Computation of Floer Invariant of $(2, 2n)$-Torus link Complement

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Abstract of the Dissertation

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A closed three manifold invariant Heegaard Floer homology was generalized to bordered Heegaard Floer homology, defined by Robert Lipshitz, Peter Ozsváth and Dylan Thurston. Bordered Heegaard Floer homology is an invariant of three manifold with connected boundary, and its variant doubly bordered Floer homology is a bi-module defined on three manifold with two disconnected boundary components. In this thesis, we compute bordered Floer homology of $(2, 2n)$-torus link complement.

Contents

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Chapter 1

Introduction

In recent years, the Heegaard Floer theory was fascinated many low-dimensional topologist. P. Ozsváth and Z. Szábo developed the Heegaard Floer invariant of closed 3-manifold and led to breakthrough in low dimensional topology. It was also proved to be equivalent to three-dimensional Seiberg-Witten Floer homology [4] that achieves one of its initial motivation of development. Moreover, Heegaard Floer homology turned out to be useful in defining knot and link invariant([10], [13], [11]), namely *knot Floer homology* and *link Floer homology*. In particular, knot Floer homology and Heegaard Floer homology of three manifold obtained by integral surgery on knot turned out to be closely related([13], [12]). For the link surgery case, the relation was discovered but more complicated than the knot case([8]).

More recently, Lipshitz, Ozsvath and Thurston extended the theory to the 3-manifold with nonempty boundary. *Bordered Floer homology* was first introduced by R. Lipshitz in [5], which consists of two different modules; $\widehat{CFD}$

and $\widehat{CFA}$. Its homotopy type is an topological invariant of three manifold with *connected* boundary and diffeomorphism type of its boundary. In addition, one can recover the Heegaard Floer homology of closed three manifold by taking "$\mathcal{A}_\infty$ tensor product" of $\widehat{CFA}$ and $\widehat{CFD}$.

Bordered Floer homology of three manifold with genus one is also related to knot Floer homology. In [6], they described an algorithm to recover $\widehat{CFD}(S^3\backslash\nu(K))$ from knot Floer homology CFK^- with arbitrary framing. This enables to compute Heegaard Floer homology of surgered manifold by taking $\mathcal{A}_\infty$ tensor product with solid torus.

In [7], they have generalized bordered Floer homology to *doubly bordered Floer homology*. As the name suggests, it associates three manifold with two boundary component to three different types of bimodules; $\widehat{CFDA}$, $\widehat{CFDD}$, and $\widehat{CFAA}$. It is natural to consider $S^3\backslash\nu(L)$, where L is a link with two components. In this thesis, we give first calculation of $\widehat{CFDD}(S^3\backslash\nu(L))$, where L is $(2, 2n)$-torus link.

Chapter 2

Backgrounds on Bordered Floer Theory

In this section, we will quickly define algebraic preliminaries of the bordered Floer homology.

2.1 $\mathcal{A}_\infty$-module

Roughly speaking, $\mathcal{A}_\infty$-module is a right module M on $\mathcal{A}_\infty$-algebra A, with a set of maps $m : M \otimes \mathcal{T}^*(A) \to M$ such that m^2 vanishes. We will assign $\mathcal{A}_\infty$-module to a link complement $S^3 \backslash \nu L$. Most definitions are introduced in [6], chapter 2.

Definition 2.1.1 *Let $\mathbb{F}$ be a field of characteristic two. Then* $\mathcal{A}_\infty$-algebra *is a $\mathbb{F}$-vector space A with multilinear maps $\mu_i : A^{\otimes i} \to A$ for $i \geq 1$ satisfying* *compatibility condition*

$$\sum_{i+j=n+1}\sum_{l=1}^{n-j+1}\mu_i(a_1\otimes\cdots\otimes a_{l-1}\otimes\mu_j(a_l\otimes\cdots\otimes a_{l+j-1})\otimes a_{l+j}\otimes\cdots\otimes a_n)=0$$

for all $n\geq 1$. An $\mathcal{A}_\infty$-algebra with unit 1 is said to be strictly unital *if $\mu_2(1,a)=\mu_2(a,1)=a$ and $\mu_i(a_1\otimes\cdots\otimes a_i)=0$ if $i\neq 2$ and $a_j=1$ for some j.*

We combine maps μ_i into a single map $\mu:\mathcal{T}^*(A)\to A$.

Above relation can be considered as follows. We define a map $\overline{D}:\mathcal{T}^*(A)\to\mathcal{T}^*(A)$ by

$$\overline{D}(a_1\otimes\cdots\otimes a_n)=\sum_{j=1}^{n}\sum_{l=1}^{n-j+1}a_1\otimes\cdots\otimes\mu_j(a_l\otimes\cdots\otimes a_{l+j-1})\otimes\cdots\otimes a_n,$$

then the compatibility condition is symbolically written as $\mu\circ\overline{D}=0$ or $\overline{D}\circ\overline{D}=0$.

Definition 2.1.2 A right $\mathcal{A}_\infty$-module $\mathcal{M}$ *is a $\mathbb{F}$-vector space M with maps $m_i:M\otimes A^{\otimes(i-1)}\to M$ for $i\geq 1$, satisfying compatibility conditions*

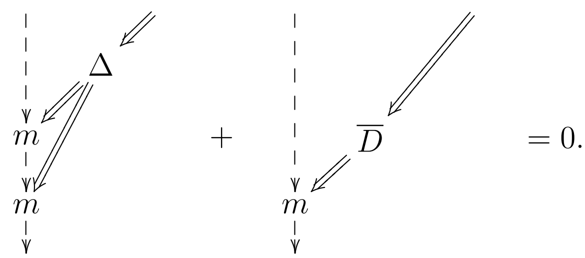
$$\begin{aligned}0 &= \sum_{i+j=n+1}m_i(m_j(\mathbf{x}\otimes a_1\otimes\cdots\otimes a_{j-1})\otimes\cdots\otimes a_{n-1})\\ &\quad+\sum_{i+j=n+1}\sum_{l=1}^{n-j}m_i(\mathbf{x}\otimes a_1\otimes\cdots\otimes a_{l-1}\otimes\mu_j(a_l\otimes\cdots\otimes a_{l+j-1})\otimes\cdots\otimes a_{n-1})\end{aligned}$$

An $\mathcal{A}_\infty$-module is said to be strictly unital *if for any $\mathbf{x}\in M$, $m_2(\mathbf{x},1)=\mathbf{x}$ and $m_i(\mathbf{x}\otimes a_1\otimes\cdots\otimes a_{i-1})=0$ for all $i>2$ and some $a_j=1$.*

Instead of spelling out compatibility conditions, we draw diagram for notational convenience. To do so, first we define a comultiplication

$$\Delta(a_1 \otimes \cdots \otimes a_n) = \sum_{m=0}^{n} (a_1 \otimes a_m) \otimes (a_{m+1} \otimes \cdots \otimes a_n).$$

The map m_i takes 1 element from M and $i-1$ elements from A. The combination of maps m_i is denoted m, and compatibility condition in Definition 2.2 can be described by diagram below.



A dashed line represents input from M, and double line from $\mathcal{T}^*(\mathcal{A})$.

2.2 Type D structure

First we fix a unital *dg* algebra $\mathcal{A}$ with differential $\mu_1 : \mathcal{A} \to \mathcal{A}$ and multiplication $\mu_2 : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$. Let N be a $\mathbb{F}_2$-vector space equipped with a map $\delta^1 : N \to \mathcal{A} \otimes N$ satisfying a compatibility condition as follows.

$$(\mu_2 \otimes \mathbb{I}_N) \circ (\mathbb{I}_{\mathcal{A}} \otimes \delta^1) \circ \delta^1 + (\mu_1 \otimes \mathbb{I}_N) \circ \delta^1 = 0$$

We call a pair (N, δ^1) *Type D structure over* $\mathcal{A}$.

We define maps $\delta^k : N \to \mathcal{A}^{\otimes k} \otimes N$ inductively

$$\begin{aligned}\delta^0 &= \mathbb{I}_N \\ \delta^i &= (\mathbb{I}_{\mathcal{A}^{\otimes(i-1)}} \otimes \delta^1) \circ \delta^{i-1}\end{aligned}$$

Thus we have a map $\delta : N \to \mathcal{T}^*(\mathcal{A}) \otimes N$ defined to be

$$\delta(\mathbf{x}) := \sum_{i=0}^{\infty} \delta^i(\mathbf{x})$$

The algebra we consider later has following properties. First, it has trivial differential, so the compatibility condition is $(\mu_2 \otimes \mathbb{I}_N) \circ (\mathbb{I}_{\mathcal{A}} \otimes \delta^1) \circ \delta^1 = 0$. Second, it has a subset of orthogonal idempotents $\mathcal{I} = \{\iota_1, \iota_2\} \subset \mathcal{A}$ and the unit element $1 \in \mathcal{A}$ which is sum of the idempotents; that is, $\iota_1 + \iota_2 = 1$. Lastly, N can be written as direct sum of 2 subspaces N_1 and N_2 with left $\mathcal{I}$-action such that

- ι_i acts trivially on N_j if $i = j$
- $\iota_i N_j = 0$ if $i \neq j$.

2.3 The Strands Algebra

One of the main concern of the bordered Floer homology is associating a type D structure on bordered Heegaard diagram. Here, the *dg* algebra of this type D structure is called *strands algebra*. Roughly speaking, strands algebra is a *dg* algebra imposed on the boundary of 3 manifold, whose generators are "Reeb" chords on the boundary of its Heegaard diagram. Detailed description can be found on chapter 3 of [6].

Let F be a surface of genus k, and choose a *preferred disk* $D \subset F$ and a point z on ∂D. We consider a a handle decomposition of F as follows. D is a zero-handle of F and mark $4k$ points on the boundary of D away from z. These $4k$ points are partitioned into pairs so that gluing of $2k$ 2 dimensional 1-handles on those pairs gives a surface F, i.e, the boundary of handlebody after attaching 1-handles is a circle. Conventionally we denote the data a *pointed matched circle* $\mathcal{Z} = (Z, \mathbf{a}, M, z)$, where $Z = \partial D$ is an oriented circle, $M : \{1, \cdots, 4k\} \to \{1, \cdots, 2k\}$ a matching data and $\mathbf{a} = \{a_1, \cdots, a_{4k}\}$ $4k$ points on the boundary.

In the chapter 3 of [6], the detailed description on constructing strands algebra is given. However, since we will be mainly interested on the torus boundary case, we focus on the surface F of genus 1.

If F is a torus, then we have 4 points $\mathbf{a} = \{a_1, \cdots, a_4\}$ and z on the boundary of its preferred disk D. Cutting open $Z = \partial D$ on z, so that 4 points on the boundary Z be labeled a_1, a_2, a_3, and a_4 along the orientation of Z. The matching $M : \{1, 2, 3, 4\} \to \{1, 2\}$ sends $M(1) = M(3) = 1$ and $M(2) = M(4) = 2$. See [Figure 2.1]. The diagram denotes the core of 1-handles attached on the pairs a_1^a and a_2^a, respectively.

Sometimes we will denote $a_i \lessdot a_j$ if $i < j$.

There are 6 Reeb chords $\rho_1, \rho_2, \rho_3, \rho_{12}, \rho_{23}, \rho_{123}$ starting and ending on $a_1, \cdots, a_4$. Each Reeb chords travels as follows.

- ρ_1 starts at a_1 and ends at a_2.
- ρ_2 starts at a_2 and ends at a_3.
- ρ_3 starts at a_3 and ends at a_4.
- ρ_{12} starts at a_1 and ends at a_3.
- ρ_{23} starts at a_2 and ends at a_4.
- ρ_{123} starts at a_1 and ends at a_4.

There are constant Reeb chords $\bar{\iota}_i$, $(i = 1, \cdots, 4)$ as well, by defining $\bar{\iota}_i$ to be a constant chord starting and ending at a_i. We define *idempotents* $\iota_1 = \bar{\iota}_1 + \bar{\iota}_3$ and $\iota_2 = \bar{\iota}_2 + \bar{\iota}_4$. Products between Reeb chords can be considered as concatenation of chords. More precisely, $\rho_1\rho_2 = \rho_{12},\ \rho_2\rho_3 = \rho_{23},\ \rho_1\rho_{23} = \rho_{12}\rho_3 = \rho_{123}$. Concatenations between Reeb chords and idempotents are also well defined. For example, $\iota_1\rho_1 = \rho_1\iota_2 = \rho_1$ and so on. For any two chords ρ_i and ρ_j such that the ending point of ρ_i is different from the starting point of ρ_j, $\rho_i\rho_j$ is defined to be zero.

Remark 2.3.1 *Some readers can consider an element that contains more than one Reeb chords. In fact, the originial definition given by [6] is that the strands algebra $\mathcal{A}(\mathcal{Z})$ associated to pointed matched circle $\mathcal{Z} = (Z, \mathbf{a}, M, z)$ can be decomposed into*

$$\mathcal{A}(\mathcal{Z}) = \bigoplus_{i=-k}^{k} \mathcal{A}(\mathcal{Z}, i)$$

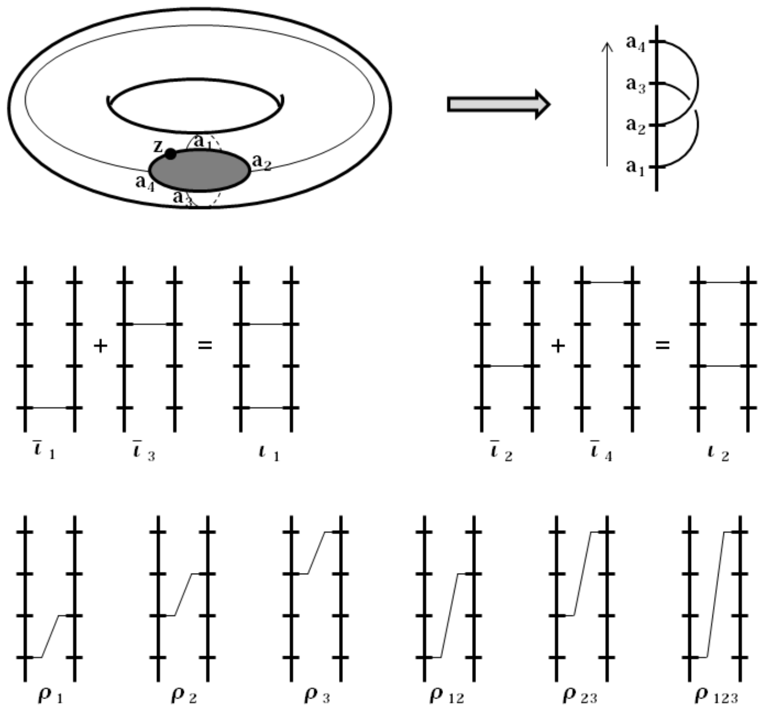


Figure 2.1: The figure on the top left corner represents torus with a preferred disk and two 2-dimensional 1 handles attached to it. The figure on the top right depicts boundary of the preferred disk after cutting open at the point z, with matching data given by 1 handles. Idempotents on the middle row are given as sum of two constant chords on the matching. The bottom row has 6 Reeb nonconstant chords.

where k is the genus of surface given by $\mathcal{Z}$ and $\mathcal{A}(\mathcal{Z}, i)$ is a summand whose number of Reeb chords equals $i + k$. However, $\mathcal{A}(\mathcal{Z}, i)$ acts trivially on $\widehat{CFA}$ and $\widehat{CFD}$ unless $i = 0$. ([6], Chapter 6). Thus we will be mainly interested in summand $\mathcal{A}(\mathcal{Z}, 0)$ part only.

Definition 2.3.2 Strands algebra of torus $\mathcal{A}(T)$ *is generated over* $\mathbb{F}_2$ *by* $\rho_1, \cdots, \rho_{123}$ *and idempotents* ι_1, ι_2 *with* $\iota_1 + \iota_2 = 1$*, whose product is defined to be a concatenation of chords.*

We let $\mathcal{I}$ be a set of idempotents of $\mathcal{A}(T)$.

Remark 2.3.3 *In [6], more general description of strands algebra is given. In fact, strands algebra has a differential that "uncrosses" each inversions of given Reeb chord. However, interesting differential appears only if the genus of surface $F \geq 2$. That means the differential of the torus algebra is trivial.*

2.4 Heegaard Floer Homology of Closed 3-Manifold

Let Y be a closed three manifold and Σ be its Heegaard surface of genus g. In [9], Heegaard Floer homology of Y was originally defined as Lagrangian intersection Floer theory on a symmetric product of Heegaard surface $\mathrm{Sym}^g(\Sigma)$, which is a symplectic manifold whose symplectic structure and almost complex structure derived from those of original Heegaard surface Σ. Let $\boldsymbol{\alpha} = \{\alpha_1, \cdots, \alpha_g\}$ and $\boldsymbol{\beta} = \{\beta_1, \cdots, \beta_g\}$ be attaching circles on Σ, thus circles $\alpha_i \in \boldsymbol{\alpha}$ are mutually non-intersecting(same is true for $\beta_i \in \boldsymbol{\beta}$). α_i and β_j may intersect, and if so, they intersect transversally. The symmetric product $\mathbb{T}_\alpha := \alpha_1 \times \cdots \times \alpha_g$ and $\mathbb{T}_\beta := \beta_1 \times \cdots \times \beta_g$ are Lagrangian submanifold in $\mathrm{Sym}^g(\Sigma)$, and their intersections are g-tuple of points $\mathbf{x} = \{x_1, \cdots, x_g\} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$ where each α_i and β_j contains exactly one $x_l \in \mathbf{x}$.

In Lagrangian Floer intersection theory, it is natural to consider a holomorphic disk between intersections. In this Heegaard Floer setting, the holomorphic disk flowing between $\mathbf{x}, \mathbf{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$ should satisfy following:

- $u : \mathbb{D} = [0,1] \times i\mathbb{R} \to \mathrm{Sym}^g(\Sigma)$ be J-holomorphic map, where J is a generic choice of almost-complex structure on $\mathrm{Sym}^g(\Sigma)$.

- $u(\{1\} \times \mathbb{R}) \subset \mathbb{T}_\alpha$ and $u(\{0\} \times \mathbb{R}) \subset \mathbb{T}_\beta$.

- $\lim_{t \to -\infty} u(s+it) = \mathbf{x}$ and $\lim_{t \to +\infty} u(s+it) = \mathbf{y}$.

$\mathbb{R}$ acts on holomorphic curve by reparametrization. We denote the moduli space of holomorphic strips up to reparametrization $\mathcal{M}_J(\mathbf{x}, \mathbf{y})$. If we drop the holomorphic condition, the set of all such maps are denoted $\boldsymbol{\pi}_2(\mathbf{x}, \mathbf{y})$, and such map is called *Whitney disk* between $\mathbf{x}$ and $\mathbf{y}$. Any element of $\boldsymbol{\pi}_2(\mathbf{x}, \mathbf{x})$ is *periodic domain.*

Suppose that there is a Whitney disk flows from $\mathbf{x}$ to $\mathbf{y}$. This condition is equivalent to say that there is a curve on Y that is trivial in terms of homology $H_1(Y)$. More precisely, let a be a path from $\mathbf{x}$ to $\mathbf{y}$ in $\mathbb{T}_\alpha$ and b in $\mathbb{T}_\beta$. The difference $a - b$ gives a loop in $\mathrm{Sym}^g(\Sigma)$, and furthermore a loop in Y using following identification.

$$\frac{H_1(\mathrm{Sym}^g(\Sigma))}{H_1(\mathbb{T}_\alpha) \oplus H_1(\mathbb{T}_\beta)} \cong \frac{H_1(\Sigma)}{[\alpha_1], \cdots, [\alpha_g], [\beta_1], \cdots, [\beta_g]} \cong H_1(Y; \mathbb{Z})$$

Let us denote $\epsilon(\mathbf{x}, \mathbf{y})$ the image of $a - b$ in $H_1(Y; \mathbb{Z})$. Then $\epsilon(\mathbf{x}, \mathbf{y}) = 0$ iff $\boldsymbol{\pi}_2(\mathbf{x}, \mathbf{y}) \neq \phi$. This fact enables to read Whitney disk from Heegaard diagram.

In order to see existence of holomorphic curves, we also need to define *multiplicity* and *domain.* Following definitions are from definition 2.13 of [9].

Definition 2.4.1 *Fix a point $z \in \Sigma - \alpha_1 - \cdots - \alpha_g - \beta_1 \cdots - \beta_g$. Let n_z be a* *map which sends a Whitney disk u to the algebraic intersection number*

$$n_z(u) = \sharp u^{-1}(\{z\} \times \mathrm{Sym}^{g-1}(\Sigma)).$$

Definition 2.4.2 *Let $\mathcal{D}_1, \cdots, \mathcal{D}_m$ be closures of the components of $\Sigma - \alpha_1 - \cdots - \alpha_g - \beta_1 - \cdots - \beta_g$. Given a Whitney disk $u : \mathbb{D} \to \mathrm{Sym}^g(\Sigma)$, the* domain associated to u *is the formal linear combination of $\{\mathcal{D}_i\}_{i=1}^m$:*

$$\mathcal{D}(u) = \sum_{i=1}^{m} n_{z_i}(u)\mathcal{D}_i$$

where $z_i \in \mathcal{D}_i$ are the points in the interior of $\mathcal{D}_i$. If all the coefficients $n_{z_i} \geq 0$, then we write $\mathcal{D}(u) \geq 0$.

When $g = 1$, it is not hard to visualize such a holomorphic disk from Heegaard diagram, since $\mathrm{Sym}^g(\Sigma) = \Sigma$. In fact, any nonnegative bigon domain(thus enclosed by α_1 and β_1 and has 2 intersections are actually $\mathbf{x}$ and $\mathbf{y}$, respectively) has a holomorphic curve of its Whitney disk. By Riemann Mapping Theorem, existence of the holomorphic disk flowing from $\mathbf{x}$ to $\mathbf{y}$ is clear. However, finding holomorphic map when $g > 1$ is not easy in general.

Remark 2.4.3 *The nonnegativeness of coefficients of domain is crucial, because in the neighborhood of each $\{z_i\} \times \mathrm{Sym}^{g-1}(\Sigma)$, where $z_i \in \mathcal{D}_i$, we are assuming integrable almost complex structure so the holomorphic disk u has to meet it non-negatively.*

Remark 2.4.4 *Following proposition (Lemma 3.6, [9]) is extremely useful in computation of holomorphic disk.*

Lemma 2.4.5 *Given holomorphic disk $u \in \mathcal{M}_J(\mathbf{x}, \mathbf{y})$, there is a g-folded branched covering space $p : \widehat{\mathbb{D}} \to \mathbb{D}$ and holomorphic map $\widehat{u} : \widehat{\mathbb{D}} \to \Sigma$, with the property that for each $z \in \mathbb{D}$, $u(z) = \widehat{u} \circ p^{-1}(z)$.*

Let us confine ourselves that $g = 2$ and Y is a rational homology sphere, in fact S^3. Choose a point $z \in \Sigma - \alpha_1 - \cdots - \alpha_g - \beta_1 - \cdots - \beta_g$. Let $\mathbf{x} = \{x_1, x_2\}$ $\mathbf{y} = \{y_1, y_2\}$. Suppose there is a rectangular domain $\mathcal{D} = \sum_i n_{z_i} \mathcal{D}_i$, whose vertices are alternating between x_i and y_j in any direction ($i, j = 1, 2$). If this is the case, then if fact this is the only domain connecting $\mathbf{x}$ and $\mathbf{y}$ with $n_z(\mathcal{D}) = 0$, since it is rational homology sphere. In addition, let us assume that all coefficients are 1. Then there is a biholomorphic map $\widehat{u}$ between rectangular domain and a unit disk $\widehat{\mathbb{D}} \subset \mathbb{C}$. Letting $p : \widehat{\mathbb{D}} \to \mathbb{D}$ by $z \mapsto z^2$, we can use lemma above to ensure there is a holomorphic map $u : \mathbb{D} \to \mathrm{Sym}^2(\Sigma)$ obtained from $\widehat{u}$. Hence, if two generators $\mathbf{x}$ and $\mathbf{y}$ has only rectangular connecting domains on the Heegaard diagram, computation gets easier.

Heegaard Floer homology $\widehat{HF}$ is defined on a *pointed Heegaard diagram* $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, z)$ where z is a point on Σ away from curves $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$. Let $\mathfrak{S}(\mathcal{H})$ be a set of g-tuple of points, and $\sharp\mathcal{M}(\phi)$ be a signed number of points of moduli space $\mathcal{M}(\phi)$ of holomorphic disks in homotopy class ϕ. If the dimension of $\mathcal{M}$ is not zero, set $\sharp\mathcal{M}(\phi) = 0$. Heegaard Floer homology is generated by $\mathfrak{S}(\mathcal{H})$ over $\mathbb{F}_2$, whose differential ∂ is defined by

$$\widehat{\partial}\mathbf{x} = \sum_{\substack{\mathbf{y} \in \mathfrak{S}(\mathcal{H}) \\ \phi \in \pi_2(\mathbf{x}, \mathbf{y}), n_z(\phi) = 0}} \sharp(\mathcal{M}(\phi)) \cdot \mathbf{y}$$

A priori, the differential may not be a finite sum, which is the case when there are infinitely many holomorphic disks of one dimensional moduli space flowing between generators. To ensure the differential to be a finite sum, we need Heegaard diagram to be admissible.

Definition 2.4.6 *Heegaard diagram is* admissible *if every nontrivial periodic domain has both positive and negative multiplicities.*

Suppose $\boldsymbol{\pi}_2(\mathbf{x}, \mathbf{y})$ is nonempty, then there is domain D corresponding to the element of $\boldsymbol{\pi}_2(\mathbf{x}, \mathbf{y})$. Proposition 2.15 of [9] shows $\boldsymbol{\pi}_2(\mathbf{x}, \mathbf{y}) \cong \mathbb{Z} \oplus H^1(Y; \mathbb{Z})$. If diagram is not admissible, then one can find a positive periodic domain corresponding to generators of summands. This possibly results infinitely many holomorphic disks connecting generators.

Lipshitz invented a different approach of Heegaard Floer homology using cylindrical formulation without introducing a symmetric product of Heegaard surface Σ, and that still gives same result. One should consult [5] for the detailed description. We spell out the construction only.

The 4-manifold $W = \Sigma \times [0, 1] \times \mathbb{R}$ is required for his construction. We also consider projections maps

$$\begin{aligned}
\pi_\Sigma &: W \to \Sigma \\
\pi_{\mathbb{D}} &: W \to [0,1] \times \mathbb{R} \\
\pi_{\mathbb{R}} &: W \to \mathbb{R}
\end{aligned}$$

Instead of using holomorphic strip, we use a Riemann surface S with boundary. On the boundary there are g "-"-punctures $\{p_1, \cdots p_g\}$ and g "+"-punctures $\{q_1, \cdots, q_g\}$. Choosing an appropriate almost complex structure J on $\Sigma \times [0,1] \times \mathbb{R}$, we will consider following proper J-holomorphic curve $u : (S, \partial S) \to (\Sigma \times [0,1] \times \mathbb{R}, (\boldsymbol{\alpha} \times 1 \times \mathbb{R}) \cup (\boldsymbol{\beta} \times 0 \times \mathbb{R}))$.

- The source S is smooth (not nodal).
- The map u is an embedding.
- $u(\partial S) \subset (\boldsymbol{\alpha} \times \{1\} \times \mathbb{R}) \cup (\boldsymbol{\beta} \times \{0\} \times \mathbb{R})$
- There is no component of S on which $\pi_{\mathbb{D}} \circ u$ is constant.
- For each i, $u^{-1}(\alpha_i \times \{1\} \times \mathbb{R})$ and $u^{-1}(\beta_i \times \{0\} \times \mathbb{R})$ each consist of exactly one component of $\partial S \backslash \{p_1, \cdots, p_g, q_1, \cdots, q_g\}$.
- $\pi_{\mathbb{D}} \circ u$ is g-folded branched cover.
- At each $-$-puncture q, $\lim_{z \to q} (\pi_{\mathbb{R}} \circ u)(z) = -\infty$.
- At each $+$-puncture q, $\lim_{z \to q} (\pi_{\mathbb{R}} \circ u)(z) = +\infty$.

- $\pi_\Sigma \circ u$ does not cover the region of Σ adjacent to z.
- For each $t \in \mathbb{R}$ and each $i = 1, \cdots, g-k$, $u^{-1}(\alpha_i \times \{1\} \times \{t\})$ consists of exactly one point.
- The *energy* of u is finite in the sense of [1].

We call that curve u is *connecting* $\mathbf{x}$ and $\mathbf{y}$. As before $\epsilon(\mathbf{x}, \mathbf{y}) = 0$ condition ensures the existence of topological map for given domain. Lipshitz proves that homology class $H_2(\Sigma \times [0,1], \boldsymbol{\alpha} \times \{1\} \cup \boldsymbol{\beta} \times \{0\})$ is isomorphic to homotopy class $\boldsymbol{\pi}_2(\mathbf{x}, \mathbf{y})$ from the previous definition. The homology class will be also denoted $\boldsymbol{\pi}_2(\mathbf{x}, \mathbf{y})$, by abusing notation. We will count holomorphic curves in homology classes in $H_2(\Sigma \times [0,1], \boldsymbol{\alpha} \times \{1\} \cup \boldsymbol{\beta} \times \{0\})$, and denote moduli space of holomorphic curves $\mathcal{M}(\mathbf{x}, \mathbf{y})$. We define Heegaard Floer homology (in cylindrical setup), a $\mathbb{F}_2$-vector space generated by $\mathfrak{S}(\mathcal{H})$ with similarly defined differential as below.

$$\widehat{\partial}\mathbf{x} = \sum_{\substack{\mathbf{y} \in \mathfrak{S}(\mathcal{H}) \\ \phi \in \pi_2(\mathbf{x}, \mathbf{y}), n_z(\phi) = 0}} \sharp(\mathcal{M}(\phi)) \cdot \mathbf{y},$$

where $\sharp\mathcal{M}(\phi)$ counts holomorphic curves in homology class ϕ. See [5] for the detailed explanation.

2.5 Bordered Heegaard Floer Homology

We briefly recall the construction of bordered Heegaard diagram of connected boundary case, then consider two disconnected boundary component case later. Topics discussed here can be found in [6] in more generalized setup.

However, we will be mainly focusing on torus boundary case.

Let Σ be a surface of genus g with a puncture p and k be an integer less less than or equal to g. We will sometimes regard the puncture as a circle boundary. There are set of pairwise disjoint circles $\boldsymbol{\beta} = \{\beta_1, \cdots, \beta_g\}$ on Σ, and set of pairwise disjoint curves $\boldsymbol{\alpha} = \{\alpha^c_1, \cdots, \alpha^c_{g-k}, \alpha^a_1, \cdots, \alpha^a_{2k}\}$. α^c_i are circles on Σ, but α^a_j are arcs so that $\partial\alpha^a_j$ are on the boundary of Heegaard surface Σ. We also put a point z on the $\partial\Sigma$ away from endpoints of α^a_j.

Curves in $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ may intersect transversely. Lastly, $\Sigma \setminus \boldsymbol{\alpha}$ and $\Sigma \setminus \boldsymbol{\beta}$ are connected. Then the boundary of Σ is a pointed matched circle. That is, $Z = \partial\Sigma$, $\mathbf{a} = \{\alpha^a_j \cap Z\}^{2k}_{j=1}$, M be a matching that matches two points on Z connected by arcs, and z is a point on the boundary. $\Sigma \setminus \boldsymbol{\beta}$ is connected implies that the boundary of handlebody after attaching 1-handles along the matched points is a circle.

Definition 2.5.1 *A quadruple $\mathcal{H} = (\Sigma, \alpha, \beta, z)$ is called a* pointed Heegaard diagram.

Construction of three manifold $Y(\mathcal{H})$ from pointed Heegaard diagram $\mathcal{H}$ is as follows: First, thicken the Heegaard surface $[0,1] \times \overline{\Sigma}$, and attaching 3 dimensional 2 handles to $\beta_i \times \{1\} \times \overline{\Sigma}$ and $\alpha^c_j \times \{0\} \times \overline{\Sigma}$. Identifying the boundary of Heegaard surface to a pointed matched circle $\mathcal{Z}$ as described above, we get a parametrization of the boundary surface. More precisely, the resulting surface $\partial Y(\mathcal{H})$ consists of three components:

- a surface D_1, which is obtained after attaching 2 handles β_i to $\{1\} \times \overline{\Sigma}$

- a surface $D_2 = [0,1] \times \partial\overline{\Sigma}$
- a surface D_3, which is obtained after attaching 1 handles α_j^c to $\{0\} \times \overline{\Sigma}$

D_1 is a disk, D_2 is an annulus and D_3 is genus-g surface with a puncture. Of course $\partial Y(\mathcal{H}) = D_1 \cup D_2 \cup D_3$. This data can be considered to be a parametrization of genus g surface with preferred disk D_1 with parametrization is explicitly given by α_i^a. We denote the boundary surface $F(\mathcal{Z})$.

We now define *generator* of bordered Floer homology. Let $\mathbf{x} = \{x_1, \cdots, x_g\}$ such that

- each β circle contains exactly one x_i
- each α circle contains exactly one x_i
- each α arc contains at most one x_i.

We again denote $\mathfrak{S}(\mathcal{H})$ a set of all generators of Heegaard diagram $\mathcal{H}$.

We also need slightly change Lipshitz's formulation to define homology class, domains and admissibility for the bordered Floer homology. First of all, consider following homology group.

$$H_2(\overline{\Sigma} \times I_s \times I_t, ((S_{\boldsymbol{\alpha}} \cup S_{\boldsymbol{\beta}} \cup S_{\partial}) \times I_t) \cup (G_{\mathbf{x}} \times \{-\infty\}) \cup (G_{\mathbf{y}} \times \{\infty\})), \quad (2.1)$$

where

$$
\begin{aligned}
I_s &= [0,1] \\
I_t &= [-\infty,\infty] \\
S_{\boldsymbol{\alpha}} &= \boldsymbol{\alpha} \times \{1\} \\
S_{\boldsymbol{\beta}} &= \boldsymbol{\beta} \times \{0\} \\
S_{\partial} &= (\partial\overline{\Sigma}\backslash z) \times I_s \\
G_{\mathbf{x}} &= \mathbf{x} \times I_s \\
G_{\mathbf{y}} &= \mathbf{y} \times I_s
\end{aligned}
$$

The homology class will be also denoted $\boldsymbol{\pi}_2(\mathbf{x},\mathbf{y})$.

Definition 2.5.2 *A* homology class connecting $\mathbf{x}$ to $\mathbf{y}$*, denoted* $\boldsymbol{\pi}_2(\mathbf{x},\mathbf{y})$*, is element of above homology group. Moreover, projecting a homology class* $B \in \boldsymbol{\pi}_2(\mathbf{x},\mathbf{y})$ *onto* $\overline{\Sigma}$ *gives linear combination of components of* $\overline{\Sigma}\backslash(\boldsymbol{\alpha}\cup\boldsymbol{\beta})$*. We call such linear combinations* domain *of* B*.*

By abusing notation, B refers to a domain or homology class.

Remark 2.5.3 *Conventionally, we will be only interested in the domain whose coefficient of the region containing* z *equals zero, because this theory only interested in* $\widehat{HF}$*. Under this hypothesis, group of periodic domains* $\boldsymbol{\pi}_2(\mathbf{x},\mathbf{x})$ *is isomorphic to* $H_2(Y,\partial Y)$ *([6], Lemma 4.18).*

A boundary of domain B consists of three different kinds of domains; $\partial^{\alpha}B$, $\partial^{\beta}B$ and $\partial^{\partial}B$, contained in $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$ and $\partial\overline{\Sigma}$ respectively. If $\partial^{\partial}B = 0$, then we call B is *provincial.*

Definition 2.5.4 *A bordered Heegaard diagram is* admissible *if every periodic domain has both positive and negative coefficients. If every provincial domain has both positive and negative coefficients, it is called* provincially admissible.

Admissibility is a stronger condition then provincial admissibility. Every admissible diagram is provincially admissible, but this inclusion is strict.

To define bordered Floer homology, we will use cylindrical formulation with slight change. First fix two generators $\mathbf{x} = \{x_1, \cdots, x_g\}$ and $\mathbf{y} = \{y_1, \cdots, y_g\}$, and a set of Reeb chords $\{\rho_1, \cdots, \rho_l\}$. Let S be a Riemann surface with boundary. On the boundary of S, there are g $+$-punctures, g $-$-punctures, and l e punctures corresponding to Reeb chords. More precisely, each $-$-puncture corresponding to x_i and $+$- puncture to y_i, and each e puncture corresponding to Reeb chords. The conditions imposed on u carry on and there are some additional conditions required to handle the punctures on boundary.

- u extends to a proper map $u_{\overline{e}} : S_{\overline{e}} \to \Sigma_{\overline{e}} \times [0, 1] \times \mathbb{R}$, where $S_{\overline{e}}$ and $\Sigma_{\overline{e}}$ denote result of east punctures filled.
- For each $t \in \mathbb{R}$ and each $i = 1, \cdots 2k$, $u^{-1}(\alpha_i^a \times \{1\} \times \{t\})$ consists of at most one point. (*strong boundary monotonicity* condition)

e-punctures can be ordered by the map $\pi_{\mathbb{R}} \circ u$, so Reeb chords can be ordered as well. Thus we can rearrange the sequence of Reeb chords $\boldsymbol{\rho} = \{\rho_{i_1}, \cdots, \rho_{i_l}\}$.

Remark 2.5.5 *If genus g of boundary is greater than 1, each ρ_{i_j} may not be a single chord. In fact, ρ_{i_j} can be a sequence of chords.*

Fix $\mathbf{x}$ and $\mathbf{y}$. Let $B \in \pi_2(\mathbf{x}, \mathbf{y})$ be a homology class, where $\pi_2(\mathbf{x}, \mathbf{y})$ is a set of homology classes connecting $\mathbf{x}$ and $\mathbf{y}$. We also fix a sequence of Reeb chords $\boldsymbol{\rho}$. Now define a moduli space of curves $\overline{\mathcal{M}}^B(\mathbf{x}, \mathbf{y}; \boldsymbol{\rho})$, and denote $\mathcal{M}^B(\mathbf{x}, \mathbf{y}; \boldsymbol{\rho}) := \overline{\mathcal{M}}^B(\mathbf{x}, \mathbf{y}; \boldsymbol{\rho})/\mathbb{R}$ to be a reduced moduli space. Sometimes we will call *a pair* $(B, \boldsymbol{\rho})$ *is compatible.*

$\widehat{CFD}(\mathcal{H})$ is type D structure generated over $\mathbb{F}_2$ by set of all generators $\mathfrak{S}(\mathcal{H})$. We set a pointed matched circle $-\mathcal{Z} = (-\partial\Sigma,\ \mathbf{a} = \{\partial\alpha_1^a, \cdots, \partial\alpha_{2k}^a\},\ z)$. Here, the negative sign denotes opposite orientation induced from Σ.

From now on, we will focus more on the torus boundary case. Then $-\mathcal{Z} = (-\partial\Sigma,\ \mathbf{a} = \{\partial\alpha_1^a, \partial\alpha_2^a\},\ z)$ is the pointed matched circle, and the torus algebra $\mathcal{A}(T) = \mathcal{A}(-\mathcal{Z})$. Let $\iota_i \in \mathcal{A}(T)$ be an idempotent associated to α_i^a, $i = 1, 2$, respectively. To get a module structure, the left $\mathcal{I}$-action is defined as follows.

$$\iota_1\mathbf{x} = \begin{cases} \mathbf{x} & \text{if } \mathbf{x} \text{ does not occupy the arc } \alpha_1^a \\ 0 & \text{otherwise} \end{cases}$$

$$\iota_2\mathbf{x} = \begin{cases} \mathbf{x} & \text{if } \mathbf{x} \text{ does not occupy the arc } \alpha_2^a \\ 0 & \text{otherwise} \end{cases}$$

As an $\mathcal{A}(T)$-module,

$$\widehat{CFD}(\mathcal{H}) := \mathcal{A}(T) \otimes_{\mathcal{I}} \mathfrak{S}(\mathcal{H})$$

Type D structure map $\delta^1 : \widehat{CFD}(\mathcal{H}) \to \mathcal{A}(T) \otimes \widehat{CFD}(\mathcal{H})$ is given by counting pseudo holomorphic curves in $\mathcal{M}^B(\mathbf{x}, \mathbf{y}; \boldsymbol{\rho})$ whose expected dimension equals 0. (Equivalently, the *index* $\mathrm{ind}(B, \boldsymbol{\rho})$ equals 1.)

We define the map δ^1 by

$$\delta^1(\mathbf{x}) := \sum_{\mathbf{y} \in \mathfrak{S}(\mathcal{H})} \sum_{\substack{B \in \pi_2(\mathbf{x}, \mathbf{y}) \\ \{\boldsymbol{\rho} | \mathrm{ind}(B, \boldsymbol{\rho}) = 1\}}} \sharp(\mathcal{M}^B(\mathbf{x}, \mathbf{y}; \boldsymbol{\rho}))(-\rho_{i_1}) \cdots (-\rho_{i_l}) \otimes \mathbf{y}.$$

Note that the negative signs before torus algebra elements denote orientations of associated Reeb chords have been reversed.

The number $\sharp(\mathcal{M}^B(\mathbf{x}, \mathbf{y}; \boldsymbol{\rho}))$ of pseudo holomorphic curves is counted modulo 2. This sum is a finite sum, if the Heegaard diagram is provincially admissible. Chapter 6 of [6] proves that this map is in fact a type D structure map.

Again, a question on finiteness of the differential map δ^1 arises. Finiteness is guaranteed if the bordered Heegaard diagram is admissible. However, to compute $\widehat{CFD}$, provincial admissibility condition is enough. In fact, a domain adjacent to the boundary whose coefficient is different from 1 cannot contribute to differential. If the coefficient is greater than 1, it leads to same Reeb chords appear twice or more in the differential and its product must be zero.

Expected dimension of $\mathcal{M}^B(\mathbf{x}, \mathbf{y}; \boldsymbol{\rho})$, or index $\mathrm{ind}(B, \boldsymbol{\rho})$ can be computed from the diagram. First, we define *Euler measure* of a region in $\Sigma \backslash (\boldsymbol{\alpha} \cup \boldsymbol{\beta})$ to be a Euler characteristic of the region minus ($\frac{1}{4} \times \{$number of 90° corners$\} - \frac{1}{4} \times \{$number of 270° corners$\}$). Then we extend it to domain so it to be additive under union, and denote it $e(B)$. For a homology class $B \in \boldsymbol{\pi}_2(\mathbf{x}, \mathbf{y})$, where generators $\mathbf{x} = \{x_1, \cdots, x_g\}$ and $\mathbf{y} = \{y_1, \cdots, y_g\}$, we define

$$n_{\mathbf{x}}(B) = \sum_{i=1}^{g} \frac{1}{4} \{ \text{ sum of four coefficients of domains surrounding } x_i \}.$$

$n_{\mathbf{y}}(B)$ is similar.

A sequence of Reeb chords $\boldsymbol{\rho}$ also affects the index. Recall $\mathcal{Z} = (Z, \mathbf{a}, M, z)$ denoted pointed matched circle. For a single Reeb chord ρ in $(Z \backslash z, \mathbf{a})$, then we let $\rho^- \in \mathbf{a}$ denote the initial endpoint and $\rho^+ \in \mathbf{a}$ the final endpoint of ρ.

Definition 2.5.6 Linking of ρ_1 and ρ_2, *denoted $L(\rho_1, \rho_2)$, is defined as follows.*

$$L(\rho_1, \rho_2) = \begin{cases} 1/2 & \text{if } \rho_1^+ = \rho_2^- \\ -1/2 & \text{if } \rho_2^+ = \rho_1^- \\ 0 & \text{if } \rho_1 \cap \rho_2 = \phi \text{ or } \rho_1 \subset \rho_2 \text{ or } \rho_2 \subset \rho_1 \text{ or } \rho_1 = \rho_2 \\ 1 & \text{if } \rho_1^- \lessdot \rho_2^- \lessdot \rho_1^+ \lessdot \rho_2^+ \\ -1 & \text{if } \rho_2^- \lessdot \rho_1^- \lessdot \rho_2^+ \lessdot \rho_1^+. \end{cases}$$

Then a quantity $\iota(\boldsymbol{\rho})$, where $\boldsymbol{\rho} = (\rho_1, \cdots, \rho_l)$, is

$$\iota(\boldsymbol{\rho}) := -\frac{l}{2} + \sum_{i<j} L(\rho_i, \rho_j).$$

Now we are ready to give formula of index of homology class B compatible to $\boldsymbol{\rho}$([6], Definition 5.61).

$$\mathrm{ind}(B, \boldsymbol{\rho}) := e(B) + n_{\mathbf{x}}(B) + n_{\mathbf{y}}(B) + |\boldsymbol{\rho}| + \iota(\boldsymbol{\rho})$$

where $|\boldsymbol{\rho}|$ is the length of the sequence $\boldsymbol{\rho}$.

We can also define right $\mathcal{A}(T)$-module $\widehat{CFA}(\mathcal{H})$. This time we set a pointed matched circle $\mathcal{Z} = (\partial\Sigma,\ \mathbf{a} = \{\partial\alpha_1^a, \partial\alpha_2^a\},\ z)$ whose orientation is induced from Σ. The torus algebra $\mathcal{A}(T) = \mathcal{A}(\mathcal{Z})$. The right $\mathcal{A}(T)$-module $\widehat{CFA}(\mathcal{H})$ is strictly unital ($\mathcal{A}_\infty$) algebra, generated by $\mathfrak{S}(\mathcal{H})$ over $\mathbb{F}_2$ with right $\mathcal{I}$-action

$$\mathbf{x}\iota_1 = \begin{cases} \mathbf{x} & \text{if } \mathbf{x} \text{ does occupy the arc } \alpha_1^a \\ 0 & \text{otherwise} \end{cases}$$

$$\mathbf{x}\iota_2 = \begin{cases} \mathbf{x} & \text{if } \mathbf{x} \text{ does occupy the arc } \alpha_2^a \\ 0 & \text{otherwise} \end{cases}$$

Then there are maps

$$m_{n+1} : \widehat{CFA}(\mathcal{H}) \otimes_{\mathcal{I}} \mathcal{A} \otimes_{\mathcal{I}} \cdots \otimes_{\mathcal{I}} \mathcal{A} \to \widehat{CFA}(\mathcal{H})$$

defined by

$$m_{n+1}(\mathbf{x}, \rho_1, \cdots, \rho_n) = \sum_{y \in \mathfrak{S}(\mathcal{H})} \sum_{\substack{B \in \boldsymbol{\pi}_2(\mathbf{x},\mathbf{y}) \\ \mathrm{ind}(B,\boldsymbol{\rho})=1}} \sharp(\mathcal{M}^B(\mathbf{x}, \mathbf{y}; \rho_1, \cdots, \rho_n))\mathbf{y}.$$

This map is proven to be satisfying $\mathcal{A}_\infty$ compatibility conditions([6], chapter 7).

Usually $\widehat{CFA}$ has richer structure than $\widehat{CFD}$. For example, if a sequence $\boldsymbol{\rho} = \{\rho_1, \cdots, \rho_n\}$ of a compatible pair $(B, \boldsymbol{\rho})$ has two consecutive Reeb chords ρ_i and ρ_{i+1} such that $\rho_i \cdot \rho_{i+1} = 0$, $\widehat{CFD}$ considers terms related the compatible pair zero. However, such compatible pair may result nontrivial term in $\widehat{CFA}$.

2.6 Doubly Bordered Heegaard Floer Homology

If a three manifold Y has connected boundary, the surface $F(\mathcal{Z})$, preferred disk D and a point on the ∂D determine parametrization of the boundary of Y. We write this data a triple $(F(\mathcal{Z}),\ D,\ z)$. However if Y has two disconnect boundary components, we need to fix two surfaces $(F_1(\mathcal{Z}_1),\ D_1,\ z_1)$ and $(F_2(\mathcal{Z}_2),\ D_2,\ z_2)$. Fixing a framed arc pointing into D_i at z_i, $i = 1, 2$, we drill a tunnel along the framed arc so that we get a single boundary surface whose genus is the sum of genuses of $F_1(\mathcal{Z}_1)$ and $F_2(\mathcal{Z}_2)$. The the bimodule of the doubly bordered three manifold Y is defined via $\widehat{CFD}$ of the drilled manifold, namely Y_{dr}.

Definition 2.6.1 *An* arced bordered Heegaard diagram *with two boundary component is a tuple* $(\Sigma, \alpha, \beta, \mathbf{z})$ *satisfying:*

- $\overline{\Sigma}$ *is a compact, genus* g *surface with 2 boundary component* $\partial_L\overline{\Sigma}$ *and* $\partial_R\overline{\Sigma}$*.*

- $\boldsymbol{\beta}$ *is* g*-tuple of pairwise disjoint curves in the interior of* $\overline{\Sigma}$*.*

- $\boldsymbol{\alpha} = \{\boldsymbol{\alpha}^{a,L} = \{\alpha_1^{a,L}, \cdots, \alpha_{2l}^{a,L}\},\ \boldsymbol{\alpha}^{a,R} = \{\alpha_1^{a,R}, \cdots, \alpha_{2r}^{a,R}\},\ \boldsymbol{\alpha}^c = \{\alpha_1^c, \cdots, \alpha_{g-l-r}^c\}\}$, *is a collection of pairwise disjoint embedded arcs with boundary on* $\partial_L\overline{\Sigma}$ *(the* $\alpha_i^{a,L}$*), arcs with boundary on* $\partial_R\overline{\Sigma}$ *(the* $\alpha_i^{a,R}$*), and circles (the* α_i^c*) in the interior of* $\overline{\Sigma}$*.*

- $\mathbf{z}$ *is a path in* $\overline{\Sigma}\backslash(\boldsymbol{\alpha} \cup \boldsymbol{\beta})$ *between* $\partial_L\overline{\Sigma}$ *and* $\partial_R\overline{\Sigma}$*.*

As usual, we denote $\mathcal{Z}_L$ (respectively, $\mathcal{Z}_R$) be a pointed matched circle on the left (respectively, on the right).

Construction of doubly bordered three manifold from an arced bordered Heegaard diagram is as follows. First cut open Σ along the path $\mathbf{z}$. Since $\mathbf{z}$ is connecting to boundaries $\partial_L\overline{\Sigma}$ and $\partial_R\overline{\Sigma}$ of $\overline{\Sigma}$, the resulting diagram $\mathcal{H}_{dr}$ is a Heegaard diagram of single boundary. Thicken $\mathcal{H}_{dr}$ and attaching 3-dimensional 2-handles on it, we get a bordered manifold Y_{dr} with single boundary component. The boundary can be decomposed into three pieces: $F_1(\mathcal{Z}_L)\backslash D_1$, $F_2(\mathcal{Z}_R)\backslash D_2$, and an annulus A. If we glue another 3-dimensional 2 handles along the annulus, we obtain the required doubly bordered three manifold.

There are three types of bimodule; $\widehat{CFAA}$, $\widehat{CFDA}$, $\widehat{CFDD}$. We will be interested in $\widehat{CFDD}$ and $\widehat{CFAA}$ only. Before giving definitions of these bimodules, we introduce algebraic preliminaries of them.

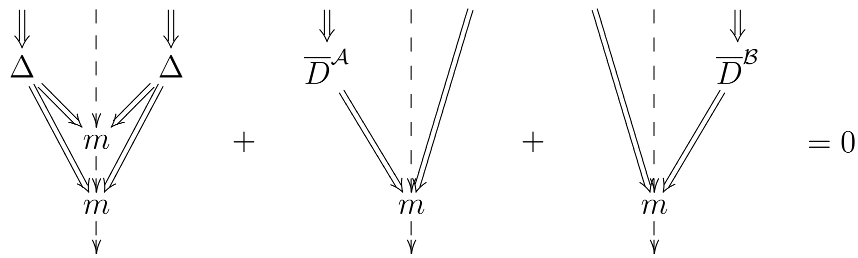
Definition 2.6.2 *Let $\mathcal{A}$ and $\mathcal{B}$ be $\mathcal{A}_\infty$-algebras over $\mathbb{F}$.* $\mathcal{A}_\infty$-bimodule ${}_{\mathcal{A}}M_{\mathcal{B}}$ *over $\mathcal{A}$ and $\mathcal{B}$ consists of $(\mathbb{F},\mathbb{F})$-bimodule M and maps*

$$m_{i,1,j} : A^{\otimes i} \otimes M \otimes B^{\otimes j} \to M$$

such that following compatibility condition holds.

$$\begin{aligned} 0 = &\sum_{\substack{k+l=i+1 \\ \lambda+\eta=j+1}} m_{k,1,\lambda}(m_{l,1,\eta}(\mathbf{x}, a_1^L \otimes \cdots \otimes a_{l-1}^L, a_1^R \otimes \cdots \otimes a_{\lambda-1}^R), a_l^L \otimes \cdots \otimes a_{i-1}^L, a_\lambda^R \otimes \cdots \otimes a_{j-1}^R) \\ &+ \sum_{k+l=i+1} \sum_{n=1}^{i-l} m_{k,1,j}(\mathbf{x}, a_1^L \otimes \cdots \otimes a_{n-1}^L \otimes \mu_l(a_n^L \otimes \cdots \otimes a_{n+l-1}^L) \otimes \cdots \otimes a_{i-1}^L, a_1^R \otimes \cdots \otimes a_{j-1}^R) \\ &+ \sum_{\lambda+\eta=j+1} \sum_{n=1}^{j-\eta} m_{i,1,\lambda}(\mathbf{x}, a_1^L \otimes \cdots \otimes a_{i-1}^L, a_1^R \otimes \cdots \otimes a_{n-1}^R \otimes \mu_l(a_n^R \otimes \cdots \otimes a_{n+l-1}^R) \otimes \cdots \otimes a_{j-1}^R) \end{aligned}$$

for all i and j. By denoting $m = \sum_{i,j} m_{i,1,j}$, the compatibility condition can *be drawn in diagram as below.*



As in section 3.1, a dashed line represents module element, and a double line represents element from tensor algebra $\mathcal{T}^*(A)$.

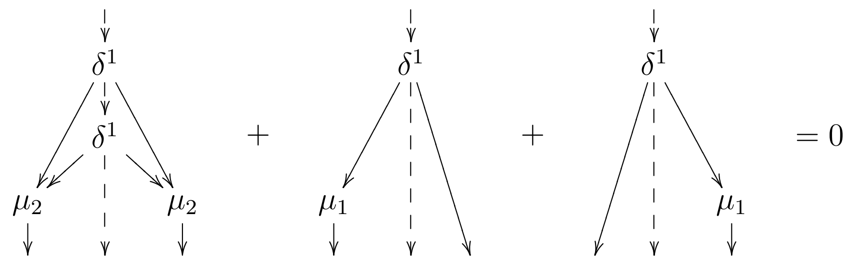
Definition 2.6.3 *Let* $\mathcal{A}$ *and* $\mathcal{B}$ *are* $\mathcal{A}_\infty$*-algebras over* $\mathbb{F}$*.* $\mathcal{A}_\infty$-bimodule ${}^{\mathcal{A}}M^{\mathcal{B}}$ *over* $\mathcal{A}$ *and* $\mathcal{B}$ *consists of* $(\mathbb{F},\mathbb{F})$*-bimodule* M *and maps*

$$\delta^1 : M \to A \otimes M \otimes B$$

satisfying following compatibility condition.

$$((\mu_2^L,\mu_2^R)\otimes\mathbb{I}_M)\circ((\mathbb{I}_A,\mathbb{I}_B)\otimes\delta^1)\circ\delta^1+((\mu_1^L,\mathbb{I}_B)\otimes\mathbb{I}_M)\circ\delta^1+((\mathbb{I}_A,\mu_1^R)\otimes\mathbb{I}_M)\circ\delta^1=0$$

Again, the compatibility condition is drawn in diagram as below.



A generating set of bimodules $\widehat{CFDD}(\mathcal{H})$ and $\widehat{CFAA}(\mathcal{H})$ is same as the set $\mathfrak{S}(\mathcal{H}_{dr})$ of generators of drilled diagram, which will be denoted $\mathfrak{S}(\mathcal{H})$. For given two generators $\mathbf{x}$ and $\mathbf{y}$, a homology class $\boldsymbol{\pi}_2(\mathbf{x},\mathbf{y})$ connecting $\mathbf{x}$ and $\mathbf{y}$ is defined in similar manner. Likewise, a domain of homology class $B \in \boldsymbol{\pi}_2(\mathbf{x},\mathbf{y})$ is a linear combination of components of $\overline{\Sigma}\backslash(\boldsymbol{\alpha}\cup\boldsymbol{\beta})$. By convention, we do not count homology classes or domains that crosses the region contains the path $\mathbf{z}$.

A boundary of domain of homology class B is union of left and right boundaries; that is, $\partial^\partial B = \partial^\partial_L B \cup \partial^\partial_R B$. We let

$$\begin{aligned}
\boldsymbol{\pi}_2^\partial(\mathbf{x},\mathbf{y}) &= \{B \in \boldsymbol{\pi}_2(\mathbf{x},\mathbf{y}) | \partial^\partial B = 0\} \\
\boldsymbol{\pi}_2^{\partial_L}(\mathbf{x},\mathbf{y}) &= \{B \in \boldsymbol{\pi}_2(\mathbf{x},\mathbf{y}) | \partial^\partial_L B = 0\} \\
\boldsymbol{\pi}_2^{\partial_R}(\mathbf{x},\mathbf{y}) &= \{B \in \boldsymbol{\pi}_2(\mathbf{x},\mathbf{y}) | \partial^\partial_R B = 0\}.
\end{aligned}$$

We denote elements of the homology classes above *provincial domain, left-provincial domain, right-provincial domain* respectively.

A doubly bordered Heegaard diagram is *provincially admissible* if the bordered diagram $\mathcal{H}_{dr}$ is admissible. Moreover, we call a diagram is *left* (respectively *right*) admissible if every nontrivial left-provincial periodic domain $B \in \boldsymbol{\pi}_2^{\partial_L}(\mathbf{x},\mathbf{x})$ (respectively right-provincial periodic domain $B \in \boldsymbol{\pi}_2^{\partial_R}(\mathbf{x},\mathbf{x})$) has positive and negative coefficients.

We now turn to moduli space of curves. Let $\mathbf{x}$ and $\mathbf{y}$ be generators, and $B \in \boldsymbol{\pi}_2(\mathbf{x},\mathbf{y})$ be homology class connecting them. Once the homology class B is fixed, there is a compatible pair $(B, \boldsymbol{\rho})$. The ordered set of Reeb chords $\boldsymbol{\rho}$ has both left and right Reeb chords. Consider a union of two ordered sets of Reeb chords $\overrightarrow{\rho^L} \coprod \overrightarrow{\rho^R}$, where $\overrightarrow{\rho^L}$ (respectively $\overrightarrow{\rho^R}$) consists of left Reeb chords (respectively right Reeb chords). An ordered set of Reeb chords $\boldsymbol{\rho}$ said to *interleave* $\overrightarrow{\rho^L} \coprod \overrightarrow{\rho^R}$, if $\rho = \overrightarrow{\rho^L} \coprod \overrightarrow{\rho^R}$ as a set and the orderings of $\overrightarrow{\rho^L}$ and $\overrightarrow{\rho^R}$ agree with the orderings induced by $\boldsymbol{\rho}$. We will sometimes use $(B, \overrightarrow{\rho^L}, \overrightarrow{\rho^R})$ and $\mathcal{M}^B(\mathbf{x},\mathbf{y};\overrightarrow{\rho^L},\overrightarrow{\rho^R})$ to denote the compatible pair and its moduli space. The expected dimension of moduli space of $\mathcal{M}^B(\mathbf{x},\mathbf{y};\overrightarrow{\rho^L},\overrightarrow{\rho^R})$, or $\mathrm{ind}(B,\overrightarrow{\rho^L},\overrightarrow{\rho^R})$ is computed by formula given above, but it also can be written in terms of $\overrightarrow{\rho^L}$ and $\overrightarrow{\rho^R}$ as below.

$$\mathrm{ind}(B,\boldsymbol{\rho}) = e(B) + n_{\mathbf{x}}(B) + n_{\mathbf{y}}(B) + |\overrightarrow{\rho^L}| + |\overrightarrow{\rho^R}| + \iota(\overrightarrow{\rho^L}) + \iota(\overrightarrow{\rho^R})$$

Now we are ready to associate two types of bimodules on doubly bordered Heegaard diagram. First we define $\widehat{CFDD}(\mathcal{H})$.

The left boundary $-\partial_L\overline{\Sigma}$ with a point $z_L = \partial_L\overline{\Sigma} \cup \mathbf{z}$, whose orientation is opposite from the induced orientation, can be considered as a pointed matched circle; i.e, we let $-\mathcal{Z}_L = (-\partial_L\overline{\Sigma}, \mathbf{a}_L = \{\partial\alpha_1^{a,L}, \partial\alpha_2^{a,L}\}, z_L)$ is the pointed matched circle. Then the we get the torus algebra $\mathcal{A}(T) = \mathcal{A}(-\mathcal{Z}_L)$ on the left boundary. Construction of the torus algebra on the right boundary is also similar.

There is an idempotent action on $\mathfrak{S}(\mathcal{H})$. Recall that the torus algebra has a subset $\mathcal{I}$ of idempotent elements, namely $\mathcal{I}_L := \{\iota_1, \iota_2\} \subset \mathcal{A}(-\mathcal{Z}_L)$ and $\mathcal{I}_R := \{\jmath_1, \jmath_2\} \subset \mathcal{A}(-\mathcal{Z}_R)$. The left and right idempotent action is defined to be,

$$\iota_i \jmath_j \mathbf{x} = \begin{cases} \mathbf{x} & \text{if } \alpha_i^{a,L} \text{ and } \alpha_j^{a,R} \text{ are not occupied by } \mathbf{x} \\ 0 & \text{otherwise} \end{cases}$$

Then the map $\delta^1 : \mathfrak{S}(\mathcal{H}) \to \mathcal{A}(-\mathcal{Z}_L) \otimes_{\mathcal{I}_L} \mathcal{A}(-\mathcal{Z}_R) \otimes_{\mathcal{I}_R} \mathfrak{S}(\mathcal{H})$ is similarly defined by taking summation on all possible holomorphic representatives of compatible pair $(B, \boldsymbol{\rho})$.

$$\delta^1(\mathbf{x}) := \sum_{\mathbf{y} \in \mathfrak{S}(\mathcal{H})} \sum_{\substack{B \in \boldsymbol{\pi}_2(\mathbf{x},\mathbf{y}) \\ \text{ind}(B; \overrightarrow{\rho^L}, \overrightarrow{\rho^R}) = 1}} \sharp(\mathcal{M}^B(\mathbf{x}, \mathbf{y}; \overrightarrow{\rho^L}, \overrightarrow{\rho^R}))(-\rho_{i_1}^L) \cdots (-\rho_{i_l}^L) \otimes (-\rho_{j_1}^R) \cdots (-\rho_{j_m}^R) \otimes \mathbf{y}$$

where $\overrightarrow{\rho^L} = \{\rho_{i_1}^L, \cdots, \rho_{i_l}^L\}$ and $\overrightarrow{\rho^R} = \{\rho_{j_1}^R \cdots \rho_{j_l}^R\}$. Provincial admissibility ensures that this sum is finite.

$\widehat{CFAA}(\mathcal{H})$ is an $\mathcal{A}_\infty$-bimodule over left and right $\mathcal{A}_\infty$ algebras $\mathcal{A}(\mathcal{Z}_L)$ and $\mathcal{A}(\mathcal{Z}_R)$ also generated by $\mathfrak{S}(\mathcal{H})$. The idempotent action is opposite from $\widehat{CFDD}$ case.

$$\mathbf{x}\iota_i\jmath_j = \begin{cases} \mathbf{x} & \text{if } \alpha_i^{a,L} \text{ and } \alpha_j^{a,R} \text{ are occupied by } \mathbf{x} \\ 0 & \text{otherwise} \end{cases}$$

Then following $\mathcal{A}_\infty$ module map

$$m_{i,1,j} : \mathfrak{S}(\mathcal{H}) \otimes_{\mathcal{I}_L} \underbrace{\mathcal{A}(\mathcal{Z}_L) \otimes \cdots \otimes \mathcal{A}(\mathcal{Z}_L)}_{i \text{ times}} \otimes_{\mathcal{I}_R} \underbrace{\mathcal{A}(\mathcal{Z}_R) \otimes \cdots \otimes \mathcal{A}(\mathcal{Z}_R)}_{j \text{ times}} \to \mathfrak{S}(\mathcal{H})$$

satisfies compatibility condition([7]).

$$m_{i,1,j}(\mathbf{x}; \rho_{i_1}^L, \cdots \rho_{i_l}^L, \rho_{j_1}^R, \cdots, \rho_{j_m}^R) = \sum_{\mathbf{y} \in \mathfrak{S}(\mathcal{H})} \sum_{\substack{B \in \boldsymbol{\pi}_2(\mathbf{x}, \mathbf{y}) \\ \operatorname{ind}(B, \overrightarrow{\rho^L}, \overrightarrow{\rho^R}) = 1}} \sharp(\mathcal{M}^B(\mathbf{x}, \mathbf{y}; \overrightarrow{\rho^L}, \overrightarrow{\rho^R}))\mathbf{y}$$

Chapter 3

Computation of $\widehat{CFDD}$ of $(2, 2n)$ Torus link

3.1 Schubert normal form and diagram of 2-bridge link complement

We will be mainly interested in 2-bridge link, so it is useful to mention *Schubert normal form* of 2-bridge link(or knot). Let p be an even positive integer and q be an integer such that $0 < q < p$ and $\gcd(p, q) = 1$. Let us consider a circle with $2p$ marked point on its boundary. Choose a point and label it a_0. Label other points $a_1, \cdots, a_{2p-1}$ in clockwise direction. Then connect a_i and a_{2p-i} with straight line, $i = 1, \cdots, p-1$. Finally connect a_0 and a_p with underbridge, a straight line that crosses below other straight lines.

Now consider two copies of such circle. Draw arcs between these two cir-

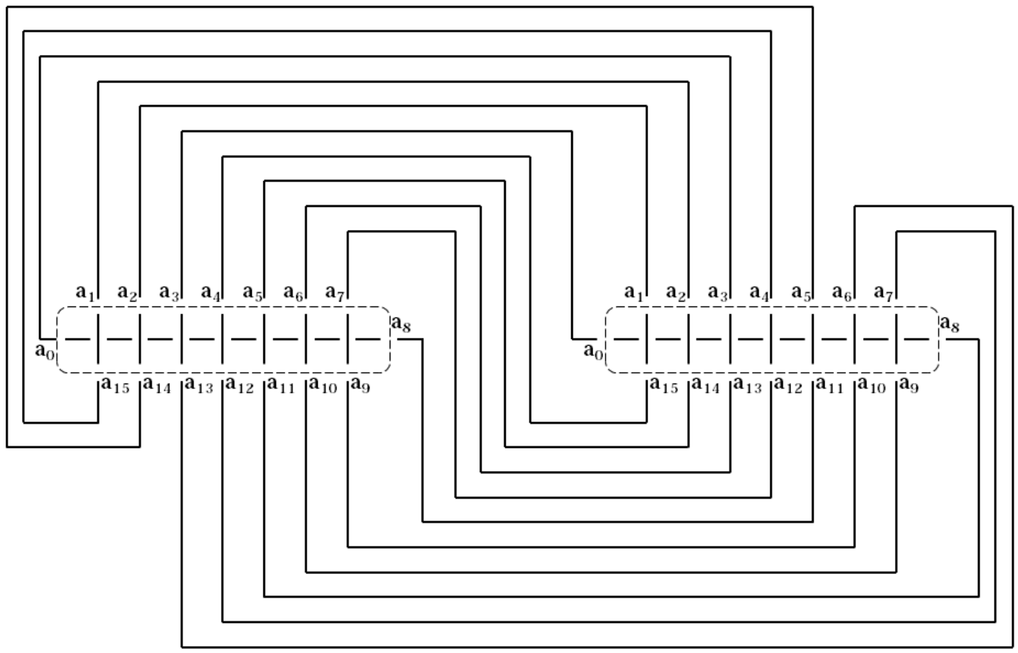


Figure 3.1: Schubert normal form of $S(8,3)$-link. According to Thistlethwaite's table, it is $L5a1$ link.

cles, so that each arc is connecting a_i from on one circle and a_{q-i} on the other(the labeling is modulo $2p$). These arcs should not intersects any other straight lines nor other arcs. The resulting diagram gives a link that we denote $S(p,q)$. The diagram is called *Schubert normal form* of the link. See [Figure 3.1]. More detailed description, especially about the Schubert normal form of 2-bridge knot can be found in chapter 2 of [13].

Recall that 2-bridge link L is a link in S^3 that admits a link diagram with two maxima and two minima. Given such a link diagram we can also construct Schubert normal form. Let B_1 and B_2 be small neighborhoods of those two maxima. Consider $(S^3 \backslash \nu L) \backslash (B_1 \cup B_2)$. Drilling a tunnel connecting B_1 and B_2 gives a three-manifold Y with single boundary and the boundary is a genus 2 surface.

The boundary of Y can be viewed as 2-sphere with 4 punctures attached to 2 tubes. For simplicity we assume the two punctures on the left corresponds to the link component on the left and vice versa. Each component has its longitude, and it passes on the 4-punctured sphere as a straight line segment connecting two punctures of the component.

This genus 2 surface divides S^3 into two pieces and one of them may not be a handlebody. Now we apply isotopy on the tubes on the surface so that the two pieces of S^3 are both 2-handlebodies. During this procedure, the part of the longitudes on the 2-sphere become 2 arcs connecting corresponding punctures on the sphere. Regard the 4-punctured sphere as $D^2 \cup \{\infty\}$ with 4 marked points, then the longitudes are 2 arcs connecting 2 marked points of each component on plane. Finally draw two "underpasses" connecting corresponding marked points. See [Figure 3.2].

In this section, we compute a bimodule of torus link complement. $(2, 2n)$ torus link is a 2 bridge link which can be embedded on unknotted torus, whose number of crossing is $2n$. See [Figure 3.3]. $(2, 2n)$-torus link can be written as $S(2n, 1)$ link in Schubert normal form.

We will construct doubly bordered Heegaard diagram of $(2, 2n)$ torus link

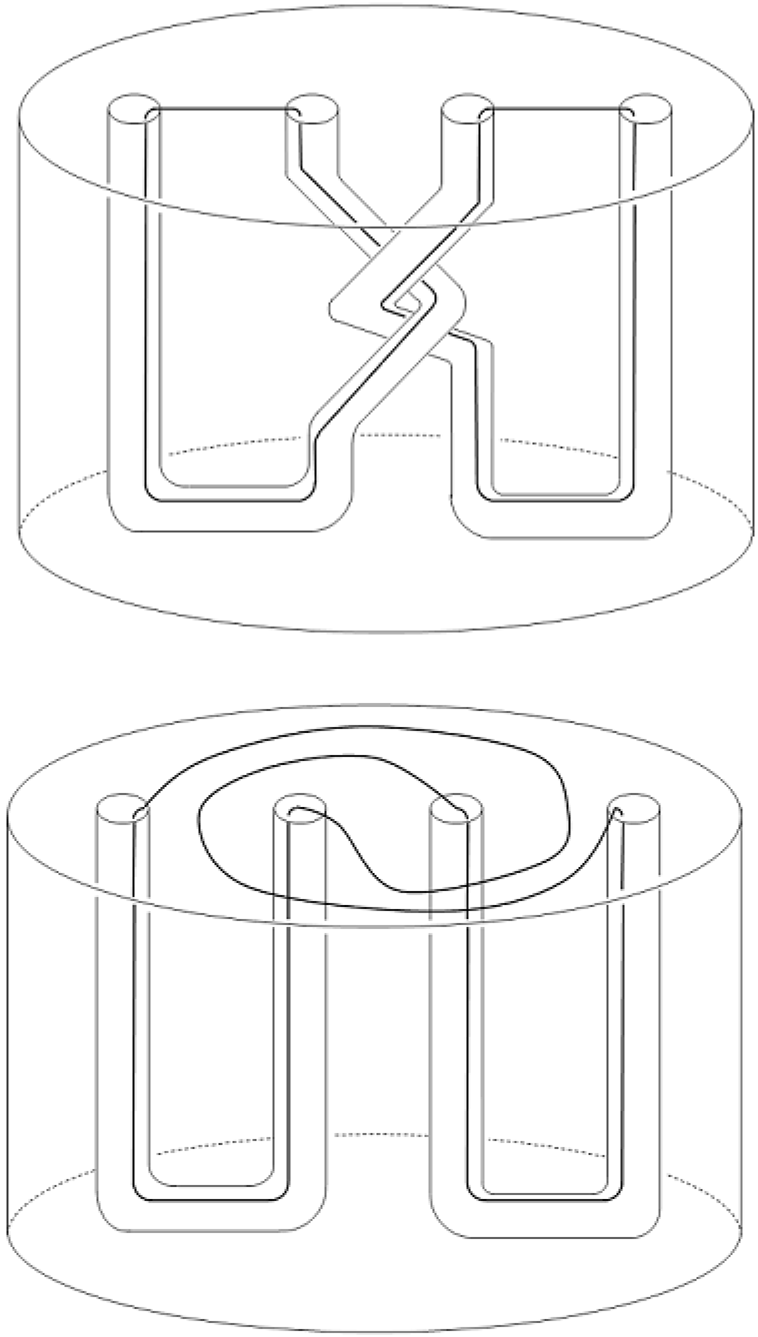


Figure 3.2: The picture above is a genus two surface obtained by digging a tunnel between two components of $S^3 \backslash \nu L$. Below is obtained after untwist braid in the middle.

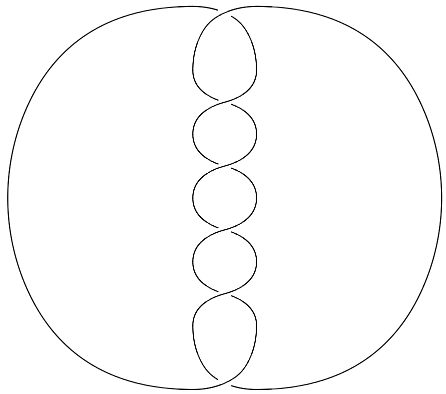


Figure 3.3: $(2,6)$-torus link. In general, $(2,2n)$-torus link has $2n$ crossings in its alternating projection.

complement from Schubert normal form. Consider Schubert normal form of the link, which consists of genus 2 surface Σ and two longitudes α_L and α_R. The genus 2 surface Σ has two underpasses that can be considered as annuli. Each annulus has a circle that generates its first homology class, and we denote these two generators μ_L and μ_R. Let β_L and β_R be circles on Σ such that each circle crosses μ_L or μ_R, respectively. We consider μ_L and μ_R as meridians of components of the link. Each meridians intersects its corresponding longitude at a single point. Puncture the intersections so that we get a doubly bordered Heegaard diagram. Thus we get $\alpha_1^{a,L}$ and $\alpha_1^{a,R}$ from α_L and α_R, and $\alpha_2^{a,L}$ and $\alpha_2^{a,R}$ from μ_L and μ_R. To maintain notational consistency with [7], we will use β_1 for β_L, β_2 for β_R, and $\partial_L\Sigma$ and $\partial_R\Sigma$ for left and right punctures. Lastly, choose a domain whose boundary is adjacent to $\partial_L\Sigma$ and $\partial_R\Sigma$, and draw an (framed) arc connects $\partial_L\Sigma$ and $\partial_R\Sigma$ that lies on the chosen domain. See [Figure 3.5].

Remark 3.1.1 *Reader should be aware that to connect left and right punctures*

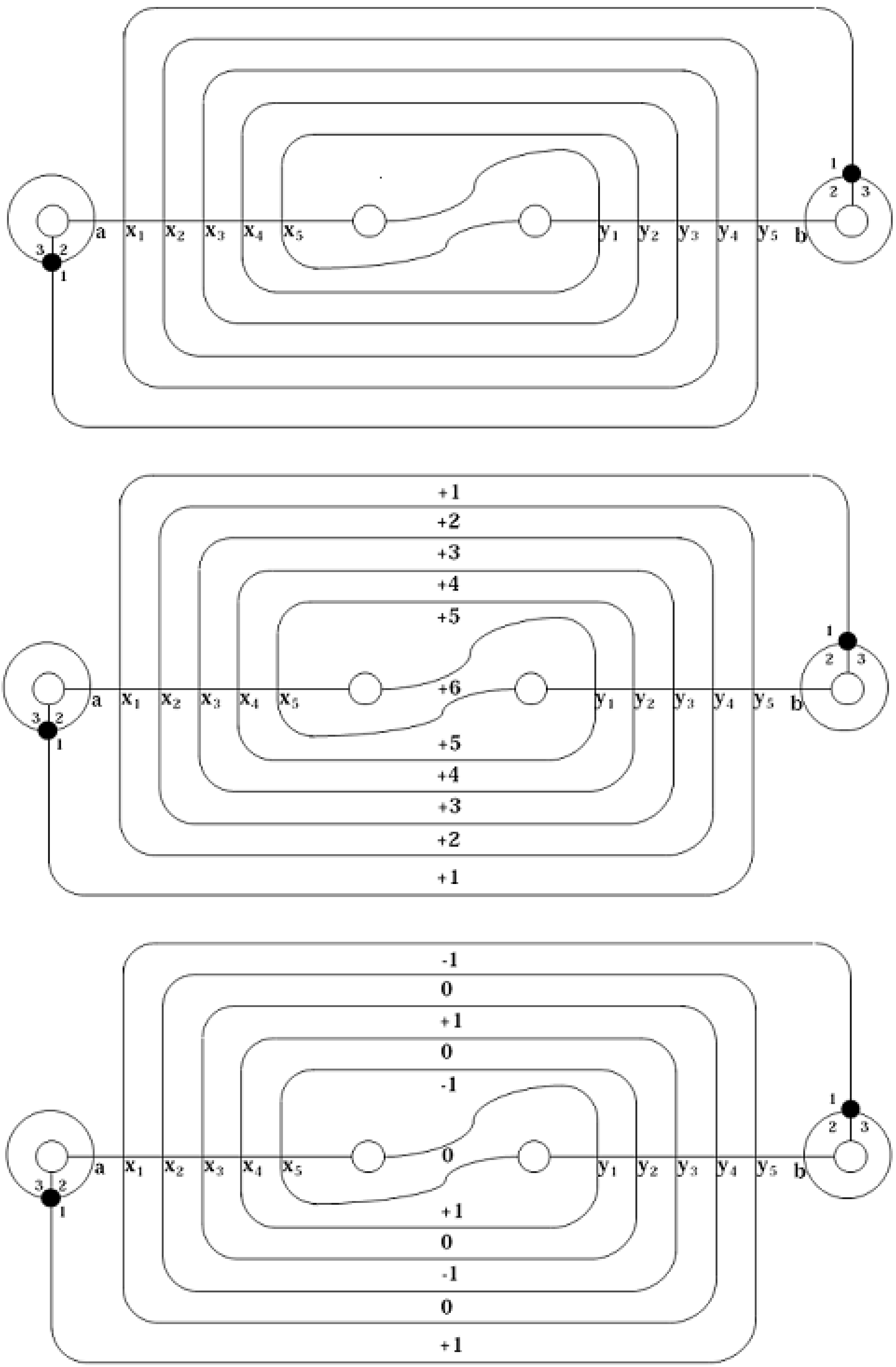


Figure 3.4: Top figure is a diagram of (2,6) torus link, where black dots represents left and right punctures. Middle and bottom figures shows two linearly independent periodic domains of the diagram.

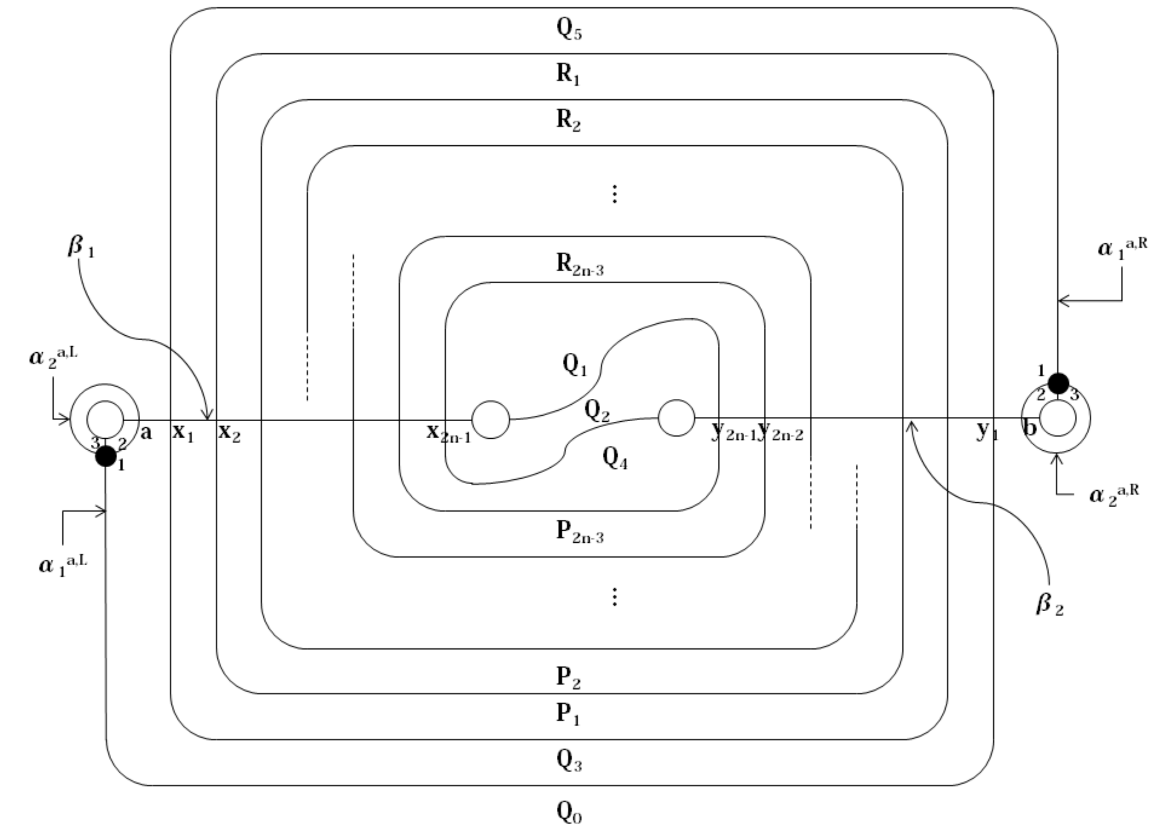


Figure 3.5: A general diagram of $(2, 2n)$ torus link. Domain Q_0 has framed arc. The orientation on the boundaries is opposite from usual "right hand" orientation.

with an (framed) arc is not always possible. In fact, a domain that is adjacent to both punctures does not exist except for the $(2, 2n)$ *torus link case. To fix this, choose* μ_L *or* μ_R *and apply a finger move on the chosen meridian along the longitude so that the resulting puncture is on the domain that is adjacent to the another puncture.*

3.2 Computation of map δ^1 of $\widehat{CFDD}$

Now we will compute $\widehat{CFDD}(\mathcal{H})$, where $\mathcal{H}$ is Heegaard diagram of $(2,2n)$-torus link complement. The Heegaard diagram is given in [Figure 6].

Periodic domain First we investigate periodic domains $\pi_2(\mathbf{x},\mathbf{x})$. It is well known that $\pi_2(\mathbf{x},\mathbf{x}) \cong H_2(Y(\mathcal{H}),\partial Y(\mathcal{H})) \cong \mathbb{Z}\oplus\mathbb{Z}$, by Meyer-Vietoris sequence. Thus there are two linearly independent periodic domain in the diagram. Recall that homology group $\boldsymbol{\pi}_2(\mathbf{x},\mathbf{x}) \cong H_2(Y,\partial Y)$ from Remark 2.13. The proof can be found in [5], Lemma 2.6.1. or [6], Lemma 4.18. In their proof, they use the isomorphism

$$\boldsymbol{\pi}_2(\mathbf{x},\mathbf{x}) \cong H_2(\Sigma' \times [0,1], (\overline{\boldsymbol{\alpha}} \times \{1\}) \cup (\boldsymbol{\beta} \times \{0\}))$$

where $\Sigma' = (\overline{\Sigma}/\partial\overline{\Sigma})\backslash\{z\}$. The isomorphism given above is proved by investigating long exact sequence of pair $(\Sigma' \times [0,1], (\overline{\boldsymbol{\alpha}} \times \{1\}) \cup (\boldsymbol{\beta} \times \{0\}))$. That is,

$$\begin{aligned} \cdots \rightarrow\ & \underbrace{H_2(\Sigma' \times [0,1])}_{\cong 0} \rightarrow H_2(\Sigma' \times [0,1], (\overline{\boldsymbol{\alpha}} \times \{1\}) \cup (\boldsymbol{\beta} \times \{0\})) \\ \rightarrow\ & H_1((\overline{\boldsymbol{\alpha}} \times \{1\}) \cup (\boldsymbol{\beta} \times \{0\})) \rightarrow H_1(\Sigma'). \end{aligned}$$

Thus periodic domain $\boldsymbol{\pi}_2(\mathbf{x},\mathbf{x}) \cong \ker(H_1(\overline{\boldsymbol{\alpha}}/\partial\overline{\boldsymbol{\alpha}}) \oplus H_1(\boldsymbol{\beta}) \rightarrow H_1(\overline{\Sigma}/\partial\overline{\Sigma}))$. This isomorphism enables us to find periodic domain from given diagram by choosing right combinations of $\overline{\boldsymbol{\alpha}}$ and $\boldsymbol{\beta}$ curves such that sum of their image in $H_1(\overline{\Sigma}/\partial\overline{\Sigma})$ equals zero. We briefly describe how to find periodic domain from such combinations. Explicitly, first choose any orientation on the longitude $\alpha_1^{a,L}$ ($\alpha_1^{a,R}$, respectively). This induces orientation of β_1 (β_2, respectively) follows. For example, if orientation of $\alpha_1^{a,L}$ is in counterclockwise direction, then the orientation of β_1 is from right to left in the diagram. Then we impose coefficient zero to the outermost region that contains the framed arc. Starting from the outermost region, we impose regions adjacent to it according to following rule. Suppose we have two adjacent region A and B such that coefficient of A equals l and coefficient of B is not determined. If we can reach region B from region A by crossing a curve of multiplicity k from right to left(notion of "left" and "right" is justified since we have orientation of curves), we give the region B coefficient $k+l$; otherwise we give coefficient $-k+l$. If we can give coefficients to all regions in this way consistently, then the orientations given to curves $\overline{\boldsymbol{\alpha}}$ and $\boldsymbol{\beta}$ is boundary in $H_1(\overline{\Sigma}/\partial\overline{\Sigma})$.

Since there are two possible choices of orientations of longitudes up to sign, we found two generators of $\pi_2(\mathbf{x},\mathbf{x})$. Then the periodic domains are,

$$Q_3 + Q_5 + \sum_{i=1}^{2n-3}(i+1)(P_i + R_i) + (n+2)(Q_1 + Q_4) + (n+3)Q_2$$

and

$$Q_3 - Q_5 + \sum_{i=1}^{2n-3}\frac{(1+(-1)^i)}{2}(P_i - R_i) + Q_4 - Q_1.$$

See also [Figure 3.4].

Thus this diagram is provincially admissible; in fact, there is no provincial periodic domain here.

Generators According to the labeling given in the diagram, There are $2n^2 + 2n$ generators and classified into 4 groups.

$$\begin{cases} \mathbf{x_i y_j} & \text{where } i \text{ and } j \text{ have same parity} \\ \mathbf{ay_i} & \text{where } i \text{ is even} \\ \mathbf{x_i b} & \text{where } i \text{ is even} \\ \mathbf{ay_i}, \mathbf{x_j b} & \text{where } i \text{ and } j \text{ are odd} \end{cases}$$

From now on, we will disregard generators of last kind because of following reason. The main purpose of the bordered Floer homology is to compute Heegaard Floer homology of three manifold obtained by taking boundary sum. In link complement case, we take boundary sum with solid tori. Typically bordered Heegaard diagram of solid tori is a genus one surface with a puncture, equipped with $\boldsymbol{\beta} = \{\beta_1\}$ and $\overline{\boldsymbol{\alpha}} = \{\alpha_1^a, \alpha_2^a\}$. In particular, these α_i^a arcs are glued to $\alpha_j^{a,L}$ or $\alpha_i^{a,R}$ of doubly bordered diagram. Every generator of the diagram of solid tori occupying exactly one α arc, therefore after gluing two diagrams the generators of last kind cannot appear in the generators of resulting diagram.

Remark 3.2.1 *As we have decomposed strands algebra $\mathcal{A}(\mathcal{Z})$ into direct sum of $\mathcal{A}(\mathcal{Z}, i)$, $i \in \{-1, 0, 1\}$ (see Remark 2.3.1), we can decompose $\widehat{CFDD}(\mathcal{H})$*

as follows.

$$\widehat{CFDD}(\mathcal{H}) = \bigoplus_{i=-1}^{1} \widehat{CFDD}(\mathcal{H}, i)$$

where

- $\widehat{CFDD}(\mathcal{H}, -1)$ *consists of generator that occupies* $\alpha_1^{a,R}$ *and* $\alpha_2^{a,R}$*.*

- $\widehat{CFDD}(\mathcal{H}, +1)$ *consists of generator that occupies* $\alpha_1^{a,L}$ *and* $\alpha_2^{a,L}$*.*

- $\widehat{CFDD}(\mathcal{H}, 0)$ *consists of all other generators.*

First three groups of generators belong to $\widehat{CFDD}(\mathcal{H}, 0)$*, but last group of generators does not.*

Clearly on the summand $\widehat{CFDD}(\mathcal{H}, 0)$ *has contribution to tensor product with* $\widehat{CFA}$ *or* $\widehat{CFD}$*, considering the only nontrivial algebra of* $\widehat{CFA}$ *and* $\widehat{CFD}$ *is* $\mathcal{A}(\mathcal{Z}, 0)$*. Moreoever, since* $\mathcal{A}(\mathcal{Z}, -1)$ *and* $\mathcal{A}(\mathcal{Z}, +1)$ *are quasi-isomorphic to* $\mathbb{F}_2$*([6], Example 3.25), so any invertible bimodule over either of these algebras is also quasi-isomorphic to* $\mathbb{F}_2$*([7], 10.0).*

From now on, we will be only interested in the generators belong to $\widehat{CFDD}(\mathcal{H}, 0)$. The number of such generators are $2n^2$, so we will be working on $2n^2$ generators placed as in [Figure 3.6].

Domains We will consider domains that contributes to the differential δ^1. First obvious condition is domain should have at most four corners, thus it can have two or four corners. In [Figure 6] there is no bigon domain, so let us consider a connected rectangular domain with four corners first. In order

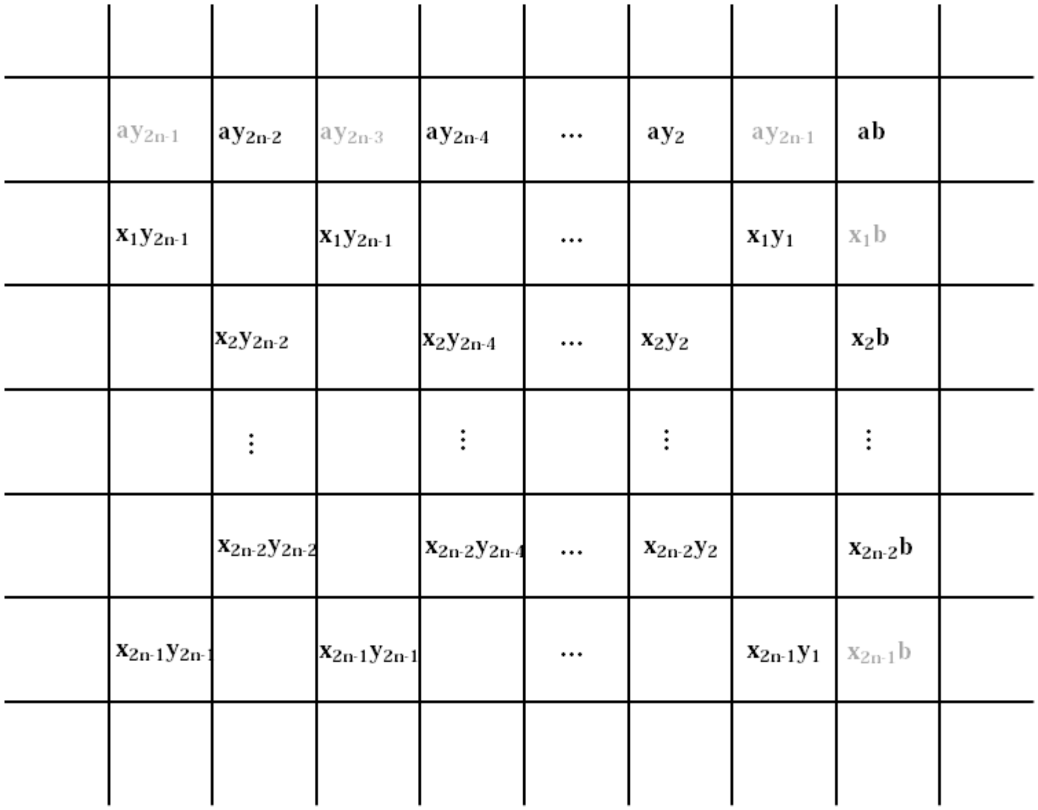


Figure 3.6: Generators of $\widehat{CFDD}(\mathcal{H})$. Blurred generators do not belong to $\widehat{CFDD}(\mathcal{H},0)$. $\mathbf{ay}_{\text{odd}}$ are in $\widehat{CFDD}(\mathcal{H},+1)$ and $\mathbf{x}_{\text{odd}}\mathbf{b}$ are in $\widehat{CFDD}(\mathcal{H},-1)$.

to get such domains, typically we need to stack up regions as follows. First begin with any provincial region (that is not adjacent to boundaries). Then one can extend the region by choosing a region adjacent to it. For example, if one begin with R_i, then may extend it by adding another provincial domain $R_{i\pm1}$ or $P_{i\pm1}$. The former way of extending region or domain, we will call the region or domain is *horizontally extended*. The later will called to be *vertically extended*. It is worth pointing out that a provincial region cannot be extended horizontally and vertically at the same time, because in such cases Maslov index cannot be one(see [Figure 3.7]). Also we need to consider non-rectangular

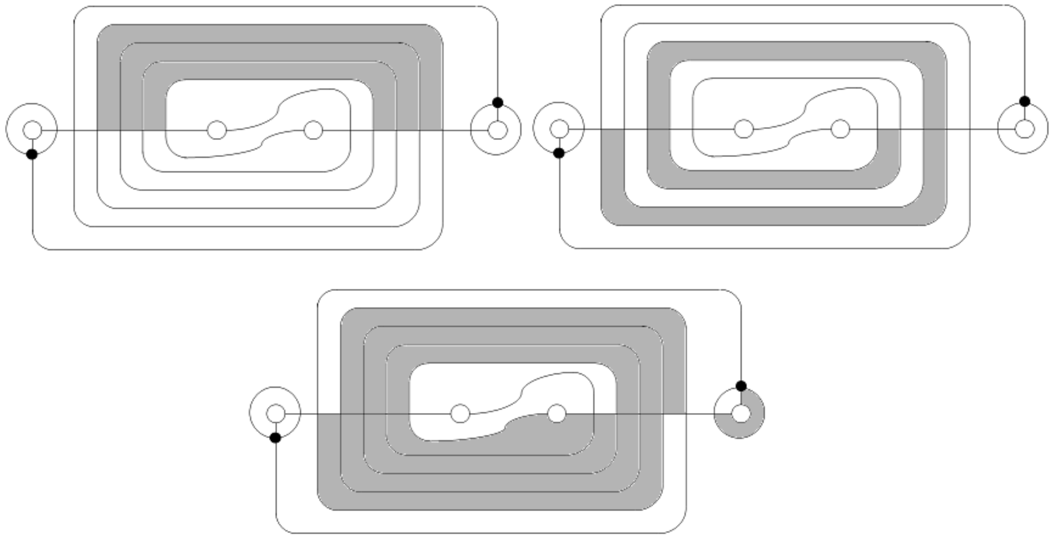


Figure 3.7: Top left : horizontally extended domain. Top right : vertically extended. Bottom : extended in both ways. Such domain has Maslov index different from one.

domains. For example, annular domain or genus 2 domain. Annular domains that have holomorphic representatives also arise, but they do not contribute to the map δ^1. The genus 2 domains have to contain $Q_1 + Q_2 + Q_4$ and it can be interpreted as annular domain or rectangular domain. Explanation is given later in this section.

For the differential $\delta^1 : \mathfrak{S}(\mathcal{H}) \to \mathcal{A}(-\partial_L \overline{\Sigma}) \otimes \mathcal{A}(-\partial_R \overline{\Sigma}) \otimes \mathfrak{S}(\mathcal{H})$ on $\widehat{CFDD}(\mathcal{H})$, it acts on a generator $\mathbf{x} \in \mathfrak{S}(\mathcal{H})$ typically as $\delta^1(\mathbf{x}) = \sum \rho_I \otimes \sigma_J \otimes \mathbf{y}$, where $I, J \in \{\phi, 1, 2, 3, 12, 23, 123\}$. Here ρ_I means an algebra element comes from the left boundary strands algebra and σ_J right strands algebra. To investigate δ^1 actions on generators, it is convenient to classify the resulting terms by its strands algebra elements.

Remark 3.2.2 *A priori, one may consider a differential that gives product of multiple algebra elements. For instance, $\rho_I \rho_J$ where $I, J \in \{1, 2, 3, 12, 23, 123\}$. Clearly, if these Reeb chords cannot be concatenated then its product equals zero and has no contribution to δ^1.*

Algebra element 1 We should find all provincial domains. We claim that only rectangular domains contribute to the differential δ^1.

Lemma 3.2.3 *Every non-rectangular domain with* $\mathrm{ind}(B, \boldsymbol{\rho}) = 1$*, its sequence of Reeb chords* $\boldsymbol{\rho}$ *is nonempty.*

proof Suppose there is a non-rectangular provincial domain(in this case, an annulus) that has nontrivial contribution to differential δ^1. Then the number of corners of the domain must be two. This claim is justified by considering number of different types of corners. Since the number of corners of any domain should not exceed four, there are only 5 possibilities;

- four $270°$ corners
- four $90°$ corners
- three $270°$ corners and one $90°$ corner
- one $270°$ corner and three $90°$ corners
- two $270°$ corners and two $90°$ corners.

Since the domain was assumed to be provincial, it must be a combination of regions $P_1, \cdots, P_{2n-3}$ and $R_1, \cdots R_{2n-3}$. Considering index formula $e(A) +$

$n_{\mathbf{x}}(A)+n_{\mathbf{y}}(A)$, indices of first three cases cannot be one. Likewise we can easily rule out the last case. The fourth case does not exist by following reason; since the shape of domain is annulus, the 270° corner must be on the boundary of the domain. Then the other boundary must have two 90° corner. If not, i.e, if one boundary component has all three 90° corners, then there cannot be a holomorphic involution interchanging inner and outer boundaries. Thus, the one boundary has two 90° corners and the other boundary has one 90° corner and one 270° corner. Especially the boundary that has two 90° corners should consist of one $\overline{\boldsymbol{\alpha}}$ curve and one $\boldsymbol{\beta}$ curve, and the intersections have to be 90°. However such a boundary cannot be obtained by any combination of domains in [Figure 3.5]. $\square$

Therefore, $P_1, \cdots, P_{2n-3}$ and $R_1, \cdots R_{2n-3}$ are only domains not adjacent to the boundaries, so extending these regions horizontally or vertically is the only possibility to get provincial domains. Such combinations of extension can be written explicitly as below.

$$
\begin{aligned}
& P_i, \quad P_i + R_{i+1} + P_{i+2}, \cdots \\
& P_i + P_{i+1} + P_{i+2}, \quad P_i + \cdots + P_{i+4}, \cdots, \quad P_1 + \cdots P_{2n-3}, \\
& R_i, \quad R_i + P_{i+1} + R_{i+2}, \cdots \\
& R_i + R_{i+1}, R_{i+2}, \cdots, \quad R_1 + \cdots + R_{2n-3}
\end{aligned}
$$

All of these domains are rectangular so each of these domains contribute nontrivial differential with algebra element 1. In terms of generators,

$$\mathbf{x_i y_j} \mapsto \begin{cases} \mathbf{x_{j-1} y_{i+1}} + \mathbf{x_{i+1} y_{j-1}} & \text{if } j-i>2 \\ \mathbf{x_{j+1} y_{i-1}} + \mathbf{x_{i-1} y_{j+1}} & \text{if } i-j>2 \\ \mathbf{x_{i+1} y_{j-1}} & \text{if } j-i=2 \\ \mathbf{x_{i-1} y_{j+1}} & \text{if } i-j=2 \\ 0 & \text{if } i=j \end{cases}$$

Some parts of these differentials are depicted in [Figure 3.8].

Algebra element ρ_1 and σ_1. Domain Q_3 is adjacent to algebra element ρ_1. By the nature of type D structure, any domain whose multiplicity of Q_3 is greater than 1 cannot contribute nontrivial differential. By similar argument as in previous case, we list possible domains as below.

$$Q_3, \quad Q_3 + P_1 + P_2, \quad Q_3 + P_1 + P_2 + P_3 + P_4, \cdots$$

All such domains are extended horizontally. On the other hand,

$$Q_3 + R_1 + P_2, \quad Q_3 + R_1 + P_2 + R_3 + P_4, \cdots$$

which are extended vertically.

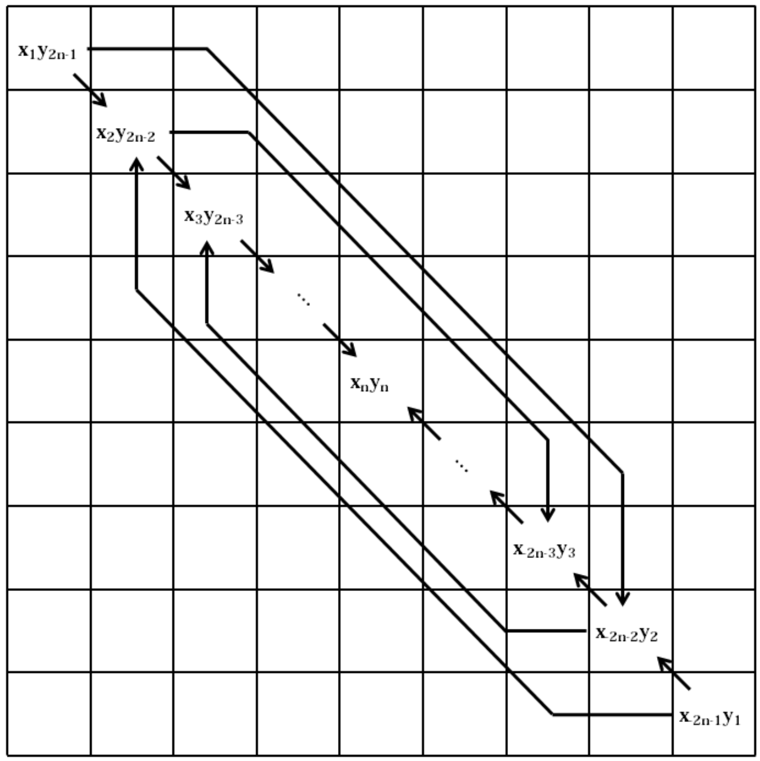


Figure 3.8: Differentials induced by provincial domains.

These domains are all quadrilateral thus dimension of moduli space and number of holomorphic curves (modulo 2) are obvious, as written below.

$$\mathbf{ay_{2k}} \mapsto \begin{cases} \rho_1 \otimes (\mathbf{x_1 y_{2k-1}} + \mathbf{x_{2k-1} y_1}) & \text{if } k \neq 1 \\ \rho_1 \otimes \mathbf{x_1 y_1} & \text{otherwise.} \end{cases}$$

Differentials involving σ_1 can be found in parallel manner, by using symmetry of the diagram.

$$\mathbf{x_{2k} b} \mapsto \begin{cases} \sigma_1 \otimes (\mathbf{x_{2k-1} y_1} + \mathbf{x_1 y_{2k-1}}) & \text{if } k \neq 1 \\ \sigma_1 \otimes \mathbf{x_1 y_1} & \text{otherwise.} \end{cases}$$

Algebra element ρ_3 and σ_3. Similarly, domains adjacent to ρ_3 are all listed

$$Q_1, \quad Q_1 + R_{2n-3} + R_{2n-2}, \quad Q_1 + R_{2n-3} + R_{2n-4} + R_{2n-5} + R_{2n-6}, \cdots$$

and,

$$Q_1 + P_{2n-3} + R_{2n-4}, \quad Q_1 + P_{2n-3} + R_{2n-4} + P_{2n-5} + R_{2n-6}, \cdots$$

Domains adjacent to σ_3 are similar. We get differentials as below.

$$\mathbf{ay_{2k}} \mapsto \begin{cases} \rho_3 \otimes (\mathbf{x_{2k+1}y_{2n-1}} + \mathbf{x_{2n-1}y_{2k+1}}) & \text{if } k \neq n-1 \\ \rho_3 \otimes \mathbf{x_{2n-1}y_{2n-1}} & \text{otherwise.} \end{cases}$$

$$\mathbf{x_{2k}b} \mapsto \begin{cases} \sigma_3 \otimes (\mathbf{x_{2n-1}y_{2k+1}} + \mathbf{x_{2k+1}y_{2n-1}}) & \text{if } k \neq n-1 \\ \sigma_3 \otimes \mathbf{x_{2n-1}y_{2n-1}} & \text{otherwise.} \end{cases}$$

Algebra element $\rho_2 \otimes \sigma_2$. A domain Q_2 adjacent to ρ_2 is adjacent to σ_2 as well. So this is the one and only domain occurs an algebra element $\rho_2 \otimes \sigma_2$. Thus we have $\mathbf{x_{2n-1}y_{2n-1}} \mapsto \rho_2 \otimes \sigma_2 \otimes \mathbf{ab}$.

Algebra element $\rho_3 \otimes \sigma_1$ and $\rho_1 \otimes \sigma_3$. There are two domains contributes $\rho_3 \otimes \sigma_1$; those are $Q_1 + R_1 + R_2 + \cdots R_{2n-3} + Q_5$ and $Q_1 + P_1 + R_2 + P_3 +$

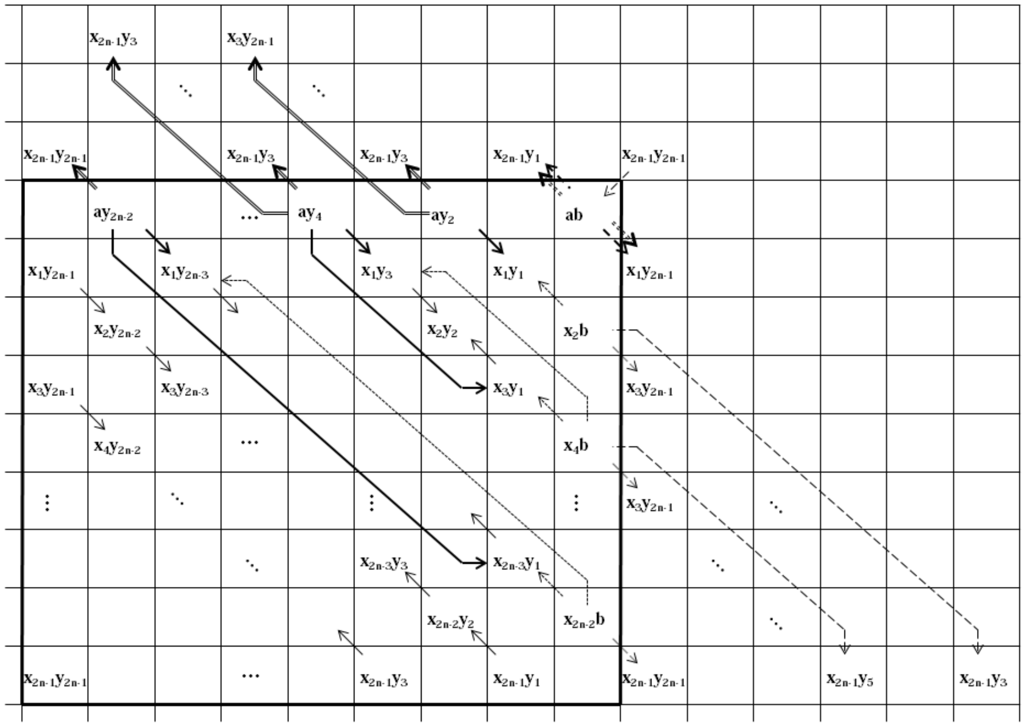


Figure 3.9: Thin solid arrows imply algebra element 1. Bold solid arrows mean algebra element ρ_1 and doubly solid arrows algebra element ρ_3. Finely dashed arrows and coarsely dashed arrows represents σ_1 and σ_3 respectively. In addition, The arrow from $\mathbf{x_{2n-1}y_{2n-1}}$ to $\mathbf{ab}$ has algebra element $\rho_2 \otimes \sigma_2$. Lastly, arrows starting from $\mathbf{ab}$ are both imply $\rho_3 \otimes \sigma_1 + \rho_1 \otimes \sigma_3$.

$R_4 + \cdots R_{2n-4} + P_{2n-3} + Q_5$. This gives $\mathbf{ab} \mapsto \rho_3 \otimes \sigma_1 \otimes (\mathbf{x_1y_{2n-1}} + \mathbf{x_{2n-1}y_1})$. Again, using symmetry of the diagram, $\mathbf{ab} \mapsto \rho_1 \otimes \sigma_3 \otimes (\mathbf{x_1y_{2n-1}} + \mathbf{x_{2n-1}y_1})$.

These differentials we have considered so far are depicted in the [Figure 3.9].

Now, we will be mostly working on differentials whose domain is non-rectangular. To find holomorphic curves of such domains we will dualize $\widehat{CFDD}$ to $\widehat{CFAA}$, so that we can use $\mathcal{A}_\infty$ structure of it and ensure existence of holomorphic curves and its count(modulo 2).

Algebra element contains ρ_{12}. To take advantage of $\mathcal{A}_\infty$ structure of $\widehat{CFAA}$, the orientation of two boundaries of Heegaard diagram has to be reversed. We denote $\overline{\rho}_I$ (respectively, $\overline{\sigma}_I$) denote the algebra element of strands algebra $\mathcal{A}(\mathcal{Z})$; that is, for a orientation reversing diffeomorphism $R : S^1\backslash\{z\} \to -S^1\backslash\{z\}$, stands algebra of left boundary maps $R_*(\rho_1) = \overline{\rho}_3$, $R_*(\rho_2) = \overline{\rho}_2$, and $R_*(\rho_3) = \overline{\rho}_1$. Right boundary is similar.

Returning to $\widehat{CFDD}$, domains contribute to ρ_{12} are $Q_2 + Q_3 + P_1 + \cdots + P_{2n-3} + Q_4$ and $Q_2 + Q_3 + R_1 + P_2 + \cdots + R_{2n-3} + Q_4$. The domain $Q_2 + Q_3 + P_1 + \cdots + P_{2n-3} + Q_4$ cannot give nonzero differential because of index reason, and the domain $Q_2 + Q_3 + R_1 + P_2 + \cdots + R_{2n-3} + Q_4$ is not a Whitney disk connecting two generators.

Algebra element contains ρ_{23}. Roughly speaking, domains that possibly contribute algebra element ρ_{23} is obtained by vertically or horizontally extending domain Q_2 so that resulting domains contain Q_1. We do not extend Q_2 horizontally and vertically at the same time to get a Maslov index one domain with at most four corners.

Case 1. We will first consider following annular domains.

$$Q_1+Q_2,$$
$$Q_1+Q_2+Q_4+R_{2n-3}+P_{2n-3}+R_{2n-4},$$
$$Q_1+Q_2+Q_4+R_{2n-3}+P_{2n-3}+R_{2n-4}+\cdots+P_{2k-1}+R_{2k},$$
$$\vdots$$

We will first consider domain Q_1+Q_2. The domain can be interpreted as $\mathcal{M}(\mathbf{ay_{2n-2}},\mathbf{ab};\overline{\rho}_{12},\overline{\sigma}_2)$. The modulo 2 count of the moduli space can be computed by using $\mathcal{A}_\infty$ relation of $m^2(\mathbf{ay_{2n-2}},\overline{\rho}_1,\overline{\rho}_2,\overline{\sigma}_2)$. Recall that $m(\mathbf{ay_{2n-2}},\overline{\rho}_1)=\mathbf{x_{2n-1}y_{2n-1}}$ and $m(\mathbf{x_{2n-1}y_{2n-1}},\overline{\rho}_2,\overline{\sigma}_2)=\mathbf{ab}$ since the associated domains are rectangles.

$$\begin{aligned}0 &= m(m(\mathbf{ay_{2n-2}},\overline{\rho}_1),\overline{\rho}_2,\overline{\sigma}_2)+m(\mathbf{ay_{2n-2}},(\overline{\rho}_1,\overline{\rho}_2),\overline{\sigma}_2)+m(m(\mathbf{ay_{2n-2}},\overline{\sigma}_2),\overline{\rho}_1,\overline{\rho}_2)\\ &= \mathbf{ab}+m(\mathbf{ay_{2n-2}},\overline{\rho}_{12},\overline{\sigma}_2)+m(m(\mathbf{ay_{2n-2}},\overline{\sigma}_2),\overline{\rho}_1,\overline{\rho}_2)\end{aligned}$$

The last term on the right hand side equals zero because $m(\mathbf{ay_{2n-2}},\overline{\sigma}_2)=0$(domain Q_2 is adjacent to Reeb chords $\overline{\rho}_2$ and $\overline{\sigma}_2$). This implies $m(\mathbf{ay_{2n-2}},\overline{\rho}_{12},\overline{\sigma}_2)=\mathbf{ab}$, hence $\sharp\mathcal{M}(\mathbf{ay_{2n-2}},\mathbf{ab};\overline{\rho}_{12},\overline{\sigma}_2)=1$.

Remark 3.2.4 *An annulus domain of such kind (i.e, outside boundary consists of both $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ curves and inside boundary $\boldsymbol{\alpha}$ curve only, and a cut on the inside boundary) always admits a holomorphic representative, since we are free to choose the length of the cut starting from the point a. so that the* *annulus admits a biholomorphic involution of it, in the sense of Lemma 9.4 of [9].*

The moduli space $\mathcal{M}(\mathbf{ay_{2n-2}}, \mathbf{ab}; \overline{\rho}_{12}, \overline{\sigma}_2) = \mathcal{M}(\mathbf{ay_{2n-2}}, \mathbf{ab}; \rho_{23}, \sigma_2)$ corresponds to $\rho_{23} \otimes \sigma_2 \otimes \mathbf{ab}$ term occurs in $\delta^1(\mathbf{ay_{2n-2}})$ in $\widehat{CFDD}$. However, the right hand side is zero by idempotent reasons. Explicitly,

$$\begin{aligned} \delta^1(\mathbf{ay_{2n-2}}) &= \rho_{23} \otimes \sigma_2 \otimes \mathbf{ab} + \cdots \\ &= \rho_{23}\iota_2 \otimes \sigma_2 \otimes \mathbf{ab} + \cdots = \rho_{23} \otimes \sigma_2 \otimes \iota_2\mathbf{ab} + \cdots \end{aligned}$$

Recall that $\iota_2\mathbf{ab} = 0$ since idempotent ι_2 occupies $\alpha_2^{a,L}$, and the arc is also occupied by $\mathbf{ab}$. Therefore the moduli space does not contribute to the differential δ^1.

Likewise, domains $Q_1 + Q_2 + Q_4 + R_{2n-3} + P_{2n-3} + R_{2n-2}$, $Q_1 + Q_2 + Q_4 + R_{2n-3} + P_{2n-3} + R_{2n-2} + \cdots + P_{2k-1} + R_{2k}$, $\cdots$ allow interpretation $\mathcal{M}(\mathbf{ay_{2j}}, \mathbf{ay_{2j+2}}; \overline{\rho}_{12}, \overline{\sigma}_2, \overline{\sigma}_1)$, whose modulo 2 count is 1. These contribute to differential between generators $\mathbf{ay_{2j}}$ and $\mathbf{ay_{2j+2}}$ with algebra element contains ρ_{23}, but all go to zero because of idempotents (There is only one exception; that is, $\mathcal{M}(\mathbf{ab}, \mathbf{ay_2}; \overline{\rho}_{12}, \overline{\sigma}_3, \overline{\sigma}_2, \overline{\sigma}_1)$. However it has no contribution in differential in $\widehat{CFDD}$ either because of idempotent action).

Remark 3.2.5 *The domains considered above allow interpretation*

$$\mathcal{M}(\mathbf{ay_{2j}}, \mathbf{ay_{2j+2}}; \overline{\rho}_{12}, \overline{\sigma}_{12}).$$

Modulo 2 count of the moduli space equals zero can be proved by considering following $\mathcal{A}_\infty$ relation.

$$\begin{aligned} 0 &= m(m(\mathbf{ay_{2j}},\overline{\rho}_{12},\overline{\sigma}_1,\overline{\sigma}_2))+m(m(\mathbf{ay_{2j}},\overline{\sigma}_1),\overline{\rho}_{12},\overline{\sigma}_2) \\ &+ m(m(\mathbf{ay_{2j}},\overline{\rho}_{12}),\overline{\sigma}_1,\overline{\sigma}_2)+m(m(\mathbf{ay_{2j}},\overline{\rho}_{12},\overline{\sigma}_1),\overline{\sigma}_2)+m(\mathbf{ay_{2j}},\overline{\rho}_{12},(\overline{\sigma}_1,\overline{\sigma}_2)) \end{aligned}$$

$m(\mathbf{ay_{2j}},\overline{\rho}_{12},\overline{\sigma}_1,\overline{\sigma}_2) = 0$ *since Maslov index is not one.* $m(\mathbf{ay_{2j}},\overline{\rho}_{12})$ *and* $m(\mathbf{ay_{2j}},\overline{\rho}_{12},\overline{\sigma}_1)$ *equal zero, because* $\overline{\sigma}_2$ *was not involved and there is no such domain corresponding to these interpretation.* $m(\mathbf{ay_{2j}},\overline{\sigma}_1)=0$ *is clear from the diagram. Thus, the last term* $m(\mathbf{ay_{2j}},\overline{\rho}_{12},(\overline{\sigma}_1,\overline{\sigma}_2)) = m(\mathbf{ay_{2j}},\overline{\rho}_{12},\overline{\sigma}_{12})$ *equals zero, too.*

Case 2. Next we will consider following domains.

$$\begin{aligned} &Q_1+Q_2+P_{2n-3}+R_{2n-4}+\cdots+R_{2k}+P_{2k-1}, \\ &Q_1+Q_2+P_{2n-3}+R_{2n-4}+\cdots+R_{2k}+P_{2k-1} \\ &\quad +Q_4+R_{2n-3}+P_{2n-4}+\cdots+P_{2l}+R_{2l-1}, \\ &\vdots \end{aligned}$$

These domains are obtained by vertically extending domain Q_2; first kind of domains have interpretation $\mathcal{M}(\mathbf{x_{2k-1}y_{2n-1}},\mathbf{x_{2k-2}b};\overline{\rho}_2,\overline{\rho}_1,\overline{\sigma}_2)$, which is essentially a rectangle. Second domains are also rectangles with moduli space $\mathcal{M}(\mathbf{x_{2k-1}y_{2l-1}},\mathbf{x_{2k-2}y_{2l}};\overline{\rho}_2,\overline{\rho}_1,\overline{\sigma}_2,\overline{\sigma}_1)$. Dualizing them, they yield algebra el-

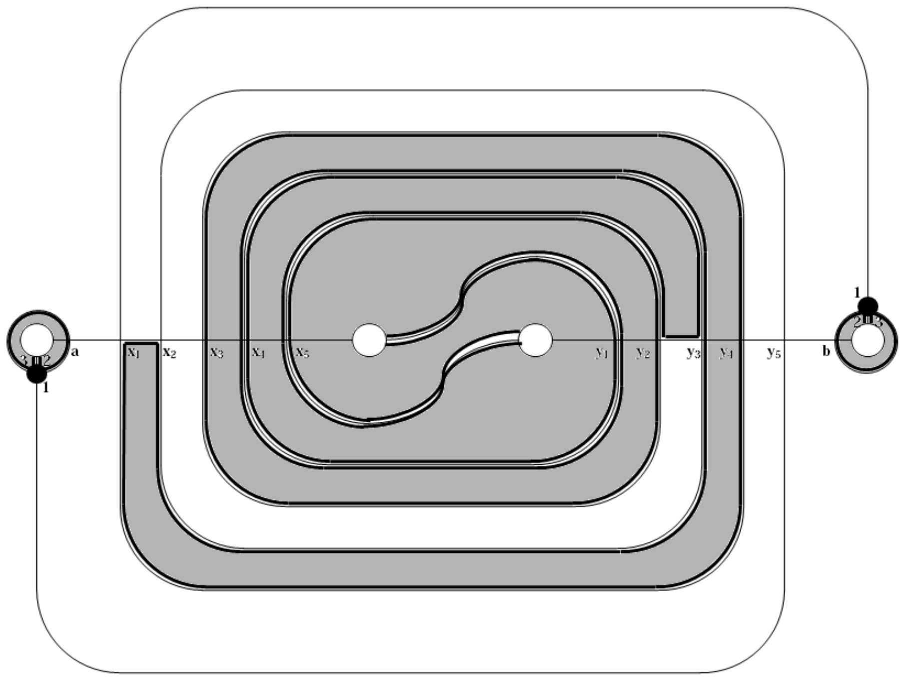


Figure 3.10: A diagram of (2,6) torus link complement. The shaded region is a domain obtained by vertically extending domain Q_2. This domain corresponds to a differential from $\mathbf{x_1 y_3}$ to $\mathbf{x_2 y_2}$. Cutting along the bold curve on the boundary of domain, the domain turns out to be rectangular.

ements $\rho_{23} \otimes \sigma_2$ and $\rho_{23} \otimes \sigma_{23}$ for the type-D structure map δ^1 in $\widehat{CFDD}$, respectively.

Remark 3.2.6 *Again, modulo 2 count of moduli space $\mathcal{M}(\mathbf{x_{2k-1}y_{2l-1}}, \mathbf{x_{2k-2}y_{2l}}; \overline{\rho}_{12}, \overline{\sigma}_{12})$ equals zero by considering similar $\mathcal{A}_\infty$ relation discussed in Remark 3.2.5.*

Case 3. Domains that possibly contribute differential with algebra element that contains ρ_{23} are obtained by horizontally extending $Q_1 + Q_2$. That is, we add $2j-1$ domains, $j = 1, \cdots, n-1$ on the top and resulting domain is $R_{2n-2j-1}+\cdots+R_{2n-3}+Q_1+Q_2$. The only possible interpretation is $\mathcal{M}(\mathbf{x_{2n-1}y_{2n-2j-1}},\mathbf{x_{2n-2j}b};\overline{\rho}_{12},\overline{\sigma}_2)$. It doe not allow holomorphic representative, because the domain does not allow holomorphic involution interchanging two boundaries.

Likewise, we consider domains obtained by adding horizontally extended domains to Q_2 on top and bottom. Consider a domain

$$Q_1+Q_2+Q_3+(R_{2n-k}+\cdots R_{2n-3})+(P_{2n-l}+\cdots+P_{2n-3}).$$

The domain is obtained by adding $k-2$ domains on top and $l-2$ domains on bottom. If $k=l$, then the domain obtained is the case that we have considered in vertically extended case above. If $k\neq l$, then two interpretations are possible. First is $\mathcal{M}(\mathbf{x_{2n-l}y_{2n-k}},\mathbf{x_{2n-k+1}y_{2n-l+1}};\overline{\rho}_{12},\overline{\sigma}_{12})$. This is a genus two domain, and modulo two count of this moduli space is zero by similar reason given in Remark 3.2.5. Second one is $\mathcal{M}(\mathbf{x_{2n-l}y_{2n-k}},\mathbf{x_{2n-k+1}y_{2n-l+1}};\overline{\rho}_{12},\overline{\sigma}_2,\overline{\sigma}_1)$(or $\mathcal{M}(\mathbf{x_{2n-l}y_{2n-k}},\mathbf{x_{2n-k+1}y_{2n-l+1}};\overline{\rho}_2,\overline{\rho}_1,\overline{\sigma}_{12})$). This is an annular interpretation, and that does not have holomorphic representative because it does not allow holomorphic involution.

Algebra element contains ρ_{123}. Domains that possibly contribute to

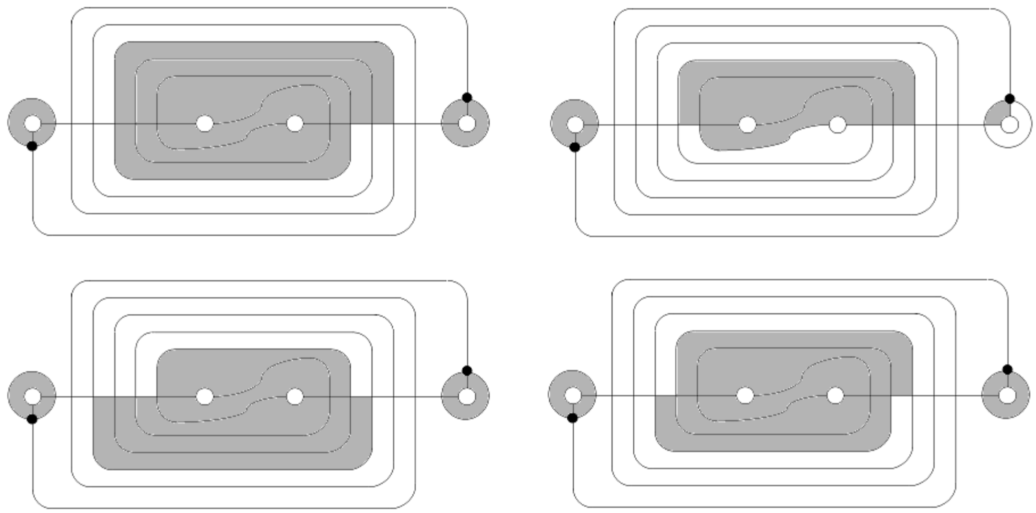


Figure 3.11: Above diagram shows examples of obtaining non-rectangular domains of $(2,6)$-torus link. Top left can be interpreted as an annular domain, but it cannot give nontrivial differential due to idempotents. Top right is obtained by horizontally extending Q_2 on top, but its only possible interpretation does not allow any holomorphic representative. Bottom left and bottom right were obtained by horizontally extending Q_2 on top and bottom. If number of regions attached on top is not equal to number of regions attached on bottom, it has two interpretations; and they do not allow holomorphic representative either(bottom left). If two numbers are equal, then the domain can be also obtained by vertically extending Q_2, which gives nontrivial differential. These four cases, in addition to the case of vertical extension of Q_2, covers all possible domains that could contribute algebra element ρ_{23}.

algebra element ρ_{123} are as follows.

$$(Q_1+Q_2+Q_3+Q_4+R_{2n-3})+R_1+P_2+R_3+\cdots+P_{2n-4}$$

$$(Q_1+Q_2+Q_3+Q_4)+P_1+\cdots P_{2n-3}$$

$$(Q_1+Q_2+Q_3+Q_4+R_{2n-3}+P_{2n-3}+R_{2n-4}+P_{2n-4}+R_{2n-5})+R_1+P_2+R_3+\cdots+P_{2n-6}$$

$$(Q_1+Q_2+Q_3+Q_4+R_{2n-3}+P_{2n-4}+R_{2n-4}+P_{2n-4}+R_{2n-5})+P_1+\cdots P_{2n-5}$$

$$\vdots$$

$$Q_1+\cdots+Q_5+P_1+\cdots+P_{2n-3}+R_1+\cdots+R_{2n-3}$$

Each of these domains are obtained by adding horizontally extended domain containing ρ_1 to the annular domain listed in algebra element ρ_{23}.

For the first two domains, the only possible interpretation is

$$\mathcal{M}(\mathbf{ay_{2n-2}}, \mathbf{x_1y_{2n-1}}; \overline{\rho}_{123}, \overline{\sigma}_2, \overline{\sigma}_1)$$

and

$$\mathcal{M}(\mathbf{ay_{2n-2}}, \mathbf{x_{2n-1}y_1}; \overline{\rho}_{123}, \overline{\sigma}_2, \overline{\sigma}_1)$$

Modulo 2 count of these moduli spaces follows by investigating coefficients of $\mathcal{A}_\infty$ relation of $m^2(\mathbf{ay_{2n-2}}, \overline{\rho}_{12}, \overline{\rho}_3, \overline{\sigma}_2, \overline{\sigma}_1)$. Since $m(\mathbf{ay_{2n-2}}, \overline{\rho}_{12}, \overline{\sigma}_2) = \mathbf{ab}$, combined with the fact $m(\mathbf{ab}, \overline{\rho}_3, \overline{\sigma}_1) = \mathbf{x_1y_{2n-1}} + \mathbf{x_{2n-1}y_1}$,

$$m(\mathbf{ay_{2n-2}}, \overline{\rho}_{123}, \overline{\sigma}_2, \overline{\sigma}_1) = \mathbf{x_1y_{2n-1}} + \mathbf{x_{2n-1}y_1}$$

This implies modulo 2 count of above moduli spaces are 1.

Similarly, the other domains (except for the last domain) give Whitney disks, and moduli spaces corresponding to the domains are $\mathcal{M}(\mathbf{ay_{2j}}, \mathbf{x_1y_{2j+1}}; \overline{\rho}_{123}, \overline{\sigma}_2, \overline{\sigma}_1)$ and $\mathcal{M}(\mathbf{ay_{2j}}, \mathbf{x_{2j+1}y_1}; \overline{\rho}_{123}, \overline{\sigma}_2, \overline{\sigma}_1)$. Each of these moduli space has count 1 modulo 2.

The moduli space of the last domain $Q_1 + \cdots + Q_5 + P_1 + \cdots + P_{2n-3} + Q_1 + \cdots + Q_{2n-3}$ can be interpreted in three ways. First, $\mathcal{M}(\mathbf{ab}, \mathbf{x_1y_1}; \overline{\rho}_{123}, \overline{\sigma}_{123})$

whose Maslov index is different from one. Second possible interpretation is

$$\mathcal{M}(\mathbf{ab}, \mathbf{x_1 y_1}; \overline{\rho}_{123}, \overline{\sigma}_3, \overline{\sigma}_2, \overline{\sigma}_1)$$

$\mathcal{A}_\infty$ relation of $m^2(\mathbf{ab}, \overline{\rho}_{12}, \overline{\rho}_3, \overline{\sigma}_3, \overline{\sigma}_2, \overline{\sigma}_1)$ gives $m(\mathbf{ab}, \overline{\rho}_{123}, \overline{\sigma}_3, \overline{\sigma}_2, \overline{\sigma}_1) = \mathbf{x_1 y_1}$, by considering $m(\mathbf{ab}, \overline{\rho}_{12}, \overline{\sigma}_3, \overline{\sigma}_2, \overline{\sigma}_1) = \mathbf{ay_2}$ and $m(\mathbf{ay_2}, \overline{\rho}_3) = \mathbf{x_1 y_1}$. Thus the modulo 2 count of the moduli space is 1. The last interpretation is

$$\mathcal{M}(\mathbf{ab}, \mathbf{x_1 y_1}; \overline{\rho}_3, \overline{\rho}_2, \overline{\rho}_1, \overline{\sigma}_3, \overline{\sigma}_2, \overline{\sigma}_1)$$

Although this interpretation cannot be obtained from previous $\mathcal{A}_\infty$ relations, existence of holomorphic curve and its modulo 2 count is quite clear from the diagram; the domain is essentially rectangular in this interpretation.

For the algebra elements of $\mathcal{A}(\mathcal{Z}_R)$, we take advantage of symmetry of the diagram. $\mathcal{A}_\infty$ relations are listed as follows.

$$\begin{aligned}
m(\mathbf{x_{2n-2}b}, \overline{\rho}_2, \overline{\sigma}_{12}) &= \mathbf{ab} \\
m(\mathbf{x_{2j}b}, \overline{\rho}_2, \overline{\rho}_1, \overline{\sigma}_{12}) &= \mathbf{x_{2j+2}b} \qquad \text{for } j = 1, \cdots, n-2 \\
m(\mathbf{ab}, \overline{\rho}_3, \overline{\rho}_2, \overline{\rho}_1, \overline{\sigma}_{12}) &= \mathbf{x_2 b} \\
m(\mathbf{x_{2j}b}, \overline{\rho}_2, \overline{\rho}_1, \overline{\sigma}_{123}) &= \mathbf{x_1 y_{2j+1}} + \mathbf{x_{2j+1} y_1} \qquad \text{for } j = 1, \cdots, n-1 \\
m(\mathbf{ab}, \overline{\rho}_3, \overline{\rho}_2, \overline{\rho}_1, \overline{\sigma}_{123}) &= \mathbf{x_1 y_1}
\end{aligned}$$

Dualizing above result, one should reverse the orientations of left and right punctures and consider idempotent restrictions. Dualized result can be summarized as,

$$
\begin{aligned}
\delta^1(\mathbf{ab}) &= \rho_{123}\sigma_{123} \otimes \mathbf{x_1 y_1} + \cdots \\
\delta^1(\mathbf{a y_{2j}}) &= \rho_{123}\sigma_{23} \otimes (\mathbf{x_1 y_{2j+1}} + \mathbf{x_{2j+1} y_1}) + \cdots \qquad \text{for } j = 1, \cdots, n-1 \\
\delta^1(\mathbf{x_{2j} b}) &= \rho_{23}\sigma_{123} \otimes (\mathbf{x_{2j+1} y_1} + \mathbf{x_1 y_{2j+1}}) + \cdots \qquad \text{for } j = 1, \cdots, n-1
\end{aligned}
$$

It is worth mention that there are three holomorphic disks contributing $\rho_{123}\sigma_{123} \otimes \mathbf{x_1 y_1}$ term in $\delta^1(\mathbf{ab})$ from following moduli spaces.

$$
\begin{aligned}
&\mathcal{M}(\mathbf{ab}, \mathbf{x_1 y_1}; \overline{\rho}_{123}, \overline{\sigma}_3, \overline{\sigma}_2, \overline{\sigma}_1), \\
&\mathcal{M}(\mathbf{ab}, \mathbf{x_1 y_1}; \overline{\sigma}_{123}, \overline{\rho}_3, \overline{\rho}_2, \overline{\rho}_1), \\
&\mathcal{M}(\mathbf{ab}, \mathbf{x_1 y_1}; \overline{\rho}_3, \overline{\rho}_2, \overline{\rho}_1, \overline{\sigma}_3, \overline{\sigma}_2, \overline{\sigma}_1).
\end{aligned}
$$

Again, differentials that yield algebra elements σ_{23} and σ_{123} can be obtained using symmetry of the diagram with exactly parallel manner.

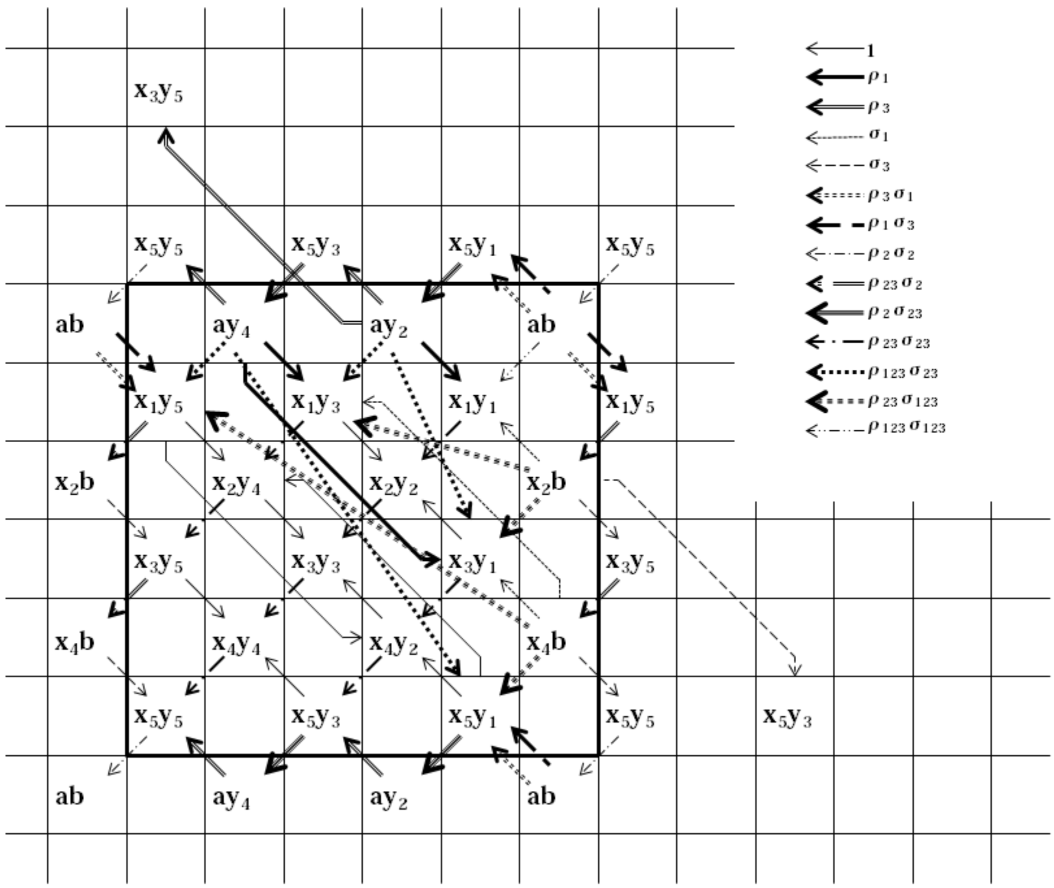


Figure 3.12: A diagram of $(2, 6)$-torus link complement, with all differentials are included.

Chapter 4

Examples

In this section, we will relate our result to known calculation for knot complements and closed 3-manifolds. These examples show how to use the algebraic structure of pairing theorem given in [6].

4.1 $\mathcal{A}_\infty$-tensor product

Originally, bordered Floer homology was invented so that $\mathcal{A}_\infty$ module and type-D structures with diffeomorphic boundaries are merged, and computes $\widehat{HF}$ of resulting 3-manifold. This procedure, namely $\mathcal{A}_\infty$ tensor product, will be described below.

Gluing Diagram Let $\mathcal{H}_i = (\overline{\Sigma}_i, \overline{\boldsymbol{\alpha}}_i, \boldsymbol{\beta}_i, z_i)$, $i = 1, 2$, be genus g_i Heegaard diagrams with single boundary components representing bordered 3 manifolds Y_1 and Y_2 respectively. We assume $-\partial Y_1$ and ∂Y_2 are diffeomorphic genus k surface. We glue two diagrams $-\mathcal{H}_1$ and $\mathcal{H}_2$ along the boundaries such that boundary points of arcs $\partial\overline{\alpha_i^a}$ and marked points z_i agree respectively. Then resulting Heegaard diagram $-\mathcal{H}_1 \cup_\partial \mathcal{H}_2$ has genus $g_1 + g_2$ with curves $\overline{\boldsymbol{\alpha}}_1 \cup \overline{\boldsymbol{\alpha}}_2$, $\boldsymbol{\beta}_1 \cup \boldsymbol{\beta}_2$ and $z_1 = z_2$. It follows from standard Morse Theory argument that the Heegaard diagram $-\mathcal{H}_1 \cup_\partial \mathcal{H}_2$ represents 3 manifold $-Y_1 \cup_\partial Y_2$.

Computing $\widehat{HF}(\mathcal{H}_1 \cup_\partial (-\mathcal{H}_2))$ from $\widehat{CFA}(\mathcal{H}_1)$ and $\widehat{CFD}(\mathcal{H}_2)$ Generators $\mathfrak{S}(\mathcal{H}_1 \cup_\partial (-\mathcal{H}_2))$ are a subset of $\mathfrak{S}(\mathcal{H}_1) \otimes \mathfrak{S}(\mathcal{H}_2)$; that is, if $\mathbf{x} \otimes \mathbf{y} \in \mathfrak{S}(\mathcal{H}_1) \otimes \mathfrak{S}(\mathcal{H}_2)$ occupies a curve of $\overline{\boldsymbol{\alpha}}_1 \cup \overline{\boldsymbol{\alpha}}_2$ more than once, then $\mathbf{x} \otimes \mathbf{y}$ has to be excluded from computation.

Let $\mathbf{x} \in \mathfrak{S}(\mathcal{H}_1)$ and $\mathbf{y} \in \mathfrak{S}(\mathcal{H}_2)$, where $\mathfrak{S}(\mathcal{H}_1)$ is a set of generators of $\widehat{CFA}$ and $\mathfrak{S}(\mathcal{H}_2)$ is a set of generators of $\widehat{CFD}$. Assume $\mathbf{x} \otimes \mathbf{y} \in \mathfrak{S}(\mathcal{H}_1 \cup_\partial (-\mathcal{H}_2))$. Then differential of the complex $\widehat{CFA}(\mathcal{H}_1) \boxtimes \widehat{CFD}(\mathcal{H}_2)$ is defined as follows.

$$\partial^{\boxtimes}(\mathbf{x} \otimes \mathbf{y}) := \sum_{k=0}^{\infty} (m_{k+1} \otimes \mathbb{I}_{\widehat{CFD}})(\mathbf{x} \otimes \delta^k(\mathbf{y})).$$

Then, there is a homotopy equivalence $\widehat{HF}(Y_1 \cup_\partial -Y_2) \cong \widehat{CFA}(\mathcal{H}_1) \boxtimes \widehat{CFD}(\mathcal{H}_2)$([6]).

We denote $\boxtimes$ a *box tensor product.*

Gluing of doubly bordered case is also similar; the only difference is the framed arc $\mathbf{z}$. If we glue doubly bordered diagram and single boundary diagram, we match marked point z from the single boundary diagram and the one end of framed arc $\mathbf{z}$. After gluing, the framed arc reduces to a marked point on the other side of boundary(if gluing two doubly bordered diagrams, then we connect two framed arc). In our example, we will be mainly interested in a type-D structure obtained by box tensor product $\widehat{CFA}(\mathcal{H}_1) \boxtimes \widehat{CFDD}(\mathcal{H}_2)$, where single boundary diagram $\mathcal{H}_1$ is glued on the right side of doubly bordered diagram $\mathcal{H}_2$. The resulting type-D structure map $(\delta')^1$ is,

$$(\delta')^1 = \sum_{k=1}^{\infty}((m_R)_{k+1} \otimes \mu_L \otimes \mathbb{I}_{\widehat{CFDD}})(\mathbf{x} \otimes \delta^k(\mathbf{y}))$$

where $\mathbf{x} \in \mathfrak{S}(\mathcal{H}_1)$ and $\mathbf{y} \in \mathfrak{S}(\mathcal{H}_2)$.

4.2 ∞-surgery on right component of link

First we will consider an ∞-surgery on the right component of $(2, 2n)$ torus link complement. Since the longitudes $\alpha_1^{a,L}$ and $\alpha_1^{a,R}$ of left and right components are passing through the β_1 and β_2 respectively, so the ∞-surgery on the right components gives a unknot complement with framing $(n-1)$. We compute $\widehat{CFD}$ of the unknot complement as follows.

Let $\mathcal{H}_{(2,2n)}$ be a doubly bordered diagram of $(2, 2n)$ torus link complement, and $\mathcal{H}_\infty$ be a single bordered diagram of solid torus. Then generators of $\mathfrak{S}(\mathcal{H}_\infty \cup_\partial \mathcal{H}_{(2,2n)})$ consists of $\mathbf{w} \otimes \mathbf{ab}$ and $\mathbf{w} \otimes \mathbf{x_{2k}b}$, $k = 1, \cdots, n-1$.

Computing $\widehat{CFA}(\mathcal{H}_\infty)$ is easy; that is,

$$m_{k+3}(\mathbf{w}, \sigma_3, \underbrace{\sigma_{23}, \cdots, \sigma_{23}}_{k\text{times}}, \sigma_2) = \mathbf{w}$$

Now we consider type-D structure of $\widehat{CFDD}(\mathcal{H}_{(2,2n)})$. We omit terms which do not appear after taking box tensor product with $\widehat{CFA}(\mathcal{H}_\infty)$, thus have no contribution in computing $\widehat{CFA}(\mathcal{H}_\infty) \boxtimes \widehat{CFDD}(\mathcal{H}_{(2,2n)})$.

$$\begin{aligned}
\delta^2(\mathbf{ab}) &= (\rho_1 \otimes \rho_{23}) \otimes (\sigma_3 \otimes \sigma_2) \otimes \mathbf{x_2 b} + \cdots \\
\delta^2(\mathbf{x_{2k} b}) &= (\rho_{23}) \otimes (\sigma_3 \otimes \sigma_2) \otimes \mathbf{x_{2k+2} b} + \cdots \quad \text{for } k = 1, \cdots n-2 \\
\delta^2(\mathbf{x_{2n-2} b}) &= (\rho_2) \otimes (\sigma_3 \otimes \sigma_2) \otimes \mathbf{ab} + \cdots
\end{aligned}$$

Thus type-D structure $(\delta')^1$ is,

$$\begin{aligned}
(\delta')^1(\mathbf{w} \otimes \mathbf{ab}) &= \mu(\rho_1 \otimes \rho_{23}) \otimes m_3(\mathbf{w}, \sigma_3, \sigma_2) \otimes \mathbf{x_2 b} \\
&= \rho_{123} \otimes \mathbf{w} \otimes \mathbf{x_2 b} \\
(\delta')^1(\mathbf{w} \otimes \mathbf{x_{2k} b}) &= \mu(\rho_{23}) \otimes m_3(\mathbf{w}, \sigma_3, \sigma_2) \otimes \mathbf{x_{2k+2} b} \\
&= \rho_{23} \otimes \mathbf{w} \otimes \mathbf{x_{2k+2} b} \quad \text{for } k = 1, \cdots n-2 \\
(\delta')^1(\mathbf{w} \otimes \mathbf{x_{2n-2} b}) &= \mu(\rho_2) \otimes m_3(\mathbf{w}, \sigma_3, \sigma_2) \otimes \mathbf{ab} \\
&= \rho_2 \otimes \mathbf{w} \otimes \mathbf{ab}
\end{aligned}$$

Compare this result with [2], Example 2.2.

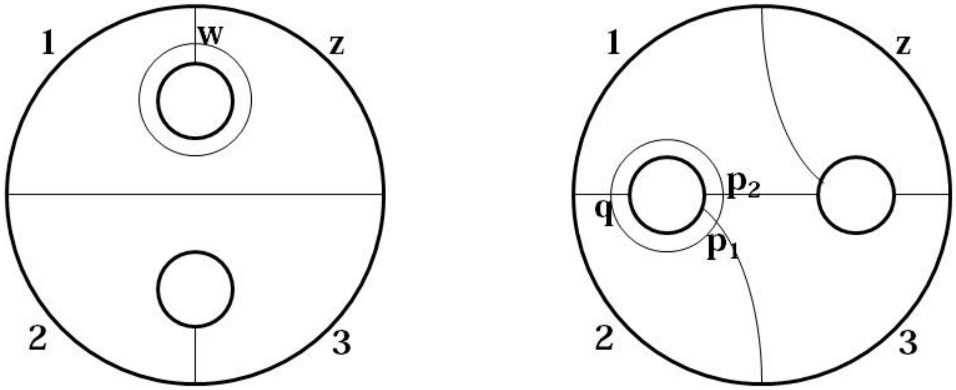


Figure 4.1: The diagram $\mathcal{H}_\infty$ on the left shows ∞-surgery on right component of the link. The diagram $\mathcal{H}_{+2}$ on the right is $+2$-surgery on the right component. The $\mathcal{A}_\infty$ relation of $\widehat{CFA}(\mathcal{H}_{+2})$is given as $m(q, \sigma_2) = p_1$, $m(p_1, \sigma_3, \sigma_2) = p_2$, and $m(p_2, \sigma_3, \sigma_2, \sigma_1) = q$.

4.3 Knot complement of trefoil

Consider $(2, 4)$ torus link. If we glue right component by solid torus of framing $+2$, then resulting diagram will be diffeomorphic to trefoil after handleslide and blow down $+1$ unknot component. A type-D structure $(N_1, (\delta_1)^1) := \widehat{CFA}(\mathcal{H}_{+2}) \boxtimes \widehat{CFDD}(\mathcal{H}_{(2,4)})$ computes,

$$\begin{tikzcd}
\mathbf{p_1} \otimes \mathbf{ab} \arrow[ddrr, dashed] & \mathbf{q} \otimes \mathbf{x_3 y_3} \arrow[l, "\rho_2"'] & \mathbf{q} \otimes \mathbf{a y_2} \arrow[l, "\rho_3"'] \arrow[d, "\rho_1"] \\
 & & \mathbf{q} \otimes \mathbf{x_1 y_1} \\
 & & \mathbf{p_2} \otimes \mathbf{ab} \arrow[u, "\rho_{123}"']
\end{tikzcd}$$

The dashed line denotes *unstable chain*, where

$$
\begin{tikzcd}
\cdots \arrow[r] & \mathbf{p_1} \otimes \mathbf{ab} \arrow[r, "\rho_{123}"'] & \mathbf{p_2} \otimes \mathbf{x_2 b} \arrow[r, "\rho_{23}"'] \arrow[d, "\rho_{23}"'] & \mathbf{q} \otimes \mathbf{x_1 y_3} \arrow[r, "\rho_{23}"'] \arrow[d, "1"'] & \mathbf{p_1} \otimes \mathbf{x_2 b} \arrow[r, "\rho_2"'] & \mathbf{p_2} \otimes \mathbf{ab} \arrow[r] & \cdots \\
& & \mathbf{q} \otimes \mathbf{x_3 y_1} \arrow[r, "1"'] & \mathbf{q} \otimes \mathbf{x_2 y_2} & & &
\end{tikzcd}
$$

We claim that the chain complex described above is homotopy equivalent to a complex $(N_2, (\delta_2)^1)$ which is identical to the complex above but unstable complex has been replaced by

$$
\cdots \to \mathbf{p_1} \otimes \mathbf{ab} \xrightarrow[\rho_{123}]{} \mathbf{p_2} \otimes \mathbf{x_2 b} \xrightarrow[\rho_{23}]{} \mathbf{q} \otimes \mathbf{x_1 y_3} \xrightarrow[\rho_{23}]{} \mathbf{p_1} \otimes \mathbf{x_2 b} \xrightarrow[\rho_2]{} \mathbf{p_2} \otimes \mathbf{ab} \to \cdots
$$

Define a map $\pi : N_1 \to N_2$ such that $\pi(\mathbf{q} \otimes \mathbf{x_3 y_1})$ and $\pi(\mathbf{q} \otimes \mathbf{x_2 y_2})$ equal zero, and otherwise acts as identity. We also define a map $\iota : N_2 \to N_1$ as an inclusion. Then $\pi \circ \iota = \mathbb{I}_{N_2}$ is obvious. In addition, a homotopy equivalence $H : N_1 \to N_1$ is given as,

$$
H(x) := \begin{cases} \mathbf{q} \otimes \mathbf{x_3 y_1} & \text{if } x = \mathbf{q} \otimes \mathbf{x_2 y_2} \\ \mathbf{q} \otimes \mathbf{x_1 y_3} + \mathbf{q} \otimes \mathbf{x_3 y_1} & \text{if } x = \mathbf{q} \otimes \mathbf{x_1 y_3} \\ \mathbf{p_2} \otimes \mathbf{x_2 b} & \text{if } x = \mathbf{p_2} \otimes \mathbf{x_2 b} \\ 0 & \text{otherwise.} \end{cases}
$$

which extends as a $\mathcal{A}(T)$-equivariant map. Then it is clear that $\iota \circ \pi = (\delta_1)^1 \circ H + H \circ (\delta_1)^1$.

Remark 4.3.1 *Compare above result with section 11.5 of [6], from which they* *spelled out an algorithm to recover* $\widehat{CFD}(S^3 \backslash \nu K)$ *from* CFK^-*. According to their notation, the length of unstable chain is 3 (the number of generators between two outermost ones). This length is closely related to the framing of knot complement and concordance invariant* $\tau(K)$*(see equation(11.18) from [6]). In our case, the framing of left component of link was originally -1, but handleslide procedure has added +4 and therefore the framing is 3. Since* $\tau(\mathit{Trefoil}) = 1$ *is less than the framing, the length of unstable chain agrees with the framing. Theorem A.11 from [6] has the precise description of relation between* $\tau(K)$ *and unstable chain.*

4.4 (n_1, n_2)-surgery on Hopf link

Hopf link is $(2, 2)$ torus link. If n_1 and n_2 are two positive integers such that $n_1 n_2 \neq 1$, then (n_1, n_2)-surgery on Hopf link results a lens space $L(n_1 n_2 - 1, n_1)$. Heegaard Floer homology of the lens space has $n_1 n_2 - 1$ generators whose differentials equal zero.

Diagram of Hopf link complement is easy. In addition, $\alpha_1^{a,L}$(respectively, $\alpha_1^{a,R}$) does not intersect β_1(respectively, β_2), therefore gluing the diagram with $\mathcal{H}_{n_1}^L$ and $\mathcal{H}_{n_2}^R$ will result closed Heegaard diagram of lens space $L(n_1 n_2 - 1, n_1)$. The $\mathcal{A}_\infty$ relation of $\widehat{CFA}(\mathcal{H}_m)$ is as follows(see [Figure 4.1]).

$$
\begin{aligned}
m(q, \rho_2) &= p_1 \\
m(p_i, \rho_3, \underbrace{\rho_{23}, \cdots, \rho_{23}}_{j \text{ times}}, \rho_2) &= p_{i+j+1} \\
m(p_m, \rho_3, \rho_2, \rho_1) &= q
\end{aligned}
$$

$\widehat{CFDD}(S^3 \backslash \nu(\text{Hopf link}))$ has two generators $\mathbf{ab}$ and $\mathbf{x_1y_1}$. Its type-D structure is given as below.

$$\begin{aligned} \delta^1(\mathbf{ab}) &= (\rho_1 \otimes \sigma_3 + \rho_3 \otimes \sigma_1 + \rho_{123} \otimes \sigma_{123}) \otimes \mathbf{x_1y_1} \\ \delta^1(\mathbf{x_1y_1}) &= \rho_2 \otimes \sigma_2 \otimes \mathbf{ab} \end{aligned}$$

Remark 4.4.1 *See [7], Proposition 10.1. Note that Hopf link complement is* $T^2 \times [0,1]$ *and it is exactly an identity module described in [7].*

Let p_i^L and q^L(p_j^R and q^R, respectively) be points of bordered Heegaard diagram $\mathcal{H}_{n_1}^L$ attached to the left ($\mathcal{H}_{n_2}^R$ attached to the right, respectively). Then we have following $n_1n_2 + 1$ generators of $\widehat{CFA}(\mathcal{H}_{n_1}^L) \boxtimes \widehat{CFA}(\mathcal{H}_{n_2}^R) \boxtimes \widehat{CFDD}(S^3 \backslash \nu(\text{Hopf link}))$.

$$\begin{aligned} &p_i^L \otimes p_j^R \otimes \mathbf{ab} \qquad i = 1, \cdots, n_1 \text{ and } j = 1, \cdots, n_2 \\ &q^L \otimes q^R \otimes \mathbf{x_1y_1}. \end{aligned}$$

The only nontrivial differential is,

$$\partial^{\boxtimes}(q^L \otimes q^R \otimes \mathbf{x_1y_1}) = m(q^L, \rho_2) \otimes m(q^R, \sigma_2) \otimes \mathbf{ab} = p_1^L \otimes p_1^R \otimes \mathbf{ab}.$$

Thus the homology of $\widehat{CFA}(\mathcal{H}_{n_1}^L) \boxtimes \widehat{CFA}(\mathcal{H}_{n_2}^R) \boxtimes \widehat{CFDD}(S^3 \backslash \nu(\text{Hopf link}))$ has $n_1n_2 - 1$ generators as expected.

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