Regularized Geometry of the Loop Space

A Dissertation presented

by

Cheng Hao

to

The Graduate School

in Partial Fulfillment of the

Requirements

for the Degree of

Doctor of Philosophy

in

Mathematics

Stony Brook University

August 2015

Stony Brook University

The Graduate School

Cheng Hao

We, the dissertation committee for the above candidate for the

Doctor of Philosophy degree, hereby recommend

acceptance of this dissertation

Mikhail Movshev - Dissertation Advisor
Associate Professor, Department of Mathematics

Marcus Khuri - Chairperson of Defense
Associate Professor, Department of Mathematics

Oleg Viro
Professor, Department of Mathematics

Alexander Abanov
Associate Professor, Department of Physics

This dissertation is accepted by the Graduate School

Charles Taber
Dean of the Graduate School

Abstract of the Dissertation

Regularized Geometry of the Loop Space

by

Cheng Hao

Doctor of Philosophy

in

Mathematics

Stony Brook University

2015

Interest in nonlinear sigma models with target space a symmetric space of positive curvature comes from both mathematics and physics. In this dissertation we focus on the two-dimensional nonlinear sigma model with target space the k-dimensional projective space $\mathbb{P}^k$. We study the properties of a finite-dimensional approximation, which we denote by $\mathcal{L}_N(\mathbb{P}^k)$, to the loop space $\mathcal{L}(\mathbb{P}^k)$, and construct resolutions of singularities for these truncated loop spaces. We also look into the geometries of such spaces and propose a conjecture about the mass gap of the Schrödinger operator H on $\mathcal{L}_N(\mathbb{P}^k)$ with analytic formula.

To my parents and all my teachers.

Contents

Acknowledgements

I would like to express my sincere gratitude to my advisor, Michael Movshev, for all his patience, enlightenment and invaluable encouragement through out my time as his student. I have learnt not only mathematics from him, but also life wisdom. I will miss our meetings at cafe Grounded, and do hope we can find time to still talk about mathematics and other aspects of life in the future.

I would like to thank Professors Marcus Khuri, Oleg Viro and Alexander Abanov for serving on my dissertation committee and providing valuable comments and feedback.

I would like to thank fellow students Matthew Young, Jimmy Mathews, Dingxin Zhang, Kenneth Knox, Jun Wen, Jingchen Niu, Zili Zhang and Zheng Zhang for all the interesting discussions in mathematics, and all my friends at Stony Brook who made my memories at Stony Brook lovely and joyful.

Finally, I would like to thank my parents, Hongzhuan Hao and Xuelian Li. None of this would have been possible without their continued unconditional support and endless love.

Chapter 1

Introduction

In this paper we study the regularized geometry of the loop space in the complex projective space $\mathbb{P}^k$, with the goal of providing a framework to study the mass gap problem for the sigma model with target space $\mathbb{P}^k$.

1.1 Mass Gap

The motivation and interest in mass gaps come from the needs in quantization of field theories. Soon after an understanding of quantum mechanics at the subatomic level of nature developed at the beginning of the twentieth century, it became clear by the late 1920s that an internally coherent account of nature must incorporate quantum concepts for fields as well as for particles to describe the interactions between them. The distinction between fields and particles breaks down as in the case of fields we have to deal with infinite degrees of freedom. One of the most challenging Quantum Field Theories in modern physics to describe the behavior of elementary particles is Yang-Mills theory. Formally it is an extension of the quantum electrodynamics (QED), a gauge theory of U(1) group. The nonabelian gauge group in YM theory encodes the particle interactions. Gauge theories originated from the work of Hermann Weyl, who first proposed the idea of using connections as fields.

In the QFT Lagrangian

$$L = \frac{\partial\phi}{\partial x^i}\frac{\partial\phi}{\partial x^i} + m^2\phi^2 + \sum_{i=3}^{\infty}\frac{g_i\phi^i}{i!}$$

the coefficient m is interpreted as the mass of the field. In the YM theory Lagrangian this term is zero. In QED (a close relative of YM) this term also vanishes. On the other hand QCD describes protons, that do have mass. Hence we have the mass gap problem. The mass gap problem was solved heuristically by physicists.

1.2 Sigma model

While the problem of providing a mathematically rigorous formalism for the quantum Yang-Mills theory on $\mathbb{R}^4$ for any compact simple gauge group G and proving that the theory has a mass gap remains to be one of the most profound unsolved problems, there has been interesting progress on various related problems.

There are two classes of quantum field theories which are believed to bear a close similarity to the Yang-Mills theory. One class of models are the supersymmetric Yang-Mills theories, which has great similarity to Yang-Mills but is significantly simpler. These are modifications of Yang-Mills theory that includes various fermionic and other fields to incorporate supersymmetry. There has been significant progress in the past decades in this direction, notably the celebrated Seiberg-Witten theory, which is then applied to and has profound influence in four-dimensional topology.

In this dissertation we focus on the second class, namely the sigma model with target space a symmetric space of positive Riemannian curvature.

The Schrödinger operator H for a sigma model is defined on the loop space in the target space of the model. The Schrödinger operator is the operator corresponding to the total energy of the system. It is in the form

$$H = -\Delta + U,$$

where Δ is the Laplacian operator giving the kinetic energy, and U is the potential energy. When the target space of the theory is the complex projective space

$$\mathbb{P}^k = S^{2k+1}/S^1,$$

the Schrödinger operator H has to act on $L^2(\mathcal{L}(\mathbb{P}^k))$, where $\mathcal{L}(\mathbb{P}^k)$ is the loop space in $\mathbb{P}^k$,

$$\mathcal{L}(\mathbb{P}^k) = \mathrm{Maps}(S^1, \mathbb{P}^k) = \left\{\sigma : S^1 \to \mathbb{P}^k\right\}.$$

The potential energy U is given by the norm square of the vector field ρ corresponding to the infinitesimal rotation of a loop $\sigma(\theta) \mapsto \sigma(\theta + \epsilon)$,

$$U = (\rho, \rho).$$

The main difficulty is to define $L^2(\mathcal{L}(\mathbb{P}^k))$ and the action of H.

A standard reference for loop groups is given by the book of Pressley and Segal [13].

1.3 Regularized geometry of the loop space

Mathematically, a mass gap $\sigma(H)$ of a theory, if it exists, is the difference between the first and the second positive eigenvalues of the Schrödinger operator H. We must keep in mind that the standard scheme of defining self-adjoint extension of an operator doesn't work in infinite dimensions. Though $\mathcal{L}(\mathbb{P}^k)$ is equipped with the Wiener measure, suitable for defining $L^2(\mathcal{L}(\mathbb{P}^k))$, there is no obvious candidate for $C^\infty(\mathcal{L}(\mathbb{P}^k))$ on which H is defined. Because of this difficulty we exploit the idea of using some finite-dimensional truncations, which we denote by $\mathcal{L}_N(\mathbb{P}^k)$, to approximate $\mathcal{L}(\mathbb{P}^k)$. Let H_N denote the Schrödinger operator on $\mathcal{L}_N(\mathbb{P}^k)$ defined by the induced metric from $\mathcal{L}(\mathbb{P}^k)$ and the restriction of the potential. In this language, the problem of finding the mass gap for the sigma model becomes finding the gap estimates $\sigma_N = \sigma(H_N)$ for $N = 1, 2, 3, \ldots$ and the limit as N goes to infinity.

We now give an analytic formulation of the mass gap conjecture for the truncated loop space which we owe to A. Kapustin.

Let $\mathbb{P}^k = S^{2k+1}/S^1$ be the k-dimensional projective space, where S^{2k+1} is a sphere of radius R. Let $\mathcal{L}_N(\mathbb{P}^k) \subset \mathcal{L}(\mathbb{P}^k) = \mathrm{Maps}(S^1, \mathbb{P}^k)$ be the N-th-truncated loop space of the projective space (Definition 5). To define a metric (equation (4.1)) on the space $\mathrm{Maps}(S^1, \mathbb{P}^k)$ we need to fix a metric not only on $\mathbb{P}^k$ but also on S^1, which is given by (up to diffeomorphism) its length L.

Conjecture 1. *There is a constant b_0 such that*

$$\sigma_N = \lambda_2(N) - \lambda_1(N) \geq \frac{N^2}{L} e^{-b_0 R^2}, \ N >> 0.$$

where $\lambda_1(N)$ and $\lambda_2(N)$ are the first and the second eigenvalues of the Schrödinger operator H_N on $\mathcal{L}_N(\mathbb{P}^k)$.

The problem of finding lower bounds for the gap $\sigma(H) = \lambda_2 - \lambda_1$ in terms of geometrical quantities of the underlying manifold has been studied by many authors using partial differential equations. To name a few, see Chen [3, 4], Li [9], Li-Yau [10, 11], Singer-Wong-Yau-Yau [14] and Wang [15].

These works formulate estimates for $\sigma(H)$ in terms of geometry quantities (Ricci curvatures, volume, diameter and so on) of the underlying manifold. Here are some relevant definitions: Let (M^m, g_{ij}) be an m-dimensional compact Riemannian manifold with metric g_{ij} and $\partial M \neq \emptyset$. Let Δ be the Laplacian operator associated to g_{ij} on M. We define on M a Schrödinger operator by $\Delta - q(x)$, where $q(x) \in C^2(M)$.

We consider the following Neumann eigenvalue problem:

$$\Delta u - qu = -\eta u \text{ in } M,$$
$$\frac{\partial u}{\partial \nu} = 0 \text{ on } \partial M.$$

Theorem (R. Chen, see [3] Theorem 1)**.** *Let M^m, $m \geq 3$, be an m-dimensional compact manifold with boundary ∂M and $\omega(q) = \sup q - \inf q$ denote the oscillation of q over M. Suppose that the Ricci curvature of M satisfies $R_{ij} + Kg_{ij}$ is positive-definite, $K \geq 0$, and the second fundamental form α_{ij} of ∂M with respect to the outward pointing unit normal ν satisfies $\alpha_{ij} + Hg_{ij}$ positive-definite, $H \geq 0$. Suppose that ∂M also satisfies the "interior rolling r-ball condition" with r chosen small (see [4], [15] for the definition of the "rolling ball condition" and the choice of r). Then the gap $\Gamma_k = \eta_k - \eta_1$ of k-th Neumann eigenvalues η_k and η_1 of M satisfies*

$$\Gamma_k \geq \alpha_1 k^{\frac{2}{m}}$$

for all $k = 2, 3, \ldots$ and some explicitly computable constant α_1 depending on m, K, H, r, $\omega(q)$ and the diameter of M.

This motivates the study of the differential geometry of the truncated loop spaces.

1.4 Dissertation work

We briefly summarize the contents of each chapter.

In Chapter 2 we define our main objects $\mathcal{L}_N(\mathbb{P}^k)$, the truncated loop spaces. We present two definitions. The first definition is given by equations, which will be useful in computing geometrical quantities of $\mathcal{L}_N(\mathbb{P}^k)$. We first define $\mathbb{P}^k$ as the subspace of $(k+1)$-dimensional Hermitian matrices that satisfies some equations. In this way loops in $\mathbb{P}^k$ can be expanded using Fourier series with coefficients in Hermitian matrices, satisfying the defining equation of $\mathbb{P}^k$. We define $\mathcal{L}_N(\mathbb{P}^k)$ to be the space of such Fourier polynomials with degree less than or equals to N (Definition 5).

The spaces $\mathcal{L}_N(\mathbb{P}^k)$ are generally not smooth. To find its smooth locus, we give an alternative definition of $\mathcal{L}_N(\mathbb{P}^k)$, call it $\tilde{\mathcal{L}}_N(\mathbb{P}^k)$. To define it, we complexify both the source space and the target space and consider the space of degree N holomorphic maps between the complexified spaces satisfying certain tangency conditions to the divisor at infinity in the target space (Definition 6). Then in the spirit of P. Georgieva and A. Zinger [6], [7], we take $\tilde{\mathcal{L}}_N(\mathbb{P}^k)$ to be the subspace of maps that satisfy some reality conditions. We show that $\tilde{\mathcal{L}}_N(\mathbb{P}^k)$ and $\mathcal{L}_N(\mathbb{P}^k)$ is related by

$$\tilde{\mathcal{L}}_N(\mathbb{P}^k) = \mathcal{L}_N(\mathbb{P}^k) \setminus \mathcal{L}_{N-1}(\mathbb{P}^k),$$

and $\tilde{\mathcal{L}}_N(\mathbb{P}^k)$ is the smooth locus of $\mathcal{L}_N(\mathbb{P}^k)$. The definition of $\tilde{\mathcal{L}}_N(\mathbb{P}^k)$ also enable us to compute the dimension of the truncated loop space using homological methods.

The loop group $\mathcal{L}(\mathrm{U}(k+1))$ is acting on $\mathcal{L}(\mathbb{P}^k)$:

$$\mathcal{L}(\mathrm{U}(k+1)) \times \mathcal{L}(\mathbb{P}^k) \longrightarrow \mathcal{L}(\mathbb{P}^k). \tag{1.1}$$

We take a closer look at the relation between $\mathcal{L}_N(\mathbb{P}^k)$ and $\mathcal{L}_{N-1}(\mathbb{P}^k)$ in Chapter 3, in which we prove the following theorem:

Theorem. *There is a connected component $G(k+1)$ in the space of smooth homomorphisms* $\mathrm{Hom}(S^1, \mathrm{U}(k+1))$ *such that the conjugation map*

$$\mu : G(k+1) \times \mathcal{L}_{N-1}(\mathbb{P}^k) \longrightarrow \mathcal{L}_N(\mathbb{P}^k),$$

which is the restriction of (1.1)*, is onto, and is one to one at generic points.*

The manifold $G(k+1)$ is smooth and is isomorphic to the projective space $\mathbb{P}^k$. As a corollary, we obtain the main result of this paper, a resolution of singularity for $\mathcal{L}_N(\mathbb{P}^k)$:

$$\mu^N : \mathbb{P}^k \times \underbrace{G \times \cdots \times G}_{\mathrm{N}} \longrightarrow \mathcal{L}_N(\mathbb{P}^k).$$

In Chapter 4 we focus on the geometry of $\mathcal{L}_N(\mathbb{P}^k)$. We first give a precise formula for the potential in the Schrödinger operator using the ordinary L^2 metric on the loop space. Then we compute the Riemann curvature tensor for the loop space $\mathcal{L}(\mathbb{P}^k)$ using the loop group action by $\mathcal{L}(\mathrm{U}(k+1))$. To be able to use results from partial differential equations to estimate the gap we also compute estimates for the volume and the diameter of $\mathcal{L}_N(\mathbb{P}^k)$ via the fibration structure established in the previous chapter.

In Chapter 5 we examine the first nontrivial truncated loop space $\mathcal{L}_1(\mathbb{P}^k)$. We give an explicit construction of $\mathcal{L}_1(\mathbb{P}^k)$ using Pauli matrices. We show that the smooth piece of $\mathcal{L}_1(\mathbb{P}^k)$ is isomorphic to the product $(-1,1) \times \mathrm{SU}(k+1)/\mathrm{U}(k-1)$. Using this construction we can transform the mass gap problem for the one-dimensional Schrödinger operator on the nonhomogeneous piece to a Sturm-Liouville equation, and prove the existence of the spectral gap using techniques from ODE.

Chapter 2

The loop space $\mathcal{L}(\mathbb{P}^k)$

In this chapter we define our main objects $\mathcal{L}_N(\mathbb{P}^k)$, the finite dimensional approximations to the loop space $\mathcal{L}(\mathbb{P}^k)$. In Section 2.1 we give a real algebraic structure on the k-dimensional projective space $\mathbb{P}^k$. In Section 2.2 we use the result of Section 2.1 to define the finite dimensional truncations $\mathcal{L}_N(\mathbb{P}^k)$ of the loop space $\mathcal{L}(\mathbb{P}^k)$, which are ordered by inclusions:

$$\mathcal{L}_0(\mathbb{P}^k) \subset \mathcal{L}_1(\mathbb{P}^k) \subset \cdots \subset \mathcal{L}_N(\mathbb{P}^k) \subset \cdots \mathcal{L}(\mathbb{P}^k).$$

In Section 2.3 we find a complexification $\mathcal{L}_N(\mathbb{P}^k)^{\mathbb{C}}$ of $\mathcal{L}_N(\mathbb{P}^k)$, and use it to calculate the dimension and smooth locus of $\mathcal{L}_N(\mathbb{P}^k)$ by homological methods in Section 2.4.

2.1 The real algebraic structure on $\mathbb{P}^k$

As explained in the introduction, we would like to define the truncated loop spaces $\mathcal{L}_N(\mathbb{P}^k)$ using equations in order to study its geometry. To this end, we first give a description of the target space $\mathbb{P}^k$ by equations in $\mathbb{R}^{(k+1)^2}$.

Proposition 2. Let $\mathbb{P}^k = \mathbb{C}^{k+1} - \{0\}/\mathbb{C}^*$ be the k-dimensional complex projective space. Then we have diffeomorphism

$$\mathbb{P}^k \simeq \{M \in V_{\text{herm}}^{k+1} \mid M^2 = R^2 I, \text{tr}M = (k-1)R\},$$

where $R > 0$, V_{herm}^{k+1} is the space of $(k+1)$-dimensional hermitian matrices defined with respect to the standard hermitian product on $\mathbb{C}^{k+1}$, and I the identity matrix.

Proof. The unitary group $\mathrm{U}(k+1)$ acts by conjugation on

$$V_{\text{herm}}^{k+1} = \left\{A \in \text{Mat}(k+1, \mathbb{C}) | A = A^*\right\},$$

the space of $(k+1)$-dimensional hermitian matrices. Since Hermitian matrices have real eigenvalues, two Hermitian matrices are in the same orbit if and only if they have the same set of eigenvalues. Therefore each orbit is characterized by a collection of $k+1$ real eigenvalues.

If an orbit consists of matrices having 2 distinct real eigenvalues λ_1, λ_2 with multiplicities 1 and k respectively, the stabilizer of the element $M = \mathrm{diag}\{\lambda_1, \lambda_2, \ldots, \lambda_2\}$ in the orbit is $\mathrm{U}(1) \times \mathrm{U}(k)$, diagonally embedded in $\mathrm{U}(k+1)$. Therefore, the orbit is diffeomorphic to $\mathrm{U}(k+1)/\mathrm{U}(1) \times \mathrm{U}(k)$, which is then diffeomorphic to $\mathbb{P}^k$.

For each positive real number R, we look at the orbit $\mathcal{O}_{R,-R}$ consists of matrices with k eigenvalues equal to R and one equal to $-R$. Matrices in this orbit can be characterized by algebraic equations:

$$M^2 = R^2 I, \ \mathrm{tr} M = (k-1)R, \tag{2.1}$$

where I is the identity matrix. Therefore, we have a diffeomorphism

$$\mathbb{P}^k \simeq \mathcal{O}_{R,-R} = \{M \in V^{k+1}_{\mathrm{herm}} \mid M^2 = R^2 I, \mathrm{tr} M = (k-1)R\}. \tag{2.2}$$

for any $R > 0$. $\square$

Note that V^{k+1}_{herm} is isomorphic to the linear space $\mathbb{R}^{(k+1)^2}$. Therefore Proposition 2 defines $\mathcal{L}_N(\mathbb{P}^k)$ as a real algebraic subvariety in $\mathbb{R}^{(k+1)^2}$.

As a corollary of the diffeomorphism between $\mathbb{P}^k$ and orbits of the $\mathrm{U}(k+1)$ action on Hermitian matrices in the proof above, we have the following embedding:

Corollary 3. *Let $[z] = [z_0 : \cdots : z_k]$ be the homogeneous coordinate of $\mathbb{P}^k$. The map*

$$\phi : [z] \longmapsto (z_i \bar{z}_j), \ |z| = 1, \tag{2.3}$$

gives an embedding of $\mathbb{P}^k$ into V^{k+1}_{herm}.

Proof. The image of ϕ corresponds to the orbit of matrices with one eigenvalue 1 and k eigenvalues of 0. $\square$

Remark 4. $\mathbb{P}^k$ is a symplectic manifold with respect to the Fubini-Study metric. The group $\mathrm{U}(k+1)$ acts on $\mathbb{P}^k$ by symplectic diffeomorphisms. The map (2.3) (composed with multiplication by $\mathfrak{i}$) is the moment map for this action.

2.2 Loop space $\mathcal{L}_N(\mathbb{P}^k)$

In this section we give a characterization of the loop space $\mathcal{L}(\mathbb{P}^k)$ and define finite dimensional approximations of it, which will be the main objects in this paper.

Let $\mathcal{L}(\mathbb{P}^k) = \mathrm{Maps}(S^1, \mathbb{P}^k)$ be the loop space, that is the space of smooth maps equipped with compact-open topology, in the complex projective space $\mathbb{P}^k$. Using the definition of $\mathbb{P}^k$ by equations in Proposition 2, $\mathcal{L}(\mathbb{P}^k)$ can be identified with a subspace of $\mathrm{Maps}(S^1, V_{\mathrm{herm}}^{k+1})$ consisting of maps $n(z)$ such that equations

$$n^2(z) = R^2 I, \ \operatorname{tr}(n(z)) = (k-1)R, \tag{2.4}$$

are satisfied for any $z \in S^1$.

Note that loops in $\mathcal{L}(\mathbb{P}^k)$ can be regarded as V_{herm}^{k+1}-valued Fourier series satisfying equations (2.4) and certain convergence conditions. We define $\mathcal{L}_N(\mathbb{P}^k)$ to be the intersection of $\mathcal{L}(\mathbb{P}^k)$ with the space of trigonometric polynomials Trig_N,

$$\mathrm{Trig}_N = \left\{ n(z) = V + \sum_{n=1}^{N} A_n \cos(n\theta) + \sum_{n=1}^{N} B_n \sin(n\theta) \;\middle|\; A_n, B_n \in V_{\mathrm{herm}}^{k+1} \right\}, \tag{2.5}$$

where $z = e^{\mathrm{i}\theta}$.

Definition 5. $\mathcal{L}_N(\mathbb{P}^k)$ is defined to be the subspace of elements in Trig_N satisfying equations (2.4). That is,

$$\mathcal{L}_N(\mathbb{P}^k) = \left\{ n(z) \in \mathrm{Trig}_N \;\middle|\; n^2(z) = R^2 I, \ \operatorname{tr}(n(z)) = (k-1)R, \text{ for all } z \in S^1 \right\}.$$

The space $\mathcal{L}_N(\mathbb{P}^k)$ is a (generally nonsmooth) real algebraic subvariety in Trig_N. The latter is isomorphic to a linear space $\mathbb{R}^{(k+1)^2} \times (\mathbb{R}^{(k+1)^2} \otimes \mathbb{C})^{\times N}$. Relations between V, A_n's and B_n's can be derived from the Fourier coefficients U, C_n's and D_n's of $n(z)^2$

$$n(z)^2 = U + \sum_{n=1}^{2N} C_n \cos(n\theta) + \sum_{n=1}^{2N} D_n \sin(n\theta),$$

which are quadratic polynomials in matrices V, A_n, B_n:

$$
\begin{aligned}
U = &\ V^2 + \frac{1}{2}\sum_{n=1}^{N}(A_n^2 + B_n^2),\\
C_l = &\ VA_l + A_lV + \frac{1}{2}\sum_{t-s=l}\Big((A_tA_s + A_sA_t) + (B_tB_s + B_sB_t)\Big)\\
&+ \frac{1}{2}\sum_{t+s=l}(A_tA_s - B_tB_s),\ 1 \le l \le N,\\
D_l = &\ VB_l + B_lV + \frac{1}{2}\sum_{t-s=l}\Big((B_tA_s + A_sB_t) - (A_tB_s + B_sA_t)\Big)\\
&+ \frac{1}{2}\sum_{t+s=l}(A_tB_s + B_tA_s),\ 1 \le l \le N,\\
C_l = &\ \frac{1}{2}\sum_{t+s=l}(A_tA_s - B_tB_s),\ N < l \le 2N,\\
D_l = &\ \frac{1}{2}\sum_{t+s=l}(A_tB_s + B_tA_s),\ N < l \le 2N.
\end{aligned}
\tag{2.6}
$$

$\mathcal{L}_N(\mathbb{P}^k)$ is then defined by equations

$$
\begin{aligned}
&U = R^2I,\ C_l = 0,\ D_l = 0,\ 1 \le l \le 2N,\\
&\mathrm{tr}V = (k-1)R,\ \mathrm{tr}A_n = 0,\ \mathrm{tr}B_n = 0,\ 1 \le n \le N.
\end{aligned}
\tag{2.7}
$$

Another way to represent elements in Trig_N is by using matrix valued Laurent polynomials

$$
n(z) = \sum_{n=-N}^{N} K_n z^n,
\tag{2.8}
$$

where K_n's satisfy

$$
K_{-n} = K_n^*
\tag{2.9}
$$

to ensure the hermitian conditions on trigonometric coefficients of $n(z)$. Coefficients in the Laurent polynomial and trigonometric expansions of $n(z)$ are related by

$$
K_n = \frac{1}{2}(A_n + \mathrm{i}B_n),\ K_{-n} = \frac{1}{2}(A_n - \mathrm{i}B_n).
\tag{2.10}
$$

Defining equations for $\mathcal{L}_N(\mathbb{P}^k)$ in terms of K_n's become

$$
\begin{aligned}
&\sum_{s+t=l} K_sK_t = \delta_{l,0}R^2I,\ -2N \le l \le 2N,\\
&\mathrm{tr}K_0 = (k-1)R,\ \mathrm{tr}K_s = 0, s \ne 0.
\end{aligned}
\tag{2.11}
$$

We will be using the two expansions of elements in $\mathcal{L}_N(\mathbb{P}^k)$ interchangeably throughout the paper.

Note that the equations for C_{2N} and D_{2N} in (2.6) are equal to zero on $\mathcal{L}_{N-1}(\mathbb{P}^k)$. Therefore we have

$$\mathcal{L}_{N-1}(\mathbb{P}^k) \subset \mathcal{L}_N(\mathbb{P}^k)^{\text{sing}}$$

and

$$\mathcal{L}_N(\mathbb{P}^k)^{\text{smooth}} \subset \mathcal{L}_N(\mathbb{P}^k) \setminus \mathcal{L}_{N-1}(\mathbb{P}^k),$$

where $\mathcal{L}_N(\mathbb{P}^k)^{\text{sing}}$ and $\mathcal{L}_N(\mathbb{P}^k)^{\text{smooth}}$ stands for the singular and the smooth locus of $\mathcal{L}(\mathbb{P}^k)$ respectively. We will prove in Section 2.4 that the smooth locus is actually equal to $\mathcal{L}_N(\mathbb{P}^k) \setminus \mathcal{L}_{N-1}(\mathbb{P}^k)$.

2.3 An alternative definition for $\mathcal{L}_N(\mathbb{P}^k)$

We defined $\mathcal{L}_N(\mathbb{P}^k)$ in Section 2.2 by equations. This approach will be useful in calculations of various geometric quantities of $\mathcal{L}_N(\mathbb{P}^k)$. In this section, we would like to give a definition of $\tilde{\mathcal{L}}_N(X)$ that applies to a more general manifold X in a coordinate-free fashion. We will then use it to compute the virtual dimension of $\mathcal{L}_N(\mathbb{P}^k)$ and determine its smooth locus in the next section.

Inspired by constructions in real Gromov-Witten theory [6], we would like to define $\tilde{\mathcal{L}}_N(X) \subset \mathrm{Maps}(S^1, X)$ to be the set of real maps in $\mathrm{Maps}_{\mathrm{hol},2N}(\mathbb{P}^1, Y)$ satisfying certain tangency conditions where Y is a complexification of X and $\mathrm{Maps}_{\mathrm{hol},2N}(\mathbb{P}^1, Y)$ is the set of holomorphic maps from $\mathbb{P}^1$ to Y of degree $2N$.

2.3.1 Terminology

We first set up the terminologies. An *involution* τ on a smooth manifold Y is a diffeomorphism $\tau : Y \to Y$ such that $\tau \circ \tau = \mathrm{id}_Y$. An involution τ on a complex manifold (Y, J) is said to be *antiholomorphic* if $\tau_* \circ J = -J \circ \tau_*$, where J is the complex structure on Y, and τ_* is the derivative of τ.

Let Y be a complex manifold equipped with an antiholomorphic involution τ, the set of *real* points is defined to be the fixed point set

$$Y^\tau = \{z \in Y \mid \tau(z) = z\}.$$

A projective complex manifold Y is said to be a *complexification* of a smooth manifold X, if there exists an antiholomorphic involution τ on Y such that X is diffeomorphic to the set of real points of Y

$$X \simeq Y^\tau.$$

Let (Y_1, τ_1) and (Y_2, τ_2) be complex manifolds with antiholomorphic involutions, a holomorphic map

$$\phi : (Y_1, \tau_1) \longrightarrow (Y_2, \tau_2)$$

is said to be *real* if

$$\phi \circ \tau_1 = \tau_2 \circ \phi. \tag{2.12}$$

The involutions τ_1 and τ_2 on Y_1 and Y_2 respectively together define an involution ρ on $\mathrm{Maps}_{\mathrm{hol}}(Y_1, Y_2)$ by

$$\rho(\phi) = \tau_2 \circ \phi \circ \tau_1.$$

The reality condition in (2.12) becomes

$$\rho(\phi) = \phi$$

in terms of ρ. We use

$$\big(\mathrm{Maps}_{\mathrm{hol}}(Y_1, Y_2)\big)^{\rho} = \Big\{\phi \in \mathrm{Maps}_{\mathrm{hol}}(Y_1, Y_2) \;\Big|\; \rho(\phi) = \phi\Big\}.$$

to denote the set of real holomorphic maps from Y_1 to Y_2.

2.3.2 Construction of $\tilde{\mathcal{L}}_N(X)$

We now can construct the space $\tilde{\mathcal{L}}_N(X)$ as follows. Let X be a smooth manifold and Y a complexification of X, i.e. X is the set of real points of the projective complex manifold (Y, τ), where τ is an antiholomorphic involution on Y. An additional piece of data necessary in our construction is a τ-invariant divisor $D \subset Y$, such that $D \cap X = \emptyset$. Let

$$\mathrm{Maps}_{0,\infty,D,N}(\mathbb{P}^1, Y) \subset \mathrm{Maps}_{\mathrm{hol},2N}(\mathbb{P}^1, Y)$$

denote the space of holomorphic maps ϕ from $\mathbb{P}^1$ to Y of degree $2N$ such that $\phi([\mathbb{P}^1]) \cap [D] = 2N$ and the degrees of tangency of $\phi(\mathbb{P}^1)$ to D at points $\phi(0)$ and $\phi(\infty)$ are N.

Let σ be the antiholomorphic involution on $\mathbb{P}^1$ defined by $\sigma(z) = \frac{1}{\bar{z}}$, which swaps 0 and ∞ in $\mathbb{P}^1$, and fixes the unit circle. From the degree consideration, $\phi^{-1}(D) = \{0, \infty\}$ for ϕ in $\mathrm{Maps}_{0,\infty,D,N}(\mathbb{P}^1, Y)$, and $\mathrm{Maps}_{0,\infty,D,N}(\mathbb{P}^1, Y)$ is stable under the involution

$$\rho(\phi) = \tau \circ \phi \circ \sigma$$

on $\mathrm{Maps}_{\mathrm{hol},2N}(\mathbb{P}^1, Y)$.

Definition 6. $\tilde{\mathcal{L}}_N(X)$ is defined to be real maps in $\mathrm{Maps}_{0,\infty,D,N}(\mathbb{P}^1, Y)$, that is,

$$\tilde{\mathcal{L}}_N(X) := \big(\mathrm{Maps}_{0,\infty,D,N}(\mathbb{P}^1, Y)\big)^{\rho}.$$

2.3.3 Relation between $\tilde{\mathcal{L}}_N(\mathbb{P}^k)$ and $\mathcal{L}_N(\mathbb{P}^k)$

We now see how this definition is related to our definition of $\mathcal{L}_N(\mathbb{P}^k)$ from Section 2.2. To apply the definition of $\tilde{\mathcal{L}}_N(X)$ to $X = \mathbb{P}^k$, we first need to choose a complexification Y of $\mathbb{P}^k$.

Proposition 7. Let $Y = \mathbb{P}^k \times \mathbb{P}^k$, then Y is a complexification of $\mathbb{P}^k$. More precisely, there exists an antiholomorphic involution τ on Y such that Y^τ is isomorphic to $\mathbb{P}^k$.

Proof. A map $f : V_1 \to V_2$ from a complex vector space to another is said to be *antilinear* if

$$f(\lambda x + y) = \bar{\lambda} f(x) + f(y),$$

for all $\lambda \in \mathbb{C}$ and $x, y \in V_1$.

Let V be a complex space with hermitian inner product $v_1 \cdot v_2$. In our case $V = \mathbb{C}^{k+1}$ together with the standard hermitian inner product. The inner product then defines an antilinear isomorphism between V and its dual space

$$\alpha : V \longrightarrow V^*. \tag{2.13}$$

On the space of pairs of lines $(l_1, l_2) \in \mathbb{P}(V) \times \mathbb{P}(V^*)$, α in (2.13) defines an antiholomorphic involution by the formula

$$\tau : (l_1, l_2) \longmapsto (\alpha^{-1}(l_2),\ \alpha(l_1)). \tag{2.14}$$

The embedding $\iota : \mathbb{P}(V) \to \mathbb{P}(V) \times \mathbb{P}(V^*),\ l \mapsto (l,\ \alpha(l))$ then induces an isomorphism

$$\iota : \mathbb{P}(V) \simeq (\mathbb{P}(V) \times \mathbb{P}(V^*))^\tau.$$

between $\mathbb{P}(V)$ and the fix point set of τ. $\square$

Note that since Proposition 2 gives a definition of $\mathbb{P}^k$ by equations in $\mathbb{R}^{(k+1)^2}$, we can also complexify $\mathbb{P}^k$ using homogenous equations in $\mathbb{P}^{(k+1)^2-1}$. Appendix A contains an exposition of such construction.

Another piece of data we need in our construction is a divisor at infinity defined by

$$D = \Big\{ (l_1, l_2) \in \mathbb{P}(V) \times \mathbb{P}(V^*) \ \Big|\ \langle l_1, l_2 \rangle = 0 \Big\},$$

where $\langle ., . \rangle$ is the canonical pairing between V and V^*. Divisor D is a partial flag space $\big\{F_1 \subset F_{n-1}\big\}$, where $F_1 = l_1$ and F_{n-1} is the hyperplane in V defined by equation $\langle l_1, l_2 \rangle = 0$. The divisor satisfies

$$(\mathbb{P}(V) \times \mathbb{P}(V^*))^\tau \cap D = \emptyset,$$

since for any vector $v \in l$, we have $\langle l, \alpha(l)\rangle \sim v\cdot v > 0$, where $\sim$ stands for equivalence up to a nonzero scalar. Therefore, the image of $\mathbb{P}(V)$ in $\mathbb{P}(V) \times \mathbb{P}(V^*)$ does not intersect D.

The Laurent polynomial $n(z)$ in (2.8) defines the map $\mathbb{C}^* \to Y \setminus D$, where Y is a projective variety. We can extend n to a map $n : \mathbb{P}^1 \to Y$, the tangency condition at $z = 0$ reflects the degree of the Laurent polynomial, and the tangency at $z = \infty$ comes from the ρ-symmetry. Therefore, $\tilde{\mathcal{L}}_N(\mathbb{P}^k)$ contains Laurent polynomials with degree exactly N, that is,

$$\tilde{\mathcal{L}}_N(\mathbb{P}^k) = \mathcal{L}_N(\mathbb{P}^k) \setminus \mathcal{L}_{N-1}(\mathbb{P}^k). \tag{2.15}$$

2.3.4 Generalization to Grassmannians

The construction in the previous section admits generalizations to Grassmannians. Let V be the same as in Section 2.3.3 and $Gr(s, V)$ the Grassmannian of s-dimensional planes in V. The manifold $Gr(s, V) \times Gr(s, V^*)$ has a similar antiholomorphic involution

$$\tau : (l_1, l_2) \to (\alpha^{-1}(l_2), \alpha(l_1)).$$

as in (2.14), where (l_1, l_2) are now s-planes and α being extended to the exterior powers of V. The divisor at infinity is defined by equation $\det(\langle v_i, w^j\rangle) = 0$, where $\{v_i\}$ is a basis of l_1, $\{w^j\}$ is a basis of l_2. Note that $\det(\langle v_i, \alpha v_j\rangle) > 0$ because α is positive-definite. An s-plane l_2 in V^* is the same thing as a $(\dim V - s)$-plane m in V. Condition $\det(\langle v_i, w^j\rangle) = 0$ is the same as non-transversality of l_1 and m, which is equivalent to l_1 and m contain a non-trivial common subspace.

2.4 Smooth locus and virtual dimension of $\tilde{\mathcal{L}}_N(\mathbb{P}^k)$

We defined $\tilde{\mathcal{L}}_N(X)$ to be the real locus of $\mathrm{Maps}_{0,\infty,D,N}(\mathbb{P}^1, Y)$ in Definition 6 in the previous section. Equivalently, we have

$$\tilde{\mathcal{L}}_N(X)^{\mathbb{C}} = \mathrm{Maps}_{0,\infty,D,N}(\mathbb{P}^1, Y),$$

where $\tilde{\mathcal{L}}_N(X)^{\mathbb{C}}$ stands for the complexification of $\tilde{\mathcal{L}}_N(X)$. The tangent space $T_\phi\big(\tilde{\mathcal{L}}_N(X)^{\mathbb{C}}\big)$ at $\phi(z) \in \tilde{\mathcal{L}}_N(X)^{\mathbb{C}}$ can be computed using homological methods, which we present in this section, then use it to find the smooth locus of $\mathcal{L}_N(\mathbb{P}^k)$.

2.4.1 Tangent space

It follows from the first order deformation theory [5] that the tangent space to $\mathrm{Maps}_{\mathrm{hol},2N}(\mathbb{P}^1, Y)$ at a map $\phi(z)$ is given by

$$T_\phi\big(\mathrm{Maps}_{\mathrm{hol},2N}(\mathbb{P}^1, Y)\big) = H^0(\mathbb{P}^1, \phi^* T_Y).$$

Let F be a subsheaf of the tangent bundle T_Y whose local sections are tangential to D. More precisely, let g be a local function that defines the divisor D, that is, D is locally defined by $g = 0$. A local section ξ is in F, if $\xi g(x) = s_\xi(x) g(x)$ for some function $s_\xi(x)$, where x is a local coordinate on Y and ξg is the Lie derivative of g with respect to ξ.

Lemma 8. The tangent space to $\tilde{\mathcal{L}}_N(X)^{\mathbb{C}} = \mathrm{Maps}_{0,\infty,D,N}(\mathbb{P}^1, Y)$ at $\phi(z)$ equals to the set of global sections of $\phi^* F$ on $\mathbb{P}^1$. That is,

$$T_\phi\big(\tilde{\mathcal{L}}_N(X)^{\mathbb{C}}\big) = H^0(\mathbb{P}^1, \phi^* F).$$

Proof. Recall that a holomorphic map ϕ is in $\mathrm{Maps}_{0,\infty,D,N}(\mathbb{P}^1, Y)$ if $\deg\phi = 2N$, $\phi^{-1}(D) = \{0, \infty\}$, and the degrees of tangency of $\phi(\mathbb{P}^1)$ to D at points $\phi(0)$ and $\phi(\infty)$ are N. In terms of the defining equation g of the divisor, the tangency condition for the map $\phi : \mathbb{P}^1 \to Y$ can be formulated as

$$g(\phi(z)) = z^N f(z),$$

where $f(z)$ is a holomorphic function and z is a local coordinate near $0 \in \mathbb{P}^1$. A local variation $\phi(z) \mapsto \phi(z) + \epsilon\xi(\phi(z))$ preserves tangency condition because

$$\frac{d}{d\epsilon} g(\phi(z) + \epsilon\xi(\phi(z)))\Big|_{\epsilon=0} = \xi g(\phi(z)) = s_\xi(\phi(z)) g(\phi(z)) = z^N s_\xi(\phi(z)) f(\phi(z)).$$

On the other hand, any infinitesimal deformation of a map $\phi : \mathbb{P}^1 \to Y$ that satisfies the tangency condition is a global section of the pullback $\phi^* F$, which completes the proof. $\square$

2.4.2 Virtual dimension

Using a perfect obstruction theory [5], we know that the actual (complex) dimension at a point $\phi(z)$ is always greater than or equal to

$$h^0(\mathbb{P}^1, \phi^* F) - h^1(\mathbb{P}^1, \phi^* F). \tag{2.16}$$

Note that (2.16) equals to the Euler characteristic $\chi(\mathbb{P}^1, \phi^* F)$ which is a topological invariant that does not depend on ϕ.

Definition 9. The quantity in (2.16) is defined to be the (complex) *virtual dimension* of the moduli space $\text{Maps}_{0,\infty,D,N}(\mathbb{P}^1, Y)$, which we denote by *vdim*.

Since the virtual dimension of the moduli space $\text{Maps}_{0,\infty,D,N}(\mathbb{P}^1, Y)$ equals to the Euler characteristic, it can be computed using Riemann-Roch under the assumption that the divisor D is smooth.

Theorem 10. *We have the following formula for the virtual dimension of $\tilde{\mathcal{L}}_N(X)^{\mathbb{C}}$*

$$\text{vdim}(\tilde{\mathcal{L}}_N(X)^{\mathbb{C}}) = \dim_{\mathbb{C}} Y + \langle c_1(\phi^* T_Y), n([\mathbb{P}^1])\rangle - 2N.$$

Proof. The sheaf F defined as before fits into a short exact sequence

$$0 \longrightarrow F \longrightarrow T_Y \longrightarrow N_{D/Y} \longrightarrow 0.$$

of sheaves on Y. Here $N_{D/Y}$ is the normal bundle of D in T_Y. The pullback of this sequence to $\mathbb{P}^1$ is

$$0 \longrightarrow \phi^* F \longrightarrow \phi^* T_Y \longrightarrow S_N(0) \oplus S_N(\infty) \longrightarrow 0,$$

where $S_N(0)$ and $S_N(\infty)$ are skyscraper sheaves supported at 0 and ∞. At these points local modules are isomorphic to $\mathbb{C}[z]/(z^{N-1})$.

By Riemann-Roch, we have

$$\chi(\mathbb{P}^1, \phi^* T_Y) = \dim_{\mathbb{C}} Y + \langle c_1(\phi^* T_Y), n([\mathbb{P}^1])\rangle.$$

Therefore,

$$\begin{aligned}
\text{vdim}(\tilde{\mathcal{L}}_N(X)^{\mathbb{C}}) &= \chi(\mathbb{P}^1, \phi^* F) \\
&= \chi(\mathbb{P}^1, \phi^* T_Y) - \chi(\mathbb{P}^1, S_N(0) \oplus S_N(\infty)) \\
&= \dim_{\mathbb{C}} Y + \langle c_1(\phi^* T_Y), n([\mathbb{P}^1])\rangle - 2N.
\end{aligned}$$

□

We can use Theorem 10 to calculate the virtual dimension of $\tilde{\mathcal{L}}_N(\mathbb{P}^k)^{\mathbb{C}}$.

Corollary 11. *The virtual dimension of $\tilde{\mathcal{L}}_N(\mathbb{P}^k)^{\mathbb{C}}$ is given by*

$$\text{vdim}(\tilde{\mathcal{L}}_N(\mathbb{P}^k)^{\mathbb{C}}) = 2k + 2Nk. \tag{2.17}$$

Proof. In the case $X = \mathbb{P}^k$, $Y = \mathbb{P}^k \times \mathbb{P}^k$,

$$c_1(T_Y) = c_1(\det\, T_Y) = \mathcal{O}_{\mathbb{P}^k}(k+1) \otimes \mathcal{O}_{\mathbb{P}^k}(k+1)$$

following the Euler sequence.

By τ-reality of ϕ, the class $\phi_*[\mathbb{P}^1]$ has bidegree $(N, N) \in H_2(\mathbb{P}^k \times \mathbb{P}^k, \mathbb{Z})$. Thus

$$\phi^*(\mathcal{O}_{\mathbb{P}^k}(1) \otimes \mathcal{O}_{\mathbb{P}^k}(1)) = \mathcal{O}_{\mathbb{P}^1}(2N),$$

and

$$\begin{aligned} c_1(\phi^* T_Y) &= \phi^*(\mathcal{O}_{\mathbb{P}^k}(k+1) \otimes \mathcal{O}_{\mathbb{P}^k}(k+1)) \\ &= \mathcal{O}_{\mathbb{P}^1}(2N(k+1)). \end{aligned}$$

Therefore

$$\begin{aligned} \mathrm{vdim}(\tilde{\mathcal{L}}_N(\mathbb{P}^k)^{\mathbb{C}}) &= \dim_{\mathbb{C}} Y + \langle c_1(\phi^* T_Y), n([\mathbb{P}^1])\rangle - 2N \\ &= 2k + 2N(k+1) - 2N \\ &= 2k + 2Nk \end{aligned}$$

□

2.4.3 The smooth locus

Recall that in Section 2.2 we showed that $\mathcal{L}_N(\mathbb{P}^k)^{\mathrm{smooth}}$, the smooth locus of $\mathcal{L}_N(\mathbb{P}^k)$, is a subset of $\mathcal{L}_N(\mathbb{P}^k) \backslash \mathcal{L}_{N-1}(\mathbb{P}^k) = \tilde{\mathcal{L}}_N(\mathbb{P}^k)$. We now show that they actually coincide.

Theorem 12. *The smooth locus of $\mathcal{L}_N(\mathbb{P}^k)$ is $\tilde{\mathcal{L}}_N(\mathbb{P}^k)$.*

The theorem is based on the following proposition.

Proposition 13. The first cohomology $H^1(\mathbb{P}^1, \phi^* F)$ vanishes.

Proof. The paring-preserving action of $\mathrm{SL}(k+1)$ on $V \times V^*$ has two orbits in $Y = \mathbb{P}(V) \times \mathbb{P}(V^*)$: D and $Y \setminus D$. The generators of $\mathfrak{sl}_{k+1}$ are global sections of F and F is generated by these sections. By abuse of notations we denote the sheaf $\mathcal{O}_Y \otimes \mathfrak{sl}_{k+1}$ also by $\mathfrak{sl}_{k+1}$. We have a short exact sequence

$$0 \longrightarrow \mathrm{Ker} \longrightarrow \mathfrak{sl}_{k+1} \xrightarrow{p} F \longrightarrow 0$$

where Ker is the kernel of the projection p. It defines a short exact sequence of the pullbacks

$$0 \longrightarrow \phi^*\mathrm{Ker} \longrightarrow \phi^*\mathfrak{sl}_{k+1} \longrightarrow \phi^* F \longrightarrow 0$$

Since $\phi^*\mathfrak{sl}_{k+1}$ is a trivial sheaf, $H^1(\mathbb{P}^1, \phi^*\mathfrak{sl}_{k+1}) = H^1(\mathbb{P}^1, \mathcal{O}) \otimes \mathfrak{sl}_{k+1} = 0$. From the long exact sequence in cohomology we immediately deduce that $H^1(\mathbb{P}^1, \phi^* F) = 0$. □

Proof of Theorem 12. For any point $\phi(z)$ in $\tilde{\mathcal{L}}_N(\mathbb{P}^k)^{\mathbb{C}}$, the following inequalities hold:

$$\dim T_\phi(\tilde{\mathcal{L}}_N(\mathbb{P}^k)^{\mathbb{C}}) \geq \text{actual dimension at } \phi(z) \geq \operatorname{vdim}(\tilde{\mathcal{L}}_N(\mathbb{P}^k)^{\mathbb{C}}). \tag{2.18}$$

By Lemma 8, we have the dimension of the tangent space

$$\dim T_\phi(\tilde{\mathcal{L}}_N(\mathbb{P}^k)^{\mathbb{C}}) = h^0(\mathbb{P}^1, \phi^* F),$$

while the virtual dimension, by Definition 9, is given by

$$\operatorname{vdim}(\tilde{\mathcal{L}}_N(\mathbb{P}^k)^{\mathbb{C}}) = h^0(\mathbb{P}^1, \phi^* F) - h^1(\mathbb{P}^1, \phi^* F).$$

Proposition 13 then ensures that at any point $\phi(z)$ in $\tilde{\mathcal{L}}_N(\mathbb{P}^k)^{\mathbb{C}}$, the dimension of the tangent space equals to the virtual dimension of the moduli space $\tilde{\mathcal{L}}_N(\mathbb{P}^k)^{\mathbb{C}} = \text{Maps}_{0,\infty,D,N}(\mathbb{P}^1, Y)$. As a result of inequalities (2.18), The actual dimension and the dimension of the tangent space must coincide. Therefore, we conclude that all points in $\tilde{\mathcal{L}}_N(\mathbb{P}^k)^{\mathbb{C}}$ are smooth points.

□

Chapter 3

Resolution of singularities of $\mathcal{L}_N(\mathbb{P}^k)$

In Section 2.4 we proved that the singular locus of $\mathcal{L}_N(\mathbb{P}^k)$ is $\mathcal{L}_{N-1}(\mathbb{P}^k) \subset \mathcal{L}_N(\mathbb{P}^k)$. In this section, we take a closer a look at $\mathcal{L}_N(\mathbb{P}^k)$ and construct a resolution of singularities of $\mathcal{L}_N(\mathbb{P}^k)$.

3.1 Loop group action

Let $\mathcal{L}(\mathrm{U}(k+1))$ be the group of smooth loops in the unitary group $\mathrm{U}(k+1)$. Recall that from Section 2.2, elements in $\mathcal{L}(\mathbb{P}^k)$ are V_{herm}^{k+1}-valued Fourier series satisfying the defining equation of $\mathbb{P}^k$ in V_{herm}^{k+1}, where V_{herm}^{k+1} is the space of $(k+1)$-dimensional hermitian matrices. The loop group $\mathcal{L}(\mathrm{U}(k+1))$ acts on $\mathcal{L}(\mathbb{P}^k)$ by conjugation. By an analogy with $\mathcal{L}_N(\mathbb{P}^k)$, we define the finite dimensional truncations $\mathcal{L}_N(\mathrm{U}(k+1)) \subset \mathcal{L}(\mathrm{U}(k+1))$ and maps

$$\mathcal{L}_M(\mathrm{U}(k+1)) \times \mathcal{L}_N(\mathbb{P}^k) \longrightarrow \mathcal{L}_{N+M}(\mathbb{P}^k), \tag{3.1}$$

which are restrictions of the conjugation action of $\mathcal{L}(\mathrm{U}(k+1))$ on $\mathcal{L}(\mathbb{P}^k)$.

We embed $\mathrm{U}(k+1)$ into $\mathrm{Mat}(k+1, \mathbb{C})$. A loop is polynomial if it is given by Laurent polynomial

$$M(z) = \sum_{i=-N}^{N} M_i z^i,$$

where $M_i \in \mathrm{Mat}(k+1, \mathbb{C})$, and $\det(M(z)) = z^k$, $k \in \mathbb{Z}$. Define

$$\mathcal{L}_N(\mathrm{U}(k+1)) = \mathrm{Trig}_N(\mathrm{Mat}(k+1, \mathbb{C})) \cap \mathcal{L}(\mathrm{U}(k+1)).$$

The space $\mathcal{L}_1(\mathrm{U}(k+1))$ contains a subspace of smooth homomorphisms $\mathrm{Hom}(S^1, \mathrm{U}(k+1))$, on which $\mathrm{U}(k+1)$ acts by conjugation. Denote by $G_i(k+1)$ the orbit with representing element $\mathrm{diag}(\underbrace{z, \ldots, z}_{i}, \underbrace{1, \ldots, 1}_{k+1-i})$, that is,

$$G_i(k+1) = \Big\{ P \ \mathrm{diag}(\underbrace{z, \ldots, z}_{i}, \underbrace{1, \ldots, 1}_{k+1-i}) \ P^{-1} \mid P \in \mathrm{U}(k+1) \Big\} \subset \mathrm{Hom}(S^1, \mathrm{U}(k+1)).$$

Let

$$\mathcal{L}_{\mathrm{poly}}(\mathrm{U}(k+1)) = \bigcup_{N=0}^{\infty} \mathcal{L}_N(\mathrm{U}(k+1)),$$

Segal and Pressley [13] proved that $\cup_{i=1}^{k+1} G_i(k+1)$ generates $\mathcal{L}_{\mathrm{poly}}(\mathrm{U}(k+1))$, and $\mathcal{L}_{\mathrm{poly}}(\mathrm{U}(k+1))$ is dense in $\mathcal{L}(\mathrm{U}(k+1))$.

Denote $G_1(k+1)$ by $G(k+1)$, that is,

$$G(k+1) = \left\{ P \begin{pmatrix} z & \\ & I \end{pmatrix} P^{-1} \,\middle|\, P \in \mathrm{U}(k+1) \right\} \subset \mathrm{Hom}(S^1, \mathrm{U}(k+1)).$$

Since $G(k+1) \subset \mathcal{L}_1(\mathrm{SU}(k+1))$, the action in (3.1) when $M = 1$ induces a map

$$\mu : G(k+1) \times \mathcal{L}_{N-1}(\mathbb{P}^k) \longrightarrow \mathcal{L}_N(\mathbb{P}^k). \tag{3.2}$$

The space $G(k+1)$ plays a fundamental role in our construction of a resolution of singularities of $\mathcal{L}_N(\mathbb{P}^k)$. By abuse of terminology, we call the map μ in (3.2) the *action* of $G(k+1)$ on $\mathcal{L}_{N-1}(\mathbb{P}^k)$.

3.2 Resolution of singularities

We have the following theorem on the action of $G(k+1)$ on $\mathcal{L}_{N-1}(\mathbb{P}^k)$.

Theorem 14.

(1) *The map* μ *in* (3.2) *is onto.*

(2) μ *is one to one away from the singular locus of* $\mathcal{L}_{N-1}(\mathbb{P}^k)$.

As a corollary, we have the following resolution of singularities of $\mathcal{L}_N(\mathbb{P}^k)$.

Corollary 15. *The iterated product map*

$$\mu^N : \mathbb{P}^k \times \underbrace{G \times \cdots \times G}_{N} \longrightarrow \mathcal{L}_N(\mathbb{P}^k)$$

induced from μ gives a resolution of singularities of $\mathcal{L}_N(\mathbb{P}^k)$.

Proof. Notice that $\mathcal{L}_0(\mathbb{P}^k) = \mathbb{P}^k$. Theorem 14 proves the case for $N = 1$. The rest can be proved by induction on N. $\square$

Notice that $G(k+1)$ is diffeomorphic to $\mathbb{P}^k$, $\dim G(k+1) = 2k$. Theorem 14 and Corollary 15 is consistent with our dimension calculation in Section 2.4.

We break the proof of Theorem 14 into two parts.

Proposition 16 (Part I)**.** The map (3.2) is onto.

Proposition 17 (Part II)**.** The map (3.2) is one to one at generic points. To be more precise, let $n(z) \in \mathcal{L}_N(\mathbb{P}^k) \setminus \mathcal{L}_{N-1}(\mathbb{P}^k)$ such that $n(z) = \lambda(z).n_1(z) = \theta(z).n_2(z)$, where $\lambda(z), \theta(z) \in G(k+1)$, $n_1(z), n_2(z) \in \mathcal{L}_{N-1}(\mathbb{P}^k) \setminus \mathcal{L}_{N-2}(\mathbb{P}^k)$. Then $\lambda(z) = \theta(z)$, $n_1(z) = n_2(z)$.

Proof of Part I (Proposition 16). Let $n \in \mathcal{L}_N(\mathbb{P}^k) \setminus \mathcal{L}_{N-1}\mathbb{P}^k$, $N \geq 1$. In order to show that there exist $\lambda \in G(k+1)$ and $n' \in \mathcal{L}_{N-1}(\mathbb{P}^k)$ such that

$$n = \lambda.n',$$

it's sufficient to show that

$$\lambda^{-1}.n \in \mathcal{L}_{N-1}(\mathbb{P}^k),$$

where λ^{-1} is in form

$$\lambda^{-1} = \Omega \begin{pmatrix} \bar{z} & \\ & I \end{pmatrix} \Omega^{-1},\ \Omega \in \mathrm{SU}(k+1).$$

Then the general statement will follow because $\mathcal{L}_N(\mathbb{P}^k)$ is filtered by $\mathcal{L}_l(\mathbb{P}^k) \setminus \mathcal{L}_{l-1}(\mathbb{P}^k)$, $l = 1, \ldots, N$.

Let $n(z) = K_{-N}z^{-N} + K_{-N+1}z^{-N+1} + \cdots + K_{N-1}z^{N-1} + K_N z^N$ be an element in $\mathcal{L}_N(\mathbb{P}^k) \setminus \mathcal{L}_{N-1}\mathbb{P}^k$. Since $n(z)$ has eigenvalues R and $-R$ with multiplicities k and 1 respectively, matrix $n(z) - RI$ has rank 1. All the 2×2 minors of $n(z) - RI$ are polynomials in z, with coefficients of the highest degree terms coming from corresponding minors of K_N. Therefore all the 2×2 minors of K_N vanish and $\mathrm{rank}(K_N)=1$. Since we further have $K_N^2=0$ by relation (2.9), K_N has Jordan canonical form

$$\begin{pmatrix} 0 & 1 & & & \\ & 0 & & & \\ & & 0 & & \\ & & & \ddots & \\ & & & & 0 \end{pmatrix}.$$

As a result, there exists a matrix $\Omega \in \mathrm{SU}(k+1)$ such that

$$\Omega K_N \Omega^{-1} = \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix}, \tag{3.3}$$

where b is a non-zero $(k-1)$-dimensional row vector. Let $\Omega K_{N-1}\Omega^{-1}$ be in block form

$$\Omega K_{N-1} \Omega^{-1} = \begin{pmatrix} e & f \\ g & h \end{pmatrix}, \tag{3.4}$$

with block sizes the same as the matrix in (3.3). Then

$$\begin{aligned} & K_{N-1}K_N + K_N K_{N-1} \\ = \;& \Omega^{-1}\left[\begin{pmatrix} e & f \\ g & h \end{pmatrix}\begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix}\begin{pmatrix} e & f \\ g & h \end{pmatrix}\right]\Omega \\ = \;& \Omega^{-1}\begin{pmatrix} bg & bh+eb \\ 0 & gb \end{pmatrix}\Omega. \end{aligned} \tag{3.5}$$

K_N and K_{N-1} satisfy $K_{N-1}K_N + K_N K_{N-1} = 0$ by relation (2.9), we must therefore have

$$gb = 0.$$

Since g is a column vector and b a nonzero row vector, $gb=0$ implies $g=0$. Hence $\Omega K_{N-1}\Omega^{-1}$ is in form

$$\Omega K_{N-1} \Omega^{-1} = \begin{pmatrix} e & f \\ 0 & h \end{pmatrix}. \tag{3.6}$$

Let

$$\lambda = \Omega^{-1}\begin{pmatrix} z & \\ & I \end{pmatrix}\Omega \in G(k+1).$$

We compute the coefficients of $z^{N+1}, z^N, z^{-N}, z^{-(N+1)}$ in

$$\begin{aligned} & \lambda^{-1}.n(z) \\ = \ & \Big(\Omega^{-1}\begin{pmatrix} \bar{z} & \\ & I \end{pmatrix}\Omega\Big)\Big(\sum_{j=-N}^{N} K_j z^j\Big)\Big(\Omega^{-1}\begin{pmatrix} z & \\ & I \end{pmatrix}\Omega\Big). \end{aligned} \tag{3.7}$$

The coefficient of z^{N+1} is

$$\begin{aligned} & \Omega^{-1}\begin{pmatrix} 0 & \\ & I \end{pmatrix}\Omega K_N \Omega^{-1}\begin{pmatrix} 1 & \\ & 0 \end{pmatrix}\Omega \\ = \ & \Omega^{-1}\begin{pmatrix} 0 & \\ & I \end{pmatrix}\begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & \\ & 0 \end{pmatrix}\Omega \\ = \ & 0 \end{aligned} \tag{3.8}$$

by (3.3). The coefficient of z^N is

$$\Omega^{-1}\Big(P_1 + P_2 + P_3\Big)\Omega, \tag{3.9}$$

where

$$\begin{aligned} P_1 &= \begin{pmatrix} 1 & \\ & 0 \end{pmatrix}\Omega K_N \Omega^{-1}\begin{pmatrix} 1 & \\ & 0 \end{pmatrix}, \\ P_2 &= \begin{pmatrix} 0 & \\ & I \end{pmatrix}\Omega K_N \Omega^{-1}\begin{pmatrix} 0 & \\ & I \end{pmatrix}, \\ P_3 &= \begin{pmatrix} 0 & \\ & I \end{pmatrix}\Omega K_{N-1} \Omega^{-1}\begin{pmatrix} 1 & \\ & 0 \end{pmatrix}. \end{aligned} \tag{3.10}$$

$P_1 = P_2 = 0$ by (3.3) and $P_3 = 0$ by (3.6). Therefore the coefficient of z^N also vanishes.

Since the coefficients of $z^{-(N+1)}$ and z^{-N} are transpose conjugate to those of z^{N+1} and z^N respectively, all coefficients in front of $z^{N+1}, z^N, z^{-N}, z^{-(N+1)}$ vanishes. Hence $\lambda^{-1}.n(z) \in \mathcal{L}_{N-1}(\mathbb{P}^k)$. □

Now we prove the second part of the theorem.

Proof of Part II (Proposition 17). Let $\lambda(z), \theta(z) \in G(k+1)$, $n_1(z), n_2(z) \in \mathcal{L}_{N-1}(\mathbb{P}^k) \setminus \mathcal{L}_{N-2}(\mathbb{P}^k)$, such that

$$\lambda(z).n_1(z) = \theta(z).n_2(z). \tag{3.11}$$

By proposition 16, there exists $\lambda_i(z), \theta_i(z) \in G(K+1)$, $i = 2, \ldots, N$ and $n_1, n_2 \in \mathcal{L}_0(\mathbb{P}^k) \simeq \mathcal{L}(\mathbb{P}^k)$ such that

$$n_1(z) = \Big(\lambda_2(z) \cdots \lambda_N(z)\Big) n_1 \Big(\lambda_2(z) \cdots \lambda_N(z)\Big)^{-1},$$
$$n_2(z) = \Big(\theta_2(z) \cdots \theta_N(z)\Big) n_2 \Big(\theta_2(z) \cdots \theta_N(z)\Big)^{-1}.$$

Let $\lambda_1(z) = \lambda(z)$, $\theta_1(z) = \theta(z)$, then

$$\lambda(z).n_1(z) = \Big(\prod_{i=1}^{N} \lambda_i(z)\Big) n_1 \Big(\prod_{i=1}^{N} \lambda_i(z)\Big)^{-1},$$
$$\theta(z).n_2(z) = \Big(\prod_{i=1}^{N} \theta_i(z)\Big) n_2 \Big(\prod_{i=1}^{N} \theta_i(z)\Big)^{-1}.$$

Denote by $\Lambda(z) = \prod_{i=1}^{N} \lambda_i(z)$, $\Theta(z) = \prod_{i=1}^{N} \theta_i(z)$, the above equations can be written as

$$\begin{aligned} \lambda(z).n_1(z) &= \Lambda(z) n_1 \Lambda(z)^{-1}, \\ \theta(z).n_2(z) &= \Theta(z) n_1 \Theta(z)^{-1}. \end{aligned} \tag{3.12}$$

Recall from the beginning of the section that $G(k+1)$ consists of elements conjugate to $\mathrm{diag}(z, 1, \ldots, 1)$, we have

$$\begin{aligned} \lambda_i(z) &= P_i \begin{pmatrix} z & \\ & I \end{pmatrix} P_i^{-1}, \\ \theta_i(z) &= Q_i \begin{pmatrix} z & \\ & I \end{pmatrix} Q_i^{-1}, \end{aligned} \tag{3.13}$$

for some unitary matrices P_i, Q_i, $i = 1, \ldots, N$. Let $z = 1$ we have

$$\lambda_i(1) = \theta_i(1) = I, \ i = 1, \ldots, N.$$

Therefore

$$\Lambda(1) = \Theta(1) = I.$$

Since (3.11) holds for any $z \in S^1$, setting $z = 1$ we have

$$n_1 = n_2$$

by equations (3.12).

Plug $n_1 = n_2$ into (3.12), equation (3.11) becomes

$$\Lambda(z)n_1\Lambda(z)^{-1} = \Theta(z)n_1\Theta(z)^{-1}.$$

Rearrange the two sides, the equation becomes

$$\Theta(z)^{-1}\Lambda(z)n_1 = n_1\Theta(z)^{-1}\Lambda(z), \tag{3.14}$$

that is, $\Theta(z)^{-1}\Lambda(z)$ commutes with n_1.

Since $n_1 \in \mathcal{L}_0(\mathbb{P}^k) \simeq \mathbb{P}^k$, n_1 has the form

$$\frac{1}{R} \cdot n_1 = \Omega \begin{pmatrix} -2 & \\ & 0 \end{pmatrix} \Omega^{-1} + I, \tag{3.15}$$

for some unitary matrix Ω. Plug (3.15) into (3.14), we have

$$\Omega^{-1}\Theta(z)^{-1}\Lambda(z)\Omega \begin{pmatrix} -2 & \\ & 0 \end{pmatrix} = \begin{pmatrix} -2 & \\ & 0 \end{pmatrix} \Omega^{-1}\Theta(z)^{-1}\Lambda(z)\Omega,$$

that is, $\Omega^{-1}\Theta(z)^{-1}\Lambda(z)\Omega$ commutes with $\begin{pmatrix} -2 & \\ & 0 \end{pmatrix}$.

Therefore $\Omega^{-1}\Theta(z)^{-1}\Lambda(z)\Omega$ must be block-wise diagonal

$$\Omega^{-1}\Theta(z)^{-1}\Lambda(z)\Omega = \begin{pmatrix} \alpha(z) & \\ & * \end{pmatrix}, \tag{3.16}$$

where $\alpha(z)$ is a Laurant polynomial in z with degree less than or equal to N.

Let $e_1 = (1, 0 \ldots, 0)^{\mathrm{T}}$. Then (3.16) is equivalent to

$$\Omega^{-1}\Theta(z)^{-1}\Lambda(z)\Omega e_1 = \alpha(z)e_1.$$

Hence

$$\Lambda(z)\Omega e_1 = \alpha(z)\Theta(z)\Omega e_1.$$

Let $\omega_1 = \Omega e_1$ be the first column of Ω, we have

$$\Lambda(z)\omega_1 = \alpha(z)\Theta(z)\omega_1.$$

Notice that all matrices involved are unitary. From equations (3.12) and (3.15) we have

$$\lambda(z).n_1(z) = -2R\big(\Lambda(z)\omega_1\big)\big(\Lambda(z)\omega_1\big)^* + RI. \tag{3.17}$$

By the assumption that $\lambda(z).n_1(z)$ is a generic point in $\mathcal{L}_N(\mathbb{P}^k)$, $\lambda(z).n_1(z)$ is a degree N Laurent polynomial. While $\Lambda(z)\omega_1$ is an array of polynomials in z with degree no larger than N, equation (3.17) ensures polynomials in $\Lambda(z)\omega_1$ are of degree N. The same reasoning applies to $\Theta(z)\omega_1$. Therefore $\alpha(z)$ must be of degree 0, and we have

$$\Lambda(z)\omega_1 = \alpha\Theta(z)\omega_1, \tag{3.18}$$

where α is a constant number.

We compare the highest order terms from the two sides of the equation above.

By (3.13), each $\lambda_i(z)$ equals to

$$\lambda_i(z) = z\mathrm{Proj}_{p_i} + \mathrm{Orth}_{p_i}, i = 1, \dots, N, \tag{3.19}$$

as an operator, where p_i is the first column of P_i, P_i as in (3.13), Proj_{p_i} the projection to p_i, $\mathrm{Orth}_{pi} = \mathrm{Id} - \mathrm{Proj}_{p_i}$.

Since $\Lambda(z) = \prod_{i=1}^{n} \lambda_i(z)$, and $\Lambda(z)\omega_1$ is of degree N, the highest power term in $\Lambda(z)\omega_1$ is

$$z^N \mathrm{Proj}_{p_1} \circ \mathrm{Proj}_{p_2} \circ \cdots \circ \mathrm{Proj}_{p_N} \omega_1.$$

Similarly, the highest power term in $\Theta(z)\omega_1$ is

$$z^N \mathrm{Proj}_{q_1} \circ \mathrm{Proj}_{q_2} \circ \cdots \circ \mathrm{Proj}_{q_N} \omega_1,$$

where q_i is the first column of Q_i, Q_i as in (3.13), $i = 1, \dots, N$.

By equation (3.18), we must have

$$\mathrm{Proj}_{p_1} \circ \mathrm{Proj}_{p_2} \circ \cdots \circ \mathrm{Proj}_{p_N} \omega_1 = \alpha \mathrm{Proj}_{q_1} \circ \mathrm{Proj}_{q_2} \circ \cdots \circ \mathrm{Proj}_{q_N} \omega_1$$

Since the left hand side is a nonzero projection to p_1, right hand side a nonzero projection to q_1, p_1 and q_1 must be collinear. Therefore,

$$\mathrm{Proj}_{p_1} = \mathrm{Proj}_{q_1}.$$

Furthermore, by (3.19), Proj_{p_i} and Proj_{q_i} completely determines $\lambda_i(z)$ and $\theta_i(z)$ respectively. Hence

$$\lambda_1(z) = \theta_1(z),$$

that is

$$\lambda(z) = \theta(z).$$

Finally, since

$$n_1(z) = \lambda(z)^{-1}.n(z),$$
$$n_2(z) = \theta(z)^{-1}.n(z),$$

it follows immediately that

$$n_1(z) = n_2(z).$$

$\square$

Chapter 4

Geometry of $\mathcal{L}(\mathbb{P}^k)$

In this chapter we study the geometry of the truncated loop space $\mathcal{L}_N(\mathbb{P}^k)$. We specify the metric on $\mathcal{L}_N(\mathbb{P}^k)$ in Section 4.1, and use it to give a concrete formula for the potential energy V in the Schrödinger operator H. In Section 4.2 we compute the Riemann curvature tensor of $\mathcal{L}(\mathbb{P}^k)$ using the loop group action by $\mathcal{L}(\mathrm{U}(k+1))$. In Section 4.3 and 4.4, we find estimates for the diameter and the volume of $\mathcal{L}_N(\mathbb{P}^k)$ respectively using the fibration structure of $\mathcal{L}_N(\mathbb{P}^k)$ over $\mathcal{L}_{N-1}(\mathbb{P}^k)$ established in Chapter 3.

4.1 Metric

The space $\mathcal{L}_N(\mathbb{P}^k)$ is equipped with the ordinary L^2 metric

$$\langle \delta n(\theta), \delta n(\theta) \rangle = \frac{L}{2\pi} \int_0^{2\pi} \mathrm{tr}\big(\delta n(\theta) \delta n(\theta)^*\big) d\theta, \tag{4.1}$$

where $\delta n(\theta) \in T_n \mathcal{L}_N(\mathbb{P}^k) \subset T_n \mathcal{L}_N(V^{k+1}_{\mathrm{herm}})$, L is the length of the circle that appeared in Conjecture 1. In the following we set L equal to one. The metric is a pullback of the metric from $\mathcal{L}(\mathbb{P}^k)$.

In the coordinate of Fourier coefficients, the metric is given by

$$\langle \delta n(\theta), \delta n(\theta) \rangle = \mathrm{tr}(\mathrm{d}V \mathrm{d}V^*) + \frac{1}{2} \sum_{n=1}^{N} \Big(\mathrm{tr}(\mathrm{d}A_n \mathrm{d}A_n^*) + \mathrm{tr}(\mathrm{d}B_n \mathrm{d}B_n^*) \Big),$$

where $\delta n(\theta) = \mathrm{d}V + \sum_{n=1}^{N} (\mathrm{d}A_n \cos(n\theta) + \mathrm{d}B_n \sin(n\theta))$.

To define the Schrödinger operator H, we use a densely defined quadratic form

$$Q(f,g) = \int_{\tilde{\mathcal{L}}_N(\mathbb{P}^k)} (\nabla f \cdot \nabla \bar{g} + U f \bar{g}) \mathrm{d}vol \tag{4.2}$$

on $L^2(\tilde{\mathcal{L}}_N(\mathbb{P}^k)) \cap C_c^\infty(\tilde{\mathcal{L}}_N(\mathbb{P}^k))$. U is a potential function defined on $\mathcal{L}_N(\mathbb{P}^k)$ given by the magnitude square of the vector field corresponding to of infinitesimal rotation of a loop $n(\theta) \to n(\theta+\epsilon) = n(\theta) + \epsilon n'(\theta)$,

$$U(n(\theta)) = \frac{1}{2} \sum_{n=1}^{N} n^2 \Big(\mathrm{tr}(A_n A_n^*) + \mathrm{tr}(B_n B_n^*) \Big). \tag{4.3}$$

The Schrödinger operator H is defined by

$$Q(f,g) = (Hf, g), \tag{4.4}$$

where (f,g) is the inner product on $L^2(\tilde{\mathcal{L}}_N(\mathbb{P}^k))$,

$$(f,g) = \int_{\tilde{\mathcal{L}}_N(\mathbb{P}^k)} f \bar{g} \mathrm{d}vol.$$

By doing formal integration by part, we see that the Schrödinger is the sum $-\Delta + U$, where Δ is the Laplacian operator and V is the potential in (4.3)

4.2 Curvature tensor of $\mathcal{L}(\mathbb{P}^k)$

We compute the Riemann curvature tensors of $\mathcal{L}(\mathbb{P}^k)$ by using the fact that the loop space $\mathcal{L}(\mathbb{P}^k)$ is a homogeneous space of the loop group $\mathcal{L}(\mathrm{U}(k+1))$. The general recipe for computing the curvature tensors of a homogeneous space is as follows.

Let G be a Lie group and (M,g) a G-homogeneous Riemannian manifold. Fix a point x in M, and let H be the stabilizer of x. Then H is a closed connected Lie subgroup of G and $M \simeq G/H$. Let $\mathfrak{g}$ and $\mathfrak{h}$ be the Lie algebra of G and H, and assume that $\mathfrak{g}$ splits

$$\mathfrak{g} = \mathfrak{h} + \mathfrak{p} \tag{4.5}$$

into a direct sum of adjoint representations of $\mathfrak{h}$. We identify the tangent space T_xM with $\mathfrak{g}/\mathfrak{h} = \mathfrak{p}$. The G-invariant metric on G/H defines an inner product on $\mathfrak{p}$ and is completely characterized by it. All curvature tensors of G/H are G-invariant. Besse in [2] gives a formula for curvature tensors $R(X,Y)$ written purely in terms of the inner product $(.,.)$ on $\mathfrak{p}$, the bracket in $\mathfrak{g}$, and decomposition (4.5). We denote by $[a,b]_{\mathfrak{h}}$ the projection of $[a,b]$ onto $\mathfrak{h}$ and by $[a,b]_{\mathfrak{p}}$ the corresponding projection to $\mathfrak{p}$. According to [2], we have

$$\begin{aligned}&(R(X,Y)X,Y) = -\frac{3}{4}|[X,Y]_{\mathfrak{p}}|^2 - \frac{1}{2}([X,[X,Y]]_{\mathfrak{p}},Y) - \frac{1}{2}([Y,[Y,X]]_{\mathfrak{p}},X)\\ &+|U(X,Y)|^2 - (U(X,X),U(Y,Y)).\end{aligned} \tag{4.6}$$

where $U : \mathfrak{p}\otimes\mathfrak{p} \to \mathfrak{p}$ is defined by

$$2(U(X,Y),Z) = ([Z,X]_{\mathfrak{p}},Y) + (X,[Z,Y]_{\mathfrak{p}}).$$

When $G = \mathcal{L}\big(\mathrm{U}(k+1)\big)$, $H = \mathcal{L}\big(\mathrm{U}(1)\times\mathrm{U}(k)\big)$, we have $\mathfrak{p} = \mathcal{L}\big(\mathfrak{u}(k+1)/\mathfrak{u}(1)\oplus\mathfrak{u}(k)\big)$. A basis of $\mathfrak{u}(k+1)/\mathfrak{u}(1)\oplus\mathfrak{u}(k)$ is given by

$$\xi_j = E_{1j} - E_{j1}, \eta_j = \mathfrak{i}(E_{1j} + E_{j1}),\ j = 2,\dots,k+1,$$

where $\mathfrak{i}$ is the imaginary unit, and this gives a basis of $\mathfrak{p}$:

$$\Big\{\xi_j, \eta_j, \cos(n\theta)\xi_j, \cos(n\theta)\eta_j, \sin(n\theta)\xi_j, \sin(n\theta)\eta_j\Big\}.$$

The linear space $\mathfrak{p}$ is equipped with the inner product:

$$\int_{S^1} \mathrm{tr}(f(\theta)\bar{g}(\theta))d\theta.$$

Let $e_{ij} = (E_{ij} - E_{ji})$, $\epsilon_{ij} = \mathfrak{i}(E_{ij} + E_{ji})$, then e_{ij}, ϵ_{ij} satisfies equations

$$\begin{aligned}[][e_{ij},e_{kl}] &= \delta_{jk}e_{il} - \delta_{jl}e_{ik} - \delta_{ik}e_{jl} + \delta_{il}e_{jk},\\ [\epsilon_{ij},\epsilon_{kl}] &= -(\delta_{jk}e_{il} + \delta_{jl}e_{ik} + \delta_{ik}e_{jl} + \delta_{il}e_{jk}),\\ [e_{ij},\epsilon_{kl}] &= \delta_{jk}\epsilon_{il} + \delta_{jl}\epsilon_{ik} - \delta_{ik}\epsilon_{jl} - \delta_{il}\epsilon_{jk}.\end{aligned}$$

Note that

$$[\xi_i,\xi_j] = -e_{ij}, [\eta_i,\eta_j] = -e_{ij}, [\xi_i,\eta_j] = -\epsilon_{ij},$$

therefore

$$[\xi_i,\xi_j]_{\mathfrak{p}} = [\eta_i,\eta_j]_{\mathfrak{p}} = [\xi_i,\eta_j]_{\mathfrak{p}} = 0,$$

and equation (4.6) becomes

$$(R(X,Y)X,Y) = -\frac{1}{2}([X,[X,Y]]_{\mathfrak{p}},Y) - \frac{1}{2}([Y,[Y,X]]_{\mathfrak{p}},X).$$

We also have relations

$$
\begin{aligned}
[\xi_i, [\xi_i, \xi_j]]_{\mathfrak{p}} &= -\xi_j, \\
[\eta_i, [\eta_i, \xi_j]]_{\mathfrak{p}} &= \xi_j, \\
[\xi_i, [\xi_i, \eta_j]]_{\mathfrak{p}} &= -\eta_j, \\
[\eta_i, [\eta_i, \eta_j]]_{\mathfrak{p}} &= \eta_j,
\end{aligned}
$$

and integrals

$$
\int_0^{2\pi} \sin^2(nx)\cos^2(mx) = \begin{cases} \dfrac{\pi}{4}, & n = m, \\ \dfrac{\pi}{2}, & n \neq m, \end{cases}
$$

$$
\int_0^{2\pi} \sin^2(nx)\sin^2(mx) = \int_0^{2\pi} \cos^2(nx)\cos^2(mx) = \begin{cases} \dfrac{3\pi}{4}, & n = m, \\ \dfrac{\pi}{2}, & n \neq m. \end{cases}
$$

Using these identities the curvature tensors on $\mathcal{L}(\mathbb{P}^k)$ can be calculated explicitly. We list some of the tensors for the basis vectors here:

- For $(X, Y) = (\cos(n\theta)\xi_i, \cos(m\theta)\xi_j)$ or $(X, Y) = (\sin(n\theta)\xi_i, \sin(m\theta)\xi_j)$, we have

  $$
  (R(X, Y)X, Y) = \begin{cases} \dfrac{3\pi}{2}, & n = m, \\ \pi, & n \neq m, \end{cases}
  $$

  Similar computations applies to $(X, Y) = (\xi_i, \xi_j)$, $(\cos(n\theta)\xi_i, \sin(m\theta)\xi_j)$, $(\xi_i, \cos(n\theta)\xi_j)$ and $(\xi_i, \sin(n\theta)\xi_j)$.

- For $(X, Y) = (\cos(n\theta)\eta_i, \cos(m\theta)\eta_j)$ or $(X, Y) = (\sin(n\theta)\eta_i, \sin(m\theta)\eta_j)$, we have

  $$
  (R(X, Y)X, Y) = \begin{cases} -\dfrac{3\pi}{2}, & n = m, \\ -\pi, & n \neq m, \end{cases}
  $$

  Similar computations applies to $(X, Y) = (\eta_i, \eta_j)$, $(\cos(n\theta)\eta_i, \sin(m\theta)\eta_j)$, $(\eta_i, \cos(n\theta)\eta_j)$ and $(\eta_i, \sin(n\theta)\eta_j)$.

- When in (X, Y) one contains ξ and the other contains η the curvature tensor vanishes.

4.3 Diameter estimates

The key tool in our diameter estimate is the map (3.2). We compare the product metric on $G(k+1) \times \mathcal{L}_{N-1}(\mathbb{P}^k)$ with the pull back metric from $\mathcal{L}_N(\mathbb{P}^k)$ by μ in (3.2), and show that the latter one is dominated by the former one. To be precise, let $g_{\mathcal{L}_N}$ and $g_{\mathcal{L}_{N-1}}$ be the metric on $\mathcal{L}_N(\mathbb{P}^k)$ and $\mathcal{L}_{N-1}(\mathbb{P}^k)$ respectively, g_G the metric on $G(k+1)$, we show that the pull back metric $\mu^* g_{\mathcal{L}_N}$ on $G(k+1) \times \mathcal{L}_{N-1}(\mathbb{P}^k)$ is dominated by the product metric $g_G + g_{\mathcal{L}_{N-1}}$.

To compute $\mu^* g_{\mathcal{L}_N}$, we fix a point (λ, n) in $G(k+1) \times \mathcal{L}_{N-1}(\mathbb{P}^k)$, and let μ_* denote the induced map of μ on the tangent space

$$\mu_* : T_\lambda G(k+1) \oplus T_n \mathcal{L}_{N-1}(\mathbb{P}^k) \longrightarrow T_{\mu(\lambda,n)} \mathcal{L}_N(\mathbb{P}^k).$$

Then

$$\mu_*(\delta\lambda, \delta n) = (\delta\lambda) n \lambda^{-1} - \lambda n (\delta\lambda) + \lambda(\delta n)\lambda^{-1}, \tag{4.7}$$

where $\delta\lambda \in T_\lambda G(k+1)$ and $\delta n \in T_n \mathcal{L}_{N-1}(\mathbb{P}^k)$.

By (4.7) and triangle inequality, we have

$$\begin{aligned} |\mu_*(\delta\lambda, \delta n) = \ & |(\delta\lambda, \delta n)|_{\mu^* g_{\mathcal{L}_N}} \\ = \ & |(\delta\lambda) n \lambda^{-1} - \lambda n(\delta\lambda) + \lambda(\delta n)\lambda^{-1}| \\ \leq \ & |(\delta\lambda) n \lambda^{-1}| + |\lambda n (\delta\lambda)| + |\lambda(\delta n)\lambda^{-1}| \end{aligned}$$

where we used $|\cdot|$ to denote the norms $|\cdot|_{g_{\mathcal{L}_N}}$, $|\cdot|_{g_{\mathcal{L}_{N-1}}}$ and $|\cdot|_{g_G}$ in all three spaces without distinction, since $g_{\mathcal{L}_N}$, $g_{\mathcal{L}_{N-1}}$ and g_G are all defined by (4.1).

Notice that λ is unitary and $nn^* = R^2 I$, so we have $|(\delta\lambda) n \lambda^{-1}| = |\lambda n(\delta\lambda)| = R|\delta\lambda|$ and $|\lambda(\delta n)\lambda^{-1}| = |\delta n|$, hence

$$|(\delta\lambda, \delta n)|_{\mu^* g_{\mathcal{L}_N}} \leq 2R|\delta\lambda| + |\delta n|. \tag{4.8}$$

For any two points $(\lambda_i, n_i) \in G(k+1) \times \mathcal{L}_{N-1}(\mathbb{P}^k)$, $i = 1, 2$,

$$\begin{aligned} & d_{\mathcal{L}_N}(\mu(\lambda_1, n_1), \mu(\lambda_2, n_2)) \\ \leq \ & d_{\mathcal{L}_N}(\mu(\lambda_1, n_1), \mu(\lambda_2, n_1)) + d_{\mathcal{L}_N}(\mu(\lambda_2, n_1), \mu(\lambda_2, n_2)) \\ \leq \ & 2R d_G(\lambda_1, \lambda_2) + d_{\mathcal{L}_{N-1}}(n_1, n_2) \end{aligned}$$

From this we get an inequality of diameters

$$d(\mathcal{L}_N(\mathbb{P}^k)) \leq 2R d(G(k+1)) + d(\mathcal{L}_{N-1}\mathbb{P}^k)$$

which gives us our final estimate for diameters:

$$d(\mathcal{L}_N(\mathbb{P}^k)) \leq 2NR d(G(k+1)) + d(\mathbb{P}^k)$$

4.4 Volume estimates

Inequality (4.8) also gives an estimate for the volume of $\mathcal{L}(\mathbb{P}^k)$. Let $\{e_i\}$, $i = 1, \dots, 2k$, be an orthonormal basis of $T_\lambda G(k+1)$, $j = 1, \dots, 2Nk$, with respect to g_G, $\{\xi_j\}$ an orthonormal basis of $T_n\mathcal{L}_{N-1}(\mathbb{P}^k)$ with respect to $g_{\mathcal{L}_{N-1}}$, then by (4.8), we have

$$\begin{aligned} |e_i|_{\mu^* g_{\mathcal{L}_N}} &\leq 2R,\ i = 1, \dots, 2k, \\ |\xi_j|_{\mu^* g_{\mathcal{L}_N}} &\leq 1,\ j = 1, \dots, 2Nk, \end{aligned}$$

Therefore

$$\mu^* dvol_{g_{\mathcal{L}_N}} \leq (2R)^{2k} dvol_{g_G + g_{\mathcal{L}_{N-1}}}$$

and

$$\int_{\mathcal{L}_N(\mathbb{P}^k)} dvol_{g_{\mathcal{L}_N}} \leq (2R)^{2k} \int_{G(k+1)} dvol_{g_G} \int_{\mathcal{L}_{N-1}(\mathbb{P}^k)} g_{\mathcal{L}_{N-1}}.$$

That is,

$$\mathrm{Vol}(\mathcal{L}_N(\mathbb{P}^k)) \leq (2R)^{2k} \mathrm{Vol}(G(k+1)) \mathrm{Vol}(\mathcal{L}_{N-1}(\mathbb{P}^k)).$$

As a result, we have our estimate for volumes:

$$\mathrm{Vol}(\mathcal{L}_N(\mathbb{P}^k)) \leq (2R)^{2Nk} \mathrm{Vol}(G(k+1))^N \mathrm{Vol}(\mathbb{P}^k).$$

Chapter 5

The variety $\mathcal{L}_1(\mathbb{P}^k)$

In this section, we investigate the space $\mathcal{L}_1(\mathbb{P}^k)$, the simplest nontrivial truncated loop space in our construction. We give an explicit construction of elements in $\mathcal{L}_1(\mathbb{P}^k)$ using Pauli matrices in Section 5.1, and show that $\tilde{\mathcal{L}}_1(\mathbb{P}^k)$, the smooth locus of $\mathcal{L}_1(\mathbb{P}^k)$, is diffeomorphic to $(-1,1) \times \mathrm{SU}(k+1)/\mathrm{U}(k-1)$. In Section 5.2 we compute the pull back metric on $(-1,1) \times \mathrm{SU}(k+1)/\mathrm{U}(k-1)$ from $\tilde{\mathcal{L}}_1(\mathbb{P}^k)$ using the diffeomorphism constructed in Section 5.1 and compute the volume form with respect to the metric. In Section 5.3 we compute the Schrödinger operator H with respect to the $\mathrm{SU}(k+1)$-symmetry of $\mathcal{L}_1(\mathbb{P}^k)$, and prove the radial part of the discreteness of its spectrum.

5.1 Explicit construction of $\mathcal{L}_1(\mathbb{P}^k)$

Recall our construction in Section 2.2, the space $\mathcal{L}_1(\mathbb{P}^k)$ consists of triples of $(k+1) \times (k+1)$ hermitian matrices (A, B, V) satisfying

$$\begin{aligned} &A^2 = B^2,\ V^2 + A^2 = R^2 I, \\ &AB + BA = 0,\ VA + AV = 0,\ VB + BV = 0, \\ &\mathrm{tr}(A) = \mathrm{tr}(B) = 0,\ \mathrm{tr}(V) = (k-1)R. \end{aligned} \tag{5.1}$$

The building blocks of hermitian matrices satisfying (5.1) are Pauli matrices,

$$\sigma_1 = \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}, \sigma_2 = \begin{pmatrix} & -\mathrm{i} \\ \mathrm{i} & \end{pmatrix}, \sigma_3 = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix} \tag{5.2}$$

that satisfy

$$\sigma_i \sigma_j = -\sigma_j \sigma_i,\ i \neq j,$$

$$\sigma_1^2 = \sigma_2^2 = \sigma_3^2 = -\mathrm{i}\ \sigma_1 \sigma_2 \sigma_3 = I.$$

We first look at $\mathcal{L}_1(\mathbb{P}^1)$, the space of triplets (A, B, V) of traceless 2×2 hermitian matrices satisfying equations (5.1). The 3-dimensional group SU(2) acts on the 4-dimensional space $\mathcal{L}_1(\mathbb{P}^1)$ by simultaneous conjugations on A, B and V, generating a family of orbits parametrized by t, with representing elements

$$A = R\sqrt{1-t^2}\sigma_1, \ B = R\sqrt{1-t^2}\sigma_2, \ V = Rt\sigma_3, \ -1 < t \leq 1,$$

where σ_i's, $i = 1, 2, 3$, are Pauli matrices.

When $-1 < t < 1$, the stabilizer of each orbit consists of scalar matrices with the scalar equals to the square roots of unity, which is the center of SU(2). So an open (smooth) piece of the family of orbits is isomorphic to $(-1, 1) \times \mathrm{PSU}(2)$.

Next, for $\mathcal{L}_1(\mathbb{P}^k)$, the constraint on the trace of V forces V to be in the form

$$V = R\begin{pmatrix} t & & & & \\ & -t & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 1 \end{pmatrix}$$

after diagonalization.

From this together with our result on $\mathcal{L}_1(\mathbb{P}^1)$ we see again there is a family of orbits under the $\mathrm{SU}(k+1)$ action, with representing elements

$$A = R\sqrt{1-t^2}\begin{pmatrix} \sigma_1 & \\ & 0 \end{pmatrix}, \ B = R\sqrt{1-t^2}\begin{pmatrix} \sigma_2 & \\ & 0 \end{pmatrix}, \ V = R\begin{pmatrix} t\sigma_3 & \\ & I \end{pmatrix}, \ -1 < t \leq 1, \tag{5.3}$$

where σ_i, $i = 1, 2, 3$, are Pauli matrices in (5.2).

The stabilizer of each orbit consists of matrices in the form

$$\begin{pmatrix} (\det U)^{-\frac{1}{2}} & & \\ & (\det U)^{-\frac{1}{2}} & \\ & & U \end{pmatrix} \tag{5.4}$$

where U is a $(k-1)$-dimensional unitary matrix. The form in (5.4) describes an embedding of $\mathrm{U}(k-1)$ into $\mathrm{SU}(k+1)$.

Proposition 18. There is a map

$$alg : [-1, 1] \times \frac{\mathrm{SU}(k+1)}{\mathrm{U}(k-1)} \longrightarrow \mathcal{L}_1(\mathbb{P}^k)$$

that is a diffeomorphism away from the boundary.

In fact, both pieces in the boundary $\{-1\} \times \frac{\mathrm{SU}(k+1)}{\mathrm{U}(k-1)} \cup \{1\} \times \frac{\mathrm{SU}(k+1)}{\mathrm{U}(k-1)}$ correspond to the singular locus $\mathcal{L}_0(\mathbb{P}^k)$ in $\mathcal{L}_1(\mathbb{P}^k)$. and $\mathcal{L}_1(\mathbb{P}^k)$ is diffeomorphic to the quotient space of $[-1,1] \times \mathrm{SU}(k+1)/\mathrm{U}(k-1)$ obtained by identifying the two components of the boundary at $t=-1$ and $t=1$ after each is taking quotient to a $\mathbb{P}^k$.

5.2 Geometry of $\mathcal{L}_1(\mathbb{P}^k)$

The space $\mathcal{L}_1(\mathbb{P}^k)$ is equipped with a metric

$$g = \mathrm{tr}(\mathrm{d}V^*\mathrm{d}V) + \frac{1}{2}(\mathrm{tr}(\mathrm{d}A^*\mathrm{d}A) + \mathrm{tr}(\mathrm{d}B^*\mathrm{d}B)).$$

We would like to pull back this metric under the map alg in Proposition 18 and compare it with the standard metric on $[-1,1] \times \mathrm{SU}(k+1)/\mathrm{U}(k-1)$.

It'll be convenient to do so by fixing an orthonormal basis $\{e_v, e_a, e_b, a_{\alpha i}, b_{\alpha i} \mid \alpha = 1,2,\ i = 1,\dots,k-1\}$ of $\mathfrak{su}(k+1)/\mathfrak{u}(k-1)$, where

$$\begin{aligned} e_v = \frac{\mathfrak{i}}{\sqrt{2}}(E_{11}-E_{22}),\ e_a = \frac{\mathfrak{i}}{\sqrt{2}}(E_{12}+E_{21}),\ e_b = \frac{1}{\sqrt{2}}(E_{21}-E_{12}), \\ a_{\alpha i} = \frac{1}{\sqrt{2}}(E_{\alpha,i+2}-E_{i+2,\alpha}), b_{\alpha i} = \frac{\mathfrak{i}}{\sqrt{2}}(E_{\alpha,i+2}+E_{i+2,\alpha}). \end{aligned} \tag{5.5}$$

Computed using Mathematica, the pull back of the metric g with respect to this basis equals to

$$alg^* g = R^2\Big(\frac{1}{1-t^2}dt^2 + g_\Omega(t)\Big), \tag{5.6}$$

where

$$g_\Omega(t) = 4(1-t^2)de_v^2 + 2(1+t^2)(de_a^2+de_b^2) + 2\sum_{i=1}^{k-1}\Big[(1-t)(da_{1i}^2+db_{1i}^2) + (1+t)(da_{2i}^2+db_{2i}^2)\Big].$$

Let $G = dt^2 + g_{ref}$ be the product metric on $(-1,1) \times \dfrac{\mathrm{SU}(k+1)}{\mathrm{U}(k-1)}$, where

$$g_{ref} = de_v^2 + de_a^2 + de_b^2 + \sum_{i=1}^{k-1}\Big(da_{1i}^2 + db_{1i}^2 + da_{2i}^2 + db_{2i}^2\Big).$$

The quantity $\mathrm{d}vol_{alg^*g}/\mathrm{d}vol_G$ is equal to

$$\omega_{alg}^k(t) = 2^{2k}R^{4k}(1+t^2)(1-t^2)^{k-1}. \tag{5.7}$$

We will use this quantity to prove the discreteness of the spectrum in the next section.

5.3 Spectrum for $\mathcal{L}_1(\mathbb{P}^k)$

We described the Schrödinger operator H using the quadratic form $Q(f,g)$ in Section 4.1. The potential function U is simple on $\mathcal{L}_1(\mathbb{P}^k)$

$$U = \frac{1}{2}\Big(\mathrm{tr}(AA^*) + \mathrm{tr}(BB^*)\Big) = 2R^2(1-t^2),$$

where we used equations in (5.3).

Due to large symmetry group of the problem it is reasonable to start the analysis of H with the SU$(k+1)$-radial part. The quadratic form $Q(f,g)$ simplifies significantly when f and g are SU$(k+1)$-radial, that is, if f and g depends only on t:

$$Q_{rad}(f,g) = \int_0^1 \Big(\frac{1-t^2}{R^2} f'(t)\bar{g}'(t) + 2R^2(1-t^2)f(t)\bar{g}(t)\Big)\omega(t)dt, \tag{5.8}$$

where

$$\omega(t) = \omega_{alg}(t) = 2^{2k}R^{4k}(1+t^2)(1-t^2)^{k-1}$$

from (5.7).

Recall the desired definition of H in (4.4), applying integration by part to (5.8) gives

$$H(f) = -\omega^{-1}(pf')' + \omega^{-1}qf, \tag{5.9}$$

where

$$p = \frac{1-t^2}{R^2}\omega(t) \text{ and } q = 2R^2(1-t^2)\omega(t).$$

We are interested in defining the operator H on an appropriate domain such that H is self-adjoint, then looking into the eigenvalue problem $Hf = \lambda f$. For this purpose, let $M = \omega H$, that is,

$$Mf = -(pf')' + qf. \tag{5.10}$$

We will define M as an operator in section 5.3.1 and use it, together with the the associated maximal domain, to define the Schrödinger operator H and show its self-adjointness. In terms of M, the eigenvalue problem $H_{rad}f = \lambda f$ becomes

$$Mf = -(pf')' + qf = \lambda\omega f. \tag{5.11}$$

Equation (5.11) is a Sturm-Liouville equation, which we will look into in section 5.3.2.

5.3.1 Hilbert space and self-adjointness of H

We would like to define M as a differential operator using the differential expression in (5.10). To this end, we bring in the Hilbert space $L^2(-1,1)$, the space of all complex-valued Lebesgue square integrable functions. To make sense of the differential expression in (5.10), we need to work in a subspace $\mathscr{D} \subset L^2(-1,1)$ consisting of functions such that both f and pf' are absolutely continuous on any compact subinterval of $(-1,1)$. To be more precise, let $AC_{loc}(-1,1)$ denote the space of all complex-valued functions on $(-1,1)$ that are absolutely continuous on any compact interval $[\alpha,\beta] \subset (-1,1)$. Define

$$\mathscr{D} = \left\{ f \in L^2(-1,1) \mid f, pf' \in AC_{loc(-1,1)} \right\}.$$

As we mentioned above, $\mathscr{D}$ is the maximal domain of M in $L^2(-1,1)$.

Our goal is to define H by

$$H = \frac{1}{\omega} M, \tag{5.12}$$

and an appropriate domain on which H is a self-adjoint operator. To make sense of equation (5.12), we need further restrictions on the underlying domain.

Let $L^2_\omega(-1,1)$ be the space of Lebesgue measurable functions on $(-1,1)$ that satisfy

$$(f,f)_\omega = \int_{-1}^{1} \omega |f|^2 dt < \infty.$$

The space $L^2_\omega(-1,1)$ is a Hilbert space with respect to inner product $(f,g)_\omega = \int_{-1}^{1} \omega f \bar{g} dt$. Following [12], we define the maximal domain generated by M in $L^2_\omega(-1,1)$ to be the subspace

$$\Delta = \left\{ f \in \mathscr{D} \mid f, \frac{1}{\omega} Mf \in L^2_\omega(-1,1) \right\}.$$

To find the maximal domain in $L^2_\omega(-1,1)$ on which H can be defined and is self-adjoint, it's necessary to classify the endpoints -1, 1 in $L^2_\omega(-1,1)$. The endpoints are classified as either regular, limit-circle singular or limit-point singular. An endpoint a is a limit-circle singularity if, for all $\lambda \in \mathbb{C}$, all solutions of equation (5.11) live in the space $L^2_\omega(U)$, where U is a neighborhood of a. That is, $\omega|f|^2$ is locally integrable near a for any solution f. On contrary, the endpoint a is limit-point singularity if differential equation (5.11) has for some $\lambda \in \mathbb{C}$ a solution f such that $\omega|f|^2$ is not integrable near a.

We follow [1] to derive the following classification.

Proposition 19. The following classification of end-points of the equation (5.11) holds:

(a) $t = -1$ and $t = 1$ are limit-circle singularities if $k = 1$,

(b) $t = -1$ and $t = 1$ are limit-point singularities if $k \geq 2$.

Proof. Equation (5.11) is equivalent to

$$-f'' + \frac{q - p'}{p} f - \frac{\lambda\omega}{p} f = 0. \tag{5.13}$$

The function p^{-1} is never integrable near $t = -1$ and $t = 1$. It's clear that both $t = -1$ and $t = 1$ are singularities.

To determine if a singular point a limit circle or limit point singularity, we need to see if $\omega|f|^2$ is locally integrable near the singular point. To this end, we need to first examine the local solutions of the equation.

The formula in (5.9) defines an operator, which we denote by the same symbol, in the extended domain $H_{rad} : \mathcal{O}(\mathbb{C} \setminus \{0, -1, 1\}) \to \mathcal{O}(\mathbb{C} \setminus \{0, -1, 1\})$. Here $\mathcal{O}$ stands for complex analytic functions of parameter $z \in \mathcal{O}(\mathbb{C} \setminus \{0, -1, 1\})$. The eigenvalue problem $H_{rad} f = \lambda f$ in this space becomes an ODE:

$$-f'' - \Big(\frac{2z}{z^2 + 1} + \frac{k}{z + 1} + \frac{k}{z - 1}\Big) f' + \frac{2R^4(z^2 - 1) + R^2\lambda}{z^2 - 1} f = 0 \tag{5.14}$$

Frobenius method [8] gives local solutions to the ordinary differential equation in (5.14). In terms of the method of Frobenius, $z = -1$ and $z = 1$ are regular singularities both of characteristic exponents $1 - k$ and 0. It follows that locally equation (5.14) has a general solution of the form

$$f(z) = c_1 y_1(z - z_0) + c_2 \left(\frac{1}{(z - z_0)^{k-1}} y_2(z - z_0) + \alpha(k, R, \lambda) \ln(z - z_0) y_1(z - z_0) \right) \tag{5.15}$$

near $z = z_0$, where $z_0 = -1, 1$ and $y_1(z)$, $y_2(z)$ are analytic functions near zero.

With help of general solutions in (5.15), it's easy to see that $\omega|f|^2$ is locally integrable at either $t = -1$ or $t = 1$ for any solution f of equation (5.11) iff $k \leq 1$. □

Following the classification of singularities above, we have the following definition and theorem.

Definition 20. Define an operator $H : \Omega_k \to L^2_\omega(-1,1)$, $k = 1, 2, \dots$ by

$$Hf = \frac{1}{\omega} Mf,$$

where

1. $\Omega_1 = \left\{ f \in \Delta \mid \lim_{t \to -1^+} (pf')(t) = \lim_{t \to 1^-} (pf')(t) = 0 \right\},$
2. $\Omega_k = \Delta,\ k \geq 2.$

Theorem 21. *The operator H is self-adjoint in the Hilbert space $L^2_\omega(-1,1)$.*

Proof. We refer the reader to [12] Section 18 and [1] Theorem 6. $\square$

5.3.2 Discrete spectrum of H

We have the following spectral properties of the self-adjoint operator H.

Theorem 22. *Let H be the self-adjoint operator defined in Definition 20. Then H has the following spectral properties:*

1. *The spectrum $\sigma(H)$ is real simple and discrete, that is, the spectrum consists solely of real simple eigenvalues*

   $$\sigma(H) = \{\lambda_n \in \mathbb{R}, n \in \mathbb{N}\},$$

   with properties

   $$\begin{aligned} &\text{(i)} && \lambda_0 \geq 0, \\ &\text{(ii)} && \lambda_n < \lambda_{n+1},\ n \in \mathbb{N}, \\ &\text{(iii)} && \lim_{n \to +\infty} \lambda_n = +\infty. \end{aligned}$$

   The operator H is bounded below in $L^2_\omega(-1,1)$ with

   $$(Hf, f)_\omega \geq 0,\ f \in \Omega.$$

   Since each eigenvalue is simple, the eigenspaces satisfy

   $$\dim\{f \in \Omega : Hf = \lambda_n f\} = 1,\ n \in \mathbb{N}.$$

2. *Let the eigenfunction be denoted by $\{\psi_n \mid n \in \mathbb{N}\}$, then*

(a) The eigenfunction ψ_n satisfy boundary conditions formulated in Definition 20.

(b) The space of eigenfunctions $\{\psi_n \mid n \in \mathbb{N}\}$ is orthogonal and complete in $L^2_\omega(-1,1)$.

(c) For each n, ψ_n has exactly n isolated zeros on $(-1,1)$.

(d) For each n, the limits $\lim_{t\to x_0} \psi_n(t)$ exist and are finite, where $x_0 = 0^+$ and 1^-. The eigenfunctions ψ_n are absolutely continuous on $[-1,1]$.

Proof. The proof follows from that for the similar statements for Heun equation in [1]. □

Appendix A

Cohomology of complexification

Let k=2, and

$$A = \begin{pmatrix} x_1 & y_1 + iz_1 & y_2 + iz_2 \\ y_1 - iz_1 & x_2 & y_3 + iz_3 \\ y_2 - iz_2 & y_3 - iz_3 & x_3 \end{pmatrix} \in V$$

be a 3×3 hermitian matrix. The set of equations $\{A^2 = R^2 I, \mathrm{tr} A = R\}$ becomes

$$\left\{ \begin{array}{l} \quad x_1y_1 + x_2y_1 + y_2y_3 + z_2z_3 = y_3z_1 + x_1z_2 + x_3z_2 + y_1z_3 \\ = x_1y_2 + x_3y_2 + y_1y_3 - z_1z_3 = y_3z_1 + x_1z_2 + x_3z_2 + y_1z_3 \\ = y_1y_2 + x_2y_3 + x_3y_3 + z_1z_2 = -y_2z_1 + y_1z_2 + x_2z_3 + x_3z_3 = 0 \\ \quad x_1^2 + y_1^2 + y_2^2 + z_1^2 + z_2^2 = x_2^2 + y_1^2 + y_3^2 + z_1^2 + z_3^2 \\ = x_3^2 + y_2^2 + y_3^2 + z_2^2 + z_3^2 = R^2, \\ \quad x_1 + x_2 + x_3 = R \end{array} \right\}.$$

The complexification of $\mathbb{P}^2$ is a projective complex manifold Y given by

$$Y = \{(x_1 : \cdots : x_3 : y_1 : \cdots : y_3 : z_1 : \cdots : z_3 : w) \in \mathbb{P}^9 \mid I\},$$

where I consists of relations given by the set of equations

$$I = \left\{ \begin{array}{l} \quad x_1y_1 + x_2y_1 + y_2y_3 + z_2z_3 = y_3z_1 + x_1z_2 + x_3z_2 + y_1z_3 \\ = x_1y_2 + x_3y_2 + y_1y_3 - z_1z_3 = y_3z_1 + x_1z_2 + x_3z_2 + y_1z_3 \\ = y_1y_2 + x_2y_3 + x_3y_3 + z_1z_2 = -y_2z_1 + y_1z_2 + x_2z_3 + x_3z_3 = 0 \\ \quad x_1^2 + y_1^2 + y_2^2 + z_1^2 + z_2^2 = x_2^2 + y_1^2 + y_3^2 + z_1^2 + z_3^2 \\ = x_3^2 + y_2^2 + y_3^2 + z_2^2 + z_3^2 = R^2w^2, \\ \quad x_1 + x_2 + x_3 = Rw \end{array} \right\}.$$

The space Y can be identified with a subspace of the projectivization of non-zero complex matrices of dimension 3,

$$Y \simeq \{[A] \in (M_{3\times 3}(\mathbb{C}) - 0)/\mathbb{C}^* \mid A^2 = I, \mathrm{tr}A = 1 \text{ or } A^2 = 0, \mathrm{tr}A = 0\}.$$

The SU(3) action on $\mathbb{P}^2$ extends to an action of $\mathrm{SL}(3, \mathbb{C})$ on Y, which has two orbits

$$\mathcal{O}_1 = \mathcal{O}(\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}),\ \mathcal{O}_2 = \mathcal{O}(\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}),$$

with corresponding stabilizers

$$\mathrm{Stab}(\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}) \simeq \mathrm{SL}(2,\mathbb{C})\times\mathbb{C}^*,\ \mathrm{Stab}(\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}) = \left\{ \begin{pmatrix} a & b & c \\ 0 & d & 0 \\ 0 & e & f \end{pmatrix} \in \mathrm{SL}(3,\mathbb{C}) \right\}.$$

We compute the cohomology of Y. We do so by first computing the cohomology of $\mathcal{O}_1$ and $\mathcal{O}_2$, then put the cohomology of the three into a long exact sequence.

For $\mathcal{O}_1$, it's isomorphic to $\mathrm{SL}(3,\mathbb{C})/\mathrm{Stab}(\mathcal{O}_1)$, which is homotopy equivalent to the compact homogeneous space $\mathrm{SU}(3)/(\mathrm{SU}(3) \cap \mathrm{Stab}(\mathcal{O}_1))$. Since $\mathrm{SU}(3) \cap \mathrm{Stab}(\mathcal{O}_1) \simeq \mathrm{U}(2)$, the homogeneous space is isomorphic to $\mathbb{P}^2$. Therefore, the cohomology groups of $\mathcal{O}_1$ are

$$H^k(\mathcal{O}_1; \mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{for } k \text{ even} \\ 0 & \text{for } k \text{ odd,} \end{cases} \quad k = 0, \cdots, 4.$$

For $\mathcal{O}_2$, it's isomorphic to $\mathrm{SL}(3,\mathbb{C})/\mathrm{Stab}(\mathcal{O}_2)$, which is homotopic equivalent to the compact homogeneous space $\mathrm{SU}(3)/(\mathrm{SU}(3)\cap\mathrm{Stab}(\mathcal{O}_2))$. Since $\mathrm{SU}(3)\cap\mathrm{Stab}(\mathcal{O}_1) \simeq \mathrm{S}^1 \times \mathrm{S}^1$, the homogeneous space is a complex flag variety, and the cohomology groups of $\mathcal{O}_2$ are

$$H^k(\mathcal{O}_2; \mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{for } k = 0,\ 6 \\ \mathbb{Z}^2 & \text{for } k = 2,\ 4 \\ 0 & \text{otherwise.} \end{cases}$$

Since $\mathcal{O}_1 = Y - \mathcal{O}_2$, $\mathcal{O}_2$ is closed and of complex codimension 1, we have Thom's isomorphism $H^i(Y, \mathcal{O}_1; \mathbb{Z}) \simeq H^i(Y, Y - \mathcal{O}_2; \mathbb{Z}) \simeq H^{i-2}(\mathcal{O}_2; \mathbb{Z})$. Combine this with the long exact sequence of relative cohomology

$$\cdots H^i(Y, \mathcal{O}_1; \mathbb{Z}) \to H^i(Y; \mathbb{Z}) \to H^i(\mathcal{O}_1; \mathbb{Z}) \to H^{i+1}(Y, \mathcal{O}_1; \mathbb{Z}) \to \cdots,$$

we get a long exact sequence relating the cohomology of Y, $\mathcal{O}_1$ and $\mathcal{O}_2$

$$\cdots H^{i-2}(\mathcal{O}_2; \mathbb{Z}) \to H^i(Y; \mathbb{Z}) \to H^i(\mathcal{O}_1; \mathbb{Z}) \to H^{i-1}(\mathcal{O}_2; \mathbb{Z}) \to \cdots. \quad \text{(A.1)}$$

Plug in the cohomology groups of $\mathcal{O}_1$ and $\mathcal{O}_2$ to the long exact sequence (A.1), we obtain a long exact sequence for $H^k(Y, \mathbb{Z})$'s

$$\begin{tikzcd}
0 \arrow[r] & H^0(Y;\mathbb{Z}) \arrow[r] & \mathbb{Z} \arrow[dll] \\
0 \arrow[r] & H^1(Y;\mathbb{Z}) \arrow[r] & 0 \arrow[dll] \\
\mathbb{Z} \arrow[r] & H^2(Y;\mathbb{Z}) \arrow[r] & \mathbb{Z} \arrow[dll] \\
0 \arrow[r] & H^3(Y;\mathbb{Z}) \arrow[r] & 0 \arrow[dll] \\
\mathbb{Z}^2 \arrow[r] & H^4(Y;\mathbb{Z}) \arrow[r] & \mathbb{Z} \arrow[dll] \\
0 \arrow[r] & H^5(Y;\mathbb{Z}) \arrow[r] & 0 \arrow[dll] \\
\mathbb{Z}^2 \arrow[r] & H^6(Y;\mathbb{Z}) \arrow[r] & 0 \arrow[dll] \\
0 \arrow[r] & H^7(Y;\mathbb{Z}) \arrow[r] & 0 \arrow[dll] \\
\mathbb{Z} \arrow[r] & H^8(Y;\mathbb{Z}) \arrow[r] & 0
\end{tikzcd}.$$

From the long exact sequence we get

$$H^k(Y;\mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{for } k = 0,\ 8 \\ \mathbb{Z}^2 & \text{for } k = 2,\ 6 \\ \mathbb{Z}^3 & \text{for } k = 4 \\ 0 & \text{otherwise,} \end{cases}$$

which is consistent with our construction using $\mathbb{P}(V) \times \mathbb{P}(V^*)$ in Proposition 7.

Appendix B

Computation of injectivity for $G(k+1) \times \mathcal{L}_0(\mathbb{P}^k) \to \mathcal{L}_1(\mathbb{P}^k)$

Proposition 23. The map $G(k+1) \times \mathcal{L}_0(\mathbb{P}^k) \to \mathcal{L}_1(\mathbb{P}^k)$ is injective at generic points.

Proof. Let $P(z) = P \begin{pmatrix} z & \\ & I \end{pmatrix} P^*$, $Q(z) = Q \begin{pmatrix} z & \\ & I \end{pmatrix} Q^*$ be two elements in $G(k+1)$, where P, Q are unitary matrices, n_1, n_2 belongs to $\mathcal{L}_0$, such that

$$P(z) n_1 P(z)^* = Q(z) n_2 Q(z)^*. \tag{B.1}$$

Note that $P(1) = Q(1) = I$, the identity matrix. Let $z = 1$ in (B.1), we have

$$n_1 = n_2.$$

Therefore (B.1) becomes

$$Q(z)^* P(z) n_1 = n_1 Q(z)^* P(z). \tag{B.2}$$

Since $n_1 \in \mathcal{L}_0$, n_1 has canonical form

$$n_1 = \Omega \begin{pmatrix} -2 & \\ & 0 \end{pmatrix} \Omega^* + I, \tag{B.3}$$

plug (B.3) into (B.2), we have

$$\Omega^* Q(z)^* P(z) \Omega \begin{pmatrix} -2 & \\ & 0 \end{pmatrix} = \begin{pmatrix} -2 & \\ & 0 \end{pmatrix} \Omega^* Q(z)^* P(z) \Omega, \tag{B.4}$$

that is, $\Omega^* Q(z)^* P(z)\Omega$ commutes with $\begin{pmatrix} -2 & \\ & 0 \end{pmatrix}$.

Therefore

$$\Omega^* Q(z)^* P(z)\Omega = \Omega^* Q \begin{pmatrix} \bar{z} & \\ & I \end{pmatrix} Q^* P \begin{pmatrix} z & \\ & I \end{pmatrix} P^* \Omega$$

must be block-wise diagonal

$$\Omega^* Q \begin{pmatrix} \bar{z} & \\ & I \end{pmatrix} Q^* P \begin{pmatrix} z & \\ & I \end{pmatrix} P^* \Omega = \begin{pmatrix} \lambda(z) & \\ & * \end{pmatrix}, \tag{B.5}$$

where $\lambda(z) = 1,\ z \text{or} \bar{z}$ since it's unitary.

Let $e_1 = (1, 0, \ldots, 0)^T$, then (B.5) is equivalent to

$$\Omega^* Q \begin{pmatrix} \bar{z} & \\ & I \end{pmatrix} Q^* P \begin{pmatrix} z & \\ & I \end{pmatrix} P^* \Omega\ e_1 = \lambda(z) e_1.$$

Multiply $\begin{pmatrix} z & \\ & I \end{pmatrix} Q^* \Omega$ on both sides and denote $m = P^* \Omega\ e_1$, the equation becomes

$$Q^* P \begin{pmatrix} z & \\ & I \end{pmatrix} m = \lambda(z) \begin{pmatrix} z & \\ & I \end{pmatrix} Q^* P m. \tag{B.6}$$

Let $Q^* P$ and m be in block forms

$$Q^* P = \begin{pmatrix} a & b \\ c & d \end{pmatrix},\ m = \begin{pmatrix} m_1 \\ m_2 \end{pmatrix},$$

respectively, then (B.6) becomes

$$\begin{pmatrix} za & b \\ zc & d \end{pmatrix} \begin{pmatrix} m_1 \\ m_2 \end{pmatrix} = \lambda(z) \begin{pmatrix} za & zb \\ c & d \end{pmatrix} \begin{pmatrix} m_1 \\ m_2 \end{pmatrix}.$$

There are three possibilities for the equation to hold:

1. $m_1 \neq 0, \lambda(z) = 1, c = 0.$

2. $m_1 \neq 0, \lambda(z) = z, m_2 = 0.$

3. $m_1 = 0.$

In case (1), since $Q^*P = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is unitary, $c = 0$ implies $b = 0$. Therefore $Q^*P = \begin{pmatrix} a & \\ & d \end{pmatrix}$ is diagonal, $P = Q \begin{pmatrix} a & \\ & d \end{pmatrix}$, $P(z) = Q(z)$, the map is injective.

In case (2) and (3), m is the first column of $P^*\Omega$, either $m_1 = 0$ or $m_2 = 0$ will guarantee $R(z) = \Omega^* P \begin{pmatrix} z & \\ & I \end{pmatrix} P^*\Omega$ being diagonal. Then

$$P(z) n_1 P(z)^* = \Omega R(z) \begin{pmatrix} -2 & \\ & 0 \end{pmatrix} R(z)^* \Omega^* + I = \Omega \begin{pmatrix} -2 & \\ & 0 \end{pmatrix} \Omega^* + I = n_1 \in \mathcal{L}_0$$

which is a singular point.

Therefore, the map is injective at generic points. $\square$

Bibliography

[1] P. B. Bailey, W. N. Everitt, D. B. Hinton, and A. Zettl. Some spectral properties of the heun differential equation. In *Operator Methods in Ordinary and Partial Differential Equations*, volume 132 of *Operator Theory: Advances and Applications*, Birkhuser, Basel, 2000.

[2] A. Besse. *Einstein Manifolds.* Springer, 2007.

[3] R. Chen. Neumann eigenvalue estimate on a compact Riemannian manifold. *Proc. AMS*, 1990:961?970, 1990.

[4] R. Chen. Gap estimates of Schrödinger operator. *Pacific Journal Of Mathematics*, 178(2), 1997.

[5] O. Debarre. *Higher-Dimensional Algebraic Geometry.* Springer, 2001.

[6] P. Georgieva and A. Zinger, Orientability in Real Gromov-Witten Theory, 2013. arXiv:1308.1347v2 [math.SG] .

[7] P. Georgieva and A. Zinger, Real Gromov-Witten Theory in All Genera and Real Enumerative Geometry I, 2015. arXiv:1504.06617v1 [math.SG].

[8] E.L. Ince. *Ordinary Differential Equations.* Dover Publications Inc., New York, 1956.

[9] P. Li. A lower bound for the first eigenvalue of the Laplacian on a compact Riemannian manifold, *Indiana U. Math. J.*, 28:1013-1019, 1979.

[10] P. Li and S.T. Yau. Estimates of eigenvalues of a compact Riemannian manifold, *AMS Proc. Symp. Pure Math.*, 36:205-239, 1980.

[11] P. Li and S.T. Yau. On the parabolic heat kernel of the Schrödinger operator, *Acta Math.*, 156:153-201, 1986.

[12] M.A. Naimark. *Linear Differential Operators II.* Ungar, 1967.

[13] A. Pressley and G. Segal. *Loop Groups*, Oxford University Press, New York, 1986.

[14] I.M. Singer, B. Wong, S.S.T. Yau and S.T. Yau. An estimate of the gap of the first two eigenvalues of the Schr?odinger operator, *Ann. Scuo. Norm. Pisa, Cl. Sci.*, IV(12):319-333, 1985.

[15] J. Wang. Global heat kernel estimates, preprint, 1994.