf{U}^p \setminus \mathcal{K}^p \arrow[r, "i,f"] & \mathbf{U} \setminus \mathcal{K}. \end{tikzcd}$$

Abusing notation, let f' and i' denote the maps $\mathbf{V}' \to \mathbf{V}$ obtained by gluing the maps $f', i' : A' \to A$ with $f, i : \mathbf{U}^p \setminus \mathcal{K}^p \to \mathbf{U} \setminus \mathcal{K}$ by π and π'. Let $\mathbf{f}'$ denote the pseudo-polynomial-like map consisting of the following data:

$$\begin{tikzcd} & \mathbf{V}' \arrow[dl, "i'"'] \arrow[dr, "f'"] & \\ \mathbf{V} & & \mathbf{V} \end{tikzcd}$$

We call $\mathbf{f}'$ the *canonical renormalization* of $\mathbf{f}$ around K_j. By construction, we have $\operatorname{mod}(\mathbf{f}') = \pi/|\gamma|_{\mathbf{U}\setminus\mathcal{K}}$.

Lemma 9.4. *Suppose that $\mathbf{f} : \mathbf{U} \to \mathbf{U}$ admits a good renormalization of period p, with little Julia sets $\mathcal{K} = \bigcup_j K_j$. Let $\{\gamma_j\}_j$ denote the corresponding peripheral geodesics in $\mathbf{U} \setminus \mathcal{K}$. Then $2^{-(d-1)}|\gamma_0|_{\mathbf{U}\setminus\mathcal{K}} \leq |\gamma_j|_{\mathbf{U}\setminus\mathcal{K}} \leq 2^{d-1}|\gamma_0|_{\mathbf{U}\setminus\mathcal{K}}$.*

Proof. The pseudo-polynomial-like map consists of a holomorphic branched covering map f and a holomorphic immersion i. The proof of this lemma parallels that of lemma 9.3, with the immersion i taking the place of the inclusion map. $\square$

10 Quasisymmetric extensions and quasiconformal interpolations

Lemma 10.1. *Let C be a quasicircle, and let $A \subset \mathbb{C}$ be a topological annulus such that C is the inner boundary component of A. Let C' and A' be defined similarly. Let $f : A \to A'$ be a quasiconformal map. Then there is a quasiconformal map $F : \mathbb{C} \to \mathbb{C}$ such that $F|C = f$. If $\operatorname{mod} A \geq \mu > 0$, then $\operatorname{Dil} F$ is bounded in terms of μ and the dilatations of C and C'.*

Proof. Let G be the component of $\mathbb{C} \setminus C$ containing A. Let G' be defined similarly. Theorem 6.4 in [LV73] asserts that there exists a quasiconformal map $F : G \to G'$ with $F|C = f$ and $\operatorname{Dil} F$ controlled by μ and $\operatorname{Dil} f$. Quasiconformal reflection across C and C' completes the lemma. $\square$

Lemma 10.2. *Let $A \subset \mathbb{C}$ be a topological annulus bounded by quasicircles, with $m = \operatorname{mod} A$. Let A' and m' be defined similarly. Suppose that $f : \partial A \to \partial A'$ restricts to a quasisymmetric homeomorphism between the inner (respectively, outer) boundary components of A and A'. Then there is a quasiconformal map $F : A \to A'$ with $F|\partial A = f$ and $\operatorname{Dil} F$ controlled by the dilatations of the quasicircle boundaries, $\operatorname{Dil} f$, and $\max\{m/m', m'/m\}$.*

Proof. See proposition 2.30(b) in [BF14]. $\square$

11 Extremal length

Two good references for the topic of extremal length are [Ahl10] and [Oht70]. Given a path family Γ on a Riemann surface X, we let $\mathcal{L}(\Gamma)$ denote the extremal length of Γ. The *extremal width* of Γ is $\mathcal{W}(\Gamma) = 1/\mathcal{L}(\Gamma)$.

Lemma 11.1. *Let Γ and Γ' be path families on a Riemann surface X. If each $\gamma \in \Gamma$ contains some $\gamma' \in \Gamma'$, then $\mathcal{L}(\Gamma) \geq \mathcal{L}(\Gamma')$.*

Proof. See theorem 4-1 in [Ahl10]. $\square$

Lemma 11.2. *Let $f : X \to Y$ be a non-constant holomorphic map. Let Γ be a path family on X. Then $\mathcal{L}(f(\Gamma)) \geq \mathcal{L}(\Gamma)$.*

Proof. See theorem 2.11 and the remark preceeding theorem 2.14 in [Oht70]. Alternatively, see lemma 4.3 in [KL09c]. $\square$

Lemma 11.3. *Let $f : X \to Y$ be a K-quasiconformal map. Let Γ be a path family on X. Then $K^{-1}\mathcal{L}(\Gamma) \leq \mathcal{L}(f(\Gamma)) \leq K\,\mathcal{L}(\Gamma)$.*

Proof. See Theorem 2.2.1 in [FM07] for a proof when f is C^1. The same proof holds in general by Fuglede's theorem. $\square$

12 Weighted arc diagrams

The study of weighted arc diagrams first appeared in [Kah06].

Let S be a hyperbolic Riemann surface of finite type, without cusps. Assume that S is not homeomorphic to an annulus and that the ideal boundary of S, $\partial^I S$, is non-empty. We define $\overline{S} = S \cup \partial^I S$.

- A path $\gamma : (0, 1) \to S$ is *proper* (in S) if it admits a continuous extension $\gamma : [0, 1] \to \overline{S}$ such that $\gamma(0)$ and $\gamma(1)$ belong to $\partial^I S$. A homotopy $H : (0, 1) \times [0, 1] \to S$ is *proper* if for each t, $\gamma_t : s \mapsto H(s, t)$ is a proper path. We denote by $[\gamma] = [\gamma]_S$ the proper homotopy class of a proper path γ.

- An *arc* in S is the proper homotopy class of a proper path. We say that α is *trivial* if for any compact set $F \subset S$, there exists $\gamma \in \alpha$ such that $\gamma \subset S \setminus F$.

- Let $\mathcal{A} = \mathcal{A}(S)$ be the set of nontrivial arcs in S.

- An *arc diagram* X on S is a subset of $\mathcal{A}$ consisting of *non-crossing* arcs: For any α and β in X, there exist $\gamma \in \alpha$ and $\delta \in \beta$ such that $\gamma \cap \delta = \emptyset$.

- A *weighted arc diagram* W on S is a function $\mathcal{A}(S) \to [0, +\infty)$ such that

$$\operatorname{supp}(W) = \{\alpha \in \mathcal{A}(S) : W(\alpha) > 0\}$$

  is an arc diagram on S. If $f : R \to S$ is a proper, holomorphic map, then we define a weighted arc diagram f^*W on R by $(f^*W)(\alpha) = W(f_*\alpha)$.

- Let X and Y be weighted arc diagrams on S. Given $c \in [0, +\infty)$, we write $X \leq Y + c$ if for every $\alpha \in \operatorname{supp}(X)$, we have $X(\alpha) \leq Y(\alpha) + c$. Consequently, the set of weighted arc diagrams on S is partially ordered with respect to $\leq$.

- We define a norm on the set of weighted arc diagrams:

$$\|X\| = \sum_{\alpha \in \mathcal{A}} X(\alpha).$$

- The *canonical* weighted arc diagram W_{can} on S is defined in the following way. Choose a holomorphic covering map $\pi : \mathbb{D} \to S$. Let Γ be the group of deck transformations of π, and let $\Lambda \subset \partial\mathbb{D}$ be the limit set of Γ. Then S is conformally isomorphic to $\mathbb{D}/\Gamma$, and π extends continuously to a covering map $\pi : \overline{\mathbb{D}} \setminus \Lambda \to \overline{S}$. We can lift a representative γ of an arc α in $\mathcal{A}(S)$ to obtain a proper path $\widetilde{\gamma}$ in $\mathbb{D}$. The proper path $\widetilde{\gamma}$ joins two connected components, I and J, of $(\partial\mathbb{D}) \setminus \Lambda$, and there is a conformal map $\phi : \mathbb{D} \to (0, a + 2) \times (0, 1) \subset \mathbb{R}^2 \cong \mathbb{C}$ such that ϕ extends to a homeomorphism $(\overline{\mathbb{D}}, I, J) \to ([0, a + 2] \times [0, 1], [0, a + 2] \times \{0\}, [0, a + 2] \times \{1\})$. We define $W_{\text{can}}(\alpha) = \max\{0, a\}$. If $W_{\text{can}}(\alpha) > 0$, the conformal rectangle $\phi^{-1}((1, a+1) \times (0,1))$ embeds into S via π, and we call $R(\alpha) = \pi(\phi^{-1}((1, a+1) \times (0,1)))$ the *canonical rectangle* corresponding to α. It can be shown that $W_{\mathrm{can}}(\alpha)$ and $R(\alpha)$ are well-defined, independent of the choice of π and $\widetilde{\gamma}$. There is an obvious foliation of $R(\alpha)$ induced by the vertical foliation of $(1, a+1) \times (0,1)$. If α and β belong to $\mathrm{supp}(W_{\mathrm{can}})$, then $\alpha \neq \beta$ implies that $R(\alpha)$ and $R(\beta)$ are disjoint.

- A Borel set $\mathcal{F}$ is called a *proper lamination* on S if there are disjoint proper paths $L_\omega : (0,1) \to S$ such that $\mathcal{F} = \bigcup_\omega L_\omega((0,1))$. We can write $\mathcal{F} = \bigcup_{\alpha \in \mathcal{A}(S)} \mathcal{F}(\alpha)$, where $\mathcal{F}(\alpha) = \{L_\omega : \alpha = [L_\omega]\}$. By associating to α the extremal width $\mathcal{W}(\mathcal{F}(\alpha))$, we see that the proper lamination $\mathcal{F}$ induces a weighted arc diagram $W_{\mathcal{F}}$. Weighted arc diagrams induced by proper laminations are called *valid*. If $f : R \to S$ is a holomorphic covering map, then $f^*\mathcal{F}$ is a proper lamination on R, and $W_{f^*\mathcal{F}} = f^* W_{\mathcal{F}}$.

Lemma 12.1. *If X is an arc diagram on S, then $|X| \leq -3\chi(S)$. In particular, if S is homeomorphic to a disk minus N points, then $|X| \leq 3N$.*

Proof. Complete X to a triangulation, and use the fact that each triangle meets 3 edges, and each edge meets 2 triangles. □

12.1 Domination

Let U and V, with $U \subset V$, be hyperbolic Riemann surfaces of finite type, without cusps. Assume that U and V have non-empty ideal boundary and that neither U nor V is homeomorphic to an annulus.

Let $\gamma \subset V$ be a proper path. Then at most finitely many components of $\gamma \cap U$ are proper paths in U. Let $(\gamma_1, \ldots, \gamma_n)$ be the ordered set of these components. The *itinerary* of γ is the ordered set $I(\gamma) = ([\gamma_i])_i$, where $[\gamma_i]$ denotes the proper homotopy class of $\gamma_i \subset U$. Given a proper arc β in V and an ordered set $(\alpha_i)_i$ of proper arcs in U, we write $(\alpha_i)_i \longrightarrow \beta$ if there exists $\gamma \in \beta$ such that $I(\gamma) = (\alpha_i)_i$.

Let X be a weighted arc diagram on U, and let Y be a weighted arc diagram on V. We say that X *dominates* Y, denoted $X \multimap Y$, if we can write

$$X = \sum_i \sum_j w_{i,j} \alpha_{i,j},$$
$$Y \leq \sum_i v_i \beta_i,$$

where for each i, we have $(\alpha_{i,j})_j \longrightarrow \beta_i$ and $\sum_j w_{i,j}^{-1} \leq v_i^{-1}$. We will not deal directly with the definition of domination. Instead, domination will be appear as a hypothesis or conclusion of some of our lemmas and theorems.

The next two lemmas follow trivially from the definitions.

Lemma 12.2. *If $X + B \multimap Y$, then $X \multimap Y - \|B\|$.*

Lemma 12.3. *If $X \multimap Y \geq Z$, then $X \multimap Z$.*

12.2 Properties

Lemma 12.4. *Let W be a valid weighted arc diagram on S. Then for any arc α in $\mathcal{A}(S)$,*

$$W(\alpha) \leq \mathcal{W}(\alpha) \leq W_{\text{can}}(\alpha) + 2.$$

Proof. See lemma 3.2 in [Kah06]. □

Lemma 12.5. *If $f : U \to V$ is a holomorphic covering map of finite degree, then $W_{\text{can}}(U) = f^*(W_{\text{can}}(V))$.*

Proof. See lemma 3.3 in [Kah06]. □

Let $\mathcal{E}$ denote the set consisting of connected components of $\partial^I S$, and let $\mathcal{E}'$ be a subset of $\mathcal{E}$. An arc $\mathcal{A}(S)$ that joins elements of $\mathcal{E}'$ is called *horizontal*. An arc in $\mathcal{A}(S)$ that joins an element of $\mathcal{E}'$ and an element of $\mathcal{E} \setminus \mathcal{E}'$ is called *vertical*. We let $\mathcal{A}^{\text{h}}(S)$ denote the subset of $\mathcal{A}(S)$ consisting of horizontal arcs, and we define a weighted arc diagram $W_{\text{can}}^{\text{h}} = W_{\text{can}}|\mathcal{A}^{\text{h}}(S)$. We define $\mathcal{A}^{\text{v}}(S)$ and $W_{\text{can}}^{\text{v}}$ in the corresponding way.

Lemma 12.6. *If $f : U \to V$ is a holomorphic map such that $f_*(\mathcal{A}^{\text{h}}(U)) \subset \mathcal{A}^{\text{h}}(V)$, then $W_{\text{can}}^{\text{h}}(U) \leq f^*(W_{\text{can}}^{\text{h}}(V))$.*

Proof. See lemma 3.4 in [Kah06]. □

Lemma 12.7. *If $U \subset V$ (and $\pi_1(U) \hookrightarrow \pi_1(V)$ is surjective), then $W_{\text{can}}(U) \multimap W_{\text{can}}(V) - 6|\chi(U)|$.*

Proof. See corollary 3.10 in [Kah06]. □

12.3 Hyperbolic geometry

Let $W = W_{\text{can}}(S)$, and let $\gamma \subset S$ be a peripheral geodesic. We let $\langle W, \gamma \rangle = \sum_{\alpha \in \text{supp}(W)} W(\alpha)\langle \gamma, \alpha \rangle$, where $\langle \gamma, \alpha \rangle$ denotes the geometric intersection number of γ with α.

Lemma 12.8. *If γ is a peripheral geodesic in S, then*

$$|\gamma| = \pi\langle W, \gamma \rangle + O(p).$$

13 Moduli space and Teichmüller space

Let S be a hyperbolic Riemann surface. As a set, the *moduli space* of S, denoted $\mathrm{Mod}(S)$, consists of all hyperbolic Riemann surfaces quasiconformally equivalent to S, modulo conformal equivalence. Abusing notation, we identify a representative X with its conformal equivalence class $[X] \in \mathrm{Mod}(S)$. We equip $\mathrm{Mod}(S)$ with a metric: Given X and Y in $\mathrm{Mod}(S)$, we define

$$\mathrm{dist}(X, Y) = \inf \frac{1}{2} \log \mathrm{Dil}(\phi),$$

where the infimum is taken over all quasiconformal maps $\phi : X \to Y$.

As a set, the *reduced Teichmüller space* of S, denoted $\mathrm{Teich}^{\#}(S)$, consists of all pairs (X, f), where $f : S \to X$ is a quasiconformal map, modulo the equivalence $\sim$ defined in the following way: $(X, f) \sim (Y, g)$ if and only if $g \circ f^{-1}$ is homotopic to a conformal isomorphism $X \to Y$. Sometimes we will abuse notation by identifying (X, f) with its equivalence class $[X, f]$. We equip $\mathrm{Teich}^{\#}(S)$ with a metric: Given $[X, f]$ and $[Y, g]$ in $\mathrm{Teich}^{\#}(S)$, we define

$$\mathrm{dist}([X, f], [Y, g]) = \inf \frac{1}{2} \log \mathrm{Dil}(\phi),$$

where the infimum is taken over all quasiconformal maps $\phi : X \to Y$ homotopic to $g \circ f^{-1}$. If S has finite topology, then $\mathrm{Teich}^{\#}(S)$ is a contractible manifold of finite dimension. A standard reference for reduced Teichmüller spaces is [Ear67].

Chapter 3

Improvement of life

14 The improvement of life theorem

In this chapter, our goal is to prove an "improvement of life" theorem in the context of pseudo-polynomial-like maps with several critical points, generalizing theorem 9.1 in [Kah06].

Definition 14.1. Let $f : U \to V$ be a polynomial-like map admitting a primitive renormalization of period p around a critical point $c \in \mathrm{Crit}(f)$. Let $\mathcal{K} = \bigcup K_j$ denote the corresponding cycle of little Julia sets. We say that the p-renormalization of f under consideration is *good*, or that f admits a *good* renormalization of period p, if $f(\mathrm{Crit}(f)) \subset \mathcal{K}$. We say that a pseudo-polynomial-like map admits a *good* renormalization of period p if its polynomial-like restriction does.

Remark 14.1. This condition is automatically satisfied when there is only one critical point. It will be clear that this condition is reasonable after we prove lemma 22.2.

Theorem 14.1. *For any $\lambda > 1$ and any integer $d \geq 2$, there exists an integer $\underline{p} = \underline{p}(\lambda, d) \geq 2$ such that for any integer $\overline{p} \geq \underline{p}$, there exists $\mu = \mu(d, \overline{p}) > 0$ such that the following property holds: Let $\mathbf{f}$ be a pseudo-polynomial-like map of degree d admitting a good renormalization $\mathbf{f}'$ of period p, with $\underline{p} \leq p \leq \overline{p}$; if $\mathrm{mod}(\mathbf{f}') < \mu$, then $\mathrm{mod}(\mathbf{f}) \leq \lambda^{-1} \mathrm{mod}(\mathbf{f}')$.*

15 Combinatorics of arcs

In this section, let $f : U \to V$ be a polynomial-like map of degree d admitting a good renormalization of period p. Let $\mathcal{K} = \bigcup_{j=0}^{p-1} K_j$ denote the cycle of little

Julia sets corresponding to p, where for each $j \in \mathbb{Z}/p\mathbb{Z}$, $K_{j+1} = f(K_j)$.

15.1 Superattracting model F of f

By the straightening theorem, there is a hybrid conjugacy between f and a polynomial-like restriction of a polynomial $P : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ of degree d. Collapsing each little Julia set of P to a single point, we obtain a topological sphere, and P descends to a continuous map P_0 of degree d. The map P_0 has exactly one p-cycle that contains topological critical points, and except for the topological critical point at ∞, the remaining topological critical points land in this p-cycle after one iteration of P_0. By theorem B.1 in [McM94], P_0 is Thurston equivalent to a polynomial $F : \mathbb{C} \to \mathbb{C}$ of degree d. This polynomial is unique, up to affine conjugacy.

The map F has exactly one superattracting p-cycle, $\mathcal{O} = \{c_j\}_{j \in \mathbb{Z}/p\mathbb{Z}}$, and the finite critical points of F are either contained in this p-cycle or land in this p-cycle after one iteration of F. Let $\mathcal{D} \subset \mathbb{C}$ denote the closure of the immediate basin of $\mathcal{O}$, and let D_j denote the connected component of $\mathcal{D}$ containing c_j. Let d_j denote the local degree of F near c_j.

15.2 The tree associated with a good renormalization

Since F is hyperbolic, $K = K(F)$ is locally connected. In this case, the classical Hubbard tree of F is the legal hull of the critical points of F and their forward orbits. We will consider a slightly different tree, the legal hull $\hat{H}$ of $\mathcal{O}$. Let H be the corresponding finite, disked tree $\hat{H} \cup \mathcal{D}$. We think of the components of $\mathcal{D}$ as the vertices of H. Each component H_k of $\overline{H \setminus \mathcal{D}}$ is a contractible 1-complex.

Remark 15.1. See [DH] or [Dou93] for a detailed discussion of legal hulls and Hubbard trees.

Lemma 15.1. *$F(H) = H$.*

Proof. We can write $\hat{H}$ as a union of closed legal arcs $[a,b]_K$, where $a, b \in \mathcal{O} \cup (\hat{H} \cap \mathrm{Crit}(F))$, and the open legal arc $(a,b)_K$ satisfies $(a,b)_K \cap \mathrm{Crit}(F) = \emptyset$. Then $F([a,b]_K) = [F(a), F(b)]_K$ is the legal arc containing $F(a)$ and $F(b)$. Being a union of such legal arcs, $F(\hat{H})$ is contained in the legal hull of $F(\mathcal{O} \cup (\hat{H} \cap \mathrm{Crit}(F))) = \mathcal{O}$. Since $F(\hat{H})$ is a legally-convex set containing $\mathcal{O}$, it must contain the legal hull of $\mathcal{O}$. Hence, $F(\hat{H}) = \hat{H}$. It is obvious that $F(\mathcal{D}) = \mathcal{D}$, so $F(H) = H$. □

Definition 15.1. Given a disked tree H and a disk $D \subset H$, the *valence* of D in H, denoted $v(D, H)$, is the number of branches of H attached to D.

Lemma 15.2. *For each $j \in \mathbb{Z}/p\mathbb{Z}$, $v(D_j, H) \leq d_j \cdot v(D_{j+1}, H)$.*

Proof. Since F has finite degree, we know that $F^{-1}(H)$ is a finite, disked tree. We know that $H \subset F^{-1}(H)$, because $F(H) = H$. It follows that $v(D_j, H) \leq v(D_j, F^{-1}(H)) = d_j \cdot v(D_{j+1}, H)$. $\square$

Lemma 15.3. *$F^{-1}\mathcal{O}$ properly contains $\mathcal{O}$.*

Proof. Since $F(\mathcal{O}) = \mathcal{O}$, it is obvious that $\mathcal{O} \subset F^{-1}(\mathcal{O})$. Then we need only show that this containment is proper. Note that $p = |\mathcal{O}|$, and set $r = |F^{-1}(\mathcal{O})|$. The Euler characteristic of $\mathbb{C} \setminus \mathcal{O}$ is $\chi = 1 - p$, and the Euler characteristic of $\mathbb{C} \setminus F^{-1}(\mathcal{O})$ is $\chi' = 1 - r$. Since $F : \mathbb{C} \setminus F^{-1}(\mathcal{O}) \to \mathbb{C} \setminus \mathcal{O}$ is a covering map of degree d, we have $\chi' = d \cdot \chi$. It follows that $r = d(p - 1) + 1$. Observing that $d \geq 2$ and $p \geq 2$, we see that $r > p$. $\square$

15.3 Pulling back arcs

A path $\gamma \subset \overline{H \setminus \mathcal{D}}$ is *aligned* with H if its endpoints belong to distinct components of $\mathcal{D}$. A proper arc α in $\mathbb{C} \setminus \mathcal{D}$ is *aligned* with H if it is represented by a path aligned with H. Since any aligned arc is represented by exactly one aligned path, there is a bijection between the set of aligned paths and the set of aligned arcs. We let $\mathbf{H}$ denote the set of arcs (or paths, when it is more convenient) aligned with H.

Let S be a hyperbolic Riemann surface with finite topology. If γ and γ' are paths in S, let $\langle \gamma, \gamma' \rangle = |\gamma \cap \gamma'| \in [0, +\infty]$. Given proper arcs α and α' in S, define $\langle \alpha, \alpha' \rangle = \inf \langle \gamma, \gamma' \rangle$, where the infimum is taken over all proper paths γ and γ' in S representing α and α', respectively. If $\mathbf{A}$ and $\mathbf{A}'$ are sets of proper arcs in S, we define $\langle \mathbf{A}, \mathbf{A}' \rangle = \sum_{\alpha \in \mathbf{A}, \alpha' \in \mathbf{A}'} \langle \alpha, \alpha' \rangle$.

The *horizontal* paths and arcs in $\mathbb{C} \setminus \mathcal{D}$ are those having their endpoints in $\mathcal{D}$, and the *vertical* paths and arcs in $\mathbb{C} \setminus \mathcal{D}$ are those having one endpoint in $\mathcal{D}$ and one endpoint at $\infty \in \hat{\mathbb{C}}$. (We will ignore paths and arcs that are neither horizontal nor vertical. Such paths and arcs connect ∞ to itself.) We denote by $\mathbf{H}^{\perp}$ the set of vertical arcs α in $\mathbb{C} \setminus \mathcal{D}$ such that $\langle \alpha, \mathbf{H} \rangle = 0$. Clearly, $\mathbf{H}$ consists of the horizontal arcs α in $\mathbb{C} \setminus \mathcal{D}$ such that $\langle \alpha, \mathbf{H}^{\perp} \rangle = 0$.

Let γ be a proper path in $\mathbb{C} \setminus \mathcal{D}$. A *lift* of γ is a component of $F^{-1}(\gamma)$. Any lift of γ is a proper path in $\mathbb{C} \setminus F^{-1}(\mathcal{D})$. Since $\mathbb{C} \setminus F^{-1}(\mathcal{D}) \subset \mathbb{C} \setminus \mathcal{D}$, we can view any lift of γ as a path in $\mathbb{C} \setminus \mathcal{D}$, where it is possibly not proper. We will only be concerned with proper paths that represent non-trivial arcs in $\mathbb{C} \setminus \mathcal{D}$:

If γ represents a non-trivial arc, we let $F^*\gamma$ denote the union of proper (in $\mathbb{C} \setminus \mathcal{D}$) lifts of γ representing non-trivial arcs.

Let $\alpha = [\gamma]$ be a non-trivial, proper arc in $\mathbb{C} \setminus \mathcal{D}$. We define $F^*\alpha = \bigcup[\gamma']$, where the union is taken over all lifts $\gamma' \subset F^*\gamma$ of γ. By the homotopy lifting property, $F^*\alpha$ is independent of the choice of representative $\gamma \in \alpha$. It is possible that $F^*\alpha$ is the empty set. We say that α is *periodic* if there is an integer $\ell \geq 1$ such that $(F^*)^\ell\alpha \supset \alpha$.

Remark 15.2. The following lemma is called Pilgrim's lemma in several papers. (For an example, see [Kah06], from which the proof below was taken.)

Lemma 15.4 (Pilgrim's lemma)**.** *Let α be a horizontal arc in $\mathbb{C} \setminus \mathcal{D}$. Then α is not periodic.*

Proof. Suppose that α is periodic. Then there is an integer $\ell \geq 1$ such that $(F^*)^\ell\alpha \supset \alpha$. Let γ be the proper path in $\mathbb{C} \setminus \mathcal{D}$ representing α such that γ is geodesic for the hyperbolic metric of $\mathbb{C} \setminus \mathcal{O}$. Since $(F^*)^\ell\alpha \supset \alpha$, we can find an iterated lift $\gamma' \subset (F^*)^\ell\gamma$ that also represents α. The covering map $F^\ell : \mathbb{C} \setminus F^{-\ell}(\mathcal{O}) \to \mathbb{C} \setminus \mathcal{O}$, which is a local isometry, restricts to an isometry $\gamma' \to \gamma$. Together with lemma 15.3 and the fact that a proper arc contains a unique geodesic, we have

$$|\gamma|_{\mathbb{C}\setminus\mathcal{O}} = |\gamma'|_{\mathbb{C}\setminus F^{-\ell}\mathcal{O}} > |\gamma'|_{\mathbb{C}\setminus\mathcal{O}} \geq |\gamma|_{\mathbb{C}\setminus\mathcal{O}},$$

which is a contradiction. $\square$

Lemma 15.5. *Let α be a vertical arc in $\mathbb{C} \setminus \mathcal{D}$. If α is periodic, then $\alpha \in \mathbf{H}^\perp$.*

This lemma corresponds to lemma 4.6 in [Kah06], and the proof there applies in our case with no meaningful changes.

Proof. The vertical arc α is represented by proper paths in $\mathbb{C}\setminus\mathcal{D}$ joining D_j and ∞. We know that for a vertical arc α, the statement that $\alpha \notin \mathbf{H}^\perp$ is equivalent to $\langle \alpha, \mathbf{H} \rangle \geq 1$. Consequently, if $\alpha \notin \mathbf{H}^\perp$, then every $\gamma \in \alpha$ intersects a path aligned with H. In particular, $\gamma \cap K(F) \neq \emptyset$.

Any $\gamma \in \alpha$ can be decomposed as a concatenation of two sub-paths, one of which, γ', is the maximal segment of γ whose endpoints belong to $K(F)$. More explicitly, $\gamma' \subset \gamma$ begins with the endpoint of γ in some D_j and terminates at the last point where γ meets $K(F)$, after which the remaining segment of γ trails off to $\infty \in \hat{\mathbb{C}}$ without returning to $K(F)$. Let $\alpha' = \{\gamma' : \gamma \in \alpha\}$.

Since α is periodic, there is an integer $\ell \geq 1$ such that $(F^*)^\ell\alpha \supset \alpha$. Given $\gamma \in \alpha$, the containment $(F^*)^\ell\alpha \supset \alpha$ implies that we can choose a component

$\delta \subset (F^*)^\ell \gamma$ such that $\alpha = [\delta]$. Then $\delta' \in \alpha'$. The covering map $F^\ell : \mathbb{C} \setminus F^{-\ell}\mathcal{O} \to \mathbb{C} \setminus \mathcal{O}$ restricts to an isometry $\delta' \to \gamma'$.

Let $\mu = \inf\{|\gamma'|_{\mathbb{C}\setminus\mathcal{O}} : \alpha = [\gamma]\}$. Suppose that $\mu > 0$, and let γ' be a geodesic such that $\mu = |\gamma'|_{\mathbb{C}\setminus\mathcal{O}}$. Let δ' be the lift of γ' described in the paragraph above. Then

$$|\gamma'|_{\mathbb{C}\setminus\mathcal{O}} = |\delta'|_{\mathbb{C}\setminus F^{-\ell}(\mathcal{O})} > |\delta'|_{\mathbb{C}\setminus\mathcal{O}},$$

but this contradicts the fact that γ' has minimal length. This contradiction implies that $\mu = 0$, so α has a representative that does not return to $K(F)$ after first leaving $\mathcal{D}$. It follows that $\langle \alpha, \mathbf{H} \rangle_{\mathbb{C}\setminus\mathcal{D}} = 0$. □

Lemma 15.6. *Let α be a vertical arc in $\mathbb{C} \setminus \mathcal{D}$. Then there exists an integer $N \geq 0$ such that*

$$\bigcup_{n=0}^{N} (F^n)^* \alpha \supset \mathbf{H}^\perp.$$

This corresponds to lemma 4.7 in [Kah06], and again, there are no meaningful changes.

Proof. The proof has two steps. First, we show that by repeatedly pulling back a vertical arc, we can find a periodic arc. The previous lemma implies that this arc belongs to $\mathbf{H}^\perp$. Next, we show that by pulling back such an arc, we obtain all of $\mathbf{H}^\perp$.

We begin with the first step of the proof, pulling α back to an arc in $\mathbf{H}^\perp$. Let $\gamma \in \alpha$ be a proper path in $\mathbb{C} \setminus \mathcal{D}$ realizing $\langle \alpha, \mathbf{H} \rangle_{\mathbb{C}\setminus\mathcal{D}}$. Let γ' be a lift of γ. The covering map $F : \mathbb{C} \setminus F^{-1}\mathcal{D} \to \mathbb{C} \setminus \mathcal{D}$ restricts to a homeomorphism $F : \gamma' \to \gamma$, so

$$\langle \gamma', F^{-1}\mathbf{H} \rangle_{\mathbb{C}\setminus F^{-1}\mathcal{D}} = \langle \gamma, \mathbf{H} \rangle_{\mathbb{C}\setminus\mathcal{D}}.$$

Since $F^{-1}\mathbf{H} \supset \mathbf{H}$, we see that the number of points of intersection between γ' and $\mathbf{H}$ is at most the number of points of intersection between γ' and $F^{-1}\mathbf{H}$. Since $F^{-1}\mathcal{D} \supset \mathcal{D}$, these points of intersection belong to $\mathbb{C} \setminus \mathcal{D}$. It follows that $\langle \gamma', \mathbf{H} \rangle_{\mathbb{C}\setminus\mathcal{D}} \leq \langle \gamma', F^{-1}\mathbf{H} \rangle_{\mathbb{C}\setminus F^{-1}\mathcal{D}}$. We obtain a vertical arc $\alpha' = [\gamma']$ such that $\langle \alpha', \mathbf{H} \rangle_{\mathbb{C}\setminus\mathcal{D}} \leq \langle \alpha, \mathbf{H} \rangle_{\mathbb{C}\setminus\mathcal{D}}$.

Set $\alpha_0 = \alpha$. The procedure in the paragraph above provides us with a sequence of vertical arcs $\{\alpha_n\}$ such that for each integer $n \geq 0$, $\alpha_{n+1} \subset F^* \alpha_n$, and $\langle \alpha_{n+1}, \mathbf{H} \rangle_{\mathbb{C}\setminus\mathcal{D}} \leq \langle \alpha_n, \mathbf{H} \rangle_{\mathbb{C}\setminus\mathcal{D}}$. The sequence $\{\langle \alpha_n, \mathbf{H} \rangle_{\mathbb{C}\setminus\mathcal{D}}\}_n$ must eventually stabilize, so there exist integers $k \geq 0$ and $n_0 \geq 0$ such that $\langle \alpha_n, \mathbf{H} \rangle_{\mathbb{C}\setminus\mathcal{D}} = k$ whenever $n \geq n_0$. If $k = 0$, then $\alpha_{n_0} \in \mathbf{H}^\perp$. Otherwise, $k > 0$, and we use the fact that there are only finitely many vertical arcs β such that $\langle \beta, \mathbf{H} \rangle_{\mathbb{C}\setminus\mathcal{D}} = k$. By the pigeon-hole principle, there exists $n \geq n_0$ such that α_n is periodic, but by lemma 15.5, $\alpha_n \in \mathbf{H}^\perp$.

Now we proceed to the second step of the proof, showing that we can obtain all of $\mathbf{H}^\perp$ by pulling back any one arc in $\mathbf{H}^\perp$. Suppose that for some $j \in \mathbb{Z}/p\mathbb{Z}$, we have already obtained the maximal number of arcs in $\mathbf{H}^\perp$ emanating from D_j; the reasoning in the proof of lemma 15.2 implies that these arcs pull back to the maximal number of arcs in $\mathbf{H}^\perp$ emanating from D_{j-1}. Any arc in $\mathbf{H}^\perp$ can be pulled back to an arc in $\mathbf{H}^\perp$ emanating from a vertex of valence one. Since this is the maximal number of arcs emanating from that vertex, we will be finished after pulling this arc back $p-1$ more times. $\square$

16 Trees of complete graphs

Let $\mathbf{G} = (\{D_j\}, \mathbf{H})$ be the graph whose vertices are the disks of $\mathcal{D} = \bigcup D_j$ and whose edges are the paths of $\mathbf{H}$. For each integer $n \geq 0$, we define

- the union of disks $\mathcal{D}^n = F^{-n}(\mathcal{D}) = \bigcup_k D_k^n$, where we have enumerated the components of $\mathcal{D}^n$ by D_k^n,
- the finite disked tree $H^n = F^{-n}(H)$,
- the set $\mathbf{H}^n$ of paths aligned with H^n, and
- the graph $\mathbf{G}^n = (\{D_k^n\}_k, \mathbf{H}^n)$.

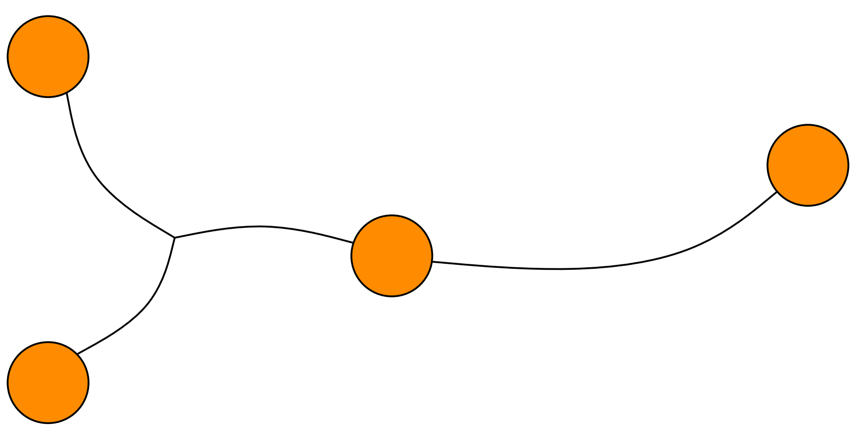

Figure 3.1: The disked tree H has a tripod on the left.

Remark 16.1. The "edges" in the tree H^n are simply connected, proper 1-complexes in $\mathbb{C} \setminus \mathcal{D}^n$. These are not the same as the edges in the graph $\mathbf{G}^n$. The edges in the graph $\mathbf{G}^n$ are obtained by opening up the 1-complexes joining

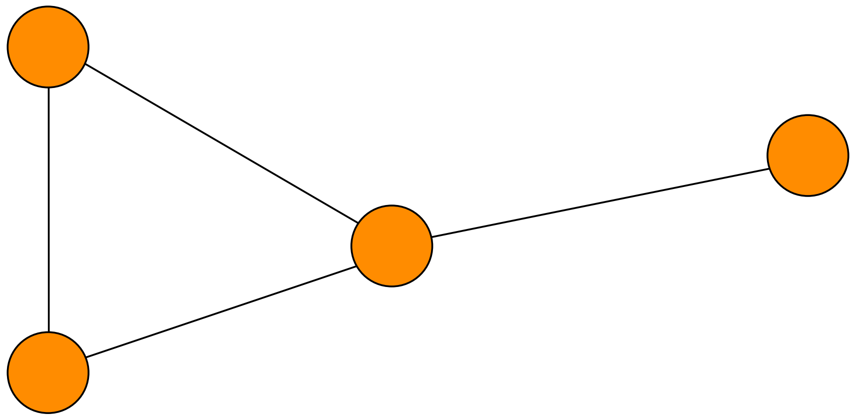

Figure 3.2: This is the abstract graph $\mathbf{G}$, where the tripod has been opened up.

disks of $\mathcal{D}^n$. (See figures 3.1 and 3.2.) For example, a tripod opens up to a triangle.

Definition 16.1. A *tree of complete graphs* is defined inductively in the following way:

- Any complete graph G is a tree of complete graphs.
- Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be trees of complete graphs. Given vertices $v_1 \in V_1$ and $v_2 \in V_2$, the one-point union $G_1 \sqcup G_2/(v_1 \sim v_2)$ is a tree of complete graphs.

Remark. The "tree" structure in a tree of complete graphs is as follows: We can think of a tree of complete graphs as a tree, in the usual sense, whose vertices are complete graphs.

Lemma 16.1. *For each integer $n \geq 0$, $\mathbf{G}^n$ is a tree of complete graphs.*

Proof. See the proof of lemma 4.1 in [Kah06]. $\square$

Definition 16.2. Let $G = (V, E)$ be a tree of complete graphs obtained through one-point unions of finitely many complete graphs G_k. Let $x, y \in V$. Let P be the union of $\{x, y\}$ and the set of $v \in V$ such that $G \setminus \{v\}$ is disconnected, and x and y belong to distinct components of $G \setminus \{v\}$. By lemma 11.10 in [Kah06], there is a unique ordering on the set S such that as an ordered set, $P = (x = v_0, v_1, \ldots, v_N = y)$, and

- for each integer $j \in \{0, \ldots, N-1\}$, v_j and v_{j+1} belong to the same complete graph $G_{k(j)}$,

- each graph in $\{G_{k(j)}\}_j$ meets at least one, but at most two, other graphs in $\{G_{k(j)}\}_j$, and
- any vertex in $\bigcup_j G_{k(j)} \subset G$ belongs to at most two graphs in $\{G_{k(j)}\}_j$.

We call the ordered set P the *chain of vertices* in G between x and y.

Now consider the tree of complete graphs $\mathbf{G}^n$. Any path γ aligned with H joins two distinct disks $D, D' \subset \mathcal{D}$. By the paragraph above, there is a unique chain $(D = D_0^n, \dots, D_N^n = D')$ in the complete graph $\mathbf{G}^n$ between D and D'.

Lemma 16.2. *For any integer $r \geq 1$ and any path γ aligned with H, the chain $(D_0^{rp}, \dots, D_N^{rp})$ in $\mathcal{D}^{rp}$ in $\mathbf{G}^{rp}$ between $\partial\gamma := \{D_0^{rp}, D_N^{rp}\} \subset \mathcal{D}$ contains at least 2^{r-1} disks belonging to $\mathcal{D}^{rp} \setminus \mathcal{D}^{(r-1)p}$.*

Proof. We claim that for any integer $r \geq 0$, if γ is aligned with H^{rp}, then γ crosses a disk of $\mathcal{D}^{(r+1)p} \setminus \mathcal{D}^{rp}$. Consider $r = 0$. Let γ be a path aligned with H. Then $F^p(\gamma) \subset H$. Then F^p maps γ over a disk of $\mathcal{D}$, because if this were not the case, then γ would represent a horizontal arc of period p. Since $\gamma \subset \mathbb{C} \setminus \mathcal{D}$, γ must cross a disk of $\mathcal{D}^p \setminus \mathcal{D}$. We will now show that the statement holds for any $r \geq 1$. Let γ be aligned with H^{rp}. Then γ joins disks of $\mathcal{D}^{rp}$. If γ does not cross any disk of $\mathcal{D}^{(r+1)p} \setminus \mathcal{D}^{rp}$, then $F^{rp}\gamma$ does not cross any disk of $\mathcal{D}^p \setminus \mathcal{D}$. Since $F^{rp}\gamma$ is aligned with H, this is a contradiction for the case $r = 0$.

The argument in the paragraph above, shows that, when $r \geq 1$ is an integer, between any two disks of $\mathcal{D}^{(r-1)p}$, there is a disk of $\mathcal{D}^{rp} \setminus \mathcal{D}^{(r-1)p}$. This implies that there is at least one disk of $\mathcal{D}^p \setminus \mathcal{D}$ between $\partial\gamma$. An obvious induction implies that there are at least 2^{r-1} disks of $\mathcal{D}^{rp} \setminus \mathcal{D}^{(r-1)p}$ between $\partial\gamma$. □

17 Restrictions of one pseudo-polynomial-like map

17.1 Restrictions of $W^{\mathrm{v+h}}_{\mathrm{can}}$

Let us return to the discussion of improvement of life: We have a pseudo-polynomial-like map $\mathbf{f} : (\mathbf{U}, \mathbf{K}) \to (\mathbf{U}, \mathbf{K})$ admitting a good renormalization of period p, with cycle of little Julia sets $\mathcal{K} \subset \mathbf{U}$. For any integer $n \geq 1$, we have the n-th iterate of $\mathbf{f}$, $\mathbf{f}^n$, which consists of the following data:

$$\begin{tikzcd} & \mathbf{U}^n \arrow[dl, "i_n"'] \arrow[dr, "f_n"] & \\ \mathbf{U} & & \mathbf{U}. \end{tikzcd}$$

We have the restriction of $\mathbf{f}$ to $\mathbf{U}^n$, which consists of the following data:

$$\begin{tikzcd} & \mathbf{U}^n \arrow[dl, "i"'] \arrow[dr, "f"] & \\ \mathbf{U}^{n-1} & & \mathbf{U}^{n-1}. \end{tikzcd}$$

Define $\widetilde{\mathcal{K}} = f^{-1}(\mathcal{K})$. We have the following diagram:

$$\begin{tikzcd} \mathbf{U}^n \setminus \widetilde{\mathcal{K}} \arrow[r, hook] \arrow[d, "f"] & \mathbf{U}^n \setminus \mathcal{K} \arrow[d, "i"] \\ \mathbf{U}^{n-1} \setminus \mathcal{K} & \mathbf{U}^{n-1} \setminus \mathcal{K}. \end{tikzcd}$$

In the diagram above, f is a covering map, so it is proper. The map i is proper on $\mathcal{K}$. The horizontal map is proper on $\mathcal{K} \cup \partial\mathbf{U}^n$.

Lemma 17.1. *For any integer $n \geq 1$, $\|W_{\text{can}}^{\text{v+h}}(\mathbf{U}\setminus\mathcal{K})\| \leq \|W_{\text{can}}^{\text{v+h}}(\mathbf{U}^n\setminus\mathcal{K})\|+6p$.*

Proof. The immersion $i_n : \mathbf{U}^n \setminus \mathcal{K} \to \mathbf{U} \setminus \mathcal{K}$ is proper on $\mathcal{K}$. Any vertical path in $\mathbf{U} \setminus \mathcal{K}$ touching K_j contains a path that lifts by i_n to a vertical path in $\mathbf{U}^n \setminus \mathcal{K}$ touching K_j. Any horizontal path touching K_j either lifts to itself or contains a path that lifts to a vertical path touching K_j. Consequently, the proper foliation $\Gamma_{\text{can}}^{\text{v+h}}$ overflows a family of paths in $\mathbf{U} \setminus \mathcal{K}$ that lifts by i_n to a proper foliation Γ on $\mathbf{U} \setminus \mathcal{K}$ consisting of horizontal and vertical paths. Applying lemma 11.1 and lemma 11.2, we obtain $\mathcal{L}(\Gamma) \leq \mathcal{L}(i_n(\Gamma)) \leq \mathcal{L}(\Gamma_{\text{can}}^{\text{v+h}})$. Then

$$\|W_{\text{can}}^{\text{v+h}}(\mathbf{U} \setminus \mathcal{K})\| = \mathcal{W}(\Gamma_{\text{can}}^{\text{v+h}}) \leq \mathcal{W}(\Gamma).$$

Since Γ is a proper foliation on $\mathbf{U}^n \setminus \mathcal{K}$, lemma 12.4, together with the fact that $|\operatorname{supp} W_{\text{can}}^{\text{v+h}}(\mathbf{U}^n \setminus \mathcal{K})| \leq 3p$, tells us that $\mathcal{W}(\Gamma) \leq \|W_{\text{can}}^{\text{v+h}}(\mathbf{U}^n \setminus \mathcal{K})\| + 2 \cdot 3p$. $\square$

17.2 The pulled-back $W_{\text{can}}^{\text{h}}$ is aligned with H

Let $\mathbf{U}^0 = \mathbf{U}$. Let $q_0 = 0$, and for each integer $n \geq 1$, define $q_{n+1} = 6d(p-1) + 3pq_n$. For each integer $n \geq 0$, let $X^n = W_{\text{can}}^{\text{h}}(\mathbf{U}^n \setminus \mathcal{K})$, and let $\hat{X}^n = X^n - q_n$.

Lemma 17.2. $\hat{X}^{n+1} \leq i^*\hat{X}^n$.

Proof. Since $i : \mathbf{U}^{n+1} \setminus \mathcal{K} \to \mathbf{U}^n \setminus \mathcal{K}$ is proper on $\mathcal{K}$, lemma 12.6 implies $X^{n+1} \leq i^*X^n$. Since $q_{n+1} \geq q_n$, we are finished. $\square$

Lemma 17.3. $f^*\hat{X}^n \multimap \hat{X}^{n+1}$.

Proof. Since $f : \mathbf{U}^{n+1} \setminus \widetilde{\mathcal{K}} \to \mathbf{U}^n \setminus \mathcal{K}$ is a covering map of degree d, we have $\chi(\mathbf{U}^{n+1} \setminus \widetilde{\mathcal{K}}) = d \cdot \chi(\mathbf{U}^n \setminus \mathcal{K}) = d(1-p)$, and $W_{\mathrm{can}}(\mathbf{U}^{n+1} \setminus \widetilde{\mathcal{K}}) = f^* W_{\mathrm{can}}(\mathbf{U}^n \setminus \mathcal{K})$. By lemma 12.7, $\mathbf{U}^{n+1} \setminus \widetilde{\mathcal{K}} \subset \mathbf{U}^{n+1} \setminus \mathcal{K}$ implies

$$W_{\mathrm{can}}(\mathbf{U}^{n+1} \setminus \widetilde{\mathcal{K}}) \multimap W_{\mathrm{can}}(\mathbf{U}^{n+1} \setminus \mathcal{K}) - 6 \cdot d(p-1).$$

Observing that horizontal arcs in $\mathbf{U}^{n+1} \setminus \mathcal{K}$ restrict to horizontal arcs in $\mathbf{U}^{n+1} \setminus \widetilde{\mathcal{K}}$, we obtain

$$W^{\mathrm{h}}_{\mathrm{can}}(\mathbf{U}^{n+1} \setminus \widetilde{\mathcal{K}}) \multimap W^{\mathrm{h}}_{\mathrm{can}}(\mathbf{U}^{n+1} \setminus \mathcal{K}) - 6 \cdot d(p-1).$$

Since f preserves the property of an arc being horizontal or vertical, we know that $W^{\mathrm{h}}_{\mathrm{can}}(\mathbf{U}^{n+1} \setminus \widetilde{\mathcal{K}}) = f^* X^n$. Applying lemma 12.3 to

$$f^*\hat{X}^n + q_n = f^* X^n \multimap X^{n+1} - 6 \cdot d(p-1),$$

we obtain

$$f^*\hat{X}^n \multimap X^{n+1} - 6 \cdot d(p-1) - \|q_n\| \geq X^{n+1} - 6 \cdot d(p-1) - 3pq_n = \hat{X}^{n+1}.$$

$\square$

Invariant horizontal arc diagrams are aligned with H

Definition 17.1. Let U and V be Riemann surfaces, with $U \subset V$. Let $\mathbf{A}$ and $\mathbf{B}$ be multiarcs on U and V, respectively. We say that $\mathbf{A} \rightsquigarrow \mathbf{B}$ if for every arc $\beta \in \mathbf{B}$, there is a set of arcs $\{\alpha_k\} \subset \mathbf{A}$ such that $(\alpha_k) \longrightarrow \beta$.

We say that a horizontal arc diagram $\mathbf{A}$ on $\mathbb{C} \setminus \mathcal{D}$ is **aligned** with H if $\langle \mathbf{A}, \mathbf{H}^\perp \rangle = 0$. We say that $\mathbf{A}$ is F**-invariant** if $F^*\mathbf{A} \rightsquigarrow \mathbf{A}$.

Lemma 17.4. *Let $\mathbf{A}$ be an F-invariant horizontal arc diagram. Then $\mathbf{A}$ is aligned with H.*

Proof. Represent the arcs of $\mathbf{A}$ by the corresponding unique proper paths in $\mathbb{C} \setminus \mathcal{D}$ that are geodesic for the hyperbolic metric of $\mathbb{C} \setminus \mathcal{O}$, and let A be the set of these geodesic segments. Observing that the paths of A are disjoint and that the endpoints in $\partial\mathcal{D}$ of two different paths of A are disjoint, we see that there is a vertical path γ in $\mathbb{C} \setminus \mathcal{D}$ such that $\langle \gamma, A \rangle = 0$.

Let $\gamma' \subset F^*\gamma$ be a component. The covering map $F : \mathbb{C} \setminus F^{-1}\mathcal{D} \to \mathbb{C} \setminus \mathcal{D}$ restricts to a homeomorphism $F : \gamma' \to \gamma$. It follows that $\langle F^*\gamma, F^{-1}(\overline{A \setminus \mathcal{D}}) \rangle = 0$. Since $F^*\mathbf{A} \to \mathbf{A}$, we have $\langle F^*\gamma, \mathbf{A} \rangle = 0$.

Repeating the procedure in the paragraph above, we find that for each integer $n \geq 0$, $\langle (F^*)^n\gamma, \mathbf{A} \rangle = 0$. By lemma 15.6, all of $\mathbf{H}^\perp$ is obtained as $\bigcup_n (F^*)^n [\gamma]_{\mathbb{C} \setminus \mathcal{D}}$, so we see that $\langle \mathbf{H}^\perp, \mathbf{A} \rangle = 0$. $\square$

Lemma 17.5. *For each integer $n \geq 3p$,* $\operatorname{supp} \hat{X}^n$ *is aligned with H.*

Proof. For each integer $n \geq 0$, set $\mathbf{A}^n = \operatorname{supp} \hat{X}^n$. Then lemma 17.3 implies $f^*\mathbf{A}^n \rightsquigarrow \mathbf{A}^{n+1}$, and lemma 17.2 implies $\mathbf{A}^{n+1} \subset \mathbf{A}^n | \mathbf{U}^{n+1} \backslash \mathcal{K}$. Since $|\mathbf{A}^0| \leq 3p$, there must be an integer m, with $0 \leq m \leq 3p$, such that $\mathbf{A}^{m+1} = \mathbf{A}^m | \mathbf{U}^{n+1} \backslash \mathcal{K}$, in which case $\mathbf{A}^m$ is f-invariant. Then $\mathbf{A}^n$ is f-invariant for each integer $n \geq m$. Applying lemma 17.4, we obtain the result. $\square$

17.3 The main inequality

In this section, we will need a superficial amount of the theory of electric circuits developed in [Kah06].

An *unplugged electric circuit* $\mathcal{C} = (G, W)$ consists of a graph $G = (V, E)$ and a function $W : E \to [0, +\infty)$ weighting the edges E. Given $v \in V$, we define $W|v = \sum_{v' \sim v} W(\{v, v'\})$, where the sum is taken over all vertices $v' \in V$ joined to v by an edge $\{v, v'\} \in E$. We will consider the unplugged electric circuits obtained in the following way. Let H be a disked tree, and let Y be a weighted arc diagram such that $\operatorname{supp}(Y)$ is aligned with H. Let $\mathbf{H}$ be the set of paths aligned with H, and let $\mathbf{G}$ be the associated tree of complete graphs. We obtain an unplugged electric circuit $\mathcal{C}_Y = (\mathbf{G}, Y)$, by letting the weight of an edge $e \in \mathbf{H}$ be $Y(e)$. (See sections 6.1 and 11.1 in [Kah06].)

Given two unplugged electric circuits $\mathcal{C}$ and $\mathcal{C}'$, there is a notion of when $\mathcal{C}'$ dominates $\mathcal{C}$, denoted $\mathcal{C}' \multimap \mathcal{C}$. In what follows, we will mostly rely on theorems that either yield domination as a consequence or exploit domination as a hypothesis. In addition to these theorems, we will only need one inequality that appears in the definition of domination.

Before we state the definition of domination for unplugged electric circuits, we point out that there is the notion of an *electric circuit* that is not unplugged; in other words, it is an unplugged electric circuit $\mathcal{C} = (G, W)$ equipped with a *battery*, which is a choice of two vertices of G. Having equipped $\mathcal{C}$ with a battery, there is a notion of the *total conductance* $\mathbf{W}(\mathcal{C}) \in [0, +\infty)$ of $\mathcal{C}$. (See section 11.6 in [Kah06] for the definition of total conductance.) For our purposes, it suffices to know the inequality that appears in the following definition.

Definition 17.2. Let $\mathcal{C} = (G, W)$ be an unplugged electric circuit. Let $\mathcal{C}' = (G', W')$ be an unplugged electric circuit, where G' is a graph obtained from G by replacing some of its edges $e \in E(G)$ with graphs $G'(e) \subset G'$. Given an edge $e \subset E(G)$, let $\mathcal{C}'(e)$ denote the restriction of $\mathcal{C}'$ to e, viewed as an electric circuit with battery ∂e. We say that $\mathcal{C}'$ *dominates* $\mathcal{C}$, denoted $\mathcal{C}' \multimap \mathcal{C}$, if for each edge $e \in E(G)$, we have

$$\mathbf{W}(\mathcal{C}'(e)) \geq W(e).$$

Lemma 17.6. *Let H and H' be disked trees, with $H' \supset H$, and let Y and Y' be weighted arc diagrams aligned with H and H', respectively. If $Y' \multimap Y$, then $\mathcal{C}_{Y'} \multimap \mathcal{C}_Y$.*

Proof. See lemma 6.1 in [Kah06]. □

Lemma 17.7. *Let $\mathcal{C} = (G, W)$ and $\mathcal{C}' = (G', W')$ be unplugged electric circuits. If $\mathcal{C}' \multimap \mathcal{C}$, then for any $D \in V(G)$, $W'|D \geq W|D$.*

Proof. See lemma 11.9 in [Kah06]. □

Lemma 17.8. *Let G be a tree of complete graphs. Let $\mathcal{C}$ be an electric circuit based on $G = (V, E)$, equipped with a battery $\{a, b\} \subset V$, and let $(a = x_0, x_1, \ldots, x_{N-1}, x_N = b)$ be the chain of vertices connecting a and b. Then*

$$\mathbf{W} \leq \bigoplus_{k=1}^{N} W|x_k.$$

Proof. Note that in the inequality above, we have taken the sum from 1 to N rather than from 0 to N. See lemma 11.11 in [Kah06]. □

Having discussed the necessary theory of electric circuits, we can begin working toward the main inequality.

Lemma 17.9. *Let Y and Z be weighted arc diagrams aligned with H such that $F^*Y \multimap Z$. If D and D' are disks of H, then $Z|D \leq 2^{d-1}Y|D'$.*

Compare with lemma 6.2 in [Kah06].

Proof. Assume that $j, k \in \mathbb{Z}/p\mathbb{Z}$ with $j < k$. By lemma 17.6, $\mathcal{C}_{F^*Y} \multimap \mathcal{C}_Z$. Then lemma 17.7 implies $Z|D_j \leq F^*Y|D_j$. Observing that

$$F^*Y|D_j = \deg(F : D_j \to D_{j+1})Y|D_{j+1},$$

and

$$Y|D_{j+1} \leq (F^*)^{k-(j+1)}Y|D_{j+1} = \deg(F^{k-(j+1)} : D_{j+1} \to D_k)Y|D_k,$$

we obtain $F^*Y|D_j \leq \deg(F^{k-j} : D_j \to D_k)Y|D_k \leq 2^{d-1}Y|D_k$, which is the desired inequality. □

The corollary below will be used in lemma 18.1.

Corollary 17.10. *Let Y and Z be weighted arc diagrams aligned with H such that $F^*Y \multimap Z$. Then*

$$\max_j (Z|D_j) \leq \frac{2^d}{p}\|Y\|.$$

Proof. From lemma 17.9, we deduce that for each $k \in \mathbb{Z}/p\mathbb{Z}$,

$$\max_j (Z|D_j) \leq 2^{d-1} Y|D_k.$$

Summing these inequalities over k, we obtain

$$p \cdot \max_j (Z|D_j) \leq 2^{d-1} \sum_{k=0}^{p-1} Y|D_k = 2^{d-1} \cdot 2\|Y\|.$$

□

Lemma 17.11 (The main inequality)**.** *Let Y and Z be weighted arc diagrams aligned with H. If $r \geq 2$ is an integer such that $(F^*)^{rp}Y \multimap Z$, then for any $\alpha \in \mathbf{H}$, $Z(\alpha) \leq 2^{d-r} \max_j Y|D_j$.*

Proof. Since Y is aligned with H, the weighted arc diagram $(F^{rp})^*Y$ is aligned with the disked tree $F^{-rp}H$. Since H is forward invariant under F, $F^{-rp}H \supset H$. By lemma 17.6, $\mathcal{C}_{(F^*)^{rp}Y} \multimap \mathcal{C}_Z$. By definition of domination for electric circuits, we have for each $\alpha \in \operatorname{supp} Z$,

$$Z(\alpha) \leq \mathbf{W}(\mathcal{C}_{(F^*)^{rp}Y}(\alpha)).$$

Let $(D_0^{rp}, \ldots, D_N^{rp})$ be the chain of disks in $\mathcal{D}^{rp} \subset \mathbf{G}^{rp}$ connecting $\partial\alpha = \{D_0^{rp}, D_N^{rp}\}$. By lemma 17.8, we have

$$\mathbf{W}(\mathcal{C}_{(F^{rp})^*Y}(\alpha)) \leq \bigoplus_{k=1}^{N} (F^*)^{rp}Y|D_k^{rp}.$$

Let I be the set of $k \in \{1, \ldots, N\}$ such that D_k^{rp} is a disk of $\mathcal{D}^{rp} \setminus \mathcal{D}^{(r-1)p}$. Then

$$\bigoplus_{k=1}^{N} (F^*)^{rp}Y|D_k^{rp} \leq \bigoplus_{k\in I} (F^*)^{rp}Y|D_k^{rp} \leq \frac{1}{|I|} \max_{k\in I} (F^*)^{rp}Y|D_k^{rp}.$$

By lemma 16.2, we know that $|I| \geq 2^{r-1}$, so $1/|I| \leq 1/2^{r-1}$. Combining all of these inequalities, we obtain

$$Z(\alpha) \leq \frac{1}{2^{r-1}} \max_{k \in I} (F^*)^{rp} Y | D_k^{rp}.$$

Observe that $F^{rp} = F^p \circ F^{(r-1)p}$ maps disks of $\mathcal{D}^{rp} \setminus \mathcal{D}^{(r-1)p}$ onto disks of $\mathcal{D}$, passing over the critical orbit $\mathcal{O}$ at most one time. More precisely, $F^{(r-1)p}$ maps any disk of $\mathcal{D}^{rp} \setminus \mathcal{D}^{(r-1)p}$ bijectively to a disk of $\mathcal{D}^p \setminus \mathcal{D}$. Applying F^p, we we pass over the cycle of disks $\mathcal{D} \supset \mathcal{O}$ at most once. Hence, if

- $D_k^{rp} \in \mathcal{D}^{rp} \setminus \mathcal{D}^{(r-1)p}$,
- $D_j \in \mathcal{D}$, and
- $F^{rp} : D_k^{rp} \to D_j$,

then

$$(F^*)^{rp} Y | D_k^{rp} = \deg(F^{rp} : D_k^{rp} \to D_j) Y | D_j \leq 2^{d-1} Y | D_j.$$

The desired inequality follows. □

18 The restricted vertical weight controls total weight

Lemma 18.1. *For any integer $n \geq 5$, we have $\|\hat{X}^{np}\| \leq 3 \cdot 2^{2d-n+4} \|\hat{X}\|$.*

Proof. It follows from lemma 17.3 that if ℓ and m are nonnegative integers such that $\ell < m$, then $(F^*)^{m-\ell} \hat{X}^\ell \multimap \hat{X}^m$. In particular, $(F^*)^{(n-4)p} \hat{X}^{4p} \multimap \hat{X}^{np}$. Applying the main inequality (lemma 17.11), we see that for any $\alpha \in \mathbf{H}$, we have

$$\hat{X}^{np}(\alpha) \leq 2^{d-(n-4)} \max_j (\hat{X}^{4p} | D_j).$$

Applying corollary 17.10 to $F^* \hat{X}^{3p} \multimap \hat{X}^{4p}$, we obtain

$$\max_j (\hat{X}^{4p} | D_j) \leq \frac{2^d}{p} \|\hat{X}^{3p}\|,$$

and combining these inequalities, we see that

$$\hat{X}^{np}(\alpha) \leq \frac{2^{2d-n+4}}{p} \|\hat{X}^{3p}\|.$$

Summing over all $\alpha \in \operatorname{supp} \hat{X}^{np}$ and using the fact that $|\operatorname{supp} \hat{X}^{np}| \le 3p$, we compute

$$\|\hat{X}^{np}\| \le 3p \cdot \frac{2^{2d-n+4}}{p} \|\hat{X}^{3p}\| \le 3 \cdot 2^{2d-n+4} \|\hat{X}^{3p}\|.$$

It follows from lemma 17.2 that $\|\hat{X}^{3p}\| \le \|\hat{X}\|$, and we are finished. $\square$

Corollary 18.2. *For any integer $n \ge 5d$, we have*

$$\|\hat{X}^{np}\| \le \frac{3}{4} \|\hat{X}\|.$$

Proof. Obviously, $3 \cdot 2^{2d-n+4} \le 3/4$ is equivalent to $2d + 6 \le n$. Since $d \ge 2$, the latter condition is satisfied when $n \ge 5d$. $\square$

Lemma 18.3. *Fix an integer $n \ge 5d$. There is a constant $Q_1 = Q_1(n, p) > 0$ such that if $\|W^{\mathrm{v+h}}_{\mathrm{can}}(\mathbf{U} \setminus \mathcal{K})\| \ge Q_1$, then*

$$\frac{1}{5} \|W^{\mathrm{v+h}}_{\mathrm{can}}(\mathbf{U} \setminus \mathcal{K})\| \le \|W^{\mathrm{v}}_{\mathrm{can}}(\mathbf{U}^{np} \setminus \mathcal{K})\|.$$

Proof. Recalling that $\hat{X}^{np} = W^{\mathrm{h}}_{\mathrm{can}}(\mathbf{U}^{np} \setminus \mathcal{K}) - q_{np}$ and $|\operatorname{supp} \hat{X}^{np}| \le 3p$, we see that $\|W^{\mathrm{h}}_{\mathrm{can}}(\mathbf{U}^{np} \setminus \mathcal{K})\| - 3pq_{np} \le \|\hat{X}^{np}\|$. Similarly, $\|\hat{X}\| \le \|W^{\mathrm{h}}_{\mathrm{can}}(\mathbf{U} \setminus \mathcal{K})\| + 3pq_0$. Together with corollary 18.2, these inequalities imply

$$\|W^{\mathrm{h}}_{\mathrm{can}}(\mathbf{U}^{np} \setminus \mathcal{K})\| \le \frac{3}{4} \|W^{\mathrm{h}}_{\mathrm{can}}(\mathbf{U} \setminus \mathcal{K})\| + C(n, p), \tag{1}$$

where $C(n, p) > 0$ is a constant depending only on n and p. Combining lemma 17.1 with (1), we see that

$$\begin{aligned}
\|W^{\mathrm{v+h}}_{\mathrm{can}}(\mathbf{U} \setminus \mathcal{K})\| &\le \|W^{\mathrm{v+h}}_{\mathrm{can}}(\mathbf{U}^{np} \setminus \mathcal{K})\| + 6p \\
&= \|W^{\mathrm{v}}_{\mathrm{can}}(\mathbf{U}^{np}) \setminus \mathcal{K})\| + \|W^{\mathrm{h}}_{\mathrm{can}}(\mathbf{U}^{np}) \setminus \mathcal{K})\| + 6p \\
&\le \|W^{\mathrm{v}}_{\mathrm{can}}(\mathbf{U}^{np} \setminus \mathcal{K})\| + \frac{3}{4} \|W^{\mathrm{h}}_{\mathrm{can}}(\mathbf{U} \setminus \mathcal{K})\| + C'(n, p),
\end{aligned}$$

where $C'(n, p) > 0$ is a constant depending only on n and p. Then

$$\begin{aligned}
\frac{1}{4} \|W^{\mathrm{v+h}}_{\mathrm{can}}(\mathbf{U} \setminus \mathcal{K})\| &\le \|W^{\mathrm{v+h}}_{\mathrm{can}}(\mathbf{U} \setminus \mathcal{K})\| - \frac{3}{4} \|W^{\mathrm{h}}_{\mathrm{can}}(\mathbf{U} \setminus \mathcal{K})\| \\
&\le \|W^{\mathrm{v}}_{\mathrm{can}}(\mathbf{U}^{np} \setminus \mathcal{K})\| + C'(n, p).
\end{aligned}$$

The result follows by taking $\|W^{\mathrm{v+h}}_{\mathrm{can}}(\mathbf{U} \setminus \mathcal{K})\| \ge 20C'(n, p) =: Q_1$. $\square$

19 Vertical weight controls total weight

19.1 The covering lemma

Let S be a topological disk in $\mathbb{C}$. Given a compact set K contained in S, let $\Gamma(S,K)$ denote the set of proper paths in $S \setminus K$ joining K and ∂S.

Let A_j, $j = 1, \ldots, N$, be disjoint islands in S. (See section 7.1 in [Kah06] for the definition of an island. For us, it suffices to know that the little Julia sets are islands.) We define $X = \mathcal{W}\,\Gamma(S, \bigcup A_j)$, $Y = \sum \mathcal{W}\,\Gamma(S, A_j)$, and $Z = \sum_j \mathcal{W}\,\Gamma(S \setminus \bigcup_{k\neq j} A_k, A_j)$. Clearly,

$$X \le Y \le Z.$$

Lemma 19.1. *Let S be a hyperbolic Riemann surface of finite topology, without cusps, having a distinguished outer boundary. Then $X = \|W^{\mathrm{v}}_{\mathrm{can}}(S)\| + O(|\chi(S)|)$.*

Proof. Write $\Gamma_X = \Gamma(S, \bigcup A_j)$. It obvious that $\mathcal{F}^{\mathrm{v}}_{\mathrm{can}} \subset \Gamma_X$. Then $\mathcal{L}(\mathcal{F}^{\mathrm{v}}_{\mathrm{can}}) \ge \mathcal{L}(\Gamma_X)$, so $\|W^{\mathrm{v}}_{\mathrm{can}}(S)\| = \mathcal{W}(\mathcal{F}^{\mathrm{v}}_{\mathrm{can}}) \le \mathcal{W}(\Gamma_X) = X$.

Let Γ denote the geodesic of S homotopic to the outer boundary. Let $\pi : A_\Gamma(S) \to S$ be the annulus cover of S associated with Γ. Then one component C of $\partial^I A_\Gamma(S)$ maps homeomorphically to the outer component of $\partial^I S$. Let Γ'_X denote the set of all lifts by π of curves in Γ_X to curves in $A_\Gamma(S)$ that start from C. Then curves of Γ'_X necessarily join the boundary components of $A_\Gamma(S)$. Let Γ_0 denote the set of all curves connecting the boundary components of $A_\Gamma(S)$.

Since holomorphic maps do not decrease extremal length, $\mathcal{L}(\Gamma'_X) \le \mathcal{L}(\Gamma_X)$. Since $\Gamma_0 \supset \Gamma'_X$, $\mathcal{L}(\Gamma_0) \le \mathcal{L}(\Gamma'_X)$. Combining the inequalities, we have $\mathcal{L}(\Gamma_0) \le \mathcal{L}(\Gamma_X)$, which gives

$$X = \mathcal{W}(\Gamma_X) \le \mathcal{W}(\Gamma_0) = \frac{1}{\pi}|\Gamma| = \|W^{\mathrm{v}}_{\mathrm{can}}(S)\| + O(p).$$

□

The following lemma is a special case of the quasi-additivity law, which is the main theorem in [KL09c].

Lemma 19.2 (Covering lemma)**.** *Fix a number $\xi \ge 1$, and let p, d, and Δ be integers such that $p \ge 2$ and $d \ge \Delta \ge 2$. There is a constant $L = L(d, p, \Delta) > 0$ satisfying the following property: Assume that*

- *U and U' are topological disks;*

- $K_1, \ldots, K_p \subset U$ *and* $K'_1, \ldots, K'_p \subset U'$ *are disjoint FJ-sets;*
- $g : (U, \bigcup K_j) \to (U', \bigcup K'_j)$ *is a branched covering map;*
- $\deg(g : U \to U') = d$*;*
- $\bigcup K'_j$ *contains the critical values of* g*;*
- *for each* $j \in \mathbb{Z}/p\mathbb{Z}$*,* K_j *is a component of* $g^{-1}(K'_j)$*;*
- *for each* $j \in \mathbb{Z}/p\mathbb{Z}$*,* $\deg(g : K_j \to K'_j) \leq \Delta$*;*
- $X \leq Y \leq Z$ *and* $X' \leq Y' \leq Z'$ *are the conformal moduli associated with* $(U, \cup K_j)$ *and* $(U', \bigcup K'_j)$*, respectively;*
- *and* $\|W^{\mathrm{v+h}}_{\mathrm{can}}(U' \setminus \cup K'_j)\| \leq \xi Y$*;*

if $Y \geq L(d, p, \Delta)$*, then* $X \leq 4\xi\Delta^2 X'$*.*

Proof. See the Covering Lemma in [Kah06]. $\square$

19.2 Applying the covering lemma

Lemma 19.3. *There are constants* $M = M(d) > 1$ *and* $Q_2 = Q_2(d, p) > 0$ *such that if* $\|W^{\mathrm{v+h}}_{\mathrm{can}}(\mathbf{U} \setminus \mathcal{K})\| \geq Q_2$*, then*

$$\|W^{\mathrm{v+h}}_{\mathrm{can}}(\mathbf{U} \setminus \mathcal{K})\| \leq M \|W^{\mathrm{v}}_{\mathrm{can}}(\mathbf{U} \setminus \mathcal{K})\|.$$

Compare lemma 7.1 in [Kah06].

Proof. Let $X \leq Y \leq Z$ and $X' \leq Y' \leq Z'$ be the conformal moduli associated with $(\mathbf{U}^{5dp}, \mathcal{K})$ and $(\mathbf{U}, \mathcal{K})$, respectively. In this proof, we will only be concerned with X, Y, and X'. Observe that $\deg(f^{5dp} : \mathbf{U}^{5dp} \to \mathbf{U}) = d^{5dp}$ and $\deg((f^p)^{5d} : \mathcal{K} \to \mathcal{K}) \leq (2^{d-1})^{5d}$. By lemma 19.1, there is a constant $L = L(p) > 0$ such that

$$\|W^{\mathrm{v}}_{\mathrm{can}}(\mathbf{U}^{5dp} \setminus \mathcal{K})\| - L \leq X \tag{2}$$

and

$$X' \leq \|W^{\mathrm{v}}_{\mathrm{can}}(\mathbf{U} \setminus \mathcal{K})\| + L. \tag{3}$$

Taking $n = 5d$ in lemma 18.3, we find a constant $Q_1 = Q_1(5d, p)$ such that if $\|W^{\mathrm{v+h}}_{\mathrm{can}}(\mathbf{U} \setminus \mathcal{K})\| \geq Q_1$, then

$$\frac{1}{5}\|W^{\mathrm{v+h}}_{\mathrm{can}}(\mathbf{U} \setminus \mathcal{K})\| \leq \|W^{\mathrm{v}}_{\mathrm{can}}(\mathbf{U}^{5dp} \setminus \mathcal{K})\|. \tag{4}$$

Combining (2) and (4), we have

$$\begin{aligned} Y \geq X &\geq \|W^{\mathrm{v}}_{\mathrm{can}}(\mathbf{U}^{5dp} \setminus \mathcal{K})\| - L \\ &\geq \frac{1}{5}\|W^{\mathrm{v+h}}_{\mathrm{can}}(\mathbf{U} \setminus \mathcal{K})\| - L. \end{aligned}$$

If $\|W^{\mathrm{v+h}}_{\mathrm{can}}(\mathbf{U} \setminus \mathcal{K})\| \geq 30L$, then the above inequality gives us

$$Y \geq \frac{1}{6}\|W^{\mathrm{v+h}}_{\mathrm{can}}(\mathbf{U} \setminus \mathcal{K})\|.$$

Now we can apply the covering lemma (lemma 19.2) with $\xi = 6$ to see that

$$X \leq 4 \cdot 6 \cdot ((2^{d-1})^{5d})^2 X'.$$

Let $C = 4 \cdot 6 \cdot ((2^{d-1})^{5d})^2$. Together with (2) and (3), this implies

$$\begin{aligned} \|W^{\mathrm{v}}_{\mathrm{can}}(\mathbf{U}^{5dp} \setminus \mathcal{K})\| &\leq X + L \leq CX' + L \\ &\leq C\|W^{\mathrm{v}}_{\mathrm{can}}(\mathbf{U} \setminus \mathcal{K})\| + (C+1)L. \end{aligned}$$

Combining this inequality with (4), we obtain

$$\frac{1}{5}\|W^{\mathrm{v+h}}_{\mathrm{can}}(\mathbf{U} \setminus \mathcal{K})\| \leq C\|W^{\mathrm{v}}_{\mathrm{can}}(\mathbf{U} \setminus \mathcal{K})\| + (C+1)L.$$

If $\|W^{\mathrm{v+h}}_{\mathrm{can}}(\mathbf{U} \setminus \mathcal{K})\| \geq 30 \cdot (C+1)L$, then

$$\frac{1}{6}\|W^{\mathrm{v+h}}_{\mathrm{can}}(\mathbf{U} \setminus \mathcal{K})\| \leq C\|W^{\mathrm{v}}_{\mathrm{can}}(\mathbf{U} \setminus \mathcal{K})\|.$$

Defining $M = 6C$, we obtain the desired inequality.

In order to obtain this inequality, we required $\|W^{\mathrm{v+h}}_{\mathrm{can}}(\mathbf{U}\setminus\mathcal{K})\|$ to be at least as big as Q_1, $30L$, and $30(C+1)L$. We can take $Q_2 = \max\{Q_1, 30L, 30(C+1)L\}$. $\square$

20 Improvement of the moduli

Lemma 20.1. *Fix integers $d, p \geq 2$. There exist $C = C(d) > 1$ and $\mu = \mu(d,p) > 0$ such that the following property is satisfied: Let $\mathbf{f}$ be a pseudo-polynomial-like map of degree d admitting a good renormalization $\mathbf{f}'$ of period p; then*

$$\mathrm{mod}(\mathbf{f}') \geq \min\{p/C \cdot \mathrm{mod}(\mathbf{f}), \mu\}.$$

Proof. First, let us fix our notation. We have two pseudo-polynomial-like maps, $\mathbf{f} : (\mathbf{U}, K) \to (\mathbf{U}, K)$ and $\mathbf{f}' : (\mathbf{U}', K') \to (\mathbf{U}', K')$. Let

$$\mathcal{K} = \bigcup_{j \in \mathbb{Z}/p\mathbb{Z}} K_j \subset \mathbf{U}$$

denote the union of little Julia sets of $\mathbf{f}$, where K_0 is the little Julia set corresponding to the canonical renormalization $\mathbf{f}'$. For each j, let $\gamma_j \subset \mathbf{U} \setminus \mathcal{K}$ denote the hyperbolic geodesic around K_j. Let $\Gamma \subset \mathbf{U} \setminus \mathcal{K}$ denote the hyperbolic geodesic separating $\partial \mathbf{U}$ from $\mathcal{K}$. Let $\gamma \subset \mathbf{U} \setminus K$ denote the core geodesic. Let $\gamma' \subset \mathbf{U}' \setminus K'$ denote the core geodesic. The following equalities are obvious:

$$\begin{aligned}
|\gamma|_{\mathbf{U}\setminus K} &= \pi / \operatorname{mod}(\mathbf{U} \setminus K), \\
|\gamma_0|_{\mathbf{U}\setminus \mathcal{K}} &= |\gamma'|_{\mathbf{U}'\setminus K'} = \pi / \operatorname{mod}(\mathbf{U}' \setminus K').
\end{aligned}$$

For convenience, set $W = W_{\text{can}}(\mathbf{U} \setminus \mathcal{K})$, and for each j, let W_j denote the restriction of W to the arcs attached to K_j. We know that

$$\|W^{\text{v+h}}_{\text{can}}\| \le \sum_j \|W_j\| \le 2\|W^{\text{v+h}}_{\text{can}}\|, \tag{5}$$

because each arc of $\operatorname{supp} W^{\text{v+h}}_{\text{can}}$ is counted at least once but at most twice in the sum $\sum \|W_j\|$.

Now, we begin the proof. By lemma 12.8, there is a constant $C = C(p) > 0$ such that for each j, $|\gamma_j| \le \pi\|W_j\| + C$. It follows that if $|\gamma_j| \ge 2C$, then

$$|\gamma_j| \le 2\pi\|W_j\|.$$

By lemma 9.4, $|\gamma_0| \le 2^{d-1}|\gamma_j|$. Consequently, if $|\gamma_0| \ge 2^d C$, then $|\gamma_j| \ge 2C$, so

$$|\gamma_0| \le 2^{d-1}|\gamma_j| \le 2^d\pi\|W_j\|.$$

Summing over all j, we obtain

$$p|\gamma_0| \le 2^d\pi \sum_j \|W_j\| \le 2^{d+1}\pi\|W^{\text{v+h}}\|, \tag{6}$$

where the last inequality comes from equation 5. By lemma 19.3, there are constants $L = L(d, p)$ and $M = M(d)$ such that $\|W^{\text{v+h}}\| \ge L$ implies $\|W^{\text{v+h}}\| \le M\|W^{\text{v}}\|$. Then

$$p|\gamma_0| \le 2^{d+1}\pi \cdot M\|W^{\text{v}}\| \le 2^{d+1}M\pi\langle W, \gamma\rangle, \tag{7}$$

where the last inequality uses the fact that $\|W^{\mathrm{v}}\| \leq \langle W, \gamma\rangle$. By lemma 12.8, $\pi\langle W,\gamma\rangle \leq |\gamma| + C$. Consequently, if $\pi\langle W,\gamma\rangle \geq 2C$, then $\pi\langle W,\gamma\rangle \leq 2|\gamma|$. Combining this with 7, we obtain

$$p|\gamma_0| \leq 2^{d+2}M|\gamma|.$$

Since $\mathbf{U}\setminus K \subset \mathbf{U}\setminus\mathcal{K}$, the Schwarz Lemma implies that $|\Gamma| \geq |\Gamma|_{\mathbf{U}\setminus\mathcal{K}}$. Since γ is the hyperbolic geodesic in $\mathbf{U}\setminus\mathcal{K}$ homotopic to Γ, we know that $|\Gamma|_{\mathbf{U}\setminus\mathcal{K}} \geq |\gamma|$. Hence, $p|\gamma_0| \leq 2^{d+2}M|\Gamma|$. This is equivalent to

$$\frac{\mathrm{mod}(\mathbf{f}')}{p} = \frac{\pi}{p|\gamma_0|} \geq \frac{\pi}{2^{d+2}M|\Gamma|} = \frac{\mathrm{mod}(\mathbf{f})}{2^{d+2}M}.$$

Finally, let us summarize the inequalities we required in the paragraph above. We required that

$$\|W^{\mathrm{v+h}}\| \geq L, \tag{8}$$
$$\pi\langle W,\gamma\rangle \geq 2C, \tag{9}$$

and $|\gamma_0| \geq 2^dC$. By (6), we see that $\|W^{\mathrm{v+h}}\| \geq p|\gamma_0|/(2^{d+1}\pi) \geq |\gamma_0|/(2^d\pi)$, so (8) is satisfied if

$$|\gamma_0| \geq 2^d\pi L.$$

By (7), we see that $\pi\langle W,\gamma\rangle \geq p|\gamma_0|/(2^{d+1}M) \geq |\gamma_0|/(2^dM)$, so (9) is satisfied if

$$|\gamma_0| \geq 2^dM\cdot 2C.$$

All of the required inequalities are satisfied if

$$|\gamma_0| \geq \max\{2^dC, 2^d\pi L, 2^dM\cdot 2C\} =: E.$$

This is equivalent to $\mathrm{mod}(\mathbf{f}') \leq \pi/E =: \mu$. □

In the lemma above, $p \geq 2$. This immediately implies the following corollary, where the constant C below is different than in the lemma above.

Corollary 20.2. *Fix integers $d, p \geq 2$. There exist $C = C(d) > 1$ and $\mu = \mu(d,p) > 0$ such that the following property is satisfied: Let $\mathbf{f}$ be a pseudo-polynomial-like map of degree d admitting a good renormalization $\mathbf{f}'$ of period p; then*

$$\mathrm{mod}(\mathbf{f}') \geq \min\{1/C\cdot\mathrm{mod}(\mathbf{f}), \mu\}.$$

The following lemma implies theorem 14.1.

Lemma 20.3. *For any $\lambda > 1$ and any integer $d \geq 2$, there exists an integer $\underline{p} = \underline{p}(\lambda, d) \geq 2$ such that for any integer $p \geq \underline{p}$, there exists $\mu = \mu(d,p) > 0$ such that the following property is satisfied: Let $\mathbf{f}$ be a pseudo-polynomial-like map of degree d admitting a good renormalization $\mathbf{f}'$ of period p, with $p \geq \underline{p}$; if $\operatorname{mod}(\mathbf{f}') < \mu$, then*

$$\operatorname{mod}(\mathbf{f}) \leq \lambda^{-1} \operatorname{mod}(\mathbf{f}').$$

Proof. Let $C = C(d) > 1$ and $\mu = \mu(d,p) > 0$ be the constants from lemma 20.1. Define $\underline{p} = C\lambda$. Assume that $p \geq \underline{p}$. By lemma 20.1, if $\operatorname{mod}(\mathbf{f}') < \mu$, then

$$\begin{aligned} \operatorname{mod}(\mathbf{f}') &\geq (p/C) \operatorname{mod}(\mathbf{f}) \\ &\geq (\underline{p}/C) \operatorname{mod}(\mathbf{f}) \\ &= \lambda \operatorname{mod}(\mathbf{f}). \end{aligned}$$

□

21 Bounds for good renormalization

21.1 Non-associativity of canonical renormalization

Canonical renormalization respects conformal geometry, but it is not associative: Iterating canonical renormalization or skipping over levels of renormalization lead to different pseudo-polynomial-like maps.

Lemma 21.1. *Let p and q be integers ≥ 2. Let $\mathbf{f} : \mathbf{U} \to \mathbf{U}$ be a pseudo-polynomial-like map admitting good renormalizations, with periods p and pq, around one of its critical points. Let $\mathcal{K}_p \subset \mathbf{U}$ denote the cycle of little Julia sets corresponding to the period p, and let $\mathcal{K}_{pq} \subset \mathcal{K}_p$ denote the cycle of little Julia sets corresponding to the period pq. Choose little Julia sets $\mathbf{K}_p \subset \mathcal{K}_p$ and $\mathbf{K}_{pq} \subset \mathcal{K}_{pq} \cap \mathbf{K}_p$. Let $\mathbf{f}_p = R_p\mathbf{f} : \mathbf{U}_p \to \mathbf{U}_p$ be the canonical p-renormalization of $\mathbf{f}$ around $\mathbf{K}_p$. Let $(\mathbf{f}_p)_q = R_q\mathbf{f}_p$ be the canonical q-renormalization of $\mathbf{f}_p$ around $\mathbf{K}_{pq}$. Let $\mathbf{f}_{pq} = R_{pq}\mathbf{f}$ be the canonical pq-renormalization of $\mathbf{f}$ around $\mathbf{K}_{pq}$. Then $\operatorname{mod}(\mathbf{f}_{pq}) > \operatorname{mod}((\mathbf{f}_p)_q)$.*

Proof. Let $\gamma_p \subset \mathbf{U} \setminus \mathcal{K}_p$ be the simple, closed geodesic going around $\mathbf{K}_p$. Let $A = A_{\gamma_p}(\mathbf{U} \setminus \mathcal{K}_p)$ be an annulus covering space, and let $\pi : A \to \mathbf{U} \setminus \mathcal{K}_p$ be a covering map. (Note that A is an abstract Riemann surface homeomorphic to an annulus.) Let $D = D_{\gamma_p} \subset \mathbf{U}$ be the open disk bounded by γ_p. Then

$\mathbf{U}_p := A \sqcup_\pi D$. (Note that $\mathbf{U}_p \cap \mathbf{U} = D$.) Consider the function $e : \mathbf{U}_p \to \mathbf{U} \setminus (\mathcal{K}_p \setminus D)$, defined by

$$e(z) = \begin{cases} \pi(z) & \text{if } z \in A, \\ z & \text{if } z \in D, \end{cases}$$

and its restriction $e|\mathbf{U}_p \setminus (D \cap \mathcal{K}_{pq}) : \mathbf{U}_p \setminus (D \cap \mathcal{K}_{pq}) \to \mathbf{U} \setminus ((\mathcal{K}_p \setminus D) \cup (D \cap \mathcal{K}_{pq}))$. It is easy to see that e and $e|\mathbf{U}_p \setminus (D \cap \mathcal{K}_{pq})$ are holomorphic immersions but not covering maps.

Let $\gamma_{pq} \subset \mathbf{U} \setminus \mathcal{K}_{pq}$ be the simple, closed geodesic going around $\mathbf{K}_{pq}$, and let $(\gamma_p)_q \subset \mathbf{U}_p \setminus (D \cap \mathcal{K}_{pq})$ be the simple, closed geodesic going around $\mathbf{K}_{pq}$. We see that $e|\mathbf{U}_p \setminus (D \cap \mathcal{K}_{pq})$ embeds $(\gamma_p)_q$ into $(\mathbf{U} \setminus \mathcal{K}_p) \cup (D \setminus \mathcal{K}_{pq})$. The inclusion $(\mathbf{U} \setminus \mathcal{K}_p) \cup (D \setminus \mathcal{K}_{pq}) \subset \mathbf{U} \setminus \mathcal{K}_{pq}$ embeds $e((\gamma_p)_q)$ into $\mathbf{U} \setminus \mathcal{K}_{pq}$, where $e((\gamma_p)_q)$ is homotopic to γ_{pq}. It follows that

$$|(\gamma_p)_q|_{\mathbf{U}_p \setminus (D \cap \mathcal{K}_{pq})} > |e((\gamma_p)_q)|_{\mathbf{U} \setminus ((\mathcal{K}_p \setminus D) \cup (D \cap \mathcal{K}_{pq}))} > |e((\gamma_p)_q)|_{\mathbf{U} \setminus \mathcal{K}_{pq}} \geq |\gamma_{pq}|_{\mathbf{U} \setminus \mathcal{K}_{pq}}.$$

Hence, $\mathrm{mod}(\mathbf{f}_{pq}) > \mathrm{mod}((\mathbf{f}_p)_q)$. □

The proof in the lemma above may be carried out under slightly more general conditions to obtain the following lemma.

Lemma 21.2. *Let $N \geq 2$ be an integer. Let $\mathbf{f}$ be a pseudo-polynomial-like map admitting good renormalizations of periods $q_1|q_2|\cdots|q_N$ around one of its critical points. Let $q_0 = 1$, let $\mathbf{g}_0 = \mathbf{f}$, and for each integer $n \in \{1, \ldots, N\}$, let $\mathbf{g}_n = R_{q_n/q_{n-1}}\mathbf{g}_{n-1}$ be a canonical q_n/q_{n-1}-renormalization of $\mathbf{g}_{n-1}$. Let $\mathbf{f}_{q_N} = R_{q_N}\mathbf{f}$ be the canonical q_N-renormalization of $\mathbf{f}$ having same Julia set as $\mathbf{g}_N$. Then* $\mathrm{mod}(\mathbf{f}_{q_N}) > \mathrm{mod}(\mathbf{g}_N)$.

Proof. As in the previous lemma, one checks that there is a holomorphic immersion that embeds the hyperbolic geodesic corresponding to $\mathbf{g}_N$ into the complement of the little Julia sets of $\mathbf{f}$ of period q_N. □

21.2 *Beau* bounds for good renormalization

The word *beau* stands for bounded and eventually universally bounded.

Theorem 21.3 (*Beau* bounds)**.** *Let B and d be integers ≥ 2. There exist $\mu = \mu(B, d) > 0$ and a function $\mathcal{N} = \mathcal{N}_{B,d} : (0, +\infty) \to \mathbb{Z}_{\geq 0}$ such that the following property is satisfied. Let $\mathbf{f} : \mathbf{U} \to \mathbf{U}$ be a pseudo-polynomial-like map of degree d admitting infinitely many good renormalizations around one of its critical points, with combinatorics bounded by B. Let $\{p_n\}_{n\geq 1}$ be the good* *renormalization periods of* $\mathbf{f}$*. For each integer* $n \geq 1$*, let* $\mathcal{K}^n \subset \mathbf{U}$ *denote the cycle of little Julia sets of* $\mathbf{f}$ *corresponding to the renormalization period* p_n*. Let* $\{\mathbf{K}^n \subset \mathcal{K}^n\}_{n\geq 1}$ *be a sequence of nested little Julia sets. For each integer* $n \geq 1$*, let* $\mathbf{f}_n = R_{p_n}\mathbf{f}$ *be the canonical* p_n*-renormalization of* $\mathbf{f}$ *around* $\mathbf{K}^n$*. If* $\operatorname{mod}(\mathbf{f}) \geq \delta > 0$*, then for any integer* $n \geq \mathcal{N}(\delta)$*,* $\operatorname{mod}(\mathbf{f}_n) \geq \mu$*.*

Proof. Let $\lambda = 10$, and let $\underline{p} = \underline{p}(\lambda, d) = \underline{p}(d) \geq 2$ be the constant from theorem 14.1. Let $k = k(d)$ be the smallest integer ≥ 2 such that $\underline{p} \leq 2^k$. Let $\overline{p} = B^k$, and let $\mu = \mu(d, \overline{p}) = \mu(B, d) > 0$ be the threshold from theorem 14.1.

Let $\mathcal{F}$ be the set of all canonical p_n-renormalizations of $\mathbf{f}$, for all $n \in \{1, \ldots, k-1\}$. Define

$$\mu' = \min\{\operatorname{mod}(\mathbf{f}') : \mathbf{f}' \in \mathcal{F}\}. \tag{10}$$

Let $\mu_0 = \min\{\delta, \mu'\}$. For notational convenience, let $\mathbf{f}_0 = \mathbf{f}$. It is obvious that for any integer n, with $0 \leq n \leq k-1$, $\operatorname{mod}(\mathbf{f}_n) \geq \mu_0$. Let $N = N(\mu, \mu_0) = N(B, d, \delta, \mu')$ be the smallest integer ≥ 0 such that $\mu/\lambda^N \leq \mu_0$.

We will show that for any integer $n \geq Nk$, $\operatorname{mod}(\mathbf{f}_n) \geq \mu$. To this end, suppose that there is an integer $n \geq Nk$ such that $\operatorname{mod}(\mathbf{f}_n) < \mu$. Let m be the greatest integer $\leq n/k$. Then $n - km \in \{0, \ldots, k-1\}$. Consider the sequence of integers

$$\{q_i = p_{n-km+ik}/p_{n-km+(i-1)k}\}_{i\geq 1} \subset [\underline{p}, \overline{p}].$$

Let $\mathbf{g}_0 = \mathbf{f}_{n-km}$, and for each integer $i \geq 1$, let $\mathbf{g}_i = R_{q_i}\mathbf{g}_{i-1}$ be the canonical q_i-renormalization of $\mathbf{g}_{i-1}$ having the same Julia set as $\mathbf{f}_{n-km+ik}$. Lemma 21.2 implies that $\operatorname{mod}(\mathbf{g}_m) < \operatorname{mod}(\mathbf{f}_n) < \mu$. Applying theorem 14.1, we see that

$$\operatorname{mod}(\mathbf{g}_{m-1}) \leq \lambda^{-1}\operatorname{mod}(\mathbf{g}_m) < \mu/\lambda < \mu.$$

We are again in a position to apply theorem 14.1. After a total of m applications of theorem 14.1, we obtain

$$\operatorname{mod}(\mathbf{f}_{n-km}) = \operatorname{mod}(\mathbf{g}_0) \leq \mu/\lambda^m \leq \mu/\lambda^N < \mu_0,$$

which is a contradiction.

We would like to define $\mathcal{N}(\delta) = Nk$, but this will not meet our requirements: N depends on μ', and μ' depends on $\mathbf{f}$. Recalling the definition of μ' in equation (10), we see that we need only bound the moduli of $\mathbf{f}'$, for all $\mathbf{f}' \in \mathcal{F}$, in terms of the modulus of $\mathbf{f}$. For each $n \in \{1, \ldots, k-1\}$, we know that

$$p_n = \frac{p_n}{p_{n-1}} \cdots \frac{p_2}{p_1} p_1 \leq B^n \leq B^{k-1}.$$

By corollary 20.2, there are constants $C = C(d) > 1$ and $\mu^* = \mu^*(d, B^{k-1}) = \mu^*(B, d)$ such that $\mu' \geq \min\{\mu^*, C^{-1} \operatorname{mod}(\mathbf{f})\} \geq \min\{\mu^*, \delta/C\} =: \hat{\mu}$. Clearly, $\hat{\mu} = \hat{\mu}(B, d, \delta)$. Let $\hat{N} = \hat{N}(\mu, \hat{\mu}) = \hat{N}(B, d, \delta)$ be the smallest integer ≥ 0 such that $\mu/\lambda^{\hat{N}} < \hat{\mu}$. We can repeat the argument in the preceding paragraphs, replacing N with $\hat{N}$ and μ_0 with $\hat{\mu}$. Defining $\mathcal{N}(\delta) = \hat{N}k$, we are finished. $\square$

Remark 21.1 (Generalization). Although the theorem above is stated for a pseudo-polynomial-like map admitting infinitely many good renormalizations around one of its critical points, infinite renormalizability is not an essential hypothesis. We have actually proven the following: Let B and d be integers ≥ 2. There exist a constant $\mu = \mu(B, d) > 0$ and a function $\mathcal{N} = \mathcal{N}_{B,d} : (0, +\infty) \to \mathbb{Z}_{\geq 0}$ such that the following property is satisfied. If $\mathbf{f}$ is a pseudo-polynomial-like map of degree d, with $\operatorname{mod}(\mathbf{f}) \geq \delta > 0$, admitting at least $N = \mathcal{N}(\delta)$ good renormalizations around one of its critical points, with combinatorics bounded by B, then for any integer $n \geq N$, we have $\operatorname{mod}(\mathbf{f}_n) \geq \mu$ whenever $\mathbf{f}_n$ is defined.

Chapter 4

Combinatorics, *a priori* bounds, and local connectivity

22 Combinatorial decomposition

Let $f : \mathbb{C} \to \mathbb{C}$ be a polynomial (or a polynomial-like map) of degree $d \geq 2$ admitting infinitely many primitive renormalizations around each of its critical points.

22.1 Around one critical point

Fix $c \in \mathrm{Crit}(f)$. The objects defined in this subsection are associated with c, but our notation will not reflect this.

Let $\mathcal{P}$ denote the set of periods p such that f is primitively p-renormalizable around c. We will write $\mathcal{P} = \{p_1, p_2, \dots\}$, where for each n, $p_n < p_{n+1}$. It can be shown that p_n divides p_{n+1}. (See proposition 3.8 in [Ino02].) Define $r_n = p_{n+1}/p_n$. Given an integer $n \geq 1$, choose domains U^n and V^n such that $f^{p_n} : U^n \to V^n$ is a polynomial-like renormalization of period p_n around c. Let $K^n = K(f^{p_n}|U^n)$ denote the corresponding Julia set. Define the following objects:

$$\begin{aligned}
\mathcal{A}^n &= \bigcup_{j=0}^{p_n-1} f^j(K^n), \\
\mathcal{C}^n &= \mathrm{Crit}(f) \cap \mathcal{A}^n, \\
\mathcal{K}^n &= U^n \cap \mathcal{A}^{n+1} = \bigcup_{j=0}^{r_n-1} (f^{p_n})^j(K^{n+1}).
\end{aligned} \tag{11}$$

Define $V_0^n = V^n$, and for each integer j, with $1 \le j \le p_n - 1$, define $V_j^n = f^j(U^n)$. Let U_j^n be the connected component of $f^{-p_n}(V_j^n)$ contained in V_j^n. Then $f^{p_n} : U_j^n \to V_j^n$ is a polynomial-like map, $\deg(f^{p_n}|U_j^n) = \deg(f^{p_n}|U^n)$, and $K(f^{p_n}|U_j^n) = f^j(K^n) = U_j^n \cap \mathcal{A}^n$. Define $K_j^n = K(f^{p_n}|U_j^n)$, and define $\mathcal{K}_j^n = U_j^n \cap \mathcal{A}^{n+1}$.

Lemma 22.1. *For any sufficiently large integer n, $\mathcal{C}^n = \mathcal{C}^{n+1}$.*

Proof. By proposition 3.9 in [Ino02], $\mathcal{C}^n \supset \mathcal{C}^{n+1}$. Since $\mathrm{Crit}(f)$ is a finite set, the sets $\mathcal{C}^n$ eventually stabilize. $\square$

Let η be the smallest integer such that $n \ge \eta$ implies $\mathcal{C}^n = \mathcal{C}^{n+1}$. All renormalizations of level $n \ge \eta$ involve the same subset of critical points of f. The following observation is obvious: If a is a critical point of $f^{p_n}|U^n$, then the orbit $\{a, f(a), \ldots, f^{p_n-1}(a)\}$ contains a critical point of f, so $f^{p_n}(a)$ belongs to the postcritical set of f. The following lemma will enable us to apply the improvement of life philosophy.

Lemma 22.2. *For each $m \ge n \ge \eta$, the critical values of $f^{p_n}|U^n$ are contained in $K^n \cap \mathcal{A}^{m+1}$.*

Proof. Let a be a critical point of $f^{p_n}|U^n$. Then $(f^{p_n})'(a) = 0$, so there is an integer j, with $0 \le j \le p_n - 1$, such that $f'(f^j(a)) = 0$. Then $f^j(a)$ is a critical point of f. Since K^n is connected, $a \in K^n$, so $f^j(a) \in f^j(K^n) \subset \mathcal{A}^n$. This shows that $f^j(a) \in \mathcal{C}^n$.

By lemma 22.1, we know that $\mathcal{C}^n = \mathcal{C}^{m+1}$, so $f^j(a) \in \mathcal{A}^{m+1}$. By carefully counting indices, we see that $f^j(K^n) \cap \mathcal{A}^{m+1} = \bigcup_{k=0}^{r_n \cdots r_m - 1} K_{j+kp_n}^{m+1}$. It follows that there is an integer ℓ, with $0 \le \ell \le r_n \cdots r_m - 1$, such that $f^j(a) \in K_{j+\ell p_n}^{m+1}$. Then $f^{p_n}(a) = f^{p_n - j}(f^j(a)) \in f^{p_n - j}(K_{j+\ell p_n}^{m+1}) = f^{(1+\ell)p_n}(K^{m+1}) \subset K^n \cap \mathcal{A}^{m+1}$. $\square$

Remark. For each $m \ge n \ge \eta$, the polynomial-like restriction of $f^{p_n}|U^n$ of period p_{m+1}/p_n around any little Julia set of $K^n \cap \mathcal{A}^{m+1}$ is good.

22.2 Partitioning the set of critical points

In this subsection, it will be necessary to indicate the dependence of the objects defined in subsection 22.1 on the critical point under consideration; for example, given a critical point c of f, we will write $\mathcal{C}^n(c)$ and $\eta(c)$.

Lemma 22.3. *Let c and c' be critical points of f. If $c' \in \mathcal{C}^{\eta(c)}(c)$, then $\eta(c) = \eta(c')$, and $\mathcal{C}^{\eta(c)}(c) = \mathcal{C}^{\eta(c')}(c')$. If $c' \notin \mathcal{C}^{\eta(c)}(c)$, then $\mathcal{C}^{\eta(c)}(c)$ and $\mathcal{C}^{\eta(c')}(c')$ are disjoint.*

Consequently, we define an equivalence relation $\sim$ on the set $\mathrm{Crit}(f)$: $c \sim c'$ if and only if c' belongs to $\mathcal{C}^{\eta(c)}(c)$. We partition $\mathrm{Crit}(f)$ into equivalence classes. Let N be the number of equivalence classes, and for each equivalence class, choose a representative critical point. Enumerate these critical points as

$$c_1, \ldots, c_N.$$

23 *A priori* bounds

We will now describe a decomposition of our original map $f : U \to V$ into polynomial-like restrictions to which the improvement of life philosophy can be applied. These restrictions are obtained by ignoring the first few renormalization levels. We will amend our notation for the objects defined in equation (11) so that, for each $k \in \{1, \ldots, N\}$ and each integer $n \geq 0$,

$$\begin{aligned} \mathcal{A}^{\eta(c_k)+n}(c_k) &\text{ becomes } \mathcal{A}^n(k), \\ \mathcal{C}^{\eta(c_k)+n}(c_k) &\text{ becomes } \mathcal{C}^n(k), \text{ and} \\ \mathcal{K}^{\eta(c_k)+n}(c_k) &\text{ becomes } \mathcal{K}^n(k). \end{aligned}$$

Similarly, $p_{\eta(c_k)+n}(c_k)$ becomes $p_n(k)$. For each $j \in \{0, \ldots, p_n(k) - 1\}$,

$$\begin{aligned} U_j^{\eta(c_k)+n}(c_k) &\text{ becomes } U_j^n(k), \\ V_j^{\eta(c_k)+n}(c_k) &\text{ becomes } V_j^n(k), \\ K_j^{\eta(c_k)+n}(c_k) &\text{ becomes } K_j^n(k), \text{ and} \\ \mathcal{K}_j^{\eta(c_k)+n}(c_k) &\text{ becomes } \mathcal{K}_j^n(k). \end{aligned}$$

In this new notation, we let $f_{k,j} = f^{p_0(k)} : (U_j^0(k), K_j^0(k)) \to (V_j^0(k), K_j^0(k))$.

By lemma 22.2, we can think of the polynomial-like restrictions $\{f_{k,j}\}_{k,j}$ of iterates of f as pseudo-polynomial-like maps admitting infinitely many good renormalizations. (These are the maps on the first level for which lemma 22.2 applies.) Provided we have a bound on combinatorics, we can apply theorem 21.3 to obtain *beau* bounds for these maps. An immediate consequence is the following theorem.

Theorem 23.1 (*A priori* bounds)**.** *Let $f : \mathbb{C} \to \mathbb{C}$ be a polynomial of degree $d \geq 2$ admitting infinitely many primitive renormalizations around each of its critical points. Let $B \geq 2$ be an integer, and assume that B bounds the relative renormalization periods of f. Consider the decomposition of f as in the paragraph above. There exists $\epsilon > 0$ such that for each $k \in \{1, \ldots, N\}$, for* *each $j \in \{0, \dots, p_0(k)-1\}$, the canonical renormalizations of the polynomial-like maps $f_{k,j}$ have moduli $\geq \epsilon$.*

Remark 23.1. In theorem 23.1, we have in mind the canonical renormalizations with periods $\{p_n(k)/p_0(k)\}_{n=1}^{\infty}$ of $f_{k,j}$. We are not thinking about the iterated canonical renormalizations. The *beau* bounds directly imply that there is an eventual lower bound on moduli that depends only on B and d. Corollary 20.2 then implies that the constant ϵ in theorem 23.1 depends only on B, d, and $\min_{k,j}\{\mathrm{mod}(f_{k,j})\}$.

Together with theorem 8.1, theorem 23.1 has the following corollary.

Corollary 23.2. *Assume the conditions stated in theorem 23.1. There exists $\epsilon > 0$ such that for each $k \in \{1, \dots, N\}$ and each integer $n \geq 0$, for each $j \in \{0, \dots, p_n(k)-1\}$, the relevant domains may be chosen so that the polynomial-like maps*

$$f^{p_n(k)} : U_j^n(k) \to V_j^n(k)$$

satisfy $\mathrm{mod}(V_j^n(k) \setminus \overline{U_j^n(k)}) \geq \epsilon$.

Remark 23.2. Again, the *beau* bounds imply that the constant ϵ in corollary 23.2 depends only on B, d, and $\min\{\mathrm{mod}(f_{k,j})\}_{k,j}$.

Remark 23.3 (Generalization). Although we have stated theorem 23.1 and corollary 23.2 in the context of polynomials admitting infinitely many primitive renormalizations around each of their critical points, infinite renormalizability around each critical point is not an essential hypothesis. We have actually proven the following: Let B and d be integers ≥ 2. Let $f : \mathbb{C} \to \mathbb{C}$ be a polynomial of degree d admitting infinitely many primitive renormalizations around $c \in \mathrm{Crit}(f)$, with combintorics bounded by B. In the notation in subsection 22.1, we have a polynomial-like map

$$f_j = f^{p_\eta} : U_j^\eta \to V_j^\eta$$

for each $j \in \{0, \dots, p_\eta - 1\}$. Each of the maps f_j has *beau* bounds in the sense of theorem 21.3. Consequently, there exists $\epsilon = \epsilon(B, d, \min\{\mathrm{mod}(f_j)\}_j) > 0$ such that the canonical renormalizations of the polynomial-like maps f_j have moduli $\geq \epsilon$. Then there exists $\delta = \delta(d, \epsilon) > 0$ such that for each integer $n \geq \eta$ and each $j \in \{0, \dots, p_n - 1\}$, the relevant domains may be chosen so that the polynomial-like maps

$$f^{p_n} : U_j^n \to V_j^n$$

satisfy $\mathrm{mod}(V_j^n \setminus \overline{U_j^n}) \geq \delta$.

Corollary 23.3. *Assume the conditions stated in theorem 23.1. We have the following:*

1. *The postcritical set $P(f)$ has bounded geometry.*
2. *As $n \to \infty$, we have $\max\{\mathrm{diam}_{\mathbb{C}}(P_j^n(k))\}_{k,j} \to 0$, where $P_j^n(k) := K_j^n(k) \cap P(f_{k,j})$.*
3. *There is no f-invariant line field on $K(f) = J(f)$.*

Proof. By the Schwarz lemma, robustness of each map in the finite set $\{f_{k,j}\}$ follows from a lower bound on some subsequence of moduli, so corollary 4.10(2) in [Ino02] applies to each of these maps. Since $P(f) = \bigcup_{k,j} P(f_{k,j})$, this proves (1). Since there are finitely many maps $\{f_{k,j}\}$ on the top level, this proves (2). Corollary 5.2 in [Ino02] applies directly to the map f, proving (3). □

Remark 23.4. The decomposition of our original map f into the maps $\{f_{k,j}\}$ was good enough for us to obtain theorem 23.1 and deduce corollary 23.3. For each $k \in \{1, \ldots, N\}$, for each integer $n \geq 0$, we know that the little Julia sets $\{K_j^n(k)\}_j$ are pairwise disjoint. However, up to this point, we do not know whether $K_j^0(k)$ and $K_{j'}^0(k')$ are disjoint when $k \neq k'$. We can quickly prove the following lemma, which tells us that on some level $n \geq 0$, the little Julia sets $\{K_j^n(k)\}_{j,k}$ are disjoint.

Lemma 23.4. *Assume the conditions stated in theorem 23.1. As $n \to \infty$, we have $\max\{\mathrm{diam}_{\mathbb{C}}(K_j^n(k))\}_{k,j} \to 0$.*

Proof. Corollary 23.3 implies that $\max\{\mathrm{diam}_{\mathbb{C}}(P_j^n(k))\}_{k,j} \to 0$. Lemma 9.1 implies that $\deg(f^{p_n}|U_j^n(k)) \leq 2^{d-1}$, where $d = \deg(f)$. Furthermore, we know that the polynomial-like map $f^{p_n} : (U_j^n(k), K_j^n(k)) \to (V_j^n(k), K_j^n(k))$ satisfies $\mathrm{mod}(V_j^n(k) \setminus \overline{U_j^n(k)}) \geq \epsilon$, where $\epsilon > 0$ is the constant from corollary 23.2. By corollary 5.10 in [McM94], there exists $C = C(d, \epsilon) > 1$ such that $\mathrm{diam}_{\mathbb{C}}(K_j^n(k)) \leq C \cdot \mathrm{diam}_{\mathbb{C}}(P_j^n(k))$, so we are finished. □

Remark 23.5. Consequently, by ignoring finitely many levels of renormalization (and renumbering all of the corresponding objects once again), we can assume that the little Julia sets $\{K_j^0(k)\}_{k,j}$ are disjoint. Consider the finitely many maps $\{f_{k,j} : (U_j^0(k), K_j^0(k)) \to (V_j^0(k), K_j^0(k))\}_{k,j}$ on the (new) top level. Disjointness of the little Julia sets $\{K_j^0(k)\}_{k,j}$ implies that the finitely many domains $\{V_j^0(k)\}_{k,j}$ can be chosen so that they are bounded by pairwise disjoint, smooth, Jordan curves. Under these conditions, we may once again use theorem 21.3 to obtain *a priori* bounds. Then theorem 23.1, corollaries 23.2 and 23.3, and lemma 23.4 still hold.

24 Local connectivity

Recall that for the polynomial f under consideration, $J(f) = K(f)$. In this section, we will prove the following theorem.

Theorem 24.1. *Assume the conditions stated in theorem 23.1. Then $K(f)$ is locally connected.*

Given a point $x \in K(f)$, we say that $K(f)$ is *locally connected* at x if there are arbitrarily small (closed) neighborhoods of x in $K(f)$. We say that $K(f)$ is *locally connected* if $K(f)$ is locally connected at each of its points. (For background on local connectivity, see [Mil06].)

24.1 Local connectivity at the critical points

Let $c \in \mathrm{Crit}(f)$. Let $\{K^n := K^n_{j_n}(k)\}_n$ be any subsequence of nested little Julia sets with $c \in \bigcap_{n=0}^{\infty} K^n$. Let $g_n := f^{p_n(k)} : (U^n_{j_n}(k), K^n) \to (V^0_{j_0}(k), K^n)$ be the corresponding polynomial-like renormalizations.

Lemma 24.2. *$K(f)$ is locally connected at c.*

Proof. We will show that c has arbitrarily small connected (closed) neighborhoods in K. Let $\delta > 0$. By lemma 23.4, there is an integer $n \geq 0$ such that $K^n \subset \delta\mathbb{D} + c$. The β-fixed points of g_n in K^n are repelling periodic points of f, so they are the landing points of some periodic, rational external rays of K. Consider the rational external rays of K landing on the β-fixed points in K^n and landing on the inverse images (under g_n) of these β-fixed points. Truncate the resulting domain by any equipotential of f to obtain a domain W. The domain W is a degenerate renormalization domain around K^n. (By degenerate, we mean that ∂W meets K^n in finitely many periodic and preperiodic points of f, but the set of points in $\overline{W}$ that do not escape $\overline{W}$ under iteration of $f^{p_n(k)}|\overline{W}$ is precisely K^n.) For each integer $m \geq 0$, let W_m be the component of $(f^{p_n(k)})^{-m}(W)$ containing K^n in its closure, and let $Y_m = \overline{W_m}$. As $m \to \infty$, the "puzzle pieces" Y_m shrink to K^n. Take any Y_N contained in $\delta\mathbb{D} + c$. Then $Y_N \cap K$ is a connected (closed) neighborhood of $c \in K$. □

24.2 Local connectivity everywhere else

Let $x \in K(f) \setminus \mathrm{Crit}(f)$. There are two possibilities, either $\{f^n(x)\}_{n=0}^{\infty}$ accumulates on $\mathrm{Crit}(f)$, or $\{f^n(x)\}_{n=0}^{\infty}$ does not accumulate on $\mathrm{Crit}(F)$. The proof of theorem 24.1 will be complete once we show that $K(f)$ is locally connected at x in both of these cases.

Lemma 24.3. *If $\{f^n(x)\}_{n=0}^{\infty}$ accumulates on* $\mathrm{Crit}(f)$*, then $K(f)$ is locally connected at x.*

Proof. As in the proof of lemma 24.2, we can construct for each $c \in \mathrm{Crit}(F)$ a sequence of puzzle pieces $\{Y_m(c)\}_{m=0}^{\infty}$, each containing c, such that

- $\mathrm{diam}_{\mathbb{C}}(Y_m(c)) \to 0$ as $m \to \infty$,
- $Y_m(c)$ contains a little Julia set of f and does not meet any other little Julia set of the same depth,
- $Y_m(c)$ is surrounded by an annulus $A_m(c)$ of definite modulus, and
- the annulus $A_m(c)$ separates $Y_m(c)$ from $P(f) \setminus Y_m(c)$.

For each m, we will call $\{Y_m(c) : c \in \mathrm{Crit}(F)\}$ the set of *critical puzzle pieces* of depth m.

For each n, let $\ell_n \geq 0$ be the first moment that $f^{\ell_n}(x)$ belongs to a critical puzzle piece $Y_n(c(n))$ of depth n. Pull this piece back univalently along the orbit $\{x, f(x), \ldots, f^{\ell_n}(x)\}$ to obtain a puzzle piece Q_n containing x. We obtain a sequence $\{Q_n\}_n$ of puzzle pieces containing x, and we want to show that $\mathrm{diam}_{\mathbb{C}}(Q_n) \to 0$ as $n \to \infty$.

Suppose that the orbit of x lands on $\mathrm{Crit}(f)$ in finite time: There is a unique $\ell \geq 1$ such that $f^{\ell}(x) = c \in \mathrm{Crit}(f)$. For some sufficiently deep level N, the puzzle pieces $\{Q_n\}_{n \geq N}$ are obtained by pulling back the puzzle pieces $\{Y_n(c)\}_{n \geq N}$ along the orbit $\{x, f(x), \ldots, f^{\ell}(x)\}$. In this case, it is clear that $\mathrm{diam}_{\mathbb{C}}(Q_n) \to 0$ as $n \to \infty$.

Now, suppose that the orbit of x never lands on $\mathrm{Crit}(f)$. We claim that $\ell_n \to \infty$ as $n \to \infty$. Otherwise, there would be a bounded subsequence of $\{\ell_n\}_n$, which we also denote by $\{\ell_n\}_n$. Then the orbit $\{f^k(x)\}_{k=0}^{\ell_n}$ would land in arbitrarily small puzzle pieces containing critical points of f, but this would imply that x lands on $\mathrm{Crit}(f)$ in finite time.

For each n, the annulus $A_n(c(n))$ separates $Y_n(c(n))$ from the rest of $P(f)$. Then the univalent map $f^{\ell_n} : Q_n \to Y_n(c(n))$ admits a univalent extension to a definitely bigger domain (in terms of the annulus $A_n(c(n))$). By the Koebe distortion theorem, $\ell_n \to \infty$ implies that $\mathrm{diam}_{\mathbb{C}}(Q_n) \to 0$. □

Lemma 24.4. *If $\{f^n(x)\}_{n=0}^{\infty}$ does not accumulate on* $\mathrm{Crit}(f)$*, then $K(f)$ is locally connected at x.*

Proof. In this case, $\{f^n(x)\}_{n=0}^{\infty}$ accumulates on a point $y \in K \setminus P(f)$. (If $\{f^n(x)\}_n$ were to accumulate on a point of $P(f)$, then robustness of the maps $\{f_{k,j}\}_{k,j}$ would imply that the orbit accumulates on some point of $\mathrm{Crit}(f)$.)

Choose a depth n such that y is in the complement of the union of all little Julia sets $\bigcup_k \mathcal{A}^n(k)$ of depth n. Consider all of the rational external rays of f landing at all of the β-fixed points and all of the corresponding "symmetric" points (the inverse images of the β-fixed points under the appropriate restrictions of the appropriate iterates of f) of all of the little Julia sets of depth n. We obtain a "puzzle piece" Q containing y such that Q is separated from $P(f)$ by an annulus A of positive modulus.

The set of integers $\ell \geq 0$ such that $f^{\ell}(x) \in Q$ is infinite. Let $\{\ell_n\}_n$ be the strictly increasing sequence of moments when $f^{\ell_n}(x) \in Q$. For each n, pull Q back along the orbit $\{x, f(x), \ldots, f^{\ell_n}(x)\}$ to obtain a puzzle piece Q_n. The univalent map $f^{\ell_n} : Q_n \to Q$ admits a univalent extension to a definitely bigger domain (in terms of the annulus A). As before, the Koebe distortion theorem allows us to conclude that $\operatorname{diam}_{\mathbb{C}}(Q_n) \to 0$. $\square$

25 Cycle trees and multi-indices

Our goal in this section will be to describe another way of labeling the little Julia sets, a way that is compatible with their nested structure.

We will use the following notation: Given an integer $q \geq 1$, we define the set $\mathbb{Z}_q = \{0, \ldots, q-1\}$.

Definition 25.1. Let $\{q_n\}_{n=0}^{\infty}$ be a strictly increasing sequence of positive integers with the property that $q_n | q_{n+1}$ for all n. For each n, let $\{K_j^n : j \in \mathbb{Z}_{q_n}\}$ be a collection of pairwise disjoint subsets of $\mathbb{C}$, and let $\mathcal{K}^n = \bigcup_{j=0}^{q_n-1} K_j^n$. Assume that

$$\mathcal{K}^0 \supset \mathcal{K}^1 \supset \mathcal{K}^2 \supset \cdots ,$$

and

$$K_0^0 \supset K_0^1 \supset K_0^2 \supset \cdots .$$

Assume that there is a map $g : \mathcal{K}^0 \to \mathcal{K}^0$ with the property

$$g(K_j^n) = K_{j+1}^n \tag{12}$$

for each n and each j, where we consider the index j in equation (12) as an element of the cyclic group $\mathbb{Z}/q_n\mathbb{Z}$. We call $(\{q_n\}, \{\mathcal{K}^n\}, g)$ a *cycle tree*.

Definition 25.2. A *multi-index* is an element of the set $\mathcal{Z} = \bigcup_{n=0}^{\infty} \mathbb{Z}^{n+1}$. (Note that this is a union of disjoint sets.) The *depth* of a multi-index α, denoted $|\alpha|$, is defined in the following way: $|\alpha| = n$ if and only if $\alpha \in \mathbb{Z}^{n+1}$. We define a partial ordering $\preceq$ on the set of multi-indices: Given multi-indices $\alpha = (j_0, \ldots, j_n)$ and $\alpha' = (j'_0, \ldots, j'_{n'})$, we write $\alpha \preceq \alpha'$ if $n \leq n'$ and

$(j_0, \ldots, j_n) = (j'_0, \ldots, j'_n)$. Given multi-indices α and α' we write $\alpha \prec \alpha'$ if $\alpha \preceq \alpha'$ and $|\alpha'| = |\alpha| + 1$

Let $\mathcal{T} = (\{q_n\}, \{\mathcal{K}^n\}, g)$ be a cycle tree. We define $r_0 = q_0$, and for each integer $n \geq 1$, we define $r_n = q_n/q_{n-1}$. For each n, define $I_n = I_n^{\mathcal{T}} = \mathbb{Z}_{r_0} \times \cdots \times \mathbb{Z}_{r_n} \subset \mathbb{Z}^{n+1}$, and let $\phi_n = \phi_n^{\mathcal{T}} : I_n \to \mathbb{Z}_{q_n}$ be the bijection defined by $\phi(j_0, \ldots, j_n) = j_0 + \sum_{k=1}^{n} q_{k-1} j_k$. Define $\mathcal{I} = \mathcal{I}^{\mathcal{T}} = \bigcup_{n=0}^{\infty} I_n \subset \mathcal{Z}$. (Again, this is a union of disjoint sets.) A multi-index α is *admissible* for $\mathcal{T}$ if $\alpha \in \mathcal{I}$.

We will only consider multi-indices in the context of cycle trees; consequently, we will always assume that multi-indices are admissible for the cycle tree under consideration.

Lemma 25.1. *$K^{\alpha} \supset K^{\alpha'}$ if and only if $\alpha \preceq \alpha'$.*

Lemma 25.2. *Fix n, i, and α, with $\phi_n(\alpha) = i$. Then $\{K_j^{n+1} : K_j^{n+1} \subset K_i^n\} = \{K^{\alpha'} : \alpha \prec \alpha'\}$.*

Lemma 25.3. *For each n, $f^{q_n}(K^{(j_0,\ldots,j_n)}) = K^{(j_0,\ldots,j_n)}$.*

Lemma 25.4. *For each n, $f^{q_n}(K^{(j_0,\ldots,j_n,j_{n+1})}) = K^{(j_0,\ldots,j_n,j_{n+1}+1)}$.*

Remark 25.1. In the notation of the previous sections, it is clear that

$$\mathcal{T}_k = (\{p_n(k)\}, \{\mathcal{A}^n(k)\}, f)$$

is a cycle tree. Consequently, we will label the little Julia sets, which are components of $\mathcal{A}^n(k)$, where n ranges over all integers ≥ 0, by multi-indices. The little Julia sets determine the corresponding polynomial-like renormalizations/restrictions or canonical renormalizations, so we will label these by multi-indices as well. Given n, j, and α, with $\phi_n^{\mathcal{T}_k}(\alpha) = j$, we can write $U^{\alpha}(k) = U_j^n(k)$, $V^{\alpha}(k) = V_j^n(k)$, $K_j^n(k) = K^{\alpha}(k)$, and $F_k^{\alpha} = F_{k,n,j}$.

Chapter 5

Rigidity

26 Collars and equidistant curves

The *standard collar function* is the decreasing homeomorphism $\eta : (0, +\infty) \to (0, +\infty)$, defined by

$$\eta(\ell) = \sinh^{-1}\left(\frac{1}{\sinh(\ell/2)}\right) = \frac{1}{2}\ln\frac{\cosh(\ell/2)+1}{\cosh(\ell/2)-1} = \ln\frac{e^{\ell/2}+1}{e^{\ell/2}-1}.$$

Let X be a hyperbolic Riemann surface, and let $\gamma \subset X$ be a simple, closed geodesic. The *standard collar neighborhood* of γ is

$$C(\gamma) = \{z \in X : \mathrm{dist}_X(z, \gamma) < \eta(|\gamma|_X)\}.$$

Theorem 26.1 (Collar theorem)**.** *Let X be a hyperbolic Riemann surface, and let $\gamma \subset X$ be a simple, closed geodesic. Then $C(\gamma)$ is an embedded annulus. If $\gamma' \subset X$ is a simple, closed geodesic such that $\gamma \cap \gamma' = \emptyset$, then $C(\gamma) \cap C(\gamma) = \emptyset$.*

The standard collar neighborhood $C(\gamma)$ of γ is foliated by simple, closed curves at a fixed hyperbolic distance from γ. We will need a notation for these curves and for sub-annuli of $C(\gamma)$ determined by these curves. Assume that $X \subset \mathbb{C}$. (This assumption is sufficient for the following definitions. More generally, we could instead assume that γ is oriented.) Then we define $\sigma : X \to \{-1, 1\}$ by

$$\sigma(z) = \begin{cases} -1 & \text{if } z \text{ belongs to the bounded component of } \mathbb{C} \setminus \gamma, \\ 1 & \text{otherwise.} \end{cases}$$

Given $T \subset [-1, 1]$, define

$$\gamma^T = \{z \in X : \sigma(z) \cdot \mathrm{dist}_X(z, \gamma)/\eta(|\gamma|_X) \in T\}.$$

Clearly, $C(\gamma) = \gamma^{(-1,1)}$. Given a real number $t \in [-1,1]$, we define $\gamma^t = \gamma^{\{t\}}$, and we call γ^t an *equidistant curve.*

Lemma 26.2. *Let $A \subset \mathbb{C}$ be an annulus, with $\operatorname{mod}(A) \geq \mu > 0$ and core geodesic $\gamma \subset A$. Then γ is a $\kappa(\mu)$-quasicircle.*

Proof. This is a consequence of the Koebe distortion theorem. See lemma 39.1(i) in [Lyu16]. □

Lemma 26.3. *Let $X \subset \mathbb{C}$ be a hyperbolic Riemann surface, and let $\gamma \subset X$ be a simple, closed geodesic. For any $t \in (-1,1)$, let $A_t \subset C(\gamma)$ be the maximal sub-annulus for which $\gamma^t \subset A_t$ is the core geodesic. Then*

- $\operatorname{mod} A_t \in (0, \operatorname{mod} C(\gamma))$ *depends only on $|t|$ and $|\gamma|_X$,*
- $\operatorname{mod} A_t \to \operatorname{mod} C(\gamma)$ *as $|t| \to 0$, and*
- $\operatorname{mod} A_t \to 0$ *as $|t| \to 1$.*

Consequently, γ^t is a quasicircle, with dilatation depending only on $|t|$ and $|\gamma|_X$.

Proof. The proof is a straightforward exercise involving the band $\mathbb{B} = \{z \in \mathbb{C} : |\operatorname{Im} z| < \pi/2\}$ and its hyperbolic metric $|dz|/\cos(\operatorname{Im} z)$. □

26.1 Convergence of hyperbolic surfaces in $\mathbb{C}$

Given a hyperbolic Riemann surface $X \subset \mathbb{C}$ and a base point $x \in X$, there is a unique universal covering map $\pi : \mathbb{D} \to X$ satisfying $\pi(0) = x$ and $\pi'(0) > 0$. We will call π the *normalized* universal covering map corresponding to (X, x). Let G denote the subgroup of $\operatorname{Aut}(\mathbb{D})$ consisting of deck transformations of π.

For each n, let $X_n \subset \mathbb{C}$ be a hyperbolic Riemann surface, and choose a base point $x_n \in X_n$. Let π_n be the normalized universal covering map corresponding to (X_n, x_n), and let G_n be the group of deck transformations of π_n. We will say that $(X_n, x_n) \to (X, x)$ *geometrically* if for every $\kappa > 1$ and $r > 0$, there exists an integer N such that whenever $n \geq N$, there is a (κ, r)-*approximate isometry* $(X_n, x_n) \dashrightarrow (X, x)$: There are open sets $U \subset X$ and $U_n \subset X_n$, with $B_X(x, r) \subset U$ and $B_{X_n}(x_n, r) \subset U_n$, and an orientation preserving homeomorphism $\phi_n : (U, x) \to (U_n, x_n)$ such that for all $a, b \in U$, we have

$$\frac{1}{\kappa} \operatorname{dist}_X(a, b) \leq \operatorname{dist}_{X_n}(\phi_n(a), \phi_n(b)) \leq \kappa \operatorname{dist}_X(a, b).$$

Theorem 26.4. *Suppose that $(X_n, x_n) \to (X, x)$ in the Carathéodory topology. Then*

1. *$\pi_n \to \pi$ in the compact-open topology.*
2. *$(X_n, x_n) \to (X, x)$ geometrically.*
3. *$G_n \to G$ in the Hausdorff topology on compact subgroups of* $\mathrm{Aut}(\mathbb{D})$*.*

Proof. See [Hej74] or [Com13] for a proof that Carathéodory convergence is equivalent to (1). See [MT98] for a proof that (2) and (3) are equivalent. Standard estimates show that as $(X_n, x_n) \to (X, x)$ in the Carathéodory topology, we have uniform convergence $\rho_{X_n} \to \rho_X$ of the hyperbolic metrics (with constant curvature -1) on compact subsets of X. It follows that restrictions of the identity map to increasingly large balls in X, centered at x, provide arbitrarily good approximate isometries. $\square$

26.2 Convergence of equidistant curves

Assume that $(X_n, x_n) \to (X, x)$ in the Carathéodory topology. Let γ be a simple, closed geodesic in X, and let γ_n be a simple, closed geodesic in X_n. Since γ and γ_n are simple, closed curves in $\mathbb{C}$, we can equip them with the induced positive orientation. Let $[\gamma]$ and $[\gamma_n]$ be the corresponding hyperbolic transformations in $\mathrm{Aut}(\mathbb{D})$, and suppose that $[\gamma_n] \to [\gamma]$.

Lemma 26.5. *$\gamma_n \to \gamma$ in the Hausdorff topology.*

Proof. Let p be the repelling fixed point of $[\gamma]$, and let q be the attracting fixed point of $[\gamma]$. Let p_n and q_n be the corresponding fixed points of $[\gamma_n]$. As $[\gamma_n] \to [\gamma]$ in $\mathrm{Aut}(\mathbb{D})$, we know that $p_n \to p$ and $q_n \to q$. Let λ denote the hyperbolic geodesic in $\mathbb{D}$ limiting on p in one direction and on q in the other. Define λ_n similarly. Then $\lambda_n \cup \{p_n, q_n\} \to \lambda \cup \{p, q\}$ in the Hausdorff topology. It follows that $\gamma_n = \pi_n(\lambda_n)$ converges to $\gamma = \pi(\lambda)$ in the Hausdorff topology. $\square$

Lemma 26.6. *For any $t \in [-1, 1]$, $\gamma_n^t \to \gamma^t$ in the C^1 topology.*

Proof. Consider an annulus cover $A_\gamma(X) \to X$. We can identify $A_\gamma(X)$ with a round annulus $\mathbb{A}(1/r, r)$, where $r = \exp(\pi^2/|\gamma|_X)$, in such a way that the induced covering map $\rho : \mathbb{A}(1/r, r) \to X$ satisfies $\rho(1) = x$, $\rho(\partial\mathbb{D}) = \gamma$, and $\deg(\rho|\partial\mathbb{D}) = 1$. Let $\rho_n : \mathbb{A}(1/r_n, r_n) \to X_n$ be the covering map defined correspondingly. As $n \to \infty$, $r_n \to r$, and it is not hard to show that $\rho_n \to \rho$ uniformly on compact subsets of $\mathbb{A}(1/r, r)$. For appropriately chosen $\lambda =$

$\lambda(t, |\gamma|_X) \in (1/r, r)$ and $\lambda_n = \lambda_n(t, |\gamma_n|_{X_n}) \in (1/r_n, r_n)$, the parametrizations $p : \partial\mathbb{D} \to \gamma^t$, defined by $p(z) = \rho(\lambda z)$, and $p_n : \partial\mathbb{D} \to \gamma_n^t$, defined by $p_n(z) = \rho_n(\lambda_n z)$, satisfy $p_n \to p$ and $p_n' \to p'$ uniformly. $\square$

26.3 Some hyperbolic geometry

Let $\mathbf{f} : (\mathbf{V}, \mathbf{K}) \to (\mathbf{V}, \mathbf{K})$ be a pseudo-polynomial-like map admitting a good renormalization of period p. Let $\mathcal{K} = \bigcup_{j=0}^{p} K_j$ denote the corresponding cycle of little Julia sets. Let $\Gamma \subset \mathbf{V} \setminus \mathbf{K}$ denote the core geodesic, and for each j, let $\gamma_j \subset \mathbf{V} \setminus \mathcal{K}$ denote the simple, closed geodesic going around K_j. (See figures 5.1 and 5.2.)

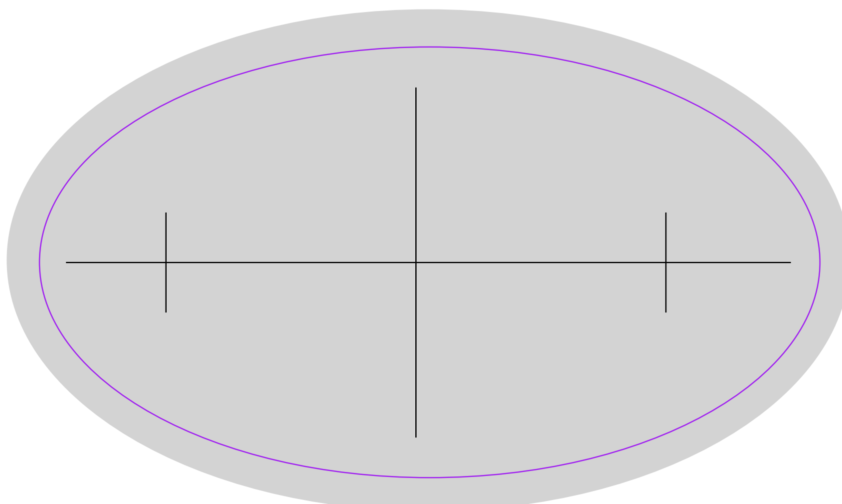

Figure 5.1: The geodesic $\Gamma \subset \mathbf{V} \setminus \mathbf{K}$ is shown in purple.

Lemma 26.7. *The curve Γ separates $\bigcup \gamma_j^1$ from $\partial\mathbf{V}$.*

Proof. Since $\mathbf{V} \setminus \mathbf{K} \subset \mathbf{V} \setminus \mathcal{K}$, we can view Γ as a simple, closed curve in $\mathbf{V} \setminus \mathcal{K}$. Let $\Gamma' \subset \mathbf{V} \setminus \mathcal{K}$ be the simple, closed geodesic in the homotopy class of $\Gamma \subset \mathbf{V} \setminus \mathcal{K}$. By theorem 26.1, Γ' separates $\bigcup \gamma_j^1$ from $\partial\mathbf{V}$.

Given a domain $D \subset \mathbb{C}$, a Borel set $E \subset \partial D$, and a point z in D, let $\omega(z, D)(E)$ be the harmonic measure of E with respect to the pointed domain (D, z). The geodesic Γ is the $1/2$-level set of $g : z \mapsto \omega(z, \mathbf{V} \setminus \mathbf{K})(\mathbf{K})$. The geodesic Γ' is the $1/2$-level set of $h : z \mapsto \omega(z, \mathbf{V} \setminus \mathcal{K})(\mathcal{K})$. Both g and h are harmonic functions on their respective domains. Since $g|\partial(\mathbf{V} \setminus \mathbf{K}) \geq h|\partial(\mathbf{V} \setminus \mathbf{K})$, we know that $g \geq h|\mathbf{V} \setminus \mathbf{K}$. Then $h|\Gamma \leq 1/2$, so Γ' cannot be closer to $\partial\mathbf{V}$ than Γ. $\square$

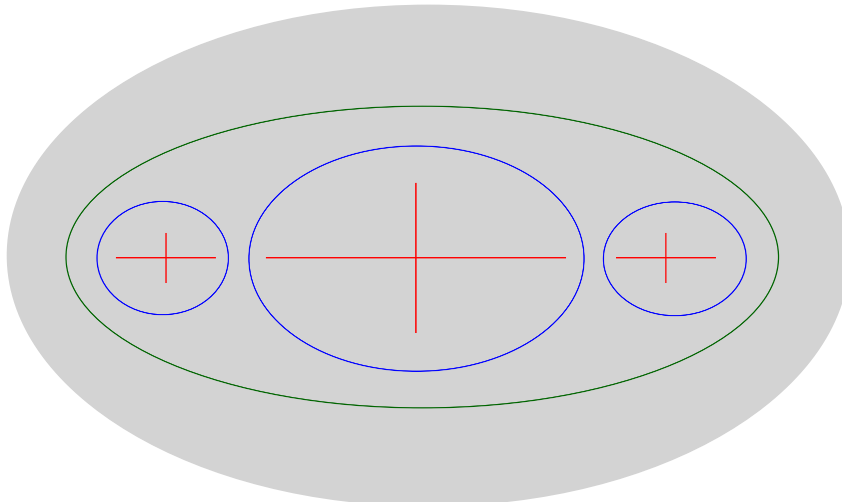

Figure 5.2: In blue, we have the equidistant curves $\bigcup_j \gamma_j^{1/2}$ associated with the simple, closed geodesics $\bigcup_j \gamma_j$ in $\mathbf{V} \setminus \mathcal{K}$. The green curve is the simple, closed geodesic in $\mathbf{V} \setminus \mathcal{K}$ homotopic to $\Gamma \subset \mathbf{V} \setminus \ \subset \mathbf{V} \setminus \mathcal{K}$.

Definition 26.1. Let $\mathbf{f} : (\mathbf{V}, \mathbf{K}) \to (\mathbf{V}, \mathbf{K})$ be a pseudo-polynomial-like map. Let $\Gamma \subset \mathbf{V} \setminus \mathbf{K}$ be the core geodesic. We let $A(\mathbf{f})$ denote the annulus $\Gamma^{(0,1/2)} \subset \mathbf{V} \setminus \mathbf{K}$.

27 One level of renormalization

27.1 The combinatorial model associated with a good renormalization

Given a polynomial-like map $f : U \to V$ admitting a good renormalization of period p, a superattracting model F of f was defined in subsection 15.1. It is a p-good polynomial, unique up to affine conjugacy. (Recall that F being p-good means that F has exactly one superattracting p-cycle, and all the critical points of F are either in this p-cycle or land on this p-cycle after one application of F.) We define $\mathrm{comb}_p(f) = F$, and by abuse of notation, we may identify F with its affine conjugacy class.

Lemma 27.1. *Fix integers $d \geq 2$ and $p \geq 1$. Up to affine conjugacy, there are only finitely many p-good polynomials of degree d.*

Proof. By corollary 3.7 in [BBL+00], there are only finitely many rational maps of degree d with postcritical set of order p. $\square$

27.2 A particular space of maps

Fix real numbers m and M, with $0 < m \leq M$. Let B, d and p be integers ≥ 2. Let $F : \mathbb{C} \to \mathbb{C}$ be a p-good polynomial of degree d. Let

$$\mathcal{X} = \mathcal{X}(d, p, B, F, m, M)$$

be the space of pseudo-polynomial-like maps $\mathbf{f}$ of degree d such that

1. $\mathbf{f}$ admits a good renormalization of period p,
2. $\mathbf{f}$ admits infinitely many good renormalizations,
3. $\mathrm{comb}_p(\mathbf{f}) = F$,
4. $\mathbf{f}$ the relative renormalization periods are bounded by B, and
5. $m \leq \mathrm{mod}(\mathbf{f}) \leq M$.

Remark 27.1. Let us explain the meaning of these conditions. Condition (1) and the upper bound in (5) allow us to define the domain $S(\mathbf{f})$ whenever $\mathbf{f} \in \mathcal{X}$. This domain is defined in terms of the hyperbolic geodesics going around the big Julia set and the little Julia sets. Condition (2) guarantees that $K(\mathbf{f})$ depends continuously on $\mathbf{f} \in \mathcal{X}$. Since our maps admit infinitely many good renormalizations, our Julia sets have no interior, so the Julia set is equal to the filled Julia set. Condition (3) guarantees that all of our maps have the same "Hubbard tree." Then we can mark the domains by the Hubbard tree, and we can choose markings of our Riemann surfaces to build a continuous map into Teichmüller space. Condition (4) guarantees that our space of maps is closed. The lower bound in condition (5) guarantees that our space of maps has compact closure. Compactness is necessary only for the proof of rigidity.

Definition 27.1. Let $\mathbf{f} : \mathbf{V} \to \mathbf{V}$ be a pseudo-polynomial-like map in $\mathcal{X}$. Let $\mathbf{K} \subset \mathbf{V}$ be the Julia set, and let $\mathcal{K} = \mathcal{K}_p = \bigcup_j K_j \subset \mathbf{K}$ be the little Julia sets corresponding to the period p. Let Γ denote the core geodesic in $\mathbf{V} \setminus \mathbf{K}$, and let γ_j denote the simple, closed geodesic in $\mathbf{V} \setminus \mathcal{K}$ going around K_j. By lemma 26.7, we know that Γ and $\bigcup \gamma_j^1$ are disjoint, and Γ surrounds $\bigcup \gamma_j^1$. We denote by $S(\mathbf{f}) = S_p(\mathbf{f})$ the domain in $\mathbb{C}$ bounded by $\Gamma \cup \bigcup \gamma_j^{1/2}$. (See figure 5.3.)

Fix a base point $\mathbf{f}_* \in \mathcal{X}$. Let $\mathrm{Mod}(\mathbf{f}_*)$ denote the moduli space $\mathrm{Mod}(S(\mathbf{f}_*))$. Let $\mathrm{Teich}(\mathbf{f}_*)$ denote the reduced Teichmüller space $\mathrm{Teich}^{\#}(S(\mathbf{f}_*))$.

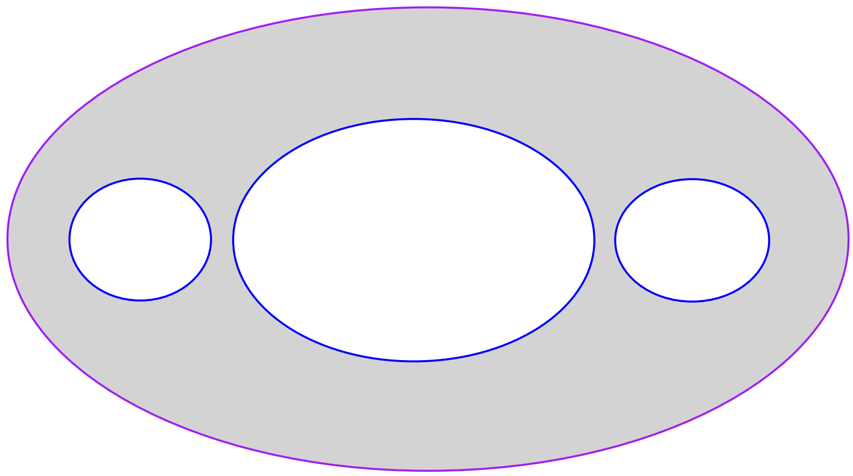

Figure 5.3: The domain $S(f)$ is bounded by $\Gamma \cup \bigcup_j \gamma_j^{1/2}$.

27.3 Continuity in moduli space

Lemma 27.2. *The map $\mathcal{X} \to \mathrm{Mod}(\mathbf{f}_*)$, defined by $\mathbf{f} \mapsto S(\mathbf{f})$, is continuous.*

Suppose that $\mathbf{f}_n \to \mathbf{f}$ in $\mathcal{X}$. For the pseudo-polynomial-like map $\mathbf{f}_n : \mathbf{V}_n \to \mathbf{V}_n$, the simple, closed geodesics $\Gamma_n \subset \mathbf{V}_n \setminus \mathbf{K}_n$ and $\gamma_{j,n} \subset \mathbf{V}_n \setminus \mathcal{K}_n$ are defined correspondingly. Let $S = S(\mathbf{f})$, and let $S_n = S(\mathbf{f}_n)$. We will show that $S_n \to S$ in $\mathrm{Mod}(\mathbf{f}_*)$.

Choose a point a in $\mathbf{V} \setminus \mathbf{K}$. (The choice of a only matters in that we need a base point to discuss Carathéodory convergence.) For all sufficiently large n, a belongs to $\mathbf{V}_n \setminus \mathbf{K}_n$.

Lemma 27.3. $(\mathbf{V}_n \setminus \mathbf{K}_n, a) \to (\mathbf{V} \setminus \mathbf{K}, a)$ *in the Carathéodory topology.*

Proof. Observing that $\mathbf{K}$ and $\mathbf{K}_n$ have empty interior, we see that $\mathbf{K}_n \to \mathbf{K}$ in the Hausdorff metric. (See [Dou94].) ☐

Lemma 27.4. $(\mathbf{V}_n \setminus \mathcal{K}_n, a) \to (\mathbf{V} \setminus \mathcal{K}, a)$ *in the Carathéodory topology.*

Proof. Let f and f_n be polynomial-like restrictions of $\mathbf{f}$ and $\mathbf{f}_n$, respectively. Consider a little Julia set $K_j \subset \mathcal{K}$, and consider a polynomial-like restriction of f^p near K_j. As $f_n \to f$, we know that $f_n^p \to f^p$ uniformly (on some compact neighborhood of K_j). Since all of the Julia sets under consideration have empty interior, we see that the Julia sets of our polynomial-like restrictions of f_n^p converge to K_j in the Hausdorff metric. ☐

We will need the following lemma:

Lemma 27.5. *Fix $m > 0$. There exist $C = C(m) > 1$ and $\epsilon_0 = \epsilon_0(m) > 0$ such that the following property holds: Given ϵ, with $0 < \epsilon < \epsilon_0$, and a C^1 embedding $f : \mathbb{R} \to \mathbb{C}$ such that $\|f - \mathrm{id}\|_{C^1} \leq \epsilon$, there is an embedding $F : \overline{\{z \in \mathbb{C} : 0 < \mathrm{Im}(z) < m\}} \to \mathbb{C}$ such that*

- $F|\mathbb{R} = f$,
- $F|(\mathbb{R} + mi) = \mathrm{id}$, *and*
- $F|\{z \in \mathbb{C} : 0 < \mathrm{Im}(z) < m\}$ *is $(1 + C\epsilon)$-quasiconformal.*

Furthermore, if f commutes with a real translation, then F does as well.

Proof. Choose $C = 2(2 + 1/m)$ and $\epsilon_0 = 1/C$. Define $F(x + iy) = (1 - y/m)f(x) + (y/m)x + iy$. The following computations are obvious:

$$\begin{aligned}
F_x(x+iy) &= f'(x) + (y/m)(1 - f'(x)),\\
F_y(x+iy) &= (1/m)(x - f(x)) + i,\\
2F_z(x+iy) &= f'(x) + 1 + (y/m)(1 - f'(x)) + i(1/m)(x - f(x)),\\
2F_{\overline{z}}(x+iy) &= f'(x) - 1 + (y/m)(1 - f'(x)) + i(1/m)(x - f(x)),\\
2|F_{\overline{z}}(x+iy)| &\leq |f'(x) - 1| + (y/m)|1 - f'(x)| + (1/m)|x - f(x)|\\
&\leq (2 + 1/m)\epsilon,\\
2|F_z(x+iy)| &\geq \big|2 - |f'(x) - 1 + (y/m)(1 - f'(x)) + i(1/m)(x - f(x))|\big|,\\
&\geq 2 - (2 + 1/m)\epsilon,\\
|\mu_F(x+iy)| &:= \left|\frac{F_{\overline{z}}(x+iy)}{F_z(x+iy)}\right| \leq \frac{(2 + 1/m)\epsilon}{2 - (2 + 1/m)\epsilon},\\
\mathrm{Dil}(F) &:= \frac{1 + \|\mu_F\|_\infty}{1 - \|\mu_F\|_\infty} \leq 1 + \frac{(2 + 1/m)\epsilon}{1 - (2 + 1/m)\epsilon} \leq 1 + C\epsilon.
\end{aligned}$$

□

Disjointness of Γ and $\bigcup \gamma_j^1$ implies that there is a real number t, $0 < t \leq 1$, such that Γ^{-t} and $\bigcup \gamma_j^1$ are disjoint.

Lemma 27.6. *Let $A \subset \mathbb{C}$ denote the annulus whose inner boundary is Γ^{-t} and outer boundary is Γ. Let $A_n \subset \mathbb{C}$ denote the annulus whose inner boundary is Γ^{-t} and outer boundary is Γ_n. There exist positive numbers $\{\epsilon_n\}_n$, with $\epsilon_n \to 0$ as $n \to \infty$, and orientation preserving homeomorphisms $\phi_n : \overline{A} \to \overline{A_n}$ such that*

- $\phi_n|A$ *is $(1 + \epsilon_n)$-quasiconformal, and*

- $\phi_n | \Gamma^{-t} = \mathrm{id}$.

Proof. By lemma 26.6, we know that $\Gamma_n^{-t} \to \Gamma^{-t}$ in the C^1 topology. Consider $\mathbf{V} \setminus \mathbf{K}$ realized as a round cylinder

$$\{z \in \mathbb{C} : |\operatorname{Im}(z)| < \pi/2\}/\langle z \mapsto z + |\Gamma|\rangle.$$

In coordinates $z = x + iy$, we see that Γ corresponds to the curve $y = 0$, and Γ^{-t} corresponds to a curve $y =$ constant. Furthermore, the geodesics Γ_n approximating Γ correspond to curves C^1 close to $y = 0$. We apply lemma 27.5. $\square$

Lemma 27.7. *Let A_j denote the annulus in $\mathbb{C}$ whose inner boundary is $\gamma_j^{1/2}$ and outer boundary is γ_j^1. Let $A_{j,n}$ denote the annulus in $\mathbb{C}$ whose inner boundary is $\gamma_{j,n}^{1/2}$ and outer boundary is γ_j^1. There exist positive numbers $\{\epsilon_{j,n}\}_n$, with $\epsilon_{j,n} \to 0$ as $n \to \infty$, and orientation preserving homeomorphisms $\phi_{j,n} : \overline{A_j} \to \overline{A_{j,n}}$ such that*

- *$\phi_{j,n} | A$ is $(1 + \epsilon_{j,n})$-quasiconformal, and*
- *$\phi_{j,n} | \gamma_j^1 = \mathrm{id}$.*

Proof. By lemma 26.6, we know that $(\gamma_{j,n})^{1/2} \to \gamma_j^{1/2}$ in the C^1 topology. Realize the annulus cover $A_{\gamma_j}(\mathbf{V} \setminus \mathcal{K})$ of $\mathbf{V} \setminus \mathcal{K}$ as a round cylinder

$$\{z \in \mathbb{C} : |\operatorname{Im}(z)| < \pi/2\}/\langle z \mapsto z + |\gamma_j|\rangle.$$

The standard collar neighborhood $C(\gamma_j)$ of γ_j embeds in this round cylinder. In coordinates $z = x+iy$, we see that $\gamma_j^{1/2}$ and γ_j^1 correspond to curves $y = c_{1/2}$ and $y = c_1$, respectively, where $c_{1/2}$ and c_1 are constants. The geodesics $\gamma_{j,n}^{1/2}$ approximating $\gamma_j^{1/2}$ correspond to curves C^1 close to $y = c_{1/2}$. We apply lemma 27.5. $\square$

To prove continuity of the S domains in moduli space, we want to show that there exist quasiconformal maps $S \to S_n$ that are arbitrarily close to being conformal. To this end, define $\psi_n : \overline{S} \to \overline{S_n}$ by

$$\psi_n(z) = \begin{cases} \phi_n(z) & \text{if } z \in \overline{A}, \\ \phi_{j,n}(z) & \text{if } z \in \overline{A_j}, \\ z & \text{if } z \in S \setminus (A \cup \bigcup_j A_j). \end{cases} \tag{13}$$

Since $\phi_n|\Gamma^{-t} = \mathrm{id}$ and $\phi_{j,n}|\gamma^1 = \mathrm{id}$, we see that ψ_n is continuous. In particular, it is an orientation-preserving homeomorphism. Furthermore, we know that

$$\psi_n : S \setminus (\Gamma^{-t} \cup \bigcup_j \gamma_j^1) \to S_n \setminus (\Gamma_n^{-t} \cup \bigcup_j \gamma_{j,n}^1)$$

is quasiconformal. Since $\Gamma^{-t} \cup \bigcup_j \gamma_j^1$ consists of finitely many analytic curves, we conclude that $\psi_n|S$ is quasiconformal. Clearly, $\mathrm{Dil}(\psi_n) \leq 1 + \max(\{\epsilon_n\} \cup \{\epsilon_{j,n}\}_j)$, and this quantity tends to 1 as $n \to \infty$. This completes the proof of lemma 27.2.

27.4 Bounded below implies bounded above

Let $\mathcal{X}^* = \mathcal{X}^*(d, p, B, F, m)$ denote the space of pseudo-polynomial-like maps obtained by replacing condition (5) in the definition of $\mathcal{X}$ with

$$\mathrm{mod}(\mathbf{f}) \geq m.$$

We will need the following lemma for the proof of rigidity.

Lemma 27.8. *There exists a real number $M = M(\mathcal{X}^*) > 0$ such that the following property is satisfied: If $\mathbf{f}'$ is a canonical p-renormalization of $\mathbf{f} \in \mathcal{X}^*$ around a little Julia set of $\mathcal{K}$, then $\mathrm{mod}(\mathbf{f}') \leq M$.*

Proof. By theorem 8.2, the space of maps $\mathcal{X}^*$ is compact, up to normalization. By lemma 27.4 and theorem 26.4, the function $\mathcal{X}^* \to (0, +\infty)$, defined by $\mathbf{f} \mapsto \min\{|\gamma_j|_{\mathbf{V}\setminus\mathcal{K}}\}_j$, is continuous. Consequently, there is a positive lower bound on the lengths of the small geodesics. This lower bound for length provides an upper bound M for the moduli of the corresponding canonical renormalizations. □

28 Two consecutive levels of renormalization

Fix real numbers m and M, with $0 < m \leq M$. Let B, d, p, and r be integers ≥ 2. Let F and G be p- and rp-good polynomials, respectively, of degree d. Let

$$\mathcal{Y} = \mathcal{Y}(d, p, r, B, F, G, m, M)$$

be the set of pseudo-polynomial-like maps $\mathbf{f}$ of degree d such that

1. $\mathbf{f}$ admits good renormalizations of periods p and rp,

2. $\mathbf{f}$ admits infinitely many good renormalizations,

3. $\mathrm{comb}_p(\mathbf{f}) = F$,

4. $\mathrm{comb}_{rp}(\mathbf{f}) = G$,

5. $\mathbf{f}$ the relative renormalization periods are bounded by B, and

6. $m \leq \mathrm{mod}(\mathbf{f}) \leq M$.

We define $T(\mathbf{f}) = S_{rp}(\mathbf{f})$. Choose a base point $\mathbf{f}_* \in \mathcal{Y}$.

Lemma 28.1. *The map $\mathcal{Y} \to \mathrm{Mod}(\mathbf{f}_*)$, defined by $\mathbf{f} \mapsto T(\mathbf{f})$, is continuous.*

Proof. This is an immediate consequence of lemma 27.2. □

29 Continuity and compatibility of markings

29.1 Markings

Let $U \subset \mathbb{C}$ be a domain bounded by at least 3, but finitely many, Jordan curves.

Definition 29.1. Let $n \geq 2$ be an integer. The *n-star*, denoted H_n, is the union $\bigcup_{k=0}^{n-1} \{re^{2\pi ik/n} : 0 \leq r < 1\}$ of half-open straight line segments joining 0 to the n-th roots of unity. The *tips* of H_n are the n-th roots of unity $\{e^{2\pi ik/n} : k \in \{0, \dots, n-1\}\}$.

By an *embedding* $H_n \to \mathbb{C}$, we mean an embedding (in the usual topological sense) that extends to an orientation preserving homeomorphism $\mathbb{C} \to \mathbb{C}$. This is equivalent to requiring that the embedding (in the usual topological sense) preserves the cyclic ordering of the tips of H_n. We denote by $\overline{H_n}$ the closure of H_n in $\mathbb{C}$. An embedding $H_n \to U$ is *proper* if it admits an extension $\overline{H_n} \to \overline{U}$ mapping the tips of H_n to ∂U. Two proper embeddings $H_n \to U$ are *equivalent* if they are homotopic through proper embeddings. These definitions extend immediately to embeddings of disjoint unions of n-stars.

Definition 29.2. A *marking* is a proper embedding $h : \bigsqcup_{i=1}^{N} A_i \to U$, where

- for each i, $A_i = H_{n_i}$ is the n_i-star,
- $h|A_i$ and $h|A_j$ are not equivalent when $i \neq j$, and
- $U \setminus h(\bigsqcup_i A_i)$ is homeomorphic to an annulus.

Definition 29.3. Let h be a marking of U, and let h' be a marking of U'. Let $\phi : U \to U'$ be an orientation preserving homeomorphism. We say that ϕ *respects the markings* if $\phi \circ h$ is equivalent to h'.

Assume that h is a marking of U. In an obvious way, we can think of the marking as a disked tree $\mathcal{T}$. (The vertices are the bounded components of $\mathbb{C} \setminus U$ and the images of $0 \in A_i$ under h. The edges are the remaining portions of the image of h.) Consider the set of orientation preserving homeomorphisms $\phi : U \to U$ respecting the marking of U. The isotopy classes of these homeomorphisms coincide with *symmetries* of the corresponding disked tree. (By such a symmetry, we mean the isotopy class, respecting the disked tree structure, of a homeomorphism $\mathcal{T} \to \mathcal{T}$ that extends to an orientation preserving homeomorphism $\mathbb{C} \to \mathbb{C}$.) The following lemma is obvious.

Lemma 29.1. *Assume that $h : \bigsqcup_{i=1}^{N} A_i$ is a marking of U, and let $\phi : U \to U$ be an orientation preserving homeomorphism.*

- *If ϕ respects the marking, then ϕ is isotopic to the identity.*
- *If the disked tree corresponding to the marking admits no non-trivial symmetries, and if $\phi(h(\bigsqcup_i A_i))$ can be continuously, properly deformed to $h(\bigsqcup_i A_i)$, then ϕ is isotopic to the identity.*

29.2 The marking induced by the Hubbard tree

Definition 29.4. Let $f : \mathbb{C} \to \mathbb{C}$ be a polynomial of degree $d \geq 2$, with connected Julia set, admitting a primitive renormalization of period $p \geq 2$ around one of its critical points. Let $\mathcal{K} = \bigcup_{j=0}^{p-1} K_j$ be the corresponding cycle of little Julia sets, enumerated according to the dynamics. For each j, let f_j be a polynomial-like restriction of f^p near K_j, and let $\boldsymbol{\beta}_j$ be the union of the β-fixed points of f_j together with their pre-images under f_j. Let $\boldsymbol{\beta} = \bigcup_j \boldsymbol{\beta}_j$. Each point of $\boldsymbol{\beta}$ is the landing point of at least 2, but finitely many, external rays of f. Let $\mathfrak{R} = \mathfrak{R}(\boldsymbol{\beta})$ be the union of these external rays together with their landing points. A component $D \subset \mathbb{C} \setminus \mathfrak{R}$ is *peripheral* if $\mathbb{C} \setminus D$ is connected. A non-peripheral component $D \subset \mathbb{C} \setminus \mathfrak{R}$ is *separating* if D and $\mathcal{K}$ are disjoint. We say that $I \subset \{0, \ldots, p-1\}$ is *admissible* if there is a separating component $D \subset \mathbb{C} \setminus \mathfrak{R}$ such that

- $K_i \cap \partial D \neq \emptyset$ for each $i \in I$, and
- $\mathcal{K} \cap \partial D = \bigcup_{i \in I} K_i \cap \partial D$.

Clearly, if such a component exists, then it is unique. We let $\mathcal{I}$ denote the set of admissible subsets of $\{0, \ldots, p-1\}$, and for each $I \in \mathcal{I}$, we let D_I denote the corresponding separating component. For each $I \in \mathcal{I}$, let H_I be the $|\mathcal{K} \cap \partial D_I|$-star. The ordering of $\mathcal{K}$ induces an ordering of $\mathcal{K} \cap \partial D_I$. Choose an embedding $h_I : H_I \to D_I$ that extends to an embedding $\overline{H_I} \to \overline{D_I}$ mapping the tips of H_I to the points of $\mathcal{K} \cap \partial D_I$, respecting the orderings. The *marking induced by the Hubbard tree* of the domain $\mathbb{C} \setminus \mathcal{K}$ is the embedding $h = \bigsqcup_{I \in \mathcal{I}} h_I : \bigsqcup_{I \in \mathcal{I}} H_I \to \mathbb{C} \setminus \mathcal{K}$.

Definition 29.5. Let X be a Riemann surface of finite type. A domain $U \subset X$ is a *compact core* if U is essential in X, U is homeomorphic to X, $\overline{U} \subset X$ is compact, and each component of ∂U is a Jordan curve.

Let f and H be as in the definition above. Let $U \subset \mathbb{C} \setminus \mathcal{K}$ be a compact core. Then U has a unique outer boundary component, and the enumeration of the little Julia sets induces an enumeration of the inner boundary components of U in the obvious way. Furthermore, the marking induced by the Hubbard tree of the domain $\mathbb{C} \setminus \mathcal{K}$ induces a marking of U (in the sense of definition 29.2), which we also denote by h, in the obvious way.

Consider the objects defined in section 27. The construction above generalizes to pseudo-polynomial-like maps in a straightforward way. Consequently, given $\mathbf{f} \in \mathcal{X}$ and a suitable dynamical enumeration (not just a cyclic order) of the little Julia sets $\mathcal{K}$, we define the corresponding marking by the Hubbard tree, denoted $h_{\mathbf{f}}$, of the domain $S(\mathbf{f})$.

Lemma 29.2. *Let $\mathbf{f} \in \mathcal{X}$. Then there is a unique homotopy class of orientation preserving homeomorphisms $S(\mathbf{f}_*) \to S(\mathbf{f})$ respecting the markings $h_{\mathbf{f}_*}$ and $h_{\mathbf{f}}$.*

Proof. The existence of the homotopy class follows from the existence of a Thurston equivalence between superattracting models of $\mathbf{f}_*$ and $\mathbf{f}$. Uniqueness, if it is not automatic, follows from lemma 29.1. □

29.3 Continuity in Teichmüller space

In light of lemma 29.2, we make the following definition: Given $\mathbf{f} \in \mathcal{X}$, let $\phi_{\mathbf{f}} : S(\mathbf{f}_*) \to S(\mathbf{f})$ be a quasiconformal map respecting the markings.

Lemma 29.3. *The map $\mathcal{X} \to \mathrm{Teich}(\mathbf{f}_*)$, defined by $\mathbf{f} \mapsto [S(\mathbf{f}), \phi_{\mathbf{f}}]$, is continuous.*

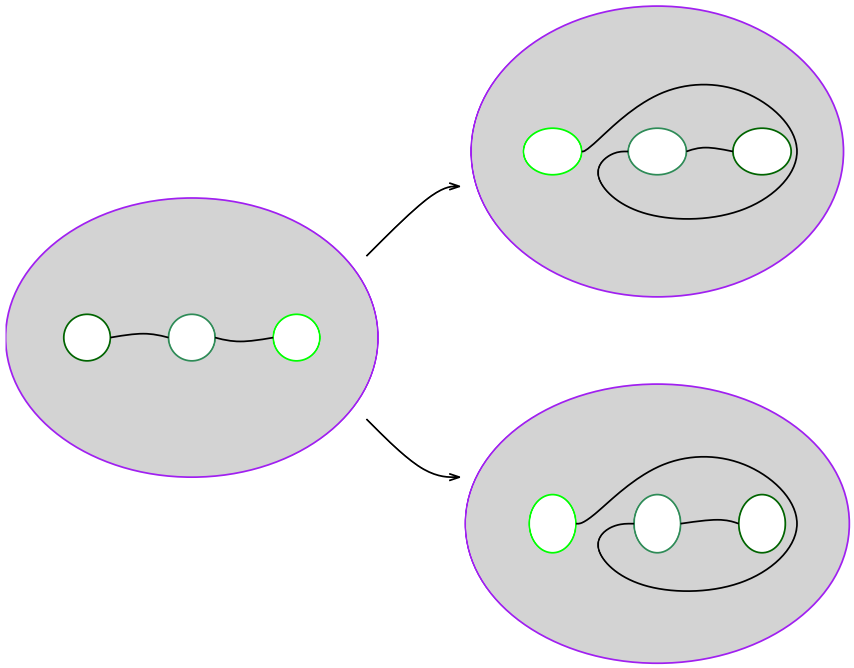

Figure 5.4: On the left, we have $S(\mathbf{f}_*)$. On the right, we have $S(\mathbf{f})$ and a nearby marked domain $S(\mathbf{f}_n)$.

Proof. Suppose that $\mathbf{f}_n \to \mathbf{f}$ in $\mathcal{X}$. Consider polynomial-like restrictions $f_n \to f$. Let $\boldsymbol{\beta} = \boldsymbol{\beta}(f)$ and $\{\boldsymbol{\beta}_n = \boldsymbol{\beta}(f_n)\}_n$ be the β-fixed points and the symmetric points in the little Julia sets. (See definition 29.4.) As $n \to \infty$, we know that $\boldsymbol{\beta}_n \to \boldsymbol{\beta}$, and all of these periodic points are repelling. Then the corresponding external rays have converging parametrizations.

Define $S = S(\mathbf{f})$, $h = h_{\mathbf{f}}$, and $\phi = \phi_{\mathbf{f}}$. For each n, define $S_n = S(\mathbf{f}_n)$, $h_n = h_{\mathbf{f}_n}$, and $\phi_n = \phi_{\mathbf{f}_n}$. From the proof of lemma 27.2, we know that the boundary curves of S_n converge to the boundary curves of S in the C^1 topology. This certainly implies Carathéodory convergence, so there exists a compact core $R \subset S$ such that for all sufficiently large n, R is also a compact core of S_n. Convergence of the external rays implies that the markings h and h_n restrict to equivalent markings on R. (See figure 5.4.) Consequently, $\phi_n \circ \phi^{-1} : S \to S_n$ is homotopic to the quasiconformal map $\psi_n : S \to S_n$ defined in equation (13). Then $[S_n, \phi_n] = [S_n, \psi_n \circ \phi]$, so

$$\operatorname{dist}([S, \phi], [S_n, \phi_n]) \leq \frac{1}{2} \log \operatorname{Dil}(\psi_n).$$

This quantity tends to 0 as $n \to \infty$ (see the proof of lemma 27.2), so we are finished. $\square$

The set of maps $\mathcal{Y}$, defined in section 28, is a subset of a set of maps $\mathcal{X}$, and the T domain is defined in terms of an S domain. Hence, we have also proven the following lemma, where $\phi_{\mathbf{f}} : T(\mathbf{f}_*) \to T(\mathbf{f})$ denotes a quasiconformal map respecting the markings.

Lemma 29.4. *The map $\mathcal{Y} \to \mathrm{Teich}(\mathbf{f}_*)$, defined by $\mathbf{f} \mapsto [T(\mathbf{f}), \phi_{\mathbf{f}}]$, is continuous.*

29.4 Compatibility of the markings

Let $\mathbf{f} \in \mathcal{Y} = \mathcal{Y}_{p,rp}$. For each $j \in \{0, \ldots, p-1\}$, let $\mathbf{f}_j$ denote the canonical p-renormalization of $\mathbf{f}$ around K_j. Let H denote the marking induced by the Hubbard tree of the domain $T(\mathbf{f}) = S_{rp}(\mathbf{f})$. Let h denote the marking induced by the Hubbard tree of the domain $S = S(\mathbf{f}) = S_p(\mathbf{f})$. For each j, let h_j denote the marking induced by the Hubbard tree of the domain $S_j = S(\mathbf{f}_j) = S_r(\mathbf{f}_j)$. The marking H restricts to a marking $H|S$ of S in an obvious way. Similarly, the marking H restricts to a marking $H|S_j$ of S_j for each j. (See figure 5.5.)

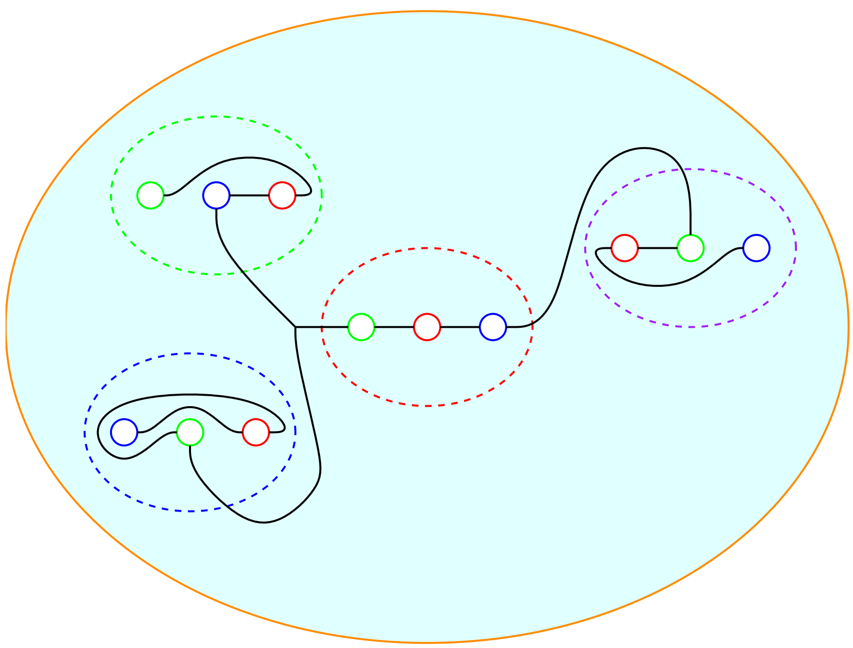

Figure 5.5: This is a marked T domain.

Lemma 29.5. *The markings h and $H|S$ of the domain S are equivalent. For each j, the markings h_j and $H|S_j$ of the domain S_j are equivalent.*

Proof. Let $\mathcal{K} = \mathcal{K}_p = \bigcup_{j=0}^{p-1} K_j$ and $\mathcal{K}' = \mathcal{K}_{rp} = \bigcup_{j=0}^{rp-1} K'_j$ denote the little Julia sets corresponding to the renormalization periods p and rp of $\mathbf{f}$, enumerated in such a way that $K_j \cap \mathcal{K}' = \bigcup_{i=0}^{r-1} K'_{j+ip}$ for each $j \in \{0, \ldots, p-1\}$. The marking H is defined using the external rays $\mathfrak{R}'$ of $\mathbf{f}$ landing on the β-fixed points and the symmetric points in the little Julia sets $\mathcal{K}'$. The marking h is defined using the external rays $\mathfrak{R}$ of $\mathbf{f}$ landing on the β-fixed points and the symmetric points in the little Julia sets $\mathcal{K}$. For each j, the marking h_j is defined using the external rays of $\mathfrak{R}'$ contained in the component of the complement of $\mathfrak{R}$ containing K_j. □

30 The proof of rigidity

Theorem 30.1. *Let f and g be polynomials of degree d admitting infinitely many primitive renormalizations around each of their critical points, with combinatorics bounded by B. If f and g are conjugate by an orientation-preserving homeomorphism, then f and g are hybrid conjugate.*

30.1 Setting things up

Consider the decomposition of f in remark 23.5. We have a special set of critical points

$$\{c_1, \ldots, c_N\} \subset \mathrm{Crit}(f).$$

For each $k \in \{1, \ldots, N\}$, we have the renormalization periods associated with c_k,

$$\{q_n(k)\}_{n=0}^{\infty},$$

which satisfy $q_{n+1}(k)/q_n(k) \leq B$ for all n. For each $j \in \{0, \ldots, q_0(k) - 1\}$, we have a polynomial-like restriction of an iterate of f,

$$f_{k,j} := f^{q_0(k)} : U_j(k) \to V_j(k).$$

The domains $\{V_j(k)\}_{k,j}$ satisfy

$$\overline{V_j(k)} \cap \overline{V_{j'}(k')} = \emptyset \text{ whenever } (k, j) \neq (k', j'). \tag{14}$$

In this paragraph, we will describe a labeling of the iterated canonical renormalizations of the maps $\{f_{k,j}\}_{k,j}$ using cycle trees. This notation respects the nested structure of the associated little Julia sets. In this way, the set of all iterated canonical renormalizations of the maps $\{f_{k,j}\}_{k,j}$ is $\{f_{k,\alpha}\}_{k,\alpha}$. Whenever we speak about a multi-index, we will implicitly require that it is admissible for the underlying cycle tree. Fix k, and consider the maps $\{f_{k,j}\}_j$. In the notation of section 23, there is an associated cycle tree,

$$\mathcal{T}_k = (\{q_n(k)\}_n, \{\mathcal{A}^n(k)\}_n, f).$$

Each map $f_{k,j}$ admits infinitely many good renormalizations (with periods $\{q_{n+1}(k)/q_0(k)\}_n$). From the point of view of the cycle tree $\mathcal{T}_k$, the maps $\{f_{k,j}\}_j$ are those labeled by multi-indices of depth 0. Let us label the iterated canonical renormalizations of the maps $\{f_{k,j}\}_j$ inductively. (In our notation, the letter j will be the index of a map on the top level. The letter α will be a multi-index. We can also think of j as a multi-index of depth 0.) If for some depth $\ell \geq 0$, the maps

$$\{f_{k,\alpha} : |\alpha| = \ell\},$$

have already been defined, then for any $\alpha' \succ \alpha$, where $|\alpha| = \ell$, we let $f_{k,\alpha'}$ be the canonical renormalization of $f_{k,\alpha}$ around $K_{\alpha'}(k) \subset \mathcal{A}^{\ell+1}(k)$.

Lemma 9.1 and theorem 21.3 imply that for each k and each α,

$$\deg(f_{k,\alpha}) \leq 2^{d-1} \text{ and } \operatorname{mod}(f_{k,\alpha}) \geq \mu_f > 0.$$

Let $\mathcal{B} : (\mathbb{C}, K(f), P(f)) \to (\mathbb{C}, K(g), P(g))$ be an orientation preserving homeomorphism conjugating f to g. There is a set of special critical points,

$$\{c'_1 = \mathcal{B}(c_1), \ldots, c'_N = \mathcal{B}(c_N)\} \subset \operatorname{Crit}(g),$$

with the same associated renormalization periods $\{q_n(k)\}_{k,n}$. Furthermore, there are polynomial-like maps

$$\{g_{k,j} := g^{q_0(k)} = \mathcal{B} \circ f_{k,j} \circ \mathcal{B}^{-1} : U'_j(k) \to V'_j(k)\}_{k,j},$$

on the top level, where $U'_j(k) = \mathcal{B}(U_j(k))$ and $V'_j(k) = \mathcal{B}(V_j(k))$ for each k and each j. The domains $\{V'_j(k)\}_{k,j}$ satisfy the disjointness property corresponding to (14). There are the iterated canonical renormalizations $\{g_{k,\alpha}\}_{k,\alpha}$, where $K(g_{k,\alpha}) = \mathcal{B}(K(f_{k,\alpha}))$ for each k and each α. Lemma 9.1 and theorem 21.3 imply that for each k and each α,

$$\deg(g_{k,\alpha}) \leq 2^{d-1} \text{ and } \operatorname{mod}(g_{k,\alpha}) \geq \mu_g > 0.$$

The domains $A(\mathbf{f})$, $S(\mathbf{f})$, and $T(\mathbf{f})$ associated with a pseudo-polynomial-like map $\mathbf{f}$ were defined in definition 26.1, definition 27.1, and section 28, respectively. (See figure 5.6.) For each k and each α, we define $A_{k,\alpha} = A(f_{k,\alpha})$,

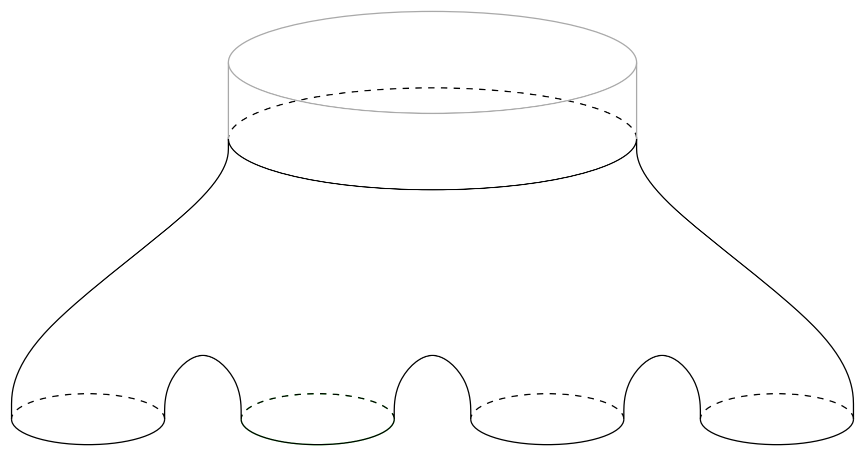

Figure 5.6: A domain $S_{k,\alpha}$ is outlined in black, and an annulus $A_{k,\alpha}$ is outlined in gray.

$S_{k,\alpha} = S(f_{k,\alpha})$, and $T_{k,\alpha} = T(f_{k,\alpha})$. We let O denote the unbounded component of $\mathbb{C} \setminus \bigcup_{k,j} \overline{A_{k,j}}$. By lemma 26.3, each component of

$$(\mathbb{C} \setminus P(f)) \setminus (O \cup \bigcup_{k,\alpha} (A_{k,\alpha} \cup S_{k,\alpha}))$$

is a quasicircle, whose dilatation is bounded in terms of μ_f.

We define the corresponding objects for g: $A'_{k,\alpha} = A(g_{k,\alpha})$, $S'_{k,\alpha} = S(g_{k,\alpha})$, and $T'_{k,\alpha} = T(g_{k,\alpha})$. We let O' denote the unbounded component of $\mathbb{C} \setminus \bigcup_{k,j} \overline{A'_{k,j}}$. By lemma 26.3, each component of

$$(\mathbb{C} \setminus P(g)) \setminus (O' \cup \bigcup_{k,\alpha} (A'_{k,\alpha} \cup S'_{k,\alpha}))$$

is a quasicircle, whose dilatation is bounded in terms of μ_g.

We have the common lower bound for the moduli of $\{f_{k,\alpha}\}_{k,\alpha}$ and $\{g_{k,\alpha}\}_{k,\alpha}$,

$$\mu = \min\{\mu_f, \mu_g\}. \tag{15}$$

30.2 Bounded geometry

Consider the relative renormalization periods

$$\{r_n(k) := q_{n+1}(k)/q_n(k)\}_{k,n}$$

for the maps $\{f_{k,\alpha}\}_{k,\alpha}$ and $\{g_{k,\alpha}\}_{k,\alpha}$. (The correspondence between n and α is $n = |\alpha|$.) By hypothesis, we know that $r_n(k) \leq B$ for all k and all n.

By lemma 9.1, we know that $\deg(f_{k,\alpha}) = \deg(g_{k,\alpha}) \leq 2^{d-1}$ for all k and all α. Then lemma 27.1 implies that there are only finitely many combinatorial models: The set of p-good polynomials of degree D, with $D \leq 2^{d-1}$ and $p \leq B$, is finite, up to affine conjugacy. Consequently, we need only consider finitely many spaces of pseudo-polynomial-like maps

$$\mathcal{X}_1^*, \ldots, \mathcal{X}_M^* \tag{16}$$

based on these combinatorial models. (See subsection 27.4.) More precisely, each $\mathcal{X}_i^*$ is equal to some space $\mathcal{X}^*(D, p, B, F, \mu)$, where $D \leq 2^{d-1}$, $p \leq B$, F is a p-good polynomial of degree D, and μ is the lower bound on moduli coming from (15).

For each k and each α, the topological conjugacy B between f and g restricts to a topological conjugacy between polynomial-like restrictions of $f_{k,\alpha}$ and $g_{k,\alpha}$. It follows that

$$\mathrm{comb}_{r_{|\alpha|}(k)}(f_{k,\alpha}) = \mathrm{comb}_{r_{|\alpha|}(k)}(g_{k,\alpha}).$$

Then $f_{k,\alpha}$ and $g_{k,\alpha}$ both belong to the same space $\mathcal{X}_i^*$, for some i. By lemma 27.8, the moduli of the canonical $r_{|\alpha|}(k)$-renormalizations of $f_{k,\alpha}$ and $g_{k,\alpha}$ are bounded above by a constant $\overline{\mu}_i = \overline{\mu}_i(\mathcal{X}_i^*)$.

Let $\overline{\mu} = \max(\{\overline{\mu}_i\}_i \cup \{\mathrm{mod}(f_{k,j})\}_{k,j} \cup \{\mathrm{mod}(g_{k,j})\}_{k,\alpha})$. Consider the spaces of pseudo-polynomial-like maps

$$\mathcal{X}_1, \ldots, \mathcal{X}_M,$$

with moduli bounded below by μ and above by $\overline{\mu}$, corresponding to (16) above. (See section 27.) Then for each k and each α, $f_{k,\alpha}$ and $g_{k,\alpha}$ belong to the same space $\mathcal{X}_i$, for some i.

Lemma 30.2. *There exists $K_1 = K_1(\mathcal{X}_1, \ldots, \mathcal{X}_M) \geq 1$ such that for each k and each α, there is a K-quasiconformal map $S_{k,\alpha} \to S'_{k,\alpha}$ in the homotopy class determined by $\mathcal{B}$.*

Proof. By theorem 8.2, the space of maps $\mathcal{X}_i$ is compact, up to normalization. By lemma 29.3, the image of $\mathcal{X}_i$ in Teichmüller space is bounded. $\square$

30.3 Building a quasiconformal map in the wrong homotopy class

We will build a quasiconformal map in the wrong homotopy class by gluing together the maps obtained in lemma 30.2, but first we need an estimate on the annuli separating the S domains:

Lemma 30.3. *The moduli of the annuli $\{A_{k,\alpha}\}_{k,\alpha}$ and $\{A'_{k,\alpha}\}_{k,\alpha}$ are bounded below and above.*

Proof. Recall that for a pseudo-polynomial-like map $\mathbf{f} : (\mathbf{V}, \mathbf{K}) \to (\mathbf{V}, \mathbf{K})$, we have $A(\mathbf{f}) = \Gamma^{(0,1/2)} \subset \mathbf{V} \setminus \mathbf{K}$. For each k and each α, we have $\underline{\mu} \leq \operatorname{mod}(f_{k,\alpha}) \leq \overline{\mu}$. Then the modulus of $A_{k,\alpha}$ is bounded above by $\overline{\mu}$ and bounded below in terms of $\underline{\mu}$. The corresponding statements also hold for $\{g_{k,\alpha}\}_{k,\alpha}$ and $\{A'_{k,\alpha}\}_{k,\alpha}$. $\square$

Consider some fixed k and α. Lemma 30.2 provides an orientation preserving homeomorphism

$$h_{k,\alpha} : \overline{S_{k,\alpha}} \to \overline{S'_{k,\alpha}}$$

such that

- $h_{k,\alpha}$ is in the homotopy class determined by $\mathcal{B}$,
- $h_{k,\alpha}|S_{k,\alpha}$ is K_1-quasiconformal, and
- the restriction of $h_{k,\alpha}$ to any component of $\partial S_{k,\alpha}$ is $M = M(K_1, \underline{\mu})$-quasisymmetric.

Lemma 30.3 (upper and lower bounds on moduli), lemma 26.3 (boundary curves are quasicircles), and lemma 10.2 (quasiconformal interpolation with controlled dilatation) imply that for each β, with $\alpha \prec \beta$, we can find an orientation preserving homeomorphism

$$a_{k,\beta} : \overline{A_{k,\beta}} \to \overline{A'_{k,\beta}}$$

such that

- $a_{k,\beta}|A_{k,\beta}$ is K_3-quasiconformal, where K_3 depends only on M, $\underline{\mu}$, and the bounds on moduli from lemma 30.3,
- restricted to $\partial S_{k,\alpha} \cap \partial A_{k,\beta}$, $a_{k,\beta} = h_{k,\alpha}$, and
- restricted to $\partial A_{k,\beta} \cap \partial S_{k,\beta}$, $a_{k,\beta} = h_{k,\beta}$.

Now, consider the domains on the top level: O, O', $\{A_{k,j}\}_{k,j}$, and $\{A'_{k,j}\}_{k,j}$. Obviously, there is an orientation preserving homeomorphism $e : \overline{O} \to \overline{O'}$ such that

- $e|O$ is quasiconformal, and

- e is in the homotopy class of homeomorphisms $O \to O'$ determined by $\mathcal{B}$.

We again apply lemmas 30.3, 26.3, and 10.2: For each k and each j, there is an orientation preserving homeomorphism $a_{k,j} : \overline{A_{k,j}} \to \overline{A'_{k,j}}$ such that

- $a_{k,j}|A_{k,j}$ is K_4-quasiconformal, where K_4 depends only on the dilatation of $e|O$, M, μ, and the bounds from lemma 30.3,
- restricted to $\partial O \cap \partial A_{k,j}$, $a_{k,j} = e$, and
- restricted to $\partial A_{k,j} \cap \partial S_{k,j}$, $a_{k,j} = h_{k,j}$.

Finally, we can define the desired quasiconformal map in the wrong homotopy class. Let $h : \mathbb{C} \setminus P_f \to \mathbb{C} \setminus P_g$ be defined by

$$h(z) = \begin{cases} e(z) & \text{if } z \in \overline{O}, \\ a_{k,\alpha}(z) & \text{if } z \in \overline{A_{k,\alpha}}, \\ h_{k,\alpha}(z) & \text{if } z \in \overline{S_{k,\alpha}}. \end{cases}$$

For purely topological reasons, $h : \mathbb{C} \setminus P(f) \to \mathbb{C} \setminus P(g)$ extends uniquely to a homeomorphism $h : (\mathbb{C}, P(f)) \to (\mathbb{C}, P(g))$. Since $P(f)$ is a Cantor set satisfying the divergence property (there is a lower bound on moduli of the annuli $\{A_{k,\alpha}\}_{k,\alpha}$), the extension $h : \mathbb{C} \to \mathbb{C}$ is quasiconformal. By construction, the homeomorphism $h|P(f) : P(f) \to P(g)$ agrees with $\mathcal{B}|P(f)$. However, there is no reason at this point that h should be homotopic to $\mathcal{B}$ relative $P(f)$.

30.4 Fixing the homotopy class

In this subsection, we make use of the topology theorems in section 31.

Reasoning similar to that in subsection 30.2 shows that we need only consider finitely many spaces of pseudo-polynomial-like maps

$$\mathcal{Y}_1, \ldots, \mathcal{Y}_{M'}$$

based on the combinatorial models reflecting two levels of renormalization, with moduli bounded below by μ and above by $\overline{\mu}$. (See section 28 for the definition of these spaces.) For each k and each α, we have

$$\operatorname{comb}_{r_{|\alpha|}(k)}(f_{k,\alpha}) = \operatorname{comb}_{r_{|\alpha|}(k)}(g_{k,\alpha}),$$
$$\operatorname{comb}_{r_{|\alpha|}(k) r_{|\alpha|+1}(k)}(f_{k,\alpha}) = \operatorname{comb}_{r_{|\alpha|}(k) r_{|\alpha|+1}(k)}(g_{k,\alpha}).$$

Then $f_{k,\alpha}$ and $g_{k,\alpha}$ belong to the same space of maps $\mathcal{Y}_i$, for some i.

Lemma 30.4. *There exists $K_2 = K_2(\mathcal{Y}_1, \dots, \mathcal{Y}_{M'}) \geq 1$ such that for each k and each α, there is a K_2-quasiconformal map $T_{k,\alpha} \to T'_{k,\alpha}$ in the homotopy class determined by $\mathcal{B}$.*

Proof. By theorem 8.2, the space of maps $\mathcal{Y}_i$ is compact, up to normalization. By lemma 29.4, the image of $\mathcal{Y}_i$ in Teichmüller space is bounded. □

Consider some fixed k and α. Lemma 30.4 provides an orientation preserving homeomorphism

$$t_{k,\alpha} : \overline{T_{k,\alpha}} \to \overline{T'_{k,\alpha}}$$

such that

- $t_{k,\alpha}$ is in the homotopy class determined by $\mathcal{B}$, and
- $t_{k,\alpha}|T_{k,\alpha}$ is K_2-quasiconformal.

Lemma 29.2 and lemma 29.5 imply that

- $t_{k,\alpha}|S_{k,\alpha}$ is homotopic to $h_{k,\alpha} : S_{k,\alpha} \to S'_{k,\alpha}$, as a map into $\mathbb{C} \setminus P(g)$, and
- for each β, with $\alpha \prec \beta$, the "restriction" of $t_{k,\alpha}$ to $S_{k,\beta}$ is homotopic to $h_{k,\beta}$, as a map into $\mathbb{C} \setminus P(g)$. (There is a unique component $R = R_{k,\beta} \subset T_{k,\alpha} \setminus S_{k,\alpha}$ that is isotopic in $\mathbb{C} \setminus P(f)$ to $S_{k,\beta}$. Consequently, the inclusion $R \hookrightarrow \mathbb{C} \setminus P(f)$ induces an isomorphism $\pi_1(R) \to \pi_1(S_{k,\beta})$. Then the homomorphisms $\pi_1(t_{k,\alpha}|R) : \pi_1(S_{k,\beta}) \to \pi_1(\mathbb{C} \setminus P(g))$ and $\pi_1(h_{k,\beta}) : \pi_1(S_{k,\beta}) \to \pi_1(\mathbb{C} \setminus P(g))$ coincide.)

Then $\phi = \phi_{k,\alpha} := h^{-1} \circ t_{k,\alpha} : T_{k,\alpha} \to \mathbb{C} \setminus P(f)$ is a quasiconformal embedding (with $\mathrm{Dil}(\phi) \leq \mathrm{Dil}(h) \cdot K_2$). By lemma 31.4, the homotopy class of ϕ is that of a composition of Dehn twists around the inner boundary components of $S_{k,\alpha}$. In other words, ϕ is homotopic to a *Dehn multi-twist*, a composition of finitely many commuting Dehn twists.

We would like to post-compose ϕ with Dehn twists around the annuli $\{A_{k,\beta} : \alpha \prec \beta\}$ so that the resulting map is quasiconformal, with controlled dilatation, and homotopic to the identity embedding $T_{k,\alpha} \to \mathbb{C} \setminus P(f)$. By lemma 30.3, we know that there is a uniform lower bound on the moduli of the annuli $\{A_{k,\beta} : \alpha \prec \beta\}$, so we need only check that there is also a uniform upper bound (independent of k and α) on the order of twisting. This is a consequence of bounded geometry (for the domains bounded by $\Gamma(\mathbf{f}) \cup \mathcal{K}_{rp}(\mathbf{f})$ corresponding to maps $\mathbf{f} \in \mathcal{Y}$) and the following theorem applied to ϕ.

Theorem 30.5. *Let $C \subset \mathbb{C}$ be a Jordan curve, and let U be the bounded component of $\mathbb{C} \setminus C$. Let $\mathcal{K} = \bigcup K_j \subset U$ be a finite union of disjoint, compact, connected sets, each containing at least 2 points. Let $Y = U \setminus \mathcal{K}$. Let $X \subset Y$ be an essentially embedded domain bounded by finitely many Jordan curves, such that $(\partial X) \cap (\partial Y) = C$. Let $\{\gamma_j\}_{j=1}^k$ be a finite set of disjoint, simple, closed, geodesics contained in X, and assume that no component of $X \setminus \bigcup \gamma_j$ is an annulus. Let $f : X \to Y$ be a quasiconformal embedding such that $f(C) = C$. Let $D = D_{\gamma_1}^{n_1} \circ \cdots \circ D_{\gamma_k}^{n_k} : X \to X$ be a Dehn multi-twist. Assume that f and D are homotopic as maps $X \to Y$. Let $n = \max\{|n_j|\}_j$. There exists $K = K(n, X, Y) \geq 1$ such that $\mathrm{Dil}(f) \geq K$. Furthermore, $K \to \infty$ as $n \to \infty$.*

Proof. Choose γ_i such that $n_i = n$. Let E_0 denote the components of ∂X outside of γ_i, and let E_1 denote the components of ∂X inside γ_i. Let $h : \overline{X} \to [0,1]$ be the unique continuous function, harmonic on X, which assumes the value 0 on E_0 and 1 on E_1. The flow lines for ∇h determine a singular foliation of X. Let Γ denote the path family made up of the nonsingular leaves of this foliation. (This is the foliation associated with the extremal distance between E_0 and E_1. See [Oht70].)

Let $d = \min\{\mathrm{diam}_{\mathbb{C}}(K_j)\}_j$. Let $\epsilon = \epsilon(Y) = d/\mathrm{diam}_{\mathbb{C}}(C) \in (0,1)$. For convenience, let us rescale the objects under consideration so that $d = 1$. (Then $\mathrm{diam}_{\mathbb{C}}(C) = 1/\epsilon$.) Clearly, this does not change ϵ.

We will find a lower bound on the extremal length of the path family $f(\Gamma)$. To this end, let δ be any path of $f(\Gamma)$. Suppose that $n \geq 2$. Then $|\delta|_{\mathbb{C}} \geq n$, because $d = 1$, and one endpoint of δ must remain on C. Using the fact that $\mathrm{diam}_{\mathbb{C}}(f(X)) = \mathrm{diam}_{\mathbb{C}}(C)$, the isodiametric inequality implies that $\mathrm{area}_{\mathbb{C}}(f(X)) \leq \pi(\mathrm{diam}_{\mathbb{C}}(C)/2)^2 = \pi/(4\epsilon^2)$. Then

$$\mathcal{L}(f(\Gamma)) \geq \frac{n^2}{\mathrm{area}_{\mathbb{C}}(f(X))} \geq 4\epsilon^2 n^2/\pi.$$

Since f is quasiconformal, we know that

$$\mathcal{L}(f(\Gamma)) \leq (\mathrm{Dil}\, f)\, \mathcal{L}(\Gamma).$$

Combining these inequalities, we obtain $\mathrm{Dil}(f) \geq 4\epsilon^2 n^2/(\pi\, \mathcal{L}(\Gamma)) =: K$. $\square$

Abusing notation, let us use the same letter h for the map we obtain after adjusting h by the appropriate Dehn twists around the annuli $\{A_{k,\alpha}\}_{k,\alpha}$ as described above. At this point, we have built a quasiconformal map $h : \mathbb{C} \setminus P(f) \to \mathbb{C} \setminus P(g)$ such that $\mathcal{B}^{-1} \circ h : \mathbb{C} \setminus P(f) \to \mathbb{C} \setminus P(f)$ satisfies the hypotheses of theorem 31.1. Hence, h is homotopic to $\mathcal{B} : \mathbb{C} \setminus P(f) \to \mathbb{C} \setminus P(g)$. We summarize our progress in the following lemma.

Lemma 30.6. *The quasiconformal map $h : (\mathbb{C}, P(f)) \to (\mathbb{C}, P(g))$ is equal to $\mathcal{B}$ on $P(f)$ and is homotopic to $\mathcal{B}$ relative $P(f)$.*

30.5 Constructing a quasiconformal conjugacy

The pullback argument (see the remark at the end of section 10.2 in [Lyu97]) allows us to promote h to a quasiconformal conjugacy of f and g. Consider polynomial-like restrictions

$$f : (U^1, K(f)) \to (U^0, K(f)) \text{ and } g : (V^1, K(f)) \to (V^0, K(f)),$$

where the domains are bounded by smooth curves. The existence of h readily implies that there is a quasiconformal map

$$h_1 : (U^0, U^1, P(f)) \to (V^0, V^1, P(g))$$

such that

- $h_1|P(f) \cup \partial U^1$ conjugates $f|P(f) \cup \partial U^1$ and $g|P(g) \cup \partial V^1$, and
- h_1 is homotopic, relative $P(f)$, to a conjugacy of f and g.

We define inductively a sequence $\{h_n : (U^0, P(f)) \to (V^0, P(g))\}_{n=1}^{\infty}$ of quasiconformal maps with uniformly bounded dilatation: Having already defined

$$h_n : (U^0, P(f)) \to (V^0, P(g))$$

for some $n \geq 1$, the map h_{n+1} is obtained by gluing a lift $\widetilde{h_n}$ of h_n to h_n along ∂U^n.

$$\begin{tikzcd}
(U^{n+1}, f^{-1}(P(f))) \arrow[r, dashed, "\widetilde{h_n}"] \arrow[d, "f"] & (V^{n+1}, g^{-1}(P(g))) \arrow[d, "g"] \\
(U^n, P(f)) \arrow[r, "h_n"] & (V^n, P(f))
\end{tikzcd}$$

For each n, we have $h_n \circ f = g \circ h_{n+1}$ on $(U^0 \setminus U^{n+1}) \cup P(f)$. Abusing notation, let us denote by h any subsequential limit of $\{h_n\}_n$. By construction, we have the following:

Lemma 30.7. *The quasiconformal map $h : U^0 \to V^0$ conjugates $f|U^1$ and $g|V^1$.*

This completes the proof that $f \sim_{\text{top}} g$ implies $f \sim_{\text{qc}} g$. By corollary 5.2 in [Ino02], we know that f and g do not carry invariant line fields on their Julia sets. Consequently, $f \sim_{\text{qc}} g$ implies $f \sim_{\text{hyb}} g$, completing the proof of rigidity.

Lemma 30.8. *The polynomials f and g are hybrid equivalent.*

31 Some topology theorems

Let $C \subset \mathbb{R}^2$ be a Cantor set, and define $X = \mathbb{R}^2 \setminus C$. Except when $\mathbb{R}^2$ is mentioned explicitly, we will view X intrinsically, forgetting that $X \subset \mathbb{R}^2$. Let $\Gamma \subset X$ be a union of disjoint, simple, closed curves such that each component of $X \setminus \Gamma$ is homeomorphic to a sphere with at least 3, but finitely many, holes. An *adjacent pair* $W \subset X$ is a domain $Y \cup \gamma \cup Z$, where $\gamma \subset \Gamma$ is a component, and Y and Z are components of $X \setminus \Gamma$ such that $\partial Y \cap \partial Z = \gamma$.

Theorem 31.1. *Let $h : X \to X$ be a homeomorphism. Suppose that for any adjacent pair $W \subset X$, there is a homotopy $W \times [0,1] \to X$ between $h|W$ and the identity map. Then h is homotopic to the identity map.*

Corollary 31.2. *Let $h : X \to X$ be a homeomorphism. Suppose that for any adjacent pair $W \subset X$, there is a domain $W' \subset X$ such that*

- *there is a homotopy $W' \times [0,1] \to X$ between $h|W'$ and the identity map, and*
- *there is a path of embeddings $i_t : W \to X$, $t \in [0,1]$, such that $i_0 = \mathrm{id}\,|W$, and $i_1(W) = W'$.*

Then h is homotopic to the identity map.

Let $S^1 = \mathbb{R}/\mathbb{Z}$ denote the unit circle.

Lemma 31.3. *Let $A = S^1 \times [0,1]$ be an annulus. Let $F : A \to A$ be an orientation-preserving homeomorphism. Let $f = F|S^1 \times \{0\}$, and let $g = F|S^1 \times \{1\}$. If $h_t : S^1 \times \{1\} \to S^1 \times \{1\}$, $t \in [0,1]$, is a homotopy such that $h_0 = g$ and $h_1 = \mathrm{id}$, then there is a homotopy $H_t : A \to A$, $t \in [0,1]$, such that*

- *$H_0 = F$,*
- *for each t, $H_t|S^1 \times \{1\} = h_t$, and*
- *for each t, $H_t|S^1 \times \{0\} = f$.*

Proof. Consider the vertical curve $V = \{0\} \times [0,1] \subset A$. Its image $F(V)$ is a curve in A joining $f(0)$ and $g(0)$. There is a unique "straight line segment" $L_0 \subset A$ joining $f(0)$ and $g(0)$, which is homotopic rel endpoints to $F(V)$. For any other point $x \in S^1 \setminus \{0\}$, there is a unique straight line segment $L_x \subset A$ joining $f(x)$ and $g(x)$, which is disjoint from L_0. Let $G : A \to A$ be the map taking (x, y) to the point on the line segment L_x at height y. Then G is an orientation-preserving homeomorphism with the same boundary values as F, and $G^{-1} \circ F$ takes V to a curve in the same homotopy class rel endpoints. Since the mapping class group of A rel ∂A is isomorphic to $\mathbb{Z}$, we conclude that G is homotopic rel ∂A to F.

We will construct a homotopy $H_t : A \to A$, $t \in [0,1]$. Define $H_0 = G$. Suppose that for some $0 \leq s < 1$, H_s has already been defined. Choose $s < T \leq 1$ such that h_s and h_T are uniformly close. For each $x \in S^1$ and $t \in (s, T]$, there is a unique straight line segment $L_{x,t} \subset A$ joining $f(x)$ and $h_t(x)$, whose distance from $H_s(x \times [0,1])$ is at most the uniform distance between h_s and h_t. We let H_t denote the map taking (x, y) to the point on the line segment $L_{x,t}$ at height y. Concatenating the homotopy we just constructed with the homotopy between G and F, we obtain the desired homotopy. $\square$

For each component $\gamma \subset \Gamma$, choose an open collar neighborhood $C(\gamma)$ containing γ. We will assume that the collar neighborhoods have been chosen so that if γ and γ' are different components of Γ, then $\overline{C(\gamma)}$ and $\overline{C(\gamma')}$ are disjoint. Adjusting h by a homotopy, we can assume that for each component $\gamma \subset \Gamma$, $h(\gamma) = \gamma$, and $h(C(\gamma)) = C(\gamma)$.

In this context, an *adjacent pair* $W \subset X$ is a domain $Y \cup \overline{C(\gamma)} \cup Z$, where $\gamma \subset \Gamma$ is a component, and Y and Z are components of $X \setminus \bigcup_{\gamma \subset \Gamma} \overline{C(\gamma)}$ such that $\partial Y \cap \partial Z = \partial C(\gamma)$. We can assume that for any adjacent pair $W \subset X$, there is an isotopy $w_t : \overline{W} \to \overline{W}$, $t \in [0,1]$, such that $w_0 = h|\overline{W}$ and $w_1 = \mathrm{id}\,|\overline{W}$.

We will define a union $\mathcal{W} \subset X$ of disjoint adjacent pairs and a union $\mathcal{C} \subset X$ of collar neighborhoods such that $\mathcal{W}$ and $\mathcal{C}$ are disjoint, $\partial \mathcal{W} = \partial \mathcal{C}$, and $\mathcal{W} \cup \overline{\mathcal{C}} = X$. Let $W^0 \subset X$ be an adjacent pair containing the unbounded component of $\mathbb{R}^2 \setminus \bigcup_{\gamma \subset \Gamma} \overline{C(\gamma)}$. Let $C^0 \subset X$ be the union of collar neighborhoods $C(\gamma)$ sharing a boundary component with W^0. We define W^n and C^n inductively: For each $n \geq 1$, let W^n be the union of adjacent pairs $W \subset X \setminus W^{n-1}$ sharing a boundary component with C^{n-1}, and let C^n be the union of collar neighborhoods $C(\gamma) \subset X \setminus C^{n-1}$ sharing a boundary component with W^n. Define $\mathcal{W} = \bigcup_n W^n$, and define $\mathcal{C} = \bigcup_n C^n$.

Proof of theorem 31.1. Let $W \subset \mathcal{W}$ be a component. By hypothesis, there is an isotopy $w_t : \overline{W} \times [0,1] \to \overline{W}$ such that $w_0 = h|\overline{W}$, and $w_1 = \mathrm{id}\,|\overline{W}$. Let $V \subset X$ be the domain such that $V \supset \overline{W}$, V is homeomorphic to W, and $\partial V \subset \Gamma$. Let $\gamma \subset \partial V$ be a component. Let $\delta = \partial C(\gamma) \cap \partial W$. Let A^δ be the closed, topological annulus bounded by δ and γ. Applying lemma 31.3 to the annulus A^δ, we see that the isotopy $w_t|\delta$ extends to a homotopy $v_t^\gamma : A^\delta \to A^\delta$, $t \in [0,1]$, satisfying

- $v_0^\gamma = h|A^\delta$,

- for each t, $v_t^\gamma|\delta = w_t$, and
- for each t, $v_t^\gamma|\gamma = h$.

Gluing the isotopy w_t with the homotopies v_t^γ, $\gamma \subset \partial V$, we obtain a homotopy $u_t : \overline{V} \to \overline{V}$, $t \in [0,1]$, defined by

$$u_t(z) = \begin{cases} w_t(z) & \text{if } z \in \overline{W}, \\ v_t^\gamma(z) & \text{if } z \in A^\delta, \end{cases}$$

which satisfies $u_0 = h|\overline{V}$, $u_1|\overline{W} = \mathrm{id}$, and for each t, $u_t|\partial V = h$.

For each component of $\mathcal{W}$, we obtain a homotopy as in the previous paragraph. Gluing together these homotopies, we obtain a homotopy $H_t : X \to X$, $t \in [0,1]$, such that $H_0 = h$, and $H_1|\overline{W} = \mathrm{id}$. Define $F = H_1$.

We will show that $F : X \to X$ is homotopic to the identity rel $\overline{\mathcal{W}}$. Let $\gamma \subset \Gamma \cap \mathcal{C}$ be a component, and let W be the adjacent pair containing γ. Since $W \backslash \mathcal{W} = \overline{C(\gamma)}$, we know that $F|W \backslash C(\gamma) = \mathrm{id}$. Since $F : X \to X$ is homotopic to h, we know that $F|W$ is homotopic to the identity. Choose a vertical curve β in $\overline{C(\gamma)}$ (such curves can be defined via a choice of homeomorphism $\overline{C(\gamma)} \to S^1 \times [0,1]$), and complete it to a proper curve $\alpha \subset W$ such that $\alpha \cap \overline{C(\gamma)} = \beta$. Since $F|W$ is homotopic to the identity, $F(\alpha)$ is properly homotopic to α. Since $F|W \setminus C(\gamma) = \mathrm{id}$, it follows that $F(\beta) \subset \overline{C(\gamma)}$ is homotopic rel endpoints to β. Then $F|\overline{C(\gamma)}$ is homotopic rel $\partial C(\gamma)$ to the identity. Since the collars $\overline{C(\gamma)}$, $\gamma \subset \Gamma'$ are disjoint, we are finished. □

We include the following theorem for completeness. It says that if the restriction of a homeomorphism to the complement of an annulus is homotopic to the identity, then the homeomorphism is homotopic to Dehn twist. (See figure 5.7.)

Theorem 31.4. *Let $W = Y_0 \cup \overline{C(\gamma)} \cup Y_1 \subset X$ be an adjacent pair. Let $f : W \to X$ be an orientation-preserving embedding such that for each $i \in \{0,1\}$, $f|Y_i$ is homotopic, by a homotopy $Y_i \times [0,1] \to X$, to the identity. Then f is homotopic, by a homotopy $W \times [0,1] \to X$, to a Dehn twist around γ.*

Proof. For each $i \in \{0,1\}$, let $\delta_i = (\partial Y_i) \cap \partial C(\gamma)$, let $V_i \subset W \setminus \gamma$ be the component containing Y_i, and let A_i be the closed, topological annulus bounded by δ_i and γ. Adjusting f by a homotopy, we can assume that $f(\gamma) = \gamma$, $f(C(\gamma)) = C(\gamma)$, and $f(W) = W$. Then the homotopy conditions can be strengthened: For each i, there is an isotopy $w_{i,t} : Y_i \cup \delta_i \to Y_i \cup \delta_i$, $t \in [0,1]$, such that $w_{i,0} = f$, and $w_{i,1} = \mathrm{id}$. For each i, lemma 31.3 implies that the isotopy $w_{i,t}|\delta_i$ extends to a homotopy $v_{i,t} : A_i \to A_i$, $t \in [0,1]$, such that

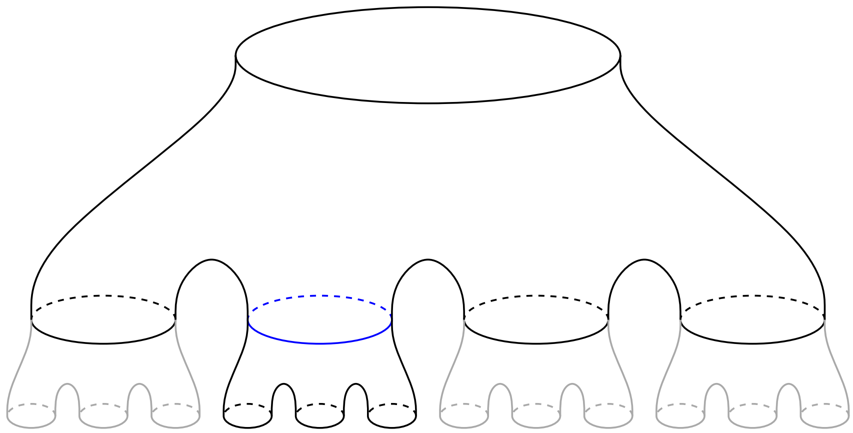

Figure 5.7: Theorem 31.4 says that if a homeomorphism from the surface illustrated above to itself is homotopic to the identity map on the black portions of the figure in the complement of the blue curve, then the homeomorphism is homotopic to a Dehn twist around the blue curve.

- $v_{i,0} = f$,
- for each t, $v_{i,t}|\delta_i = w_{i,t}$, and
- for each t, $v_{i,t}|\gamma = f$.

For each i, we can glue $w_{i,t}$ with $v_{i,t}$, and we can glue $v_{0,t}$ with $v_{1,t}$. We obtain a homotopy $H_t : W \to W$, $t \in [0, 1]$ such that $H_0 = f$, and $H_1|W \setminus C(\gamma) = \mathrm{id}$. It follows that $H_1|\overline{C(\gamma)} : \overline{C(\gamma)} \to \overline{C(\gamma)}$ is homotopic to a Dehn twist. $\square$

Bibliography

[Ahl10] Lars Ahlfors, *Conformal invariants: topics in geometric function theory*, American Mathematical Soc., 2010.

[AKLS09] Artur Avila, Jeremy Kahn, Mikhail Lyubich, and Weixiao Shen, *Combinatorial rigidity for unicritical polynomials*, Annals of Mathematics (2009), 783–797.

[BBL$^+$00] Eva Brezin, Rosemary Byrne, Joshua Levy, Kevin Pilgrim, and Kelly Plummer, *A census of rational maps*, Conformal Geometry and Dynamics **4** (2000), 35–74.

[BF14] Bodil Branner and Núria Fagella, *Quasiconformal surgery in holomorphic dynamics*, Cambridge University Press, 2014.

[BH88] Bodil Branner and John Hubbard, *The iteration of cubic polynomials Part I: The global topology of parameter space*, Acta Mathematica **160** (1988), 143–206.

[BH92] ———, *The iteration of cubic polynomials Part II: patterns and parapatterns*, Acta Mathematica **169** (1992), 229–325.

[Che10] Davoud Cheraghi, *Combinatorial rigidity for some infinitely renormalizable unicritical polynomials*, Conformal Geometry and Dynamics of the American Mathematical Society **14** (2010), 219–255.

[Com13] Mark Comerford, *The Carathéodory topology for multiply connected domains I*, Open Mathematics **11** (2013), 322–340.

[DH] Adrien Douady and John Hubbard, *Exploring the Mandelbrot set: the Orsay notes*, `http://www.math.cornell.edu/~hubbard/OrsayEnglish.pdf`.

[DH85] ———, *On the dynamics of polynomial-like mappings*, Annales scientifiques de l'École normale supérieure **18** (1985), 287–343.

[Dou93] Adrien Douady, *Descriptions of compact sets in* $\mathbb{C}$, Topological methods in modern mathematics, Publish or Perish, 1993, pp. 429–465.

[Dou94] ______, *Does a Julia set depend continuously on the polynomial?*, Proceedings of Symposia in Applied Mathematics, vol. 49, 1994, pp. 91–135.

[Ear67] Clifford Earle, *Reduced Teichmüller spaces*, Transactions of the American Mathematical Society **126** (1967), 54–63.

[FM07] Alastair Fletcher and Vladimir Markovic, *Quasiconformal maps and Teichmüller theory*, Oxford University Press, 2007.

[GS96] Jacek Graczyk and Grzegorz Świątek, *Polynomial-like property for real quadratic polynomials*, Topology Proceedings, 1996, `http://topology.auburn.edu/tp/reprints/v21/tp21004.pdf`.

[Hej74] Dennis Hejhal, *Universal covering maps for variable regions*, Mathematische Zeitschrift **137** (1974), 7–20.

[HJ98] Jun Hu and Yunping Jiang, *The Julia set of Feigenbaum quadratic polynomial*, Dynamical Systems: Proceedings of the International Conference in Honor of Professor Liao Shantao, 1998.

[Hub93] John Hubbard, *Local connectivity of Julia sets and bifurcation loci: three theorems of J.-C. Yoccoz*, Topological methods in modern mathematics, Publish or Perish, 1993.

[Ino02] Hiroyuki Inou, *Renormalization and rigidity of polynomials of higher degree*, Journal of Mathematics of Kyoto University **42** (2002), 351–392.

[Jia00] Yunping Jiang, *Infinitely renormalizable quadratic polynomials*, Transactions of the American Mathematical Society **352** (2000), 5077–5091.

[Kah06] Jeremy Kahn, *A priori bounds for some infinitely renormalizable quadratics: I. bounded primitive combinatorics*, `http://arxiv.org/abs/math/0609045`, 2006.

[KL08] Jeremy Kahn and Mikhail Lyubich, *A priori bounds for some infinitely renormalizable quadratics: II. decorations*, Annales scientifiques de lÉ. NS **41** (2008), 57–84.

[KL09a] ______, *Local connectivity of Julia sets for unicritical polynomials*, Annals of Mathematics (2009), 413–426.

[KL09b] ______, *A priori bounds for some infinitely renormalizable quadratics: III. molecules*, Complex dynamics: families and friends, A. K. Peters, 2009, pp. 229–254.

[KL09c] ______, *The quasi-additivity law in conformai geometry*, Annals of Mathematics (2009), 561–593.

[KSvS07] Oleg Kozlovski, Weixiao Shen, and Sebastian van Strien, *Rigidity for real polynomials*, Annals of Mathematics (2007), 749–841.

[LV73] Olli Lehto and Kaarlo Virtanen, *Quasiconformal mappings in the plane*, Springer, 1973.

[LvS98] Genadi Levin and Sebastian van Strien, *Local connectivity of the Julia set of real polynomials*, Annals of Mathematics (1998), 471–541.

[LY97] Mikhail Lyubich and Michael Yampolsky, *Dynamics of quadratic polynomials: complex bounds for real maps*, Annales de l'Institut Fourier **47** (1997), 1219–1255.

[Lyu97] Mikhail Lyubich, *Dynamics of quadratic polynomials, I-II*, Acta Mathematica **178** (1997), 185–297.

[Lyu16] ______, *Conformal geometry and dynamics of quadratic polynomials*, unpublished manuscript, 2016.

[McM94] Curtis McMullen, *Complex dynamics and renormalization*, Princeton University Press, 1994.

[McM96] ______, *Renormalization and 3-manifolds which fiber over the circle*, Princeton University Press, 1996.

[Mil00] John Milnor, *Local connectivity of Julia sets: expository lectures*, The Mandelbrot set, theme and variations, Cambridge University Press, 2000.

[Mil06] ______, *Dynamics in one complex variable*, Princeton University Press, 2006.

[MT98] Katsuhiko Matsuzaki and Masahiko Taniguchi, *Hyperbolic manifolds and Kleinian groups*, Clarendon Press, 1998.

[Oht70] Makoto Ohtsuka, *Dirichlet problem, extremal length, and prime ends*, Van Nostrand Reinhold Company, 1970.

[Roe00] Pascale Roesch, *Holomorphic motions and puzzles (following M. Shishikura)*, The Mandelbrot set, theme and variations, Cambridge University Press, 2000.

[Sul85] Dennis Sullivan, *Quasiconformal homeomorphisms and dynamics I. solution of the Fatou-Julia problem on wandering domains*, Annals of Mathematics **122** (1985), 401–418.

[Sul88] ______, *Bounds, quadratic differentials, and renormalization conjectures*, Proceedings of the AMS Centennial Symposium, 1988.

[Yar95] Brian Yarrington, *Local connectivity and Lebesgue measure of polynomial Julia sets*, Ph.D. thesis, State University of New York at Stony Brook, 1995, `http://www.math.stonybrook.edu/alumni/1995-Yarrington-Brian.pdf`.