Algebraic Methods in Topology and Applications

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Abstract of the Dissertation

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In statistics cumulants are defined to be functions that measure the linear independence of random variables. Cumulants can be described as functions that measure deviation of a map between algebras from being an algebra morphism. In Algebraic topology maps that are homotopic to being algebra morphisms are studied using the theory of A_∞ and C_∞ algebras. In this thesis we will explore the link between these two points of views on maps between algebras that are not algebra maps.

Contents

Chapter 1

Introduction

In statistics cumulants are defined to be functions that measure dependence of random variables. If the random variables are independent the cumulants are zero. These cumulants can be defined in general for linear maps between commutative algebras which do not respect the algebraic structure.

The first few cumulants for a map e between two commutative algebras are defined as follows.

$$k_1(a) = e(a)$$

$$k_2(a, b) = e(ab) - e(a)e(b)$$

$$k_3(a, b, c) = e(abc) - e(ab)e(c) - e(a)e(bc) - e(ca)e(b) + 2e(a)e(b)e(c)$$

In general, k_n is defined as follows.

$$k_n(a_1, a_2, \ldots, a_n) = \sum_{\pi}(|\pi| - 1)!(-1)^{|\pi|-1} \prod_{b\in\pi} e(\prod_{i\in b} a_i)$$

The sum is taken over all partitions of $\{1, \ldots, n\}$. Knowing the cumulants allows you to calculate the expectations of products. For example

$$e(ab) = k_2(a, b) + k_1(a)k_1(b)$$

$$e(abc) = k_3(a, b, c)+k_2(a, b)k_1(c)+k_2(b, c)k_1(a)+k_2(c, a)k_1(b)+k_1(a)k_1(b)k_1(c)$$

For a linear maps e between associative algebras we define Boolean cumulants to measure the deviation of e from being an algebra map [11]. The Boolean cumulants are a family of maps $K_n : V^{\otimes n} \to \mathbb{K}$ defined as follows.

$$K_1(a) = e(a)$$

$$K_2(a, b) = e(ab) - e(a)e(b)$$

$$K_3(a,b,c) = e(abc) - e(a)e(bc) - e(ab)e(c) + e(a)e(b)e(c)$$

K_n in general is given by the following formula.

$$K_n(a_1, a_2, \ldots, a_n) = \sum \pm e(a_1, \ldots, a_i)e(a_{i+1}, \ldots) \ldots e(\ldots, a_n)$$

The above sum is taken over all ordered partitions of n. The even partitions occur with negative signs and the odd partitions occur with positive signs. Expectations of products can also be computed using Boolean cumulants. If e is a map of algebras then the cumulants are all zero.

More generally cumulants can be defined using the above formulas for chain maps between differential graded algebras. Furthermore for linear maps between A_∞ algebras Boolean cumulants can be defined up to homotopy and for maps between C_∞ algebras the usual cumulants can be defined up to homotopy.

For instance, consider the differential forms $\Omega(M)$ on a manifold M and the cochains $C^*(M)$ on a discrete simplicial structure on the manifold. There is a chain map I from $\Omega(M)$ to $C^*(M)$ given by integrating forms on the simplices. This map induces an isomorphism on the cohomologies of the two complexes. The differential forms have an algebra structure given by the wedge product. An associative cup product can be defined on $C^*(M)$ but I is not a map of algebras for this product. Both the products however induce products on cohomology and the isomorphism induced by I on cohomology respects the induced products. Thus while the cumulants exist at the level of cochain complexes they vanish on cohomology.

Alternatively, the algebra structure of differential forms can be transferred to an A_∞ algebra structure on $C^*(M)$. This implies $C^*(M)$ has a product m_2 which is (infinitely) homotopic to being associative. This transfer depends on several choices including an appropriate choice of a homotopy inverse to I. For this A_∞ structure on $C^*(M)$, I is the first term of an A_∞ map. We can define Boolean cumulants of I up to homotopy. In fact for appropriate choices of homotopies, the transferred structure is C_∞ and I is the first term of a C_∞ morphism. Thus regular cumulants are defined up to homotopy. All of the above mentioned cumulants vanish for the isomorphism that is induced on cohomology.

We have the following two theorems which relate the Boolean cumulants to A_∞ morphisms and regular cumulants to C_∞ morphisms.

Theorem 1. Let A and B be two A_∞ algebras. Let p be a chain map from A to B. Let K_2, K_3 and so on be the Boolean cumulants of p defined up to homotopy. Suppose p is the first term of an A_∞ morphism $(p, p_2, p_3, \ldots)$ where $p_n : A^{\otimes n} \to B$. Then the following statements hold.

i) p_2 gives a homotopy from the second Boolean cumulant K_2 to zero. All the different ways of defining the higher Boolean cumulants K_n are also homotopic to zero using maps created by p_2 and p_1.

ii) p_3 gives a homotopy between different ways of making K_3 homotopic to zero. For all the higher Boolean cumulants, homotopies between the multiple different ways of making them homotopic to zero are homotopic to each other using p_3, p_2 and p_1.

iii) In general any cycles that are created using the homotopies $\{p_j\}_{j=1}^{n}$ are made homotopic to zero using maps made by $\{p_j\}_{j=1}^{n+1}$.

The above theorem means that if p is the first term of an A_∞ morphism then the cumulants of p completely collapse. That is, they are not only homotopic to zero, multiple homotopies are homotopic to each other. In particular, the above statement holds when A and B are differential graded associative algebras and p is a chain map which does not respect the algebra structure but is the first term of an A_∞ morphism. A similar theorem holds in case of C_∞ algebras and the regular cumulants.

Theorem 2. Let A and B be two C_∞ algebras. Let p be a chain map from A to B. Let k_2, k_3 and so on be the regular cumulants of p defined up to homotopy. Suppose p is the first term of an C_∞ morphism $(p, p_2, p_3, \ldots)$ where $p_n : A^{\otimes n} \to B$. Then the following statements hold.

i) p_2 gives a homotopy from the second cumulant k_2 to zero. All the different ways of defining the higher cumulants k_n are also homotopic to zero using maps created by p_2 and p_1.

ii) p_3 gives a homotopy between different ways of making k_3 homotopic to zero. For all the higher cumulants, homotopies between the multiple different ways of making them homotopic to zero are homotopic to each other using p_3, p_2 and p_1.

iii) In general any cycles that are created using the homotopies $\{p_j\}_{j=1}^{n}$ are made homotopic to zero by maps made using $\{p_j\}_{j=1}^{n+1}$.

Chapter 2

A_∞ and C_∞ algebras

2.1 Differential graded algebas and coalgebras

Definition 1. A *differential graded associative algebra* or a *dga* is a triple (A, d, m) such that

i) $A = \bigoplus_{n\in\mathbb{Z}} A_n$ is a graded vector space.

ii) $m : A \otimes A \to A$ is an associative product of degree zero. That is m is associative and for $a \in A_n$ and $b \in A_m$, $m(a \otimes b)$ is in A_{n+m}.

iii) $d : A \to A$ is a linear map of degree 1 (for $a \in A_n$, $d(a) \in A_{n+1}$) such that $d^2 = 0$.

iv) (Leibniz Rule) d and m satisfy the following compatibility relationship

$$d(m(a \otimes b)) = m(d(a) \otimes b) + (-1)^{|a|} m(a \otimes d(b))$$

Definition 2. An *differential graded commutative algebra* or a *dgca* is a dga (A, d, m) such that m is graded commutative. That is

$$m(a \otimes b) = (-1)^{|a||b|} m(b \otimes a)$$

Remark 1 (Koszul sign convention). Given two linear maps f and g of graded vector space we can define the $f \otimes g$ to be a map from the tensor product of the domain to the tensor product of the range. We use the Koszul sign convention when applying tensor products of linear maps. That is

$$f \otimes g(x \otimes y) = (-1)^{|x||g|}(f(x) \otimes g(y))$$

Definition 3. A *differential graded coalgebra* or a *dg-coalgebra* is a triple (C, δ, Δ) where

i) $C = \bigoplus_{n\in\mathbb{Z}} C_n$ is a graded vector space.

ii) $\Delta : C \to C \otimes C$ is a co-associative coproduct of degree zero. Coassociativity implies that

$$(\Delta \otimes 1) \circ \Delta = (1 \otimes \Delta) \circ \Delta$$

iii) $\delta : C \to C$ is a linear map of degree -1 (for $a \in C_n$, $\delta(a) \in C_{n-1}$) such that $\delta^2 = 0$.

iv) (Leibniz Rule) δ and Δ satisfy the following compatibility relationship

$$(\delta \otimes 1 + 1 \otimes \delta) \circ \Delta = \Delta \circ \delta$$

Remark 2. Sweedler's notation for coalgebras is a way of describing the coproduct Δ on an element c of C.

$$\Delta(c) = \sum_{(c)} c_{(1)} \otimes c_{(2)}$$

Definition 4. A *differential graded commutative coalgebra* or a *dgc-coalgebra* is a *dg-coalgebra* (C, δ, Δ) such that Δ is graded co-commutative. In the Sweedler notation this means

$$\Delta(c) = \sum_{(c)} c_{(1)} \otimes c_{(2)} = \sum_{(c)} c_{(2)} \otimes c_{(1)}$$

Definition 5. Let (A, d_A) and (B, d_B) be two differential graded complexes. A *chain map of degree* i is a linear map $f : A \to B$ such that $f(A_n) \subseteq B_{n+i}$ and $f \circ d_A = (-1)^i d_B \circ f$.

Remark 3. A *dga* (or a *dgca*) is in particular a cochain complex (degree 1 differential) with some additional product structure while a *dg*-coalgebra or a *dgc*-coalgebra is a chain complex (degree -1 differential) with an additional coproduct structure. For a cochain complex (A, d) the cohomology groups are defined to be

$$H^n(A) = (ker(d) \cap A_n)/(Im(d) \cap A_n)$$

Similarly the homology groups for a chain complex (C, δ) are defined to be

$$H_n(C) = (ker(\delta) \cap C_n)/(Im(\delta) \cap C_n)$$

A chain map of complexes induces a map of the same degree on the cohomology or the homology.

Remark 4 (Tensor product of *dg*-complexes is a *dg*-complex). Suppose (A, d_A) and (B, d_B) are non-negatively graded differential cochain (or chain) complexes. That is for $n < 0$, A_n and B_n are zero. Then $A \otimes B$ is also a differential graded complex. The nth grading of $A \otimes B$ is

$$(A \otimes B)_n = \bigoplus_{i+j=n} A_i \otimes B_j$$

The differential on this complex is given by $d_A \otimes 1 + 1 \otimes d_B$. Note that the Leibniz rule condition in the definition of a *dga* (or a *dg*-coalgebra) is equivalent to saying that the product (or the co-product) is a degree zero chain map of the two complexes.

Remark 5 (*Hom*-complex). Suppose (A, d_A) and (B, d_B) are two cochain (or chain) complexes. Then the complex of graded linear maps from A to B, $Hom(A, B)$ is also a differential graded complex. The grading on the complex is given by the linear maps being graded and the differential ∂ acts on a map p as

$$\partial(p) = d_B \circ p + (-1)^{|p|} p \circ d_A$$

Example 1. One of the first non-trivial example of a *dgca* is the algebra of differential forms on $\Omega^*(M)$ on a manifold M. The differential d has degree one as d of an n form is an $n+1$ form. The product m is the wedge product which has degree zero as the wedge product of an m-form with an n-form is a $(m+n)$-form. This algebra is also graded commutative as give two forms ω and η,

$$\omega \wedge \eta = (-1)^{|\omega||\eta|} \eta \wedge \omega$$

The wedge product induces the graded commutative cup product on the cohomology of the manifold.

Example 2. Consider the chains C_* on a finite simplicial decomposition of a space X. There is a coproduct map $\Delta : C_* \to C_* \otimes C_*$ called the Alexander-Whitney map which is given by the following formula on a simplex $[v_0, v_1, \ldots, v_n]$.

$$\Delta([v_0, v_1, \ldots, v_n]) = \sum_i [v_0, v_1, \ldots, v_i] \otimes [v_i, \ldots, v_n]$$

This product dualizes to an associative product μ on the cochains C^*. The associativity follows from the fact that the coproduct Δ is coassociative. Also the coproduct satisfies the co-Leibniz property with respect to the boundary operator ∂. That is, for a simplex σ

$$\Delta(\partial(\sigma)) = (1 \otimes \partial + \partial \otimes 1)(\Delta(\sigma))$$

This implies that the co-boundary map δ is a derivation of the dual product μ on the cochains C^*. Thus (C^*, δ, μ) is a differential graded algebra. Unlike the differential forms the cochains are not graded commutative, but the product μ also induces a graded commutative product on the cohomology of the space.

2.2 A_∞ algebras and morphisms

In 1963 James Stasheff defined a notion of an algebra that was associative up to 'infinite homotopy'.

Definition 6. An A_∞ algebra is a graded vector space A with a collection of linear maps

$$m_n : A[1]^{\otimes n} \to A[1]$$

such that m_n have degree 1 on and they satisfy the following equations for every n

$$\sum_{i+j=n} m_i(1 \otimes 1 \otimes \dots m_j \dots \otimes 1) = 0 \tag{2.1}$$

The equations in 2.1 imply the following statements.

- m_1 is a linear map of degree 1 that squares to zero. Thus m_1 is a differential on A.
- m_2 is a binary product and m_1 is a derivation of this binary product.
- Since m_2 is not associative, that associator $m_2(m_2 \otimes 1) - m_2(1 \otimes m_2)$ is not zero. m_3 is a map whose boundary is the associator. That is m_3 makes m_2 homotopic to being associative.
- m_n, for n larger than 3, makes cycles created by m_k, for k less than n, homotopic to zero.

The homotopies given by m_n can be described using polyhedrons described by Stasheff. For instance m_3 is a homotopy between the two terms of the associator and is described by a line. There are five different ways of combining four terms using a binary product and they correspond to five vertices of a pentagon that is used to describe m_4. The first three associahedra are described as follows.

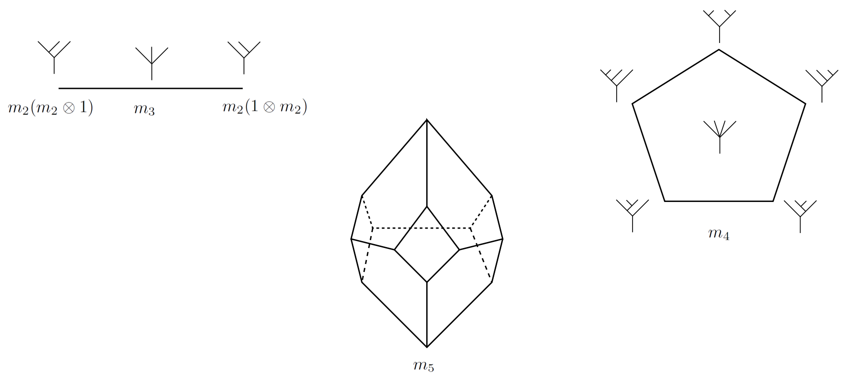

Figure 2.1:

Definition 7. For a differential graded complex V we define the *cofree conilpotent coalgebra without a co-unit* as follows.

$$T(V) = \bigoplus_{n=1}^{\infty} V^{\otimes n}$$

The coproduct of the coalgebra is defined on monomials as

$$\Delta(x_1 \otimes \ldots \otimes x_n) = \sum_{j=0}^{n} x_1 \otimes \ldots x_j \bigotimes x_{j+1} \otimes \ldots x_n$$

From this point onward, in order to avoid confusion between the two different kinds of tensor signs, we will use a comma instead of $\otimes$. For example, $x \otimes y$ in $T(V)$ will be denoted by (x, y) and

$$\Delta(x, y) = 1 \otimes (x, y) + x \otimes y + (x, y) \otimes 1$$

The grading on V gives a grading on $T(V)$. The degree of a monomial $(x_1, \ldots, x_n)$ is $|x_1| + \ldots + |x_n|$. $T(V)$ has the universal property that given any linear map f from a conilpotent coalgebra C to V there exists a unique non-counital coalgebra map $\tilde{f}$ from C to $T(V)$ such that the following diagram commutes.

$$\begin{tikzcd} & C \arrow[dl, dashed, "\exists \tilde{f}"'] \arrow[d, "f"] \\ T(V) \arrow[r, "\pi"] & V \end{tikzcd} \tag{2.2}$$

In the above diagram π, is the projection from $T(V)$ to V. Any linear map $l : V^{\otimes k} \to V$ can be extended to a coderivation $\tilde{l}$ on $T(V)$ given by the following formula.

$$\tilde{l}(x_1, \ldots, x_n) = \sum_{i=0}^{n-k} \pm(x_1, \ldots, d(x_{i+1}, \ldots, x_{i+k}), \ldots, x_n)$$

where the sign of the ith term is $(-1)^{(|x_1|+\ldots+|x_i|)(|l|)}$. The degree of $\tilde{l}$ is equal to the degree of l. Thus if V is a differential graded complex with a derivation $d : V \otimes V$ where the degree of d id 1 and $d^2 = 0$, $\tilde{d}$ is a coderivation of degree 1 on $T(V)$. $d^2 = 0$ implies $\tilde{d}^2$. Thus $(T(V), \tilde{d})$ is a differential graded coalgebra.

Let (A, d, m) be a differential graded algebra (dga). Consider the complex $A[1]$ which is A shifted down by 1. A product $m[1]$ is defined on A using the follwoing formula.

$$m[1](x[1], y[1]) = (-1)^{|x|} m(x, y)$$

$m[1]$ has a degree 1 when m has degree 0. The map d and $m[1]$ can both be lifted to coderivations $\tilde{d}$ and $\tilde{m}[1]$ on $T(A[1])$ and together give a coderivation $D = \tilde{d} + \tilde{m}[1]$. Consider the following equation.

$$D^2 = \tilde{d}^2 + \tilde{d} \circ \tilde{m}[1] + \tilde{m}[1] \circ \tilde{d} + \tilde{m}[1]^2$$

The following observations follow from straight forward computations.

i) $d^2 = 0$ if and only if $\tilde{d}^2 = 0$.

ii) d is the derivation of the product m if and only if $\tilde{d} \circ \tilde{m}[1] + \tilde{m}[1] \circ \tilde{d}$.

iii) m is associative if and only if $\tilde{m}[1]^2 = 0$.

These imply that $D^2 = 0$. Also note that $\tilde{d}$ preserves the monomial grading of $T(A[1])$ and $\tilde{m}[1]$ reduces it by 1. Thus the above three conditions have to be true if $D^2 = 0$.

In general for a graded complex V any coderivation D_V on $T(V)$ is of the form

$$D_V = \tilde{d}_1 + \tilde{d}_2 + \tilde{d}_3 \ldots$$

where d_n is a linear map from $V^{\otimes n}$ to V and $\tilde{d}_n$ are their lifts. The maps d_n are called the Taylor coefficients of D_V. The above discussion shows that a differential graded algebra is a graded complex A with a coderivation D of degree 1 on $T(A[1])$, where only the first two Taylor coefficients of D are non-zero. An A_∞ algebra is the generalization of a differential graded algebra in the following way.

Definition 8. An A_∞ *algebra* (A, D) is a graded vector space A with a coderivation D of degree 1 on $T(A[1])$ such that $D^2 = 0$. The differential graded coalgebra $(T(A[1], D)$ is called the *bar construction* of A.

A morphism of A_∞ algebras is a map that preserves this structure.

Definition 9. An A_∞ morphism from an A_∞ algebra (A, D_A) to an A_∞ algebra (B, D_B) is a map of differential graded coalgebras from $(T^cA[1], D_A)$ to $(T^cB[1], D_B)$.

Suppose $D = \tilde{m}_1 + \tilde{m}_2 + \tilde{m}_3 + \ldots$, where $m_n : A[1]^{\otimes n} \to A[1]$ are the Taylor coefficients of D. Then squaring D gives us

$$D^2 = \tilde{m}_1^2 + \tilde{m}_1 \circ \tilde{m}_2 + \tilde{m}_2 \circ \tilde{m}_1 + \tilde{m}_2^2 + \tilde{m}_1 \circ \tilde{m}_3 + \tilde{m}_3 \circ \tilde{m}_1 + \ldots$$

From monomial degree considerations we get that if $D^2 = 0$ then the following equations must hold.

$$\tilde{m}_1^2 = 0$$

$$\tilde{m}_1 \circ \tilde{m}_2 + \tilde{m}_2 \circ \tilde{m}_1 = 0$$

$$\tilde{m}_2^2 + \tilde{m}_1 \circ \tilde{m}_3 + \tilde{m}_3 \circ \tilde{m}_1 = 0$$

and so on. In general we have

$$\sum_{i+j=n} \tilde{m}_i \circ \tilde{m}_j = 0$$

Thus we get the following equivalent definition of an A_∞ algebra.

Now consider an A_∞ morphisms $P : (T^cA[1], D_A) \to (T^cB[1], D_B)$. Let $p_n : A[1]^{\otimes n} \to B[1]$ be a map given by restriction of P to $A[1]^{\otimes n}$ followed by a projection from $T^cB[1]$ to $B[1]$. As P is a map of coalgebras, it is completely determined by the maps $\{p_n\}$. For monomials of lengths one, two and three P is given using p_1, p_2 and p_3 using the following formulas.

$$P(x) = p_1(x)$$

$$P(x, y) = p_2(x, y) + (p_1(x), p_1(y))$$

$$P(x, y, z) = p_3(x, y, z) + (p_2(x, y), p_1(z)) + (p_1(x), p_2(y, z)) + (p_1(x), p_1(y), p_1(z))$$

In general for a monomial of length n

$$P(x_1, x_2, \ldots, x_n) = \sum (p_{i_i}(x_1, \ldots, x_{i_1}), p_{i_2}(\ldots), \ldots, p_{i_k}(\ldots))$$

where the sum is taken over all ordered partitions of n. Since P is an A_∞ morphism P commutes with the differentials on $T^cA[1]$ and $T^cB[1]$. This relation induces certain relations between p_n and m_n and we get the following equivalent defination for A_∞ morphisms.

Definition 10. An A_∞ *morphism* P between A_∞ algebras $(A, m_1^A, m_2^A, \ldots)$ and $(B, m_1^B, m_2^B, \ldots)$ is a collection of linear maps

$$p_n : A^{\otimes n} \to B$$

such that

$$\sum_{k=1}^{n} \sum_{n_1+\ldots+n_k=n} m_k^B(p_{n_1} \otimes \ldots \otimes p_{n_k})$$

$$= \sum_{k=1}^{n} \sum_{j=0}^{n-k} p_{n-k+1}(1 \otimes \ldots m_k^A \ldots \otimes 1)$$

2.3 C_∞ algebras and morphisms

An A_∞ algebra is a generalization of an associative algebra and an A_∞ morphism is a generalization of an algebra morphism. One way to generalize commutative associative algebras is to define C_∞ algebras and morphisms. A C_∞ algebra is an A_∞ algebra such that the maps m_n satisfy certain equations involving (q, r)-shuffles, where $q + r = n$.

Definition 11. A (q, r)-*shuffle* is a permutation σ of $(1, 2, \ldots, q+r)$ such that

- if $1 \leq i \leq j \leq q$, then $\sigma(i) \leq \sigma(j)$
- if $q + 1 \leq i \leq j \leq q + r$, then $\sigma(i) \leq \sigma(j)$

For any vector space V, the tensor coalgebra $T(V)$ also has a product μ on it called the shuffle product defined as follows.

$$\mu((x_1, \ldots, x_q) \otimes (x_{q+1}, \ldots, x_{q+r})) = \sum_{\sigma \in (q,r)-\text{shuffles}} \pm (x_{\sigma^{-1}(1)}, x_{\sigma^{-1}(2)}, \ldots, x_{\sigma^{-1}(q+r)})$$

The sign of each term is determined by the degrees of x_i and the permutation σ. For just two terms x and y

$$\mu(x \otimes y) = (x, y) + (-1)^{|x||y|}(y, x)$$

$T(V)$ with the shuffle product and the coproduct Δ defined earlier is a Hopf algebra.

Suppose (A, d, m) is a differential graded associative algebra. Then since m is an associative product A is in particular an A_∞ algebra. This implies $D = \tilde{d} + \tilde{m}[1]$ is a coderivation of the coproduct on the bar construction $T(A[1])$.

Lemma 1. *A dga A is also a dgca (that is the product m is graded commutative) if and only if $D = \tilde{d} + \tilde{m}[1]$ is a derivation of the shuffle product.*

Proof. Note that $\tilde{d}$ is already a derivation of the shuffle product. Thus if D is a derivation of the shuffle product then so is $\tilde{m}[1]$. This implies for all x and y in A

$$\tilde{m}[1](\mu(x[1] \otimes y[1]) = 0$$
$$\implies \tilde{m}[1]((x[1], y[1]) + (-1)^{(|x|-1)(|y|-1)}(y[1], x[1]) = 0$$
$$\implies (-1)^{|x|}m(x, y) + (-1)^{|x||y|-|x|-1}m(y, x) = 0$$
$$\implies m(x, y) = (-1)^{|x||y|}m(y, x)$$

Which implies that m is graded commutative.

Conversely, suppose m is graded commutative then

$$\tilde{m}[1]((\mu(x_1, \ldots, x_q) \otimes (x_{q+1}, \ldots, x_{q+r}))$$
$$= \tilde{m}[1](\sum_{\sigma \in (q,r)-\text{shuffles}} \pm(x_{\sigma^{-1}(1)}, x_{\sigma^{-1}(2)}, \ldots, x_{\sigma^{-1}(q+r)}))$$

Note that since m is graded commutative

$$(x_{\sigma^{-1}(1)}, \ldots, \tilde{m}[1](x_{\sigma^{-1}(i)}, x_{\sigma^{-1}(i+1)}), \ldots, x_{\sigma^{-1}(q+r)})$$
$$= \pm(x_{\sigma^{-1}(1)}, \ldots, \tilde{m}[1](x_{\sigma^{-1}(i+1)}, x_{\sigma^{-1}(i)}), \ldots, x_{\sigma^{-1}(q+r)})$$

If $\sigma^{-1}(i) \in 1, \ldots, q$ and $\sigma^{-1}(i+1) \in q+1, \ldots, q+r$ then both the terms in the above equality occur in $\tilde{m}[1]$ of the shuffle product and cancel out. Otherwise, since σ is a (q, r)-shuffle, $\sigma^{-1}(i+1) = \sigma^{-1}(i) + 1$. Thus

$$(x_{\sigma^{-1}(1)}, \ldots, \tilde{m}[1](x_{\sigma^{-1}(i)}, x_{\sigma^{-1}(i+1)}), \ldots, x_{\sigma^{-1}(q+r)})$$

is a term in $\mu(\tilde{m}[1] \otimes 1 + 1 \otimes \tilde{m}[1])((x_1, \ldots, x_q) \otimes (x_{q+1}, \ldots, x_{q+r}))$ and we have

$$\tilde{m}[1]((\mu(x_1, \ldots, x_q) \otimes (x_{q+1}, \ldots, x_{q+r})) =$$
$$\mu(\tilde{m}[1] \otimes 1 + 1 \otimes \tilde{m}[1])((x_1, \ldots, x_q) \otimes (x_{q+1}, \ldots, x_{q+r}))$$

and $\tilde{m}[1]$ is a derivation of the shuffle product μ which in turn means that D is a derivation of the shuffle product.

□

The above proposition motivates the following definition of a C_∞ algebra.

Definition 12. A C_∞ *algebra* is an A_∞ algebra (A, D) where D is also a derivation of the shuffle product on $T(A[1])$. Thus $(T(A[1]), D, \Delta, \mu)$ is a differential graded Hopf Algebra.

For the above definition of a C_∞ morphisms are defined as follows.

Definition 13. A C_∞ morphism from a C_∞ algebra (A, D_A) to a C_∞ algebra (B, D_B) is a map of differential graded Hopf algebras from $(T^c A[1], D_A)$ to $(T^c B[1], D_B)$.

Lemma 2. *Let (A, D) be an A_∞ algebra where $D = \tilde{m}_1 + \tilde{m}_2 + \ldots$. Then D is a derivation of the shuffle product on $T(A[1])$ if and only if for every pair (q, r) of positive integers and $x_1, x_2, \ldots, x_{q+r} \in A[1]$*

$$m_{q+r}(\mu((x_1, \ldots, x_q) \otimes (x_{q+1}, \ldots, x_{q+r})) = 0$$

Proof. Suppose D is a derivation of the product. Then

$$D(\mu((x_1, \ldots, x_q) \otimes (x_{q+1}, \ldots, x_{q+r})))$$
$$= \mu \circ (D \otimes 1 + 1 \otimes D)((x_1, \ldots, x_q) \otimes (x_{q+1}, \ldots, x_{q+r}))$$

Then by taking the projection to $A[1]$ on both sides we get

$$m_{q+r}(\mu((x_1, \ldots, x_q) \otimes (x_{q+1}, \ldots, x_{q+r}))) = 0$$

Conversely, suppose the above statement is true for all pairs (q, r) of positive integers and $x_1, x_2, \ldots, x_{q+r} \in A[1]$. For a fixed (q, r), and a (q, r) shuffle σ, consider $(\sigma^{-1}(i), \sigma^{-1}(i+1), \ldots, \sigma^{-1}(i+k))$. This is either a string of k consecutive intergers in $\{1, 2, \ldots, q\}$ or in $\{q+1, \ldots, q+r\}$, or it is a shuffle of a subset of $\{1, 2, \ldots, q\}$ and a susset of $\{q+1, \ldots, q+r\}$. By definition of $\tilde{m}_k$

$$\tilde{m}_k(\mu((x_1, \ldots, x_q) \otimes (x_{q+1}, \ldots, x_{q+r})))$$
$$= \tilde{m}_k\Big(\sum_{\sigma \in (q,r)-\text{shuffles}} \pm(x_{\sigma^{-1}(1)}, x_{\sigma^{-1}(2)}, \ldots, x_{\sigma^{-1}(q+r)})\Big)$$
$$= \sum_{\sigma \in (q,r)-\text{shuffles}} \pm(x_{\sigma^{-1}(1)}, x_{\sigma^{-1}(2)}, \ldots, m_k(\ldots), \ldots, x_{\sigma^{-1}(q+r)})$$

The above sum contains m_k applied to (q_1, r_1)-shuffles where $q_1 + r_1 = k$ which are zero by hypothesis. All the other terms that occur, also occur in

$$\mu \circ (\tilde{m}_k \otimes 1 + 1 \otimes \tilde{m}_k)((x_1, \ldots, x_q) \otimes (x_{q+1}, \ldots, x_{q+r}))$$

Thus $\tilde{m}_k$ is a derivation of μ for every k. This implies D is a derivation of μ. $\square$

Lemma 3. *Let A and B be two C_∞ algebras and let $P = (p_1, p_2, \ldots)$ be an A_∞ morphism from A to B. Then P is also a C_∞ morphism if and only if every pair (q, r) of positive integers and $x_1, x_2, \ldots, x_{q+r} \in A[1]$*

$$p_{q+r}(\mu((x_1, \ldots, x_q) \otimes (x_{q+1}, \ldots, x_{q+r})) = 0$$

Proof. Suppose P respects the shuffle product μ. Then

$$P(\mu((x_1, \ldots, x_q) \otimes (x_{q+1}, \ldots, x_{q+r}))) = \mu(P(x_1, \ldots, x_q) \otimes P(x_{q+1}, \ldots, x_{q+r}))$$

Then by taking the projection to $A[1]$ on both sides we get

$$p_{q+r}(\mu((x_1, \ldots, x_q) \otimes (x_{q+1}, \ldots, x_{q+r}))) = 0$$

The proof of the converse is similar to the proof of the previous lemma. The expression

$$P(\sum_{\sigma \in (q,r)-\text{shuffles}} \pm(x_{\sigma^{-1}(1)}, x_{\sigma^{-1}(2)}, \ldots, x_{\sigma^{-1}(q+r)}))$$

contains terms which contain p_k applied to shorter shuffles which add up to zero by hypothesis. All the other terms also appear in $\mu(P(x_1, \ldots, x_q) \otimes P(x_{q+1}, \ldots, x_{q+r}))$. This imples that P is a map of algebras.

□

The above Lemmas motivate the following alternate definitions for a C_∞ algebra and a C_∞ morphism.

Definition 14. A C_∞ *algebra* is an A_∞ algebra $(A, m_1, m_2, \ldots)$ such that for every ordered pair of positive integers (q, r) and $(x_1, x_2, \ldots, x_{q+r})$ where $x_i \in A$

$$m_{q+r}(\sum_{\sigma \in (q,r)-\text{shuffles}} (x_{\sigma^{-1}(1)}, x_{\sigma^{-1}(2)}, \ldots, x_{\sigma^{-1}(q+r)})) = 0$$

Definition 15. A C_∞ *morphism* is an A_∞ morphism $P = (p_1, p_2, \ldots)$ such that for every ordered pair of positive integers (q, r) and $(x_1, x_2, \ldots, x_{q+r})$ where $x_i \in A$

$$p_{q+r}(\sum_{\sigma \in (q,r)-\text{shuffles}} (x_{\sigma^{-1}(1)}, x_{\sigma^{-1}(2)}, \ldots, a_{\sigma^{-1}(q+r)})) = 0$$

Chapter 3

Cumulants

3.1 Commutative Cumulants vs. Boolean Cumulants

Definition 16. A *probability space* is a commutative algebra C over a field $\mathbb{K}$ and a linear function e called the *expectation* to the base field.

The expectation function does not necessarily respect the product and is not an algebra map. The cumulants of e are a family of functions k_n which can measure the deviation of e from being an algebra map. k_n takes n inputs and gives an output in the base field. These functions can be used to calculate the expectations of products of the variables. They are defined using the following recursive formulas.

$$e(a) = k_1(a)$$

$$e(ab) = k_2(a, b) + k_1(a)k_1(b)$$

$$e(abc) = k_3(a, b, c)+k_2(a, b)k_1(c)+k_1(a)k_2(b, c)+k_1(b)k_2(c, a)+k_1(a)k_1(b)k_1(c)$$

In general the expectation of the product of n variables is given by

$$e(a_1a_2 \ldots a_n) = \sum_{\pi} \prod_{B \in \pi} k_{|B|}(B)$$

where the sum is taken over all the partitions π of $1, 2, \ldots, n$. As the product on C is commutative, it can be inductively shown that $k_{|B|}(B)$ is well defined. This is because the value of k_n is independent of the order of the inputs.

The first few cumulants for a map e between two commutative algebras can be computed using the following formulas.

$$k_1(a) = e(a)$$

$$k_2(a,b) = e(ab) - e(a)e(b)$$

$$k_3(a,b,c) = e(abc) - e(ab)e(c) - e(a)e(bc) - e(ca)e(b) + 2e(a)e(b)e(c)$$

In general the k_n is given by the following formula.

$$k_n(a_1, a_2, \ldots, a_n) = \sum_{\pi} (|\pi| - 1)!(-1)^{|\pi|-1} \prod_{b \in \pi} e(\prod_{i \in b} a_i)$$

The cumulants vanish when e is a map of algebras. When the product of the space is not commutative, the cumulants cannot be defined as above.

Definition 17. An *associative probability space* is a vector space V over a field $\mathbb{K}$ with an associative product and a function E called the *expectation*.

Just like in the commutative case, the expectation is not required to satisfy any compatibility with the product m. This incompatibility can be measured using the Boolean cumulants of E which are a family of maps $K_n : V^{\otimes n} \to \mathbb{K}$. The first three Boolean cumulants can be defined.

$$K_1(a) = E(a)$$

$$K_2(a,b) = E(ab) - E(a)E(b)$$

$$K_3(a,b,c) = E(abc) - E(a)E(bc) - E(ab)E(c) + E(a)E(b)E(c)$$

K_n in general is given by the following formula.

$$K_n(a_1, a_2, , a_n) = \sum \pm E(a_1, \ldots, a_i)E(a_{i+1}, \ldots) \ldots E(\ldots, a_n)$$

The above sum is taken over all ordered partitions of n. The even partitions occur with negative signs and the odd partitions occur with positive signs. If E is a map of algebras then the cumulants are all zero.

Knowing the Boolean cumulants allows us to compute the expectation of products. For example

$$E(ab) = K_2(a,b) + K_1(a)K_2(b)$$

$$E(abc) = K_3(a,b,c) + K_1(a)K_2(b,c) + K_2(a,b)K_1(c) + 3K_1(a)K_2(b)K_1(c)$$

Example 3. Let V be the algebra of $n \times n$ matrices over $\mathbb{R}$ and E be the trace map from V to $\mathbb{R}$. Then the trace map does not respect matrix multiplication m. (V, m, E) is a probability space and the cumulants of E are non-zero.

The Boolean cumulants can be defined even in the case where the target of the expectation function is another vector space with an associative product instead of the base field. Also the vector spaces can have more structure like a differential.

Example 4. Consider the algebra of differential forms Ω^* on a manifold M and the cochains C^* on a finite simplicial decomposition of M. The differential forms are a *dgca* and the cochains have the Alexander-Whitney cup product which makes them a *dga*. Consider the map $I : \Omega^* \to C^*$ defined as follows. For a differential for ω and a simplex σ

$$I(\omega)(\sigma) = \int_\sigma \omega$$

From Stoke's theorem it follows that I is a chain map. This map induces an isomorphism on cohomology of the complexes. It does not respect the product structure at the level of complexes. However, by the de Rahm's theorem I induces an isomorphism on cohomology. The induced isomorphism is in fact a map of the algebra structures. Thus the Boolean cumulants of I are defined on chains, but they vanish on cohomology since the induced map is an algebra map.

3.2 Boolean cumulants for A_∞ algebras

The Boolean cumulants can be defined for A_∞ algebras in multiple ways up to homotopy. Suppose A and B are A_∞ algebras and E is a chain map between them. In an A_∞ algebra products of three or more variables are not well defined thus (abc) can be defined as $(ab)c$ or $a(bc)$. Thus while there is only one way to define K_1 and K_2, there are four different ways of defining K_3. All the four ways of defining K_3 are homotopic to each other since $E((ab)c)$ is homotopic to $E(a(bc))$ and $E(a)(E(b)E(c))$ is homotopic to $(E(a)E(b))E(c)$.

Lemma 4. *Suppose A and B are A_∞ algebras. The different ways of defining the cumulants are homotopic to each other via the maps m_2. Multiple homotopies given in this manner are all homotopic to each other, the homotopies of such homotopies are homotopic to each other and so on*

Proof. The terms of the cumulants that are defined only up to homotopy correspond to the vertices of Stasheff associahedra. The different ways of defining the cumulants are homotopic to each other via the edges. The two cells correspond to the homotopies of such homotopies and so on. Since the associahedra are contractible, the above lemma follows. $\square$

Chapter 4

Transfer of the A_∞ and C_∞ structure

Given an isomorphism of cochain (or chain) complexes where one of the cochain (or chain) complexes is a *dga* (or a *dgca*), the multiplicative structure can be transferred and the chain complexes are indeed both algebras and the isomorphism is an isomorphism of algebras. In Algebraic topology we often encounter situations where a map between chain complexes is not an isomorphism but it induces an isomorphism on cohomology (homology). Since we are working over field coefficients such a map has in inverse up to homotopy. Such a map is called a quasi-isomorphisms. In this chapter we will discuss how the multiplicative structure transfers over quasi-isomorphisms.

4.1 Transferring associative structure

Suppose $(A, d_A, \wedge)$ is a *dga* and (B, d_B) is a cochain complex. Suppose $p : A \to B$ is a map that induces an isomorphism on cohomology. Since we are working over field coefficients, there exists a map i from B to A such that $p \circ i$ is homotopic to identity on A and $i \circ p$ is homotopic to identity on B. When p is surjective the map i can be picked so that it is injective and $p \circ i$ is equal to identity. If p is injective then i can be picked to be surjective so that $i \circ p$ is equal to identity.

Example 5. Suppose $A = \Omega^*(M)$ is the algebra of differential forms on a smooth manifold M and suppose $B = C^*(M)$ is the cochain complex corresponding to a certain fixed regular cell decomposition of M. Consider the map I defined as follows. For a differential for ω and a cell σ of the cell decomposition

$$I(\omega)(\sigma) = \int_\sigma \omega$$

The cochains $C^*(M)$ have a canonical basis given by the cells of the decomposition. A basis element σ^* corresponding to a cell σ is a map such that

$$\sigma^*(\sigma) = 1$$

$$\sigma^*(\tau) = 0$$

for every cell $\tau \neq \sigma$.

A map i from $C^*(M)$ to $\Omega^*(M)$ can be constructed to have the following properties.

1) $i(\sigma^*)$ integrates to 1 on σ.

2) $i(\sigma^*)$ is supported only in a small neighborhood of the interior of σ.

3) $i(\sigma^*)$ integrates to zero on all cells that have the same dimension as σ but are not σ.

The map i is an inclusion of the cochains into differential forms. The map $I \circ i$ is equal to identity on $C^*(M)$ and $i \circ I$ is homotopic to identity on $\Omega^*(M)$. The homotopy h is a map of degree -1 on $\Omega^*(M)$ such that

$$dh + hd = i \circ I - id$$

h can be constructed inductively on cells and then glued together on the whole manifold.

Example 6. Suppose $(A, d, \wedge)$ is a *dga* and $B = H^*(A)$ is it's cohomology. B can be considered to be a cochain complex with the zero differential. Then since $H^*(A) = ker(d)/Im(d)$ and we are working over field coefficients,

$$ker(d) \cong H^*(A) \oplus Im(d)$$

Also, since $ker(d) \subseteq A$, there exists a subspace of $ker(d)^\perp$ so that

$$A \cong ker(d) \oplus ker(d)^\perp$$

Thus we have a decomposition for A

$$A \cong H^*(A) \oplus Im(d) \oplus ker(d)^\perp$$

For a fixed decomposition, there is an inclusion i of B into A and a projection p from A to B. Since B is actually the cohomology of A with the zero differential, both of these maps induce an isomorphism on cohomology. $p \circ i$ is identity on B. $i \circ p$ is homotopic to identity via a homotopy h that can be constructed inductively.

In both of the above examples, the composition $p \circ i$ is exactly equal to the identity, while $i \circ p$ it is homotopic to identity. We will first consider this case.

$$\begin{tikzcd} A \arrow[loop left, "h"] \arrow[r, "p"] & B \arrow[l, bend left, "i"] \end{tikzcd}$$

where $p \circ i$ is identity on B and $i \circ p - id_A = dh + hd$.

We define a binary product m_2 on B by first including the elements of B into A and then taking the product and then projecting them back on B. Thus for a and b in B

$$m_2(a, b) = p \circ \wedge(i(a), i(b))$$

The map m_2 is not associative since. Consider the associater of m_2

$$m_2(m_2(a, b), c) - m_2(a, m_2(b, c))$$

can be diagrammatically expressed as follows.

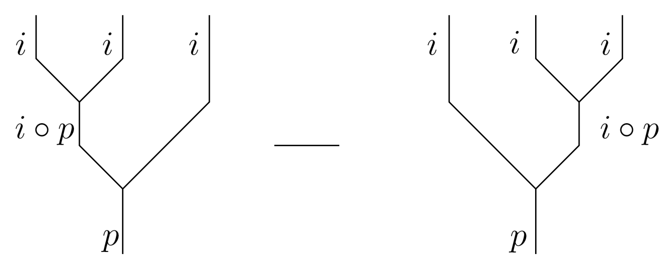

Figure 4.1:

Even though the associator is not zero it is homotopic to zero since $i \circ p$ is homotopic to identity and $\wedge$ is an associative product on A. We define the map m_3 as follows.

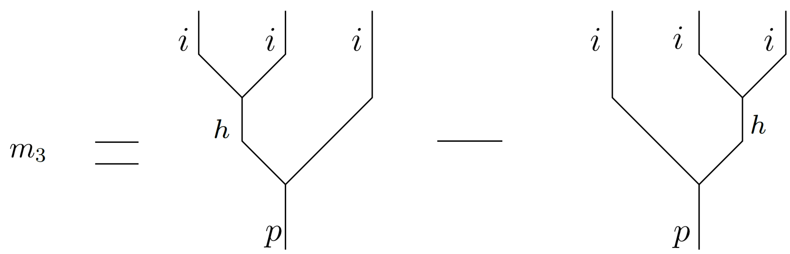

Figure 4.2:

$$
\begin{aligned}
m_3 = p \circ \wedge \circ (h \otimes id) \circ (\wedge \otimes id) \circ (i \otimes i \otimes i) \\
-p \circ \wedge \circ (id \otimes h) \circ (\wedge \otimes id) \circ (i \otimes i \otimes i)
\end{aligned}
$$

d_B, m_2, and m_3 satisfy the first three equations for an A_∞ algebra.

In general we can define m_n by taking a signed sum over rooted planer binary trees with n leaves (inputs) labeled by i, the nodes are labeled by $\wedge$, the internal edges are labeled by h, and the root is labeled by p.

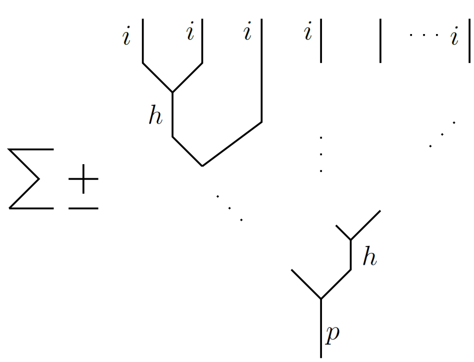

Figure 4.3:

Theorem 3 (T. V. Kadeisvili, 1980)**.** Suppose maps $m_n : B^{\otimes n} \to B$ are defined by the above formulas using a surjective quasi-isomorphism $p : A \to B$, an associative produce $\wedge$ on A which makes $(A, d, \wedge)$ a *dga*, a right inverse $i : B \to A$ of p, and a homotopy $h : A \to A$. Then $(B, d_B, m_2, m_3, \ldots)$ is an A_∞ algebra. [8]

Let us consider the case of Example 6 where A is *dga* and B is the cohomology of A. m_2 defined on B by according to the above formula is homotopic to being associative via the map m_3 defined as above. The differential on the complex $B = H^*(A)$ is zero which implies, that the associator of m_2 is zero.

$$m_2(m_2(a, b), c) - m_2(a, m_2(b, c)) = m_3 \circ 0 + 0 \circ m_3$$

m_2 is in fact the associative cup product on the cohomology of M. Even though m_2 is associative, m_3, m_4 and so on are maps that can still be defined and are usually non-zero. The definitions of these maps depend on the choice of an inclusion i of the cohomology into the differential forms and a choice of a homotopy h which makes the $i \circ p$ homotopic to identity. It is not always possible to make a choice for i and h which makes the higher products m_n zero. The products m_n serve as higher invariants of the space. These products are called the A_∞ Massey products on cohomology.

In case of the Example 5, the map I transfers the associative structure on differential forms $\Omega^*(M)$ to an A_∞ structure on the cochains $C^*(M)$. Since $\Omega^*(M)$ is also graded commutative the transferred m_2 is also graded commutative. However it is not associative and there are higher products m_3, m_4 and so on which make it A_∞. Suppose the cell decomposition that $C^*(M)$ corresponds to, is a simplicial decomposition then there is an associative product on $C^*(M)$ as described in Chapter 1 which is also an A_∞ structure. This product is associative but not graded commutative. Thus the two A_∞ structures on $C^*(M)$ are not the same however they induce the same cup product on the cohomology of M.

4.2 Transferring A_∞ or C_∞ structures

The formulas above transfer the associative product structure on A to an A_∞ structure on B across a quasi-isomorphism. This can be generalized to give formulas that transfer an A_∞ structure on A to an A_∞ structure on B. Suppose $(A, d_A, \wedge_2, \wedge_3, \ldots)$ is an A_∞ algebra and p is a quasi-isomorphism from A to a cochain complex (B, d_B). Just like in the previous section we pick a map $i : B \to A$ which is a homotopy inverse to p and a map $h : A \to A$ which makes

$i \circ p$ homotopic to identity. We define $m_n : B^{\otimes n} \to B$ using rooted planer trees with n leaves. The leaves are labeled by i, internal edges are labeled by h, and the root is labeled with p. An n-valent vertex of this tree corresponds to the map m_n.

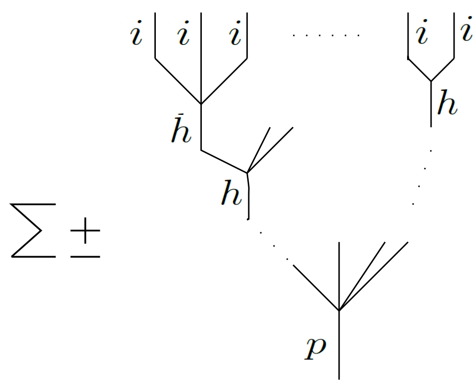

Figure 4.4:

The above formulas give us a generalization of Theorem 3.

Theorem 4 (Konstevich, Soibelman)**.** Suppose maps $m_n : B^{\otimes n} \to B$ are defined by the above formulas using a quasi-isomorphism $p : A \to B$, the maps $\wedge_n : A^{\otimes n} \to A$, which make A an A_∞ algebra, a right inverse $i : B \to A$ of p, and a homotopy $h : A \to A$. Then $(B, d_B, m_2, m_3, \ldots)$ is an A_∞ algebra. [9] [13]

In the example of differential forms, the algebra is in fact graded commutative. In this case the transferred structure defined using planer trees is in fact C_∞.

Theorem 5 (Cheng, Getzler)**.** Suppose A is a C_∞ algebra and $p : A \to B$ is a quasi-isomorphism. The transferred A_∞ structure defined on B by the Theorem of Kostevich and Soibelman is in fact a C_∞ structure. [5]

4.3 Extending a quasi-isomorphism to an A_∞ morphism

In all our examples so far the map i also has the property that $p \circ i$ is identity (homotopy retract). In this situation, starting from $i_1 = i$ we can define maps

$i_n : B^{\otimes} \to A$ using formulas that are very similar to the formulas for m_n in the figure 5.1. The sum is taken over planer trees with n leaves, the leaves are labeled by i, the internal n valent vertices correspond to the map $\wedge_n$, internal edges labeled by h and the root is also labeled by h.

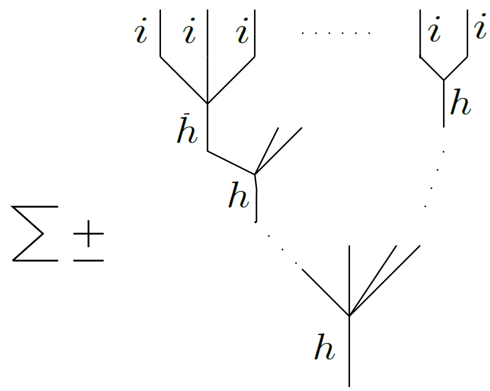

Figure 4.5:

i_n are actually the terms of an A_∞ morphism.

Theorem 6. (Konstevich and Soibelmann) Suppose A, B, p, i, and h are as in theorem 4. Further suppose that $p \circ i$ is identity on B. Then the $(i, i_1, i_2, \ldots)$ as defined by figure 4.5 is an A_∞ morphism from B with the transferred A_∞ structure to A. [9]

In fact the quasi-isomorphism p extends to an A_∞ morphism $P = (p, p_2, p_3, \ldots)$ from A to B that is an inverse of $I = (i, i_2, i_3, \ldots)$.

Theorem 7. (K. Lefevre-Hasegawa, 2003) Every A_∞ quasi isomorphism admits an inverse A_∞ quasi-isomorphism up to homotopy.

Remark 6 (constructing P in the special case of deformation retracts). In the special case where $p \circ i = id$ we can construct P such that $P \circ I = id$. This can be done since $p \circ i = id$ The inclusion of B into A gives a decomposition of A

$$A = ker(p) \oplus i(B)$$

Thus it is enough to define $p_n(a_1, a_2, \ldots, a_n)$ where a_i are either in $ker(p)$ or in $i(B)$. We define p_n to be zero whenever any a_i are in $ker(p)$. On elements of $i(B)$ we can define p_n inductively. Recall that

$$P \circ I(b_1, b_2) = P(i_2(b_1, b_2) + (i(b_1), i(b_2)) =$$

$$p(i_2(b_1, b_2)) + p_2(i(b_1), i(b_2)) + (p \circ i(b_1), p \circ i(b_2))$$

Since $P \circ I = id$ the right hand side of the above equation should be equal to (b_1, b_2). Since we know what p is, since i is an inclusion, and since $p \circ i$ is identity p_2 is well-defined by the above equation. In general

$$P \circ I(b_1, b_2, \dots b_n) = p_n(i(b_1), \dots, i(b_n))$$

$$+\text{terms involving } p_k \text{ and } i_k \text{ for } k \text{ smaller than } n$$

$$+(b_1, b_2, \dots b_n)$$

Thus p_n is inductively defined.

Chapter 5

Special case of the integration Example

5.1 Transferring structure to cochains

Suppose $C^*(M)$ is a cochain complex corresponding to a regular cell decomposition and $D^*(M)$ is a cochain complex corresponding to another finer cell decomposition. Every cell of the original cell decomposition can be written as a union of cells of the finer cell decomposition. Thus there is a map p from $D^*(M)$ to $C^*(M)$. There are projections p_D and p_C from the differential forms to $D^*(M)$ and $C^*(M)$ respectively given by integrating the forms of the cells of the complexes.

$$\begin{tikzcd} & D^*(M) \arrow[d, "p"] \\ \Omega^*(M) \arrow[ur, dashed, "p_D"] \arrow[r, "p_C"] & C^*(M) \end{tikzcd} \tag{5.1}$$

We can pick inclusions (right inverses) of $D^*(M)$ and $C*(M)$ into the differential forms. However, the transfer maps for the transferred structure might not necessarily commute. For the maps to commute it is necessary to pick the inclusions and the homotopies appropriately. We first transfer the multiplicative structure of $\Omega^*(M)$ to an A_∞ structure on $D^*(M)$. For this we pick an inclusion i_D and a homotopy h_D such that $dh_D + h_D d = i_D \circ p_D - id$

$$\begin{tikzcd} \Omega^*(M) \arrow[loop left, "h_D"] \arrow[r, "p_D"] & D^*(M) \arrow[l, bend left, "i_D"] \end{tikzcd}$$

Since the map p is a quasi-isomorphism and since we are working over field coefficients there is an inverse quasi-isomorphism i up to homotopy. Also since p is a projection i can be picked to be an inclusion such that $p \circ i$ is identity on $C^*(M)$. For this inclusion we can pick a homotopy h on $D^*(M)$ such that $dh + hd = i \circ p$. We can transfer the structure from $D^*(M)$ to $C^*(M)$ using i and h.

$$\begin{tikzcd} D^*(M) \arrow[loop left, "h"] \arrow[r, "p"] & C^*(M) \arrow[l, bend left, "i"] \end{tikzcd}$$

Consider the map $i_C = i_D \circ i$ which is an inclusion of $C^*(M)$ into the differential forms Ω.

Lemma 5. *$h_C = i_D \circ h \circ p_D + h_D$ is a homotopy from $i \circ p$ to identity. That is $dh_C + h_C d = i \circ p + id$.*

Proof.

$$dh_C + h_C d = d(i_D \circ h \circ p_D) + (i_D \circ h \circ p_D)d + dh_D + h_D d$$

Since p_D and i_D are chain maps this is equal to

$$i_D \circ dh \circ p_D + i_D \circ hd \circ p_D + dh_D + h_D d$$

Since h and h_D are homotopies this is equal to

$$i_D \circ i \circ p \circ p_D - i_D \circ p_D + i_D \circ p_D - id$$

$$= i_C \circ p_C - id$$

□

Thus we can transfer the structure from the differential forms directly to $C^*(M)$.

$$\begin{tikzcd} \Omega^*(M) \arrow[loop left, "h_C"] \arrow[r, "p_C"] & C^*(M) \arrow[l, bend left, "i_C"] \end{tikzcd}$$

Lemma 6. *The A_∞ structure on $C^*(M)$ that is transferred from $\Omega^*(M)$ is the same as the one transferred from $D^*(M)$.*

Proof. Recall that the formula for the transferred structure using i and h is as follows.

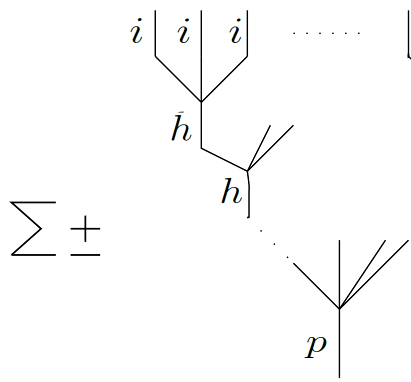

Figure 5.1:

In the above diagram the nodes of the trees correspond to the maps m_n in the structure transferred on $D^*(M)$ from the differential forms. The formulas for these are given by

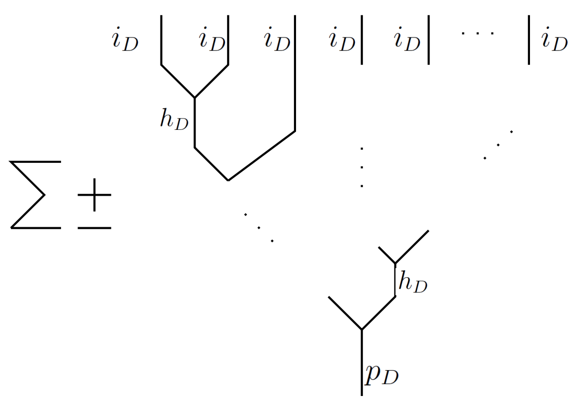

Figure 5.2:

Similarly the formulas for the transferred structure on $C^*(M)$ from the differential forms is

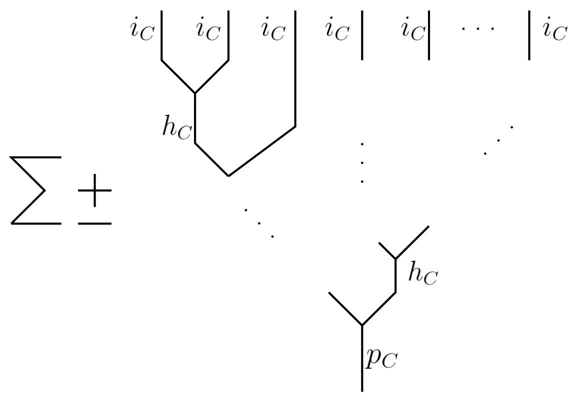

Figure 5.3:

Since $h_C = i_D \circ h \circ p_D + h_D$ and $i_C = i \circ i_D$, and also since $p_D \circ i_D = id$ we get that the above sum is obtained by replacing the nodes in the first diagram by the trees in the second diagram.

$\square$

Thus given a finite set of cochain complexes $C_1^*(M)$, $C_2^*(M)$, and so on, where $C_n^*(M)$ correspond to a finer cell decomposition than $C_{n-1}^*(M)$ we can transfer the associative structure from the differential forms in a compatible way.

5.2 A_∞ morphism from differential forms to the associative cochains

Suppose $C^*(M)$ are simplicial cochains on M. Then there is an associative product on $C^*(M)$ which is not commutative. The map p as described in the previous example given by integrating the forms on the cells is not an algebra map for this product either. In 1978 V. K. A. M. Gugenheim constructed an A_∞ morphism whose first Taylor coefficient is p [7]. This construction uses iterated integrals as defined by Kuo-Tsai Chen [2]. We will consider the special case of forms and cochains on the interval $[0, 1]$. The details of the case are worked out in the paper by Ruggero Bandiera and Florian Schaetz [1]

The 0 cochains on $[0, 1]$ are functions on the set $\{0, 1\}$ and 1 cochains are given by one generator corresponding to the one cell. We will call this generator dt. Thus a 1 cochain is of the form rdt where r is in $\mathbb{R}$. The map p

is given for a zero form by taking the restriction of the function to the points 0 and 1. On the one forms it is given as follows.

$$p(f(x)dx) = (\int_0^1 f(x)dx)dt$$

Recall that the associative cup product on the cochains is defined as follows. For two zero forms the cup product is the product of the two functions. For a zero form F and a one form rdt we have

$$F \cup rdt = F(0)rdt$$

$$rdt \cup F = 0$$

and the cup product of two one forms is zero. Note that this product is not associative and the map p is not a map of algebras. We define the map $p_n : \Omega([0,1])^{\otimes n} \to C^*([0,1]$ as follows. If any of the inputs of p_n is a zero form then p_n is zero. For n one forms

$$p_n(f_1(x)dx, f_2(x)dx \ldots, f_n(x)dx)$$

$$= (\int_{t_1 \leq t_2 \leq \ldots \leq t_n} f_1(t_1)f_2(t_2)\ldots f_n(t_n)dt_1dt_2\ldots dt_n)dt$$

$(p, p_2, p_3, \ldots)$ is an A_∞ morphism from the differential forms to the cochains.

For a general simplex $\Delta^n$of dimension n, and the map $p : \Omega(\Delta^n) \to C^*(\Delta^n)$ maps p_n can by defined using iterated integrals in a manner very similar to the case of $[0,1]$. For a simplicial decomposition of a manifold, the maps are locally defined on each simplex and can be glued together to extend the integral map to an A_∞ morphism.

Chapter 6

Main Results

6.1 Structure of an A_∞ morphism between *dga*s

Recall that between associative algebras without differentials, every A_∞ morphism is in fact an algebra morphism. This is however not necessarily the case when we consider A_∞ morphisms between *dga*s.

Suppose $(A, d_A, \wedge_A)$ and $(B, d_B, \wedge_B)$ are two differential graded algebra. Recall that by definition an A_∞ morphism is a collection of maps$(p_1, p_2, \ldots)$, $p_n : A^{\otimes n} \to B$ where which satisfy the following compatibility relations for every n.

$$\sum_{i+j=n} \wedge_B(p_i \otimes p_j) + d_B \circ p_n = \sum p_{n-1}(1 \otimes \ldots \wedge_A \ldots 1) + p_n(1 \otimes \ldots d_A \ldots \otimes 1)$$

In particular for $n = 1$ the compatibility relation is as follows.

$$\wedge_B (p_1 \otimes p_1) + d_B \circ p_2 = p_1 \circ \wedge_A + p_2(d_A \otimes 1 + 1 \otimes d_A) \tag{6.1}$$

Also recall that for a map $p_n : A^{\otimes n} \to B$, the differential of p_n in the space $Hom(A^{\otimes n}, B)$ and is defines as

$$(p_n) = d_B \circ p_n + (-1)^{n+1} p_n(1 \otimes \ldots d_A \ldots \otimes 1) \tag{6.2}$$

We call this the boundary of the map p_n. Note that since $\partial(p_1) = 0$ which implies p_1 is a chain map.

Lemma 7. *The Boolean cumulants K_2, K_3 and so on of the map $p_1 : A \to B$ are boundaries of maps that can be constructed using the map p_2.*

Proof. We will prove this lemma by induction. For a and b in A,

$$K_2(a,b) = p_1(\wedge_A(a,b) - \wedge_B(p_1(a), p_1(b)))$$

For simplicity of notation we will suppress $\wedge_A$ and $\wedge_B$. Thus the formula for the cumulants is now more familiar.

$$K_2(a,b) = p_1(ab) - p_1(a)p_1(b)$$

Thus from equations 6.1 and 6.2 we have that

$$\partial(p_2)(a,b) = K_2(a,b)$$

In general we know that

$$K_n(a_1, a_2, \ldots, a_n) = \sum_{\text{ordered partitions of } n} \pm p_1(a_1 \ldots a_i)p_1(a_{i+1} \ldots) \ldots p_1(\ldots a_n)$$

In general we can describe K_n in terms of K_{n-1} and p_1 as follows.

$$K_n(a_1, a_2, \ldots, a_n) = K_{n-1}(a_1a_2, a_3, \ldots, a_n) - p_1(a)K_{n-1}(a_2, \ldots, a_n)$$

Since K_{n-1} can be written as a boundary of some map f and $\partial(p_1) = 0$ we have

$$K_n = \partial(f \circ (\wedge_A \otimes id)) - p_1 \otimes \partial(f) = \partial(f \circ (\wedge_A \otimes id) - p_1 \otimes f)$$

This proves that all the cumulants are boundaries in the Hom-complex. □

Note that for K_3, K_4 and so on there is not a unique way to write K_n as a boundary of a map. Given that K_2 is the boundary of p_2, K_3 can be describes as the boundary of two different maps.

$$K_3(a,b,c) = \partial(p_2(ab,c) - p_1(a)p_2(b,c))$$

$$= \partial(p_2(a,bc) - p_2(a,b)p_1(c))$$

Similarly K_4 can be described as a boundary of multiple different maps.

The terms of the nth cumulant correspond the the ordered partitions of n. We associate a graph G_n to K_n. The vertices of G_n correspond to terms of K_n (or equivalently to ordered partitions of n). Two vertices are connected to each other via an edge for the corresponding partitions, one partition can be obtained from the other by splitting one of the sub strings. Note that the vertices of G_n correspond to all the different ways of combining $n-1$ ordered inputs from A using p and the binary products to give exactly one output in B. If p were an algebra map all of these ways would be equal.

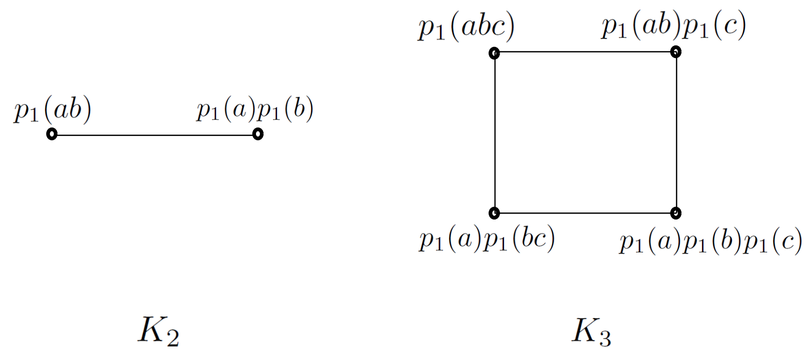

Figure 6.1:

Lemma 8. *The graph G_n is the one skeleton of an $n-1$-cube.*

Proof. We will prove this by induction. Note that K_3 is a square and recall

$$K_n(a_1, a_2, \ldots, a_n) = K_{n-1}(a_1a_2, a_3, \ldots, a_n) - p_1(a_1)K_{n-1}(a_2, \ldots, a_n)$$

By induction hypothesis the subgraphs of G_n corresponding to the above two terms are a $n-2$-cubes (as G_{n-1} is an $n-2$ cube. Edges that go between these subgraphs correspond to splitting sub-strings of the form $a_1a_2 \ldots a_i$ into a_1 and $a_2 \ldots a_i$. Thus for these edges give a one to one correspondence between the vertices of the two $n-2$ cubes. It is easy to check that the adjacent vertices in the first cube go to adjacent vertices in the second cube. Thus the graph of G_n is an $n-1$ cube.

□

If two vertices of G_n are connected by an edge then they occur with opposite signs in K_n. Also, the corresponding terms of the cumulant are the boundary of a map involving p_1 and p_2. For instance $p_1(ab)p_1(c) - p_1(a)p_1(b)p_1(c)$ is the boundary of the map $p_2(a,b)p_1(c)$ and $p_1(abc) - p_1(a)p_1(bc)$ is the boundary of $p_2(a,bc)$. This is true because the differentials are derivations of the binary product and p_1 is a chain map. Thus we can label the edges of G_n with the corresponding maps involving p_2. Thus cycles in G_n correspond to cycles in $Hom(A^{\otimes n}, B)$. For instance the following map is the sum of the maps corresponding to the four edges of G_3.

$$p_2(ab, c) - p_1(a)p_2(b, c) - p_2(a, bc) + p_2(a, b)p_1(c)$$

This map is a cycle.

Note that this map is essentially all the ways of composing the maps p_2 and p_1 withe the binary products. From the compatibility relation for p_3 we get that

$$\partial(p_3)(a, b, c) = p_2(ab, c) - p_1(a)p_2(b, c) - p_2(a, bc) + p_2(a, b)p_1(c)$$

Lemma 9. *The cycle corresponding to the squares in the cubes G_n are boundaries of maps constructed using p_3, p_2 and p_1.*

Proof. In general a square in G_n is made with four vertices which differ in partitions added at two positions. There are two cases to consider. First is when a single substring is split into three in two different ways. Both these cases and the maps that give the homotopies to zero are shown in the following diagrams

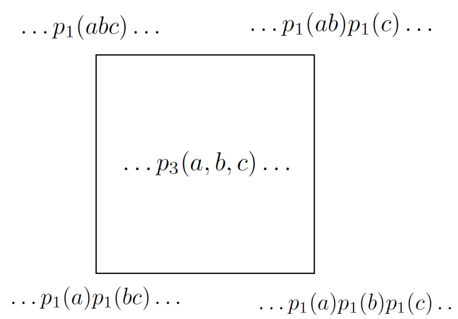

Figure 6.2:

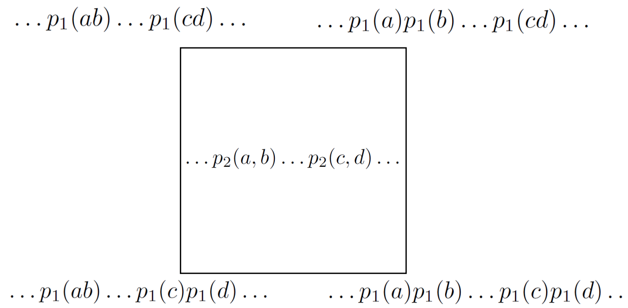

Figure 6.3:

□

Let g_n be an $n-2$-dimensional solid cube such that G_n is its one skeleton. Then from the above lemma we can associate to the 2-cells of g_n maps made using p_3 and p_2. We are now ready to state and prove our theorem in the context of associative algebras.

Theorem 8. Let $(A, \wedge_A, d_A)$ and $(B, \wedge_B, d_B)$ be two *dgas*. Let p be a chain map from A to B. Let K_2, K_3 and so on be the Boolean cumulants of p. Suppose p is the first term of an A_∞ morphism $(p, p_2, p_3, \ldots)$ where $p_n : A^{\otimes n} \to B$. Then the following statements hold.

i) p_2 gives a homotopy from the second Boolean cumulant K_2 to zero. All the higher Boolean cumulants K_n are also homotopic to zero using maps created by p_2 and p_1.

ii) p_3 gives a homotopy between different ways of making K_3 homotopic to zero. For all the higher Boolean cumulants, homotopies between the multiple different ways of making them homotopic to zero are homotopic to each other using p_3, p_2 and p_1.

iii) In general any cycles that are created using the homotopies $\{p_j\}_{j=1}^{n}$ are made homotopic to zero using maps made by $\{p_j\}_{j=1}^{n+1}$.

Proof. The previously proved lemmas prove the first two parts of this theorem. In general 2 cycles created by p_2 and p_3 correspond to 2 cycles in g_n. Consider the boundary of p_n in general. Recall that from by definition p_n satisfies the equation.

$$\sum_{k=1}^{n} \sum_{n_1+\ldots+n_k=n} m_k^B(p_{n_1} \otimes \ldots \otimes p_{n_k})$$

$$= \sum_{k=1}^{n} \sum_{j=0}^{n-k} p_{n-k+1}(1 \otimes \ldots m_k^A \ldots \otimes 1)$$

Since in this case m_k are all zero except for $k = 1$ and $k = 2$ we get

$$d(p_n) + \sum_{n_1+n_2=n} \wedge_B(p_{n_1} \otimes p_{n_2})$$

$$= \sum_{k=1}^{n} p_n(1 \otimes \ldots d \ldots \otimes 1) + \sum_{k=1}^{n-1} p_{n-1}(1 \otimes \ldots \wedge_A \ldots \otimes 1)$$

By rearranging the terms of the above equation we find $\partial(p_n)$.

$$d(p_n) - \sum_{k=1}^{n} p_n(1 \otimes \ldots d \ldots \otimes 1)$$

$$= \sum_{k=1}^{n-1} p_{n-1}(1 \otimes \ldots \wedge_A \ldots \otimes 1) - \sum_{n_1+n_2=n} \wedge_B(p_{n_1} \otimes p_{n_2})$$

Also

$$\partial(\wedge_B(p_{n_1} \otimes p_{n_2})) = \wedge_B(\partial(p_{n_1}) \otimes \partial(p_{n_2}))$$

Thus in general to a map of the type $p_{j_1} p_{j_2} \ldots p_{j_m}$ we associate a cell of dimension $j_1 + j_2 \ldots j_m - m$ which is attached in g_n to the cycle corresponding to its boundary.

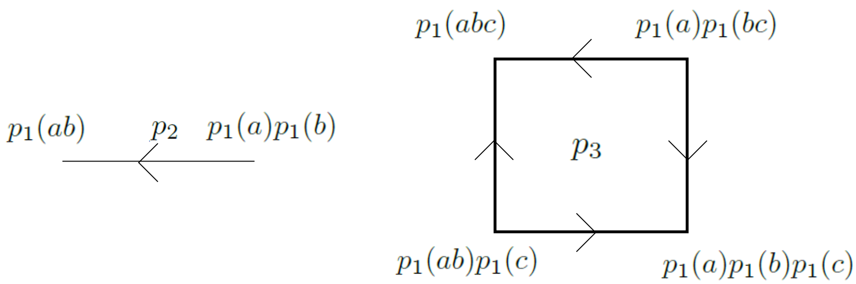

Figure 6.4: p_2 and p_3

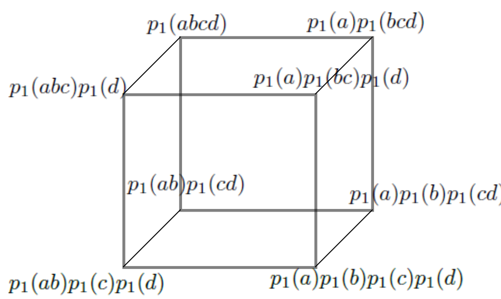

Figure 6.5: p_4

Since g_n are solid cubes, they are contractible. Also, all the cells of g_n correspond to either a function of the form $p_1 \dots p_k \dots p_1$ or a function of the form $p(a_1) \dots p_k \dots p_l \dots p(a_n)$. Thus we have that all cycles created by $\{p_k\}$ are contractible.

$\square$

6.2 C_∞ morphism between *dgca*s

Suppose A and B are also graded commutative and $(p_1, p_2, \dots)$ is a C_∞ morphism from A to B. Recall that the commutative cumulants are defined as follows.

$$k_n(a_1, a_2, \dots, a_n) = \sum_\pi (|\pi| - 1)!(-1)^{|\pi|-1} \prod_{b\in\pi} p(\prod_{i\in b} a_i)$$

Recall that k_2 is the same as the Boolean cumulant K_2 and thus from the previous section it follows that k_2 is the boundary of the map p_2.

Lemma 10. *The commutative cumulants can be describes as boundaries of maps described using* p_2

Proof. Note that the coefficients of k_n are integers that must add up to zero. Also each term in k_n corresponds to all partitions of n. Like in the previous section we associate a graph G_n whose vertices correspond to the terms of k_n with multiplicities. Edges go between a vertex α and β is the partition corresponding to β can be obtained from the partition corresponding to α by splitting one of its subsets into two. Note that since the coefficients of k_n add up to zero, G_n has even number of vertices. Also note that it is a connected graph. Any two terms corresponding to adjacent vertices in G_n are homotopic to each other via p_2 and occur in k_n with opposite signs. Thus we can take pairs of terms with opposite signs in k_n that are homotopic to each other and use that to describe k_n as a boundary.

□

The third cumulant k_3 is given by the formula

$$k_3(a, b, c) = p(abc) - p(ab)p(c) - p(bc)p(a) - p(ca)p(b) + 2p(a)p(b)p(c)$$

Thus the corresponding graph is

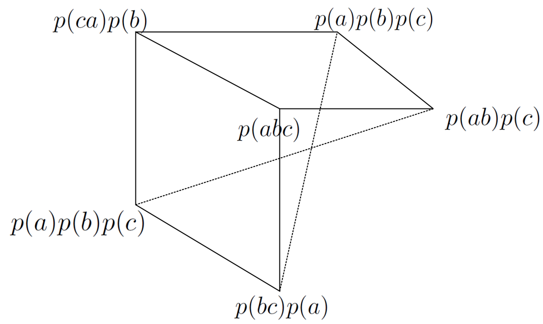

Figure 6.6:

We can now state the following theorem.

Theorem 9. Let (A, d_A) and (B, d_B) be two *dgca*s. Let p be a chain map from A to B. Let k_2, k_3 and so on be the cumulants of p. Suppose p is the first term of a C_∞ morphism $(p, p_2, p_3, \ldots)$ where $p_n : A^{\otimes n} \to B$. Then the following statements hold.

i) p_2 gives a homotopy from the second commutative cumulant k_2 to zero. All the higher cumulants k_n are also homotopic to zero using maps created by p_2 and p_1.

ii) p_3 gives a homotopy between different ways of making K_3 homotopic to zero. For all the higher Boolean cumulants, homotopies between the multiple different ways of making them homotopic to zero are homotopic to each other using p_3, p_2 and p_1.

iii) In general any cycles that are created using the homotopies $\{p_j\}_{j=1}^{n}$ are made homotopic to zero using maps made by $\{p_j\}_{j=1}^{j+1}$.

Proof. We will construct an $n-1$ dimensional cube complex c_n corresponding to k_n whose one skeleton is G_n. We first attach two cells corresponding to maps of the types that are described in figures 6.2 and 6.3. The boundaries of those maps correspond to 2-cycles in G_n since the edges in G_n. For every j less then n we attach a j-cube corresponding to maps of the form $p_1 \ldots p_{k+1} \ldots p_1$ and $p_1 \ldots p_{j_1} \ldots p_{j_2} \ldots p_1$ attached along the cells corresponding to their boundaries. Recall that

$$\partial(p_n) = d(p_n) - \sum_{k=1}^{n} p_n(1 \otimes \ldots d \ldots \otimes 1)$$

$$= \sum_{k=1}^{n-1} p_{n-1}(1 \otimes \ldots \wedge_A \ldots \otimes 1) - \sum_{n_1+n_2=n} \wedge_B(p_{n_1} \otimes p_{n_2})$$

and

$$\partial(\wedge_B(p_{n_1} \otimes p_{n_2})) = \wedge_B(\partial(p_{n_1}) \otimes \partial(p_{n_2}))$$

Thus the boundaries of the cubes correspond to the sum of lower dimensional cubes.

The complex c_n is constructed similarly to the complex g_n constructed in the previous section. Since the terms of k_n include all permutations of the inputs, c_n consists of $n-1$ cubes corresponding to p_n with permuted inputs, glued together in a certain way. Thus for some subset of permutations of j elements, we have cycles of the form

$$\ldots p_j(\sum(a_{\sigma(1)}, a_{\sigma(2)} \ldots a_{\sigma(j)}) \ldots$$

Recall that by the definition of a C_∞ morphism p_j vanishes over the sum of all shuffle permutations adding up to length j. Thus the map corresponding to the above sums is zero.

$\square$

6.3 Structure of a general A_∞ morphism

Suppose A and B are A_∞ algebras. The compatibility equation still implies that p_2 gives a homotopy between $p_1(ab)$ and $p_1(a)p_1(b)$. However we now have

$$p_1((ab)c) \neq p_1(a(bc))$$

$$\{p_1(a)p_1(b)\}p_1(c) \neq p_1(a)\{p_1(b)p_1(c)\}$$

There are a triple products m_3^A and m_3^B on A and B respectively, which makes terms homotopic to each other. When A and B were associative, there were four different ways of combining three inputs from A using p_1 and the binary products to give one output from B. When A and B are A_∞ algebras there are six different ways that are now homotopic to each other via maps involving p_2, p_1, m_2 and m_3.

Lemma 11. *The cycle created by various homotopies between the several ways of combining three inputs is homotopic to zero via the homotopy p_3.*

Proof. Thus if we made a graph G_3 with six vertices each corresponding to ways of combining n inputs, and edges corresponding to appropriate homotopies, we get a hexagon. Recall that the equation the p_3 satisfies gives the value of $\partial(p_3)$ to be

$$\begin{gathered} d(p_3) - p_3(\tilde{d}) \\ = p_2(m_2 \otimes 1 + 1 \otimes m_2) - m_2(p_1 \otimes p_2 + p_2 \otimes p_1) \\ +p_1(m_3) - m_3(p_1 \otimes p_1) \otimes p_1 \end{gathered}$$

Note that the six terms of the boundary p_3 correspond to homotopies between adjacent vertices of hexagon G_3.

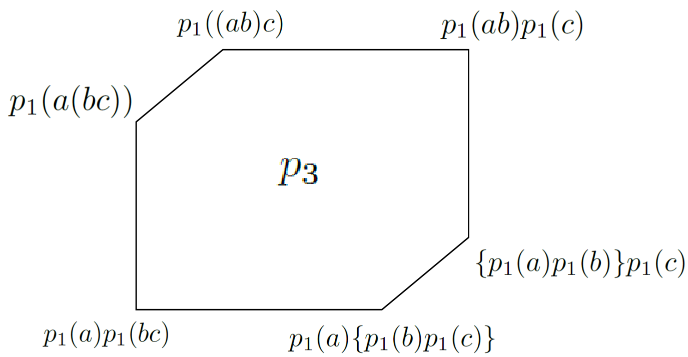

Figure 6.7:

∎

Similarly for k_4 we get the following polyhedron

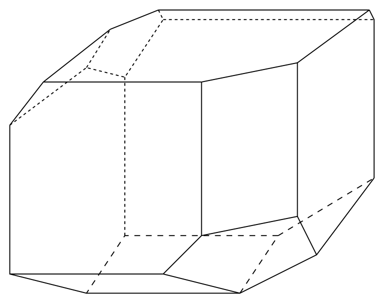

Figure 6.8:

In the context of A_∞ algebras the Boolean cumulants are only defined up to homotopy. In general for every k_n there is an $n-1$ dimensional polyhedron whose cells correspond to maps which take n inputs that are compositions of maps p_j's and m_j's.

The Boolean cumulants are defined in the context of A_∞ algebras only up to homotopy. Since the Stasheff associahedra make these different ways homotopic to each other and indeed different homotopies are homotopic to each other and so on, we have the following theorem in the context of A_∞ cumulants.

Theorem 10. Let A and B be two A_∞ algebras. Let p be a chain map from A to B. Let K_2, K_3 and so on be the Boolean cumulants of p defined up to homotopy. Suppose p is the first term of an A_∞ morphism $(p, p_2, p_3, \ldots)$ where $p_n : A^{\otimes n} \to B$. Then the following statements hold.

i) p_2 gives a homotopy from the second Boolean cumulant K_2 to zero. All the different ways of defining the higher Boolean cumulants K_n are also homotopic to zero using maps created by p_2 and p_1.

ii) p_3 gives a homotopy between different ways of making K_3 homotopic to zero. For all the higher Boolean cumulants, homotopies between the multiple different ways of making them homotopic to zero are homotopic to each other using p_3, p_2 and p_1.

iii) In general any cycles that are created using the homotopies $\{p_j\}_{j=1}^{n}$ are made homotopic to zero using maps made by $\{p_j\}_{j=1}^{n+1}$.

Proof. The proof of this theorem follows from the fact that the polyhedrons corresponding to each p_n are contractible. The cells of the polyhedrons correspond to concrete maps constructed using p_j and m_j for smaller j. $\square$

The theorem in the case of C_∞ algebras is as follows.

Theorem 11. Let A and B be two C_∞ algebras. Let p be a chain map from A to B. Let k_2, k_3 and so on be the Boolean cumulants of p defined up to homotopy. Suppose p is the first term of an C_∞ morphism $(p, p_2, p_3, \ldots)$ where $p_n : A^{\otimes n} \to B$. Then the following statements hold.

i) p_2 gives a homotopy from the second cumulant k_2 to zero. All the different ways of defining the higher cumulants k_n are also homotopic to zero using maps created by p_2 and p_1.

ii) p_3 gives a homotopy between different ways of making k_3 homotopic to zero. For all the higher Boolean cumulants, homotopies between the multiple different ways of making them homotopic to zero are homotopic to each other using p_3, p_2 and p_1.

iii) In general any cycles that are created using the homotopies $\{p_j\}_{j=1}^{n}$ are made homotopic to zero using maps made by $\{p_j\}_{j=1}^{n+1}$.

Proof. Much like in the previous cases we construct a CW-complex for every n. In the case of a C_∞ morphism between C_∞ algebras the nth complex is made of the $n-1$ dimensional polyhedrons corresponding to the Boolean cumulants in the A_∞ case. The cycles that aren't boundaries in this complex correspond to sums of p_j and m_j with permuted inputs. Recall that by the definition of C_∞ algebras we have

$$m_{q+r}\Big(\sum_{\sigma \in (q,r)-\text{shuffles}} (x_{\sigma^{-1}(1)}, x_{\sigma^{-1}(2)}, \ldots, x_{\sigma^{-1}(q+r)})\Big) = 0$$

and

$$p_{q+r}\Big(\sum_{\sigma \in (q,r)-\text{shuffles}} (x_{\sigma^{-1}(1)}, x_{\sigma^{-1}(2)}, \ldots, x_{\sigma^{-1}(q+r)})\Big) = 0$$

where μ is the shuffle product. Thus sums of cells corresponding to m_{q+r} and p_{q+r} applied to shuffle products are cycles in the CW-complex. However these maps are also boundaries since they are indeed equal to zero. Thus we can add cells corresponding to the zero map whose boundaries are the above cycles. Thus the CW-complex is indeed contractible and the corresponding maps are boundaries.

$\square$

6.4 Revisiting the A_∞ morphism between forms and associative cochains

Recall that the map $p : \Omega([0,1]) \to C^*([0,1])$ is actually the first term of an A_∞ morphism from the differential forms to the associative cochains. The maps p_n are defined by the following formula.

$$p_n(f_1(x)dx, f_2(x)dx \ldots, f_n(x)dx)$$
$$= (\int_{t_1 \le t_2 \le \ldots \le t_n} f_1(t_1)f_2(t_2)\ldots f_n(t_n)dt_1dt_2\ldots dt_n)dt$$

Note that all the one forms on the interval are exact. Suppose df_1 and df_2 are exact forms then

$$p_2(df_1, df_2) = p_2(d(f_1, df_2)) = \partial(p_2)(f_1, df_2) = K_2(f_1, df_2)$$

In general for forms df_1, df_2, df_3 and so on we have

$$p_n(df_1, df_2, \ldots, df_n) = p_n(d(f_1, df_2, \ldots, df_n)) = \partial(p_n)(f_1, df_2, \ldots, df_n)$$

The above expression is equal to

$$p_1(f_1)p_{n-1}(df_2, \ldots, df_n) \pm p_{n-1}(f_1 df_2, \ldots, df_n)$$

We can compute these quantities by induction on n. Similar analysis can be made of the A_∞ morphism between the differential forms and the cochains on an n dimensional simplex. Fewer terms would be zero in higher dimensions but we can use induction on n to compute each p_n

6.5 Conclusion: Associating CW-complexes to cumulants and maps

In the proofs of the above theorems we associated cell complexes to the cumulants of maps that were a part of some kind of a higher structure. The vertices of such cell complexes corresponded to the terms of the cumulants. The edges and faces correspond to maps provided by the higher structure, which provide appropriate homotopies. In the above theorems the cell complexes end up being contractible. However, one can imagine situations where the cell complexes have a homotopy type. Further there are several inclusions of the cell complexes associated with the nth cumulant into the cell complex associated with the $n+1$th cumulant. There are also inclusions of products of smaller dimensional cell complexes into a celcomplex corresponding to a higher dimension. Thus we have a directed system of cell complexes and we can take the direct limit of such a system.

For instance, suppose A and B are *dgcas*. Suppose $(p_1, p_2, \dots)$ is an A_∞ morphism from A to B (not necessarily a C_∞ morphism). In this situation there are cycles in the cell complex which correspond to the maps

$$p_n(\mu(x_1, \dots, x_q) \otimes (x_{q+1}, \dots, x_{q+r}))$$

where $q + r = n$ and μ is the shuffle product. The corresponding cells in the cell complex create a cycle that is in fact a sphere. The homotopy type of this cell complex is not trivial. These cycles will continue to exist through the directed system of cell complexes. The direct limit of the system of cell complexes will have a non-trivial homotopy type. Thus while the cumulants themselves are homotopic to zero and can be expressed as boundaries, there is a homotopy type associated to the cumulants which is not trivial.

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