Shintani Zeta Functions and Weyl Group Multiple Dirichlet Series

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Abstract of the Dissertation

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In recent years, substantial progresses have been made towards the development of a general theory of multiple Dirichlet series with functional equations. In this dissertation, we investigate the Shintani zeta function associated to a prehomogeneous vector space and identify it with a Weyl group multiple Dirichlet series. The example under consideration is the set of 2 by 2 by 2 integer cubes, that is the integral lattice in a certain prehomogeneous vector space acted on by three copies of GL(2). One of M. Bhargava's achievements is the determination of the corresponding integral orbits and the discovery of an extension of the Gauss's composition law for integral binary quadratic forms. We instead consider the action of a certain parabolic subgroup on the same vector space. We show there are three relative invariants that all have arithmetic meanings and completely determine the integral orbits. We prove that the associated Shintani zeta function coincides with the A_3 Weyl group multiple Dirichlet series. Lastly, we show that the set of semi-stable integral orbits maps finitely and surjectively to a certain moduli space. The last part of this dissertation is devoted to showing the connection between Shintani zeta functions of PVS and periods of automorphic forms.

To my parents

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Chapter 1

Introduction

The study of Dirichlet series in several complex variables has seen much development in recent years. Multiple Dirichlet series (MDS) can arise from metaplectic Eisenstein series, zeta functions of prehomogeneous vector spaces (PVS), height zeta functions or multiple zeta values. This list is far from exhaustive. In general, completely different techniques are involved in the study of different series arising in these different contexts. Partially for this reason, it is of great interest to identify examples of multiple Dirichlet series which lie in two or more of the areas above.

This paper investigates one such example. We study in detail a Shintani zeta function associated to a certain prehomogeneous vector space and show that it coincides with a Weyl group multiple Dirichlet series (WMDS) of the form introduced in BBCFH ([BBC$^+$06]). This latter series (in three complex variables) is also a Whittaker function of a Borel Eisenstein on the metaplectic double cover of GL_4.

The seminal work of Bhargava [Bha04a] generalizing Gauss's composition law on binary quadratic forms starts with investigating the rich structure of integer orbits of $2 \times 2 \times 2$-cubes acted on by $\mathrm{SL}_2(\mathbb{Z}) \times \mathrm{SL}_2(\mathbb{Z}) \times \mathrm{SL}_2(\mathbb{Z})$. This is the D_4 case among the list of classification of PVS given in Sato-Kimura [SK77]. Bhargava shows that the set of projective integer orbits with given discriminant has a group structure and it is isomorphic to the square product of the narrow class group of the quadratic order with that discriminant.

We begin with some definitions. Let $G_{\mathbb{C}}$ be a connected complex Lie group, usually we assume that $G_{\mathbb{C}}$ is the complexification of a real Lie group $G_{\mathbb{R}}$. A PVS $(G_{\mathbb{C}}, V_{\mathbb{C}})$ is a complex finite dimensional vector space $V_{\mathbb{C}}$ together with a holomorphic representation of $G_{\mathbb{C}}$ such that $G_{\mathbb{C}}$ has an open orbit in

$V_{\mathbb{C}}$. One of the important properties is that if $V_{\mathbb{C}}$ is a PVS for $G_{\mathbb{C}}$ then there is just one open orbit, and that orbit is dense (see [Kna02, Chapter X]). Let P be a complex polynomial function on $V_{\mathbb{C}}$. We call it a relative invariant polynomial if $P(gv) = \chi(g)P(v)$ for some rational character χ of $G_{\mathbb{C}}$ and all $g \in G_{\mathbb{C}}$, $v \in V_{\mathbb{C}}$. We say the PVS has n relative invariants if n algebraically independent relative invariant polynomials generate the invariant ring. We define the set of semi-stable points $V_{\mathbb{C}}^{ss}$ to be the subset on which no relative invariant polynomial vanishes.

Let $V_{\mathbb{Z}} = \mathbb{Z}^2 \otimes \mathbb{Z}^2 \otimes \mathbb{Z}^2$ and consider the action of $B_2'(\mathbb{Z}) \times B_2'(\mathbb{Z}) \times \mathrm{SL}_2(\mathbb{Z})$, where $B_2'(\mathbb{Z})$ is the subgroup of lower-triangular matrices in $\mathrm{SL}_2(\mathbb{Z})$ with positive diagonal elements. We denote $A \in V_{\mathbb{Z}}$ by $\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \begin{pmatrix} e & f \\ g & h \end{pmatrix}\right)$ for simplicity. The complex group $B_2'(\mathbb{C}) \times B_2'(\mathbb{C}) \times \mathrm{GL}_2(\mathbb{C})$ acting on $V_{\mathbb{C}}$ is a PVS and will have three relative invariants. They are $m(A) = ad - bc, n(A) = ag - ce$ and the discriminant $D(A) = \mathrm{disc}(A)$.

In their fundamental paper [SS74], M. Sato and T. Shintani introduced the notion of a zeta function associated to a PVS, nowadays called the Shintani zeta functions. See [Shi75] for the application to the average values of $h(d)$, the number of primitive inequivalent binary quadratic forms of discriminant d. Our primary interest will be in the Shintani zeta function in three variables associated to the PVS of $2 \times 2 \times 2$-cubes, which is defined to be:

$$Z_{\mathrm{Shintani}}(s_1, s_2, w) = \sum_{A \in \left(B_2'(\mathbb{Z}) \times B_2'(\mathbb{Z}) \times \mathrm{SL}_2(\mathbb{Z})\right) \backslash V_{\mathbb{Z}}^{ss}} \frac{1}{|D(A)|^w |m(A)|^{s_1} |n(A)|^{s_2}}.$$

We denote the partial sum of Shintani zeta function by

$$Z_{\mathrm{Shintani}}^{\mathrm{odd}}(s_1, s_2, w) = \sum_{\substack{A \in \left(B_2'(\mathbb{Z}) \times B_2'(\mathbb{Z}) \times \mathrm{SL}_2(\mathbb{Z})\right) \backslash V_{\mathbb{Z}}^{ss} \\ D(A) \text{ odd}}} \frac{1}{|D(A)|^w |m(A)|^{s_1} |n(A)|^{s_2}}.$$

We will show that it is closely related to another multiple Dirichlet series which arises in the Whittaker expansion of the Borel Eisenstein series on the metaplectic double cover of GL_4. This latter series is found to be the WMDS associated to the root system A_3 and it is of the form:

$$Z_{\mathrm{WMDS}}(s_1, s_2, w) = \sum_{D \text{ odd discriminant}} \frac{1}{|D|^w} \sum_{m,n>0} \frac{\chi_D(\hat{m})\chi_D(\hat{n})}{m^{s_1} n^{s_2}} a(D, m, n)$$

where $\hat{m}$ denotes the factor of m that is prime to the square-free part of D and χ_D is the quadratic character associated to the field extension $\mathbb{Q}(\sqrt{D})$ of $\mathbb{Q}$. A precise formula of the coefficients $a(D, m, n)$ will be given in chapter 4.

The main results of this paper, given in Theorems 1.0.1, 1.0.2 and 1.0.3 below, are as follows:

(1) We give an explicit description of the Shintani zeta function associated to the PVS of $2 \times 2 \times 2$-cubes, under the action of group $B_2' \times B_2' \times \mathrm{GL}_2$.

(2) We show how the series is related to the quadratic WMDS associated to the root system A_3.

(3) We give an arithmetic meaning to the semi-stable integer orbits of the PVS.

Theorem 1.0.1. *The Shintani zeta function associated to the* PVS *of* $2 \times 2 \times 2$*-cubes is given by:*

$$Z_{\mathrm{Shintani}}(s_1, s_2, w) = \sum_{A \in \left(B_2'(\mathbb{Z}) \times B_2'(\mathbb{Z}) \times \mathrm{SL}_2(\mathbb{Z})\right) \backslash V_{\mathbb{Z}}^{ss}} \frac{1}{|D(A)|^w |m(A)|^{s_1} |n(A)|^{s_2}}$$
$$= \sum_{D = D_0 D_1^2 \neq 0} \frac{1}{|D|^w} \sum_{m,n>0} \frac{B(D,m,n)}{m^{s_1} n^{s_2}},$$

where

$$B(D, m, n) = \sum_{\substack{d | D_1 \\ d|m,\ d|n}} d \cdot A(D/d^2, 4m/d) \cdot A(D/d^2, 4n/d),$$

where $A(d, a)$ *is denoted by the number of solutions to the congruence* $x^2 = d \ (mod\ a)$.

Theorem 1.0.2. *The Shintani zeta function can be related to the* A_3-WMDS *in the following way:*

$$Z^{\mathrm{odd}}_{\mathrm{Shintani}}(s_1, s_2, w) = 4 \frac{\zeta(s_1)}{\zeta(2s_1)} \frac{\zeta(s_2)}{\zeta(2s_2)} Z_{\mathrm{WMDS}}(s_1, s_2, w).$$

where $\zeta(s)$ *stands for the Riemann zeta function.*

Lastly, we give the arithmetic meaning to the semi-stable integer orbits of the PVS by showing that:

Theorem 1.0.3. *There is a natural map, which is a surjective and finite morphism,*

$$\begin{aligned}&(B_2'(\mathbb{Z}) \times B_2'(\mathbb{Z}) \times \mathrm{SL}_2(\mathbb{Z})) \backslash V_{\mathbb{Z}}^{ss} \\ &\to \mathrm{Iso}\backslash\{(R; I_1, I_2) : R/I_1 \cong \mathrm{N}(I_1)\mathbb{Z}, R/I_2 \cong \mathrm{N}(I_2)\mathbb{Z}\},\end{aligned}$$

where R is an oriented quadratic ring and $I_i's$ are oriented ideals in R with the norm $\mathrm{N}(I_i)$. The cardinality $n(R; I_1, I_2)$ of the fiber is equal to

$$\sigma_1\left(g.c.d.(D_1, a_1, a_2)\right),$$

where $D = D_0 D_1^2 = \mathrm{disc}(R)$ with D_0 is square-free, and $a_i = \mathrm{N}(I_i)$. It further satisfies

$$\sum_{\substack{(R; I_1, I_2)/\sim \\ \mathrm{N}(I_i) = a_i}} n(R; I_1, I_2) = B(D, a_1, a_2).$$

The organization of this paper is as follows. In chapter 2, we will review a double Dirichlet series arising from three different approaches. We show the connection between Shintani's PVS approach and the A_2-WMDS approach. In chapter 3, we will investigate the structure of integer orbits of the PVS of $2 \times 2 \times 2$-cubes, using the reduction theory, to derive the main formula in Thm 1.0.1. In chapter 4, we will show its connection to the A_3-WMDS. The main idea of the proof is to consider the generating function for the p-parts of $a(D, m, n)$. From the construction of an A_3-WMDS, the p-parts of the rational function which is invariant under the Weyl group action is given explicitly by taking the residue of the convolution of two rational functions of A_2 root system. We show it coincides with the generating function of $a(D, m, n)$. In the chapter 5, we will show the set of semi-stable integer orbits naturally encodes the arithmetic information by showing it maps finitely and surjectively to the moduli space of isomorphism classes of pairs $(R; I_1, I_2)$, where R is an oriented quadratic ring and $I_i's$ are the oriented ideals in R. We will recall the definition of orientation of a quadratic ring and its ideal. We further show that each fiber is counted by a divisor function. In the final chapter, we indicate how the approach of Shintani zeta functions of PVS can be used to study the periods of automorphic forms.

Chapter 2

A_2 Weyl Group Dirichlet series

The simplest generalization to the famous Riemann zeta function

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

is the Dirichlet series, which has the form

$$L(s, \chi_d) = \sum_{n=1}^{\infty} \frac{\chi_d(n)}{n^s},$$

where χ_d is a primitive character of $(\mathbb{Z}/d\mathbb{Z})^\times$. The basic properties of those series are meromorphic continuation and functional equations, provided in [Dav00].

Consider families of quadratic Dirichlet series χ_d with $\chi_d^2 = 1$, we can form a Dirichlet series in two variables

$$Z(s, w) = \sum_{d} \frac{L(s, \chi_d)}{d^w}.$$

This is the simplest multiple Dirichlet series with the analytic properties well established in [GH85]. Combining with the basic Tauberian technique, this leads to the mean value theorem

$$\sum_{0<d<X} L(1/2, \chi_d) \sim cX \log X,$$

where c is a non-zero constant.

In the language of automorphic representation of GL(r), Riemann zeta function and Dirichlet series can be replaced by the L-functions. In general, if π is denoted by the fixed automorphic representation of GL(r) over the filed F, the twisted L-function can be written as

$$L(s,\pi\times\chi)=\sum c(m)\chi(m)|m|^s,$$

where χ is a idèle class character. There are two basic but important questions to ask: the first is the non-vanishing of central values $L(1/2,\pi)$ or even $L(1/2,\pi\times\chi)$ for infinitely many χ. The second one is the asymptotics of moments

$$\sum_{\mathrm{cond}<X} L(s,\pi\times\chi)^k.$$

There are many results related to the above questions in the literature. One approach, as we discussed for two-variable Dirichlet series, is to put twisted L-functions into families and construct

$$Z(s,w)=\sum_d \frac{L(s,\pi\times\chi_d)a(s,\pi,d)}{|d|^w}.$$

The weight factor $a(s,\pi,d)$ is introduced so that one can establish the results such as meromorphic continuation and functional equations for Z. In general the multiple Dirichlet series has the form

$$Z(s_1,s_2,\cdots,s_r,w)=\sum_d \frac{\prod_{i=1}^r L(s_i,\pi_i\times\chi_d)a(\{s_i\}\},\{\pi_i\},d)}{|d|^w}$$

for suitable weight factors a. The method of multiple Dirichlet series is a powerful recent method in the theory of automorphic L-functions, and has so far been used to prove a number of interesting nontrivial theorems. For example, assuming a number field K containing the n-th roots of unity, the double Dirichlet series has the form

$$Z^{(n)}(s,w)=\sum_{d\neq 0}\frac{L(s,\pi\otimes\bar{\chi}_{d_0}^{(n)})\epsilon(d_0)P_d(s,\pi)}{\mathbb{N}d^w},$$

where $d=d_0d_n^n$ with d_0 n-th power-free. Here $\epsilon(d_0)$ denotes an n-th order Gauss sum corresponding to the character $\chi_{d_0}^{(n)}$ and $P_d(s,\pi)$ denotes certain correction factors. The numerator is a families of Dirichlet series in the variable s. The double Dirichlet series exhibits a functional equation in s coming from that of single Dirichlet series. On the other hand, interchanging the other of summation results in the numerator a new Dirichlet series formed from the n-th order Gauss sums. Such series arise in the theory of Eisenstein series on the n-th fold cover of GL_2 introduced by Kubota in [Kub69]. Therefore the double Dirichlet series inherits a corresponding functional equations in w. The detailed study of its analytic continuity and arithmetic consequences are carried out in [BBC+04]. A related result for $n = 3$ has been obtained by Brubaker, S. Friedberg, and J. Hoffstein in [BFH05].

One particular class of multiple Dirichlet series is Weyl group multiple Dirichlet series (WMDS). Let Φ be an irreducible root system of rank r with Weyl group W. These are multiple Dirichlet series attached to root systems Φ such that the group of functional equations is isomorphic to the Weyl group W of Φ. Classical Dirichlet series fit into this framework as examples of series of type A_1: the functional equation $s \to 1-s$ corresponds to the generator of $W = \mathbb{Z}/2\mathbb{Z}$. One reason to study Weyl group multiple Dirichlet series is that they provide tools for computations in analytic number theory, such as mean values of special values of L-functions and moments of L-functions. A second reason is that conjecturally these series are Fourier-Whittaker coefficients for Borel Eisenstein series on metaplectic groups ([BBF11a]).

For example, let $\Phi = A_r$ and $K = \mathbb{Q}$. Then the series Z has the form

$$\sum \frac{a(m_1, m_2, \cdots, m_r)}{m_1^{s_1} m_2^{s_2} \cdots m_r^{s_r}},$$

where the sum is over all positive integers m_i. If $m_1 m_2 \cdots m_r$ is odd and square free, we have

$$a(m_1, m_2, \cdots, m_r) = \left(\frac{m_1}{m_2}\right)\left(\frac{m_2}{m_3}\right)\cdots\left(\frac{m_{r-1}}{m_r}\right).$$

The coefficients satisfy the following twisted multiplicativity property

$$a(m_1 m_1', \cdots, m_r m_r') = a(m_1, \cdots, m_r) a(m_1', \cdots, m_r') \prod_{j=1}^{r-1} \left(\frac{m_j}{m_{j+1}'}\right)\left(\frac{m_j'}{m_{j+1}}\right),$$

where $(m_1 \cdots m_r, m_1' \cdots m_r') = 1$. Thus, constructing a Weyl group multiple Dirichlet series reduces to constructing its p-part coefficients

$$a(p^{k_1}, \cdots, p^{k_r}),$$

where p ranges over all primes. One method, pursued by G. Chinta and P. Gunnells in [CG07] and [CG10], uses a metaplectic analogue of the Weyl character formula to produce the p-part once they are organized into a generating function

$$\sum_{k_1,\cdots,k_r} \frac{a(p^{k_1},\cdots,p^{k_r})}{p^{k_1 s_1}\cdots p^{k_r s_r}}.$$

This technique works for all root systems, but has the disadvantage that the coefficients are not determined explicitly. The second method, developed by B. Brubaker, D. Bump, S. Friedberg, and J. Hoffstein in [BBF06], [BBFH07] and [BBF11b], prescribes the coefficients explicitly in terms of Gauss sums built using subtle combinatorics of crystal graphs. It has the disadvantage that the definitions are best understood only for Φ of type A, and that the functional equations are difficult to prove.

In the following sections we will introduce three double Dirichlet series and show that they are all essentially the same. The results are not new, Diamantis and Goldfeld in [DG11] show their relations using a type of converse theorem, and Shintani in [Shi75, §2] compares his series to Siegel's, but the computations in section will serve as a prototype for the more involved computations involving 3-variable Dirichlet series in the latter chapters.

2.1 Siegel Double Dirichlet Series

The quadratic multiple Dirichlet series first appeared in the paper of Siegel [Sie56], and its twisted Euler product with respect to one variable was given explicitly. Siegel constructed his series as the Mellin transform of an Eisenstein series of half integral weight on the congruence subgroup $\Gamma_0(4)$. In this section we will first recall the definition of Siegel's double Dirichlet series and then, using the multiplicative property of Euler products, prove it can be expressed as a sum formed from quadratic characters.

Denote by $A(d,a)$ the number of solutions to the quadratic congruence equation $x^2 = d \pmod{a}$. Then Siegel's double Dirichlet series is defined to be:

$$Z_{\text{Siegel}}(s,w) = \sum_{d\neq 0} \frac{1}{|d|^w} \sum_{a\neq 0} \frac{A(d,a)}{|a|^s}.$$

For any positive prime integer p and any integer d, we define a generating series for the p-part of the inner summation of above series to be

$$f_p(d,s) = (1-p^{-s})\sum_{l=0}^{\infty} A(d,p^l)p^{-ls} \ (p \neq 2),$$
$$f_2(d,s) = (2^s-1)\sum_{l=1}^{\infty} A(d,2^l)2^{-ls}.$$

Siegel shows in [Sie56, §4] that

$$\frac{1-\chi_d(p)p^{-s}}{1-p^{-2s}} f_p(d,s) = \begin{cases} p^{\alpha(1-2s)} + (1-\chi_d(p)p^{-s})\sum_{l=0}^{\alpha-1} p^{l(1-2s)} & (p \neq 2), \\ \frac{1+\chi_d(2)}{1+2^{-s}} + (2^{1-s}-\chi_d(2))\sum_{l=0}^{\alpha} 2^{l(1-2s)} & (p=2), \end{cases}$$

where χ_d is the quadratic character associated to the field $\mathbb{Q}(\sqrt{d})$ and $p^{2\alpha}$ is the highest power of p which divides d/d^*, d^* is the fundamental discriminant of the field $\mathbb{Q}(\sqrt{d})$. Now the inner summation of Siegel's double Dirichlet series can be expressed as a normalized quadratic L-function.

Proposition 2.1.1. *Fix an integer $d \neq 0$, then*

$$\sum_{a<0} \frac{A(d,a)}{|a|^s} = \sum_{a>0} \frac{A(d,a)}{a^s} = \zeta(2s)^{-1}\zeta(s)L(s,\chi_d)P(d,s)$$

where the last term is

$$\begin{aligned} P(d,s) =& 2^{-s}\prod_{p\neq 2}\left(p^{\alpha(1-2s)} + (1-\chi_d(p)p^{-s})\sum_{l=0}^{\alpha-1} p^{l(1-2s)}\right) \\ &\cdot \left(\frac{1+\chi_n(2)}{1+2^{-s}} + \frac{1-\chi_d(2)2^{-s}}{1+2^{-2s}}(2^s-1) + \left(2^{1-s}-\chi_n(2)\right)\sum_{l=0}^{\alpha} 2^{l(1-2s)}\right). \end{aligned}$$

Proof. The first equality of the claim follows from the fact that

$$A(d,a) = A(d,-a).$$

For the second equality, first note that the multiplicative property holds

$$A(d,m)A(d,n) = A(d,mn)$$

for any pairs of coprime positive integers m and n, by the Chinese remainder theorem. Then the results follows from two equations

$$\begin{aligned}&\frac{1-\chi_n(p)p^{-s}}{1-p^{-2s}}\left(1-p^{-s}\right)\sum_{l=0}^{\infty}A\left(d,p^l\right)p^{-ls}\\ =&p^{\alpha(1-2s)}+\left(1-\chi_n\left(p\right)p^{-s}\right)\sum_{l=0}^{\alpha-1}p^{l(1-2s)},\end{aligned}$$

and

$$\begin{aligned}&\frac{1-\chi_n(2)2^{-s}}{1-2^{-2s}}\left(2^s-1\right)\sum_{l=1}^{\infty}A\left(d,2^l\right)2^{-ls}\\ =&\frac{1+\chi_n(2)}{1+2^{-s}}+\left(2^{1-s}-\chi_n(2)\right)\sum_{l=0}^{\alpha}2^{l(1-2s)},\end{aligned}$$

as well as the multiplicative property. □

2.2 Shintani Double Dirichlet Series

Another approach to the theory of double Dirichlet series is based on the zeta function associated to the prehomogeneous vector space (PVS) developed by M. Sato and Shintani in [SS74]. In [Shi75], where the double Dirichlet series associated to the PVS of binary quadratic forms acted on by the Borel subgroup of $\mathrm{GL}_2(\mathbb{C})$ is studied in detail, the author obtains the mean values of class numbers of primitive and integral binary quadratic forms. It should be mentioned that in [Sai93] the essentially same double Dirichlet series is discovered as the zeta function associated to another PVS. In this section, we will recall the Shintani zeta function in two variables arising from the PVS approach.

Now we let $B_2'(\mathbb{C})$ be the subgroup of lower-triangular matrices in $G_{\mathbb{C}} = \mathrm{GL}_2(\mathbb{C})$ and let ρ be the representation of $\mathrm{GL}_2(\mathbb{C})$ acting on the three dimensional vector space $V_{\mathbb{C}} = \{Q(u,v) = x_1u^2 + x_2uv + x_3v^2 | (x_1, x_2, x_3) \in \mathbb{C}^3\}$ of binary quadratic forms as follows

$$\rho(g)(Q)(u,v) = Q(au + cv, bu + dv)$$

where $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. It is well known that $(B_2'(\mathbb{C}), V_{\mathbb{C}})$ is a PVS and there are two relative invariants for the action of $B_2'(\mathbb{C})$ on $V_{\mathbb{C}}$, namely,

$$P_1(Q) = x_1 = Q(1,0),$$

and the discriminant

$$P_2(Q) = \mathrm{disc}(Q) = x_2^2 - 4x_1x_3$$

of the quadratic form. These two invariants freely generate the ring of relative invariants. If we set for $b = \begin{pmatrix} b_{11} & 0 \\ b_{21} & b_{22} \end{pmatrix} \in B_2'(\mathbb{C})$

$$\chi_1(b) = b_{11}^2 \text{ and } \chi_2(b) = \det(b)^2,$$

then $P_1(b \cdot Q) = \chi_1(b)P_1(Q)$ and $P_2(b \cdot Q) = \chi_2(b)P_2(Q)$. We define the singular set S to be

$$S = \{Q \in V_{\mathbb{C}} | P_1(Q)P_2(Q) = 0\}.$$

Let $V_{\mathbb{Z}}^{ss}$ be the set of non-singular integral binary quadratic forms. Define $B_2'(\mathbb{Z})$ to be the Borel subgroup in $\mathrm{SL}_2(\mathbb{Z})$ with positive diagonal elements. For an arbitrary $K = \mathrm{SO}_2(\mathbb{R})$-invariant Schwartz function $f \in \mathcal{S}(V_{\mathbb{R}})$, the global Shintani zeta integral is defined to be

$$Z(V^{ss}, f, s, w) = \int_{B'(\mathbb{Z})\backslash B'(\mathbb{R})} \chi_1(b)^s \chi_2(b)^w \sum_{x \in V_{\mathbb{Z}}^{ss}} f(x \cdot b) db,$$

where db denotes the left invariant measure on $B'(\mathbb{R})$. Set $V_i = \{x \in V_{\mathbb{R}}, (-1)^{i-1}P(x) > 0\} (i = 1, 2)$,

$$\Phi_i(f, s, w) = \int_{V_i} f(x)|x_1|^s |P(x)|^w dx$$

$(dx = dx_1dx_2dx_3, f \in \mathcal{S}(V_{\mathbb{R}}), (s, w) \in \mathbb{C}^2)$. It is obvious that $\Phi_i(f, s, w)$ is a holomorphic function in the domain $\{(s, w) : \mathrm{Re}\ s, \mathrm{Re}\ w > 0\}$. It is easy to see that

$$Z(V^{ss}, f, s, w) = 2^{-1} \sum_{i=1}^{2} \xi_i(s, w)\Phi_i(f, s - 1, w - 1),$$

where

$$\xi_i(s,w) = \sum_{Q \in B_2'(\mathbb{Z})\backslash V_i^{ss}} \frac{1}{|Q(1,0)|^s |\mathrm{disc}(Q)|^w}$$
$$= \sum_{a \neq 0} \frac{1}{|a|^s} \sum_{\substack{0 \leq b \leq 2a-1 \\ c:(-1)^{i-1}(b^2-4ac)>0}} \frac{1}{|b^2-4ac|^w}.$$

Now the Shintani zeta function associated to the PVS $(B_2'(\mathbb{C}), V_{\mathbb{C}})$ is defined to be

$$Z_{\mathrm{Shintani}}(s,w;B_2') = \sum_{Q \in B_2'(\mathbb{Z})\backslash V_{\mathbb{Z}}^{ss}} \frac{1}{|Q(1,0)|^s |\mathrm{disc}(Q)|^w}$$
$$= \sum_{a \neq 0} \frac{1}{|a|^s} \sum_{\substack{0 \leq b \leq 2a-1 \\ c:b^2-4ac \neq 0}} \frac{1}{|b^2-4ac|^w},$$

where as before $V_{\mathbb{Z}}^{ss}$ is the semi-stable subset of (not necessarily primitive) quadratic forms with $Q(1,0) \neq 0$ and non-zero discriminant. Alternatively, we can express the Shintani zeta function as

$$\sum_{Q \in \mathrm{SL}_2(\mathbb{Z})\backslash V_{\mathbb{Z}}^{ss}} \frac{1}{|\mathrm{disc}(Q)|^w} \sum_{\gamma \in B_2'(\mathbb{Z})\backslash \mathrm{SL}_2(\mathbb{Z})/\mathrm{stab}_Q} \frac{1}{|\gamma \circ Q(1,0)|^s}.$$

For the rest of this section we suppress the B_2' from the notation. We also define

$$Z_{\mathrm{Shintani}}^{\mathrm{odd}}(s,w) = \sum_{\substack{Q \in B_2'(\mathbb{Z})\backslash V_{\mathbb{Z}}^{ss} \\ \mathrm{disc}(Q) \text{ odd}}} \frac{1}{|Q(1,0)|^s |\mathrm{disc}(Q)|^w}$$
$$= \sum_{a \neq 0} \frac{1}{|a|^s} \sum_{\substack{0 \leq b \leq 2a-1 \\ c:b^2-4ac \text{ odd}}} \frac{1}{|b^2-4ac|^w}.$$

Remark 2.2.1. *In section 5, we will show that the orbits in $B_2'(\mathbb{Z})\backslash V_{\mathbb{Z},\mathrm{primitive}}^{ss}$ parameterize the isomorphism classes of the pairs (R,I), where R is an oriented quadratic ring and I is an oriented ideal with cyclic quotient in R. We will call (R_1,I_1) and (R_2,I_2) isomorphic if there is a ring isomorphism from R_1 to R_2 preserving the orientation and sending I_1 to I_2.*

Lemma 2.2.2. *The Shintani zeta function can be written as*

$$Z_{\text{Shintani}}(s,w) = \xi_1(s,w) + \xi_2(s,w),$$

where $\xi_i(s,w) = \sum_{a,d>0} \frac{A((-1)^{i-1}d,4a)}{a^s d^w}$.

Proof. First note that under the action of $g = \begin{pmatrix} 1 & 0 \\ m & 1 \end{pmatrix}$ on the quadratic form $Q(u,v) = au^2 + buv + cv^2$, the middle coefficient b is mapped to $b + 2am$. Given non-zero integers a and d, the number of the solutions to $b^2 - 4ac = d$ with $0 \leq d \leq 2a-1$ and $c \in \mathbb{Z}$ is equal to $\frac{A(d,4a)}{2}$. So we have the equality

$$Z_{\text{Shintani}}(s,w) = \frac{1}{2} \sum_{a,d \neq 0} \frac{A(d,4a)}{|a|^s |d|^w}.$$

On the other hand, $A(d,4a) = A(d,-4a)$. Therefore,

$$Z_{\text{Shintani}}(s,w) = \sum_{a,d>0} \frac{A(d,4a)}{a^s d^w} + \sum_{a,d>0} \frac{A(-d,4a)}{a^s d^w}.$$

The result follows. □

As a corollary to Proposition 2.1.1, we can express the inner summation of the Shintani zeta function in terms of a quadratic Dirichlet L-function.

Corollary 2.2.3. *Fix* $d \neq 0$. *Then we have*

$$\sum_{a>0} \frac{A(d,4a)}{a^s} = \zeta(2s)^{-1} \zeta(s) L(s,\chi_d) P'(d,s),$$

where the last term is

$$P'(d,s) = 4^s \prod_{p \neq 2} \left(p^{\alpha(1-2s)} + \left(1 - \chi_d(p) p^{-s}\right) \sum_{l=0}^{\alpha-1} p^{l(1-2s)} \right)$$
$$\cdot \left(\frac{1 + \chi_n(2)}{1 + 2^{-s}} + \left(2^{1-s} - \chi_n(2)\right) \sum_{l=0}^{\alpha} 2^{l(1-2s)} - \frac{1 - \chi_n(2) 2^{-s}}{1 - 2^{-2s}} \left(1 - 2^{-s}\right) \right).$$

Proof. As $\sum_{a>0} \frac{A(d,4a)}{a^s} = 4^s \sum_{a>0} \frac{A(d,4a)}{(4a)^s}$, by the proof of proposition 2.1, we need only to correct the generating function at prime $p = 2$ in order to incorporate the factor 4, in which case the generating function should be

$$\sum_{l=0}^{\infty} A(d, 4 \times 2^l) 2^{-ls} = 4^s \sum_{l=2}^{\infty} A(d, 2^l) 2^{-ls}.$$

While the function on the right hand side satisfies

$$\begin{aligned} \frac{1 - \chi_d(2)2^{-s}}{1 - 2^{-2s}} (2^s - 1) \sum_{l=2}^{\infty} A(d, 2^l) 2^{-ls} = & \frac{1 - \chi_d(2)2^{-s}}{1 - 2^{-2s}} f_2(d, s) \\ & - \frac{1 - \chi_n(2)2^{-s}}{1 - 2^{-2s}} (2^s - 1) A(d, 2) 2^{-s}. \end{aligned}$$

Note that $A(d, 2) = 1$, therefore, using the multiplicative property of $A(d, \cdot)$, we have

$$\sum_{a>0} \frac{A(d, 4a)}{a^s} = 4^s \sum_{a>0} \frac{A(d, 4a)}{(4a)^s} = \zeta(2s)^{-1} \zeta(s) L(s, \chi_d) P'(d, s).$$

The result follows. $\square$

2.3 A_2-Weyl Group Multiple Dirichlet Series

A_2 Weyl group double Dirichlet series is the first example of WMDS found by Siegel in [Sie56]. This quadratic A_2 Weyl group double Dirichlet series has the form

$$Z_{A_2}(s, w) = \sum_{\substack{m>0, \\ D \text{ odd discriminant}}} \frac{\chi_D(\hat{m})}{m^s |D|^w} a(D, m),$$

where $\hat{m}$ is the factor of m that is prime to the square-free part of D and χ_D is the quadratic character associated to the field extension $\mathbb{Q}(\sqrt{D})$ of $\mathbb{Q}$. Moreover, the multiplicative factor $a(D, m)$ is defined by

$$a(D, m) = \prod_{p^k || D, p^l || m} a(p^k, p^l)$$

and

$$a(p^k, p^l) = \begin{cases} \min(p^{k/2}, p^{l/2}) & \text{if } \min(k,l) \text{ is even,} \\ 0 & \text{otherwise.} \end{cases}$$

Siegel obtained this series as the Mellin transform of a half-integral weight Eisenstein series for the congruence subgroup $\Gamma_0(4)$: Let $E^*(z,s)$ be the half-integral weight Eisenstein series of $\Gamma_0(4)$:

$$E^*(z,s) = \sum_{\Gamma_\infty \backslash \Gamma_0(4)} j_{1/2}(\gamma, z)^{-1} \Im(\gamma z)^{s/2}.$$

Maass showed its d-th Fourier coefficient is essentially the

$$L(s, \chi_d),$$

where χ_d is the quadratic character associated to $\mathbb{Q}(\sqrt{d})/\mathbb{Q}$. The Mellin transform of $E^*(z,s)$ is

$$Z(s,w) = \int_0^\infty (E^*(iy,s) - \text{constant term}) y^w \frac{dy}{y}.$$

The result is the double Dirichlet series roughly of the form

$$Z(s,w) = \sum_d \frac{L(s,\chi_d) a(s,d)}{d^w}.$$

This integral representation implies two functional equations for $Z(s,w)$, one coming from the functional equation of the Eisenstein series, and one coming from the Mellin transform, via the automorphy of the Eisenstein series. These functional equations take the form

$$Z(s,w) \to Z(1-s, w+s-1/2) \text{ and } Z(s,w) \to Z(s, 3/2-s-w).$$

They altogether generate the subgroup of functional equations isomorphic to $S_3 = W(A_2)$. Note that an extra functional equation can be recognized as the Fourier-Whittaker coefficients of a minimal parabolic metaplectic Eisenstein series of the double cover of GL_3. This fact is demonstrated for WMDS of type A in the paper [BBF11a]. This extra one swaps s and w that corresponds to the outer automorphism of the Dynkin diagram.

In view of the approach of G. Gautam and P. Gunnells in [CG07] and [CG10], the generating function of the p-part coefficients

$$\sum_{k,l\geq 0} \frac{a(p^k,p^l)}{p^{ks_1}p^{ls_2}}$$

can be expressed as

$$f_{A_2}(x_1,x_2) = \frac{(1-x_1x_2)}{(1-x_1)(1-x_2)(1-px_1^2x_2^2)},$$

where p^k is replaced by x_1 and p^l is replaced by x_2.

Regarding the relation between this quadratic A_2-WMDS and the Shintani zeta function of PVS of binary quadratic forms, we have

Theorem 2.3.1. *The Shintani zeta function $Z^{\mathrm{odd}}_{\mathrm{Shintani}}(s,w)$ and the quadratic A_2 Weyl group multiple Dirichlet series $Z_{A_2}(s,w)$ satisfy the relation:*

$$Z^{\mathrm{odd}}_{\mathrm{Shintani}}(s,w) = 2\zeta(2s)^{-1}\zeta(s)Z_{A_2}(s,w).$$

Proof. Analogous to the p-part formula of $a(p^k,p^l)$, we will show that for an odd prime p

$$A(p^k,p^l) = \begin{cases} 2a(p^k,p^l) & \text{if } k<l, \\ p^{\lfloor l/2\rfloor} & \text{otherwise.} \end{cases}$$

First consider the case when $p \neq 2$. If $k < l$ and k is an odd integer, then the congruence equation $x^2 = p^k (\mathrm{mod}\ p^l)$ reduces to the equation of $x^2 = p(\mathrm{mod}\ p^i)$ for some power i of p and there is no solution to it; while when k is an even integer, the congruence equation $x^2 = p^k(\mathrm{mod}\ p^l)$ reduces to the equation of or $x^2 = 1(\mathrm{mod}\ p^i)$ and there are two solutions to it. In both case, the number of solutions are both equal to the value of $2a(p^k,p^l)$ by its definition.

If on the other hand $k \geq l$, then the set of solutions to the congruence equation $x^2 = p^k(\mathrm{mod}\ p^l)$ is the set of multipliers of $p^{\lceil \frac{l}{2} \rceil}$, so the number of distinct solutions mod p^l is $p^{\lfloor \frac{l}{2} \rfloor}$.

Therefore, for odd prime p,

$$A(p^k,p^l) = \chi_{p^k}(\hat{p}^l)a(p^k,p^l) + \chi_{p^k}(\hat{p}^{l-1})a(p^k,p^{l-1}) = a(p^k,p^l) + a(p^k,p^{l-1}),$$

where we set the term $a(p^k,p^{l-1})$ equal to 0 when $l=0$.

Next, using Hensel's lemma, an integer d relatively prime to an odd prime p is a quadratic residue modulo any power of p if and only if it is a quadratic residue modulo p. In fact, if an integer d is prime to the odd prime p, as

$$A(d, p^l) = 2 \iff \chi_d(p) = 1 \iff A(d, p) = 2,$$

so

$$A(d, p^l) = \chi_d(p^l) + \chi_d(p^{l-1}).$$

By the prime power modulus theory [Gau66], suppose the modulus is p^l, then $p^k d$

1. is a quadratic residue modulo p^l if $k \geq l$,
2. is a non-quadratic residue modulo p^l if $k < l$ is odd,
3. is a quadratic residue modulo p^l if $k < l$ is even and d is a quadratic residue,
4. is a non-quadratic residue modulo p^l if $k < l$ is even and d is a non-quadratic residue.

Therefore, for an odd integer d prime to $p \neq 2$, we have

$$A(dp^k, p^l) = \begin{cases} 0 & \chi_d(p^l) = \text{-1 and } k < l \text{ even,} \\ A(p^k, p^l) & \text{otherwise.} \end{cases}$$

In the former case, we have:

$$A(dp^k, p^l) = 0 = \chi_{dp^k}(\hat{p}^l) a(dp^k, p^l) + \chi_{dp^k}(\hat{p}^{l-1}) a(dp^k, p^{l-1}). \tag{1}$$

In the latter case, we also have:

$$A(dp^k, p^l) = A(p^k, p^l) = \chi_{dp^k}(\hat{p}^l) a(dp^k, p^l) + \chi_{dp^k}(\hat{p}^{l-1}) a(dp^k, p^{l-1}). \tag{2}$$

Now let D be an arbitrary odd integer. Given an prime integer p, write $D = D_0 p^k$, where D_0 is prime to p. Then from the equality (1) and (2) with d replaced by D_0, it follows that

$$\sum_{l=0}^{\infty} A(D, p^l) p^{-ls} = (1 - p^{-2s})(1 - p^{-s})^{-1} \sum_{l=0}^{\infty} \chi_D(\hat{p}^l) a(D, p^l) p^{-ls}.$$

For $p = 2$, we define $\tilde{P}_2(D, s)$ by equating

$$\sum_{l=0}^{\infty} \frac{A(D, 2^{l+2})}{2^{ls}} = \tilde{P}_2(D, s)(1 - 2^{-2s})(1 - 2^{-s})^{-1} \sum_{l=0}^{\infty} \frac{\chi_D(2^l)}{2^{ls}}$$
$$= \tilde{P}_2(D, s)(1 - 2^{-2s})(1 - 2^{-s})^{-1} \sum_{l=0}^{\infty} \frac{\chi_D(2^l)a(D, 2^l)}{2^{ls}}.$$

By the multiplicative property of $A(D, \cdot)$ and that of $\chi_D(\cdot)a(D, \cdot)$,

$$\sum_{m>0} \frac{A(D, 4m)}{m^s} = \tilde{P}_2(D, s)\zeta(2s)^{-1}\zeta(s) \sum_{m>0} \frac{\chi_D(\hat{m})a(D, m)}{m^s}.$$

It remains to compute $\tilde{P}_2$. This is obtained by the next lemma.

Lemma 2.3.2. *Let D be an odd integer. With $\tilde{P}_2(D, s)$ defined in the above proposition, we have*

$$\tilde{P}_2(D, s) = \begin{cases} 2 & D \equiv 1 \pmod 4, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Write

$$\sum_{l=0} \frac{A(D, 2^{l+2})}{2^{ls}} = A(D, 4) + \sum_{l=1} \frac{A(D, 2^{l+2})}{2^{ls}},$$

and note that

$$A(D, 4) = \begin{cases} 2 & D \equiv 1 \text{ or } 5 \pmod 8, \\ 0 & \text{otherwise.} \end{cases}$$

To simplifying the second term, note that if D is an odd integer and $m = 8, 16$, or some higher power of 2, then D is a quadratic residue modulo m if and only if $D \equiv 1 (\text{mod } 8)$, therefore for $l \geq 1$,

$$A(D, 2^{l+2}) = \begin{cases} 4 & D \equiv 1 \pmod 8, \\ 0 & \text{otherwise.} \end{cases}$$

Also note that

$$\chi_D(2) = \begin{cases} 1 & D \equiv 1 \pmod 8, \\ -1 & D \equiv 5 \pmod 8, \\ 0 & \text{otherwise.} \end{cases}$$

Then direct computation gives the results. $\square$

This finishes the proof of the proposition. $\square$

Chapter 3

Shintani zeta function of Bhargava cubes

In a seminal series of papers ([Bha04a], [Bha04b], [Bha04c], [Bha08]), M. Bhargava has extended Gauss's composition law for binary quadratic forms to far more general situations. The key step in his extension is the investigation of the integral orbits of a group over $\mathbb{Z}$ acting on a lattice in a prehomogeneous vector space. One of Bhargava's achievements is the determination of the corresponding integral orbits, i.e. the determination of the $\mathrm{SL}_2(\mathbb{Z})^3$-orbits on $\mathbb{Z}^2 \otimes \mathbb{Z}^2 \otimes \mathbb{Z}^2$. In particular, he discovered that in this case, the generic integral orbits are in bijection with isomorphism classes of tuples (A, I_1, I_2, I_3) where

(a) A is an order in an étale quadratic $\mathbb{Q}$-algebra;

(b) I_1, I_2 and I_3 are elements in the narrow class group of A such that $I_1 \cdot I_2 \cdot I_3 = 1$.

In fact, Bhargava's cube arises naturally in the structure theory of the linear algebraic group of type D_4. More precisely, let G be a simply connected Chevalley group of this type. It has a maximal parabolic subgroup $P = M \cdot N$ whose Levi factor M has a derived group $M_{\mathrm{der}}(F) \cong \mathrm{SL}_2(F)^3$ and whose unipotent radical N is a Heisenberg group. The adjoint action of $M_{\mathrm{der}}(F)$ on $V = N(F)/[N(F), N(F)]$ is isomorphic to Bhargava's cube.

It is noted that the spaces of forms studied in Bhargava's papers were considered over algebraically closed fields in the fundamental work of M. Sato and T. Kimura ([SK77]) classifying prehomogeneous vector spaces. Over other fields, such as the rational number field, these spaces were more recently studied in the work of D. Wright and A. Yukie ([WY92]).

3.1 Prehomogeneous vector space of a parabolic subgroup.

Let $V_{\mathbb{Z}}$ be the $\mathbb{Z}$-module of $2 \times 2 \times 2$ integer matrices, which we also call Bhargava integeral cubes. There are three ways to form pairs of matrices by taking the opposite sides out of 6 sides. Denote them by

$$\begin{aligned} A^F = M_1 = \begin{pmatrix} a & b \\ c & d \end{pmatrix}; A^B = N_1 = \begin{pmatrix} e & f \\ g & h \end{pmatrix}, \\ A^L = M_2 = \begin{pmatrix} a & c \\ e & g \end{pmatrix}; A^R = N_2 = \begin{pmatrix} b & d \\ f & h \end{pmatrix}, \\ A^U = M_3 = \begin{pmatrix} a & e \\ b & f \end{pmatrix}; A^D = N_3 = \begin{pmatrix} c & g \\ d & h \end{pmatrix}. \end{aligned}$$

For each pair (M_i, N_i) we can associate to it a binary quadratic form by taking

$$Q_i(u, v) = \det(M_i u - N_i v).$$

Explicitly for A as above,

$$\begin{aligned} Q_1(u, v) &= u^2(ad - bc) + uv(-ah + bg + cf - de) + v^2(eh - fg), \\ Q_2(u, v) &= u^2(ag - ce) + uv(-ah - bg + cf + de) + v^2(bh - df), \\ Q_3(u, v) &= u^2(af - be) + uv(-ah + bg - cf + de) + v^2(ch - dg). \end{aligned}$$

Following Bhargava [Bha04a], we call A projective if the associated binary quadratic forms are all primitive. The action of $G_{\mathbb{Z}} = \mathrm{GL}_2(\mathbb{Z}) \times \mathrm{GL}_2(\mathbb{Z}) \times \mathrm{GL}_2(\mathbb{Z})$ on $V_{\mathbb{Z}}$ is defined by letting the g_i in $(g_1, g_2, g_3) \in G_{\mathbb{Z}}$ act on the matrix pair (M_i, N_i). It is easy to check that the actions of the three components commute with each other, thereby giving an action of the product group. For example, if $g_1 = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}$, then it acts on the pair (M_1, N_1) by

$$\begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} \cdot \begin{pmatrix} M_1 \\ N_1 \end{pmatrix}.$$

The action extends to the complex group $G_{\mathbb{C}} = \mathrm{GL}_2(\mathbb{C}) \times \mathrm{GL}_2(\mathbb{C}) \times \mathrm{GL}_2(\mathbb{C})$ on the complex vector space $V_{\mathbb{C}}$. We denote by $(G_{\mathbb{C}}, V_{\mathbb{C}})$ a complex vector space acted on by a (connected) complex group. In our setting, the $(G_{\mathbb{C}}, V_{\mathbb{C}})$ is a PVS which refers to the D_4 case discussed in [WY92]. Now we consider the Borel subgroup $B_2'(\mathbb{C}) \subset \mathrm{GL}_2(\mathbb{C})$ consisting of the lower-triangular matrices. The action of $B_2'(\mathbb{C}) \times B_2'(\mathbb{C}) \times \mathrm{GL}_2(\mathbb{C})$ induced from $G_{\mathbb{C}}$ on the vector space $V_{\mathbb{C}}$ has three relative invariants, explicitly for $A \in V_{\mathbb{C}}$, given as follows

$$\begin{aligned}
D(A) &= \mathrm{disc}(A) = (-ah + bg + cf - de)^2 - 4(ad - bc)(eh - fg), \\
m(A) &= \det(A^F) = ad - bc, \\
n(A) &= \det(A^L) = ag - ce.
\end{aligned}$$

Furthermore, we can show that

Proposition 3.1.1. *The pair $(B_2'(\mathbb{C}) \times B_2'(\mathbb{C}) \times \mathrm{GL}_2(\mathbb{C}), V_{\mathbb{C}})$ is a prehomogeneous vector space.*

Proof. Let $\mathcal{H}$ be the hypersurface in $V_{\mathbb{C}}$ defined as the zero locus of the single equation

$$\mathrm{disc}(A) \det(A^F) \det(A^L) = 0.$$

Any $A \in V_{\mathbb{C}} \backslash \mathcal{H}$ is $B_2'(\mathbb{C}) \times B_2'(\mathbb{C}) \times \mathrm{GL}_2(\mathbb{C})$ equivalent to some element with the form

$$(A^F, A^B) = \left(\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}, \begin{pmatrix} 0 & f \\ g & h \end{pmatrix} \right),$$

where $a, d, g, f \neq 0$. Furthermore, we can show that they are all in the single orbit of

$$(A^F, A^B) = \left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right).$$

This follows from finding solutions to the following system of equations, and then taking the proper scaling:

$$\begin{aligned}
&\lambda_{12} f + \lambda_{23} a + \lambda_{32} d = 0, \\
&\lambda_{13} a + \lambda_{22} f + \lambda_{32} g = 0, \\
&\lambda_{12} g + \lambda_{22} d + \lambda_{33} a = 0, \\
&\lambda_{13} d + \lambda_{23} g + \lambda_{33} d = -h - \lambda_{14} h - \lambda_{24} h - \lambda_{34} h,
\end{aligned}$$

where $\begin{pmatrix} \lambda_{i1} & \lambda_{i2} \\ \lambda_{i3} & \lambda_{i4} \end{pmatrix}$($\lambda_{i2} = 0$, for $i = 1, 2$) is in the i-th place of $B_2'(\mathbb{C}) \times B_2'(\mathbb{C}) \times \mathrm{GL}_2(\mathbb{C})$. So $(B_2'(\mathbb{C}) \times B_2'(\mathbb{C}) \times \mathrm{GL}_2(\mathbb{C}), V_{\mathbb{C}})$ is a prehomogeneous vector space. $\square$

Let $B_2'(\mathbb{Z})$ be the Borel subgroup of $B_2'(\mathbb{C})$ in $\mathrm{SL}_2(\mathbb{Z})$ with positive diagonal elements. The Shintani zeta function associated to the prehomogeneous vector space $(B_2'(\mathbb{C}) \times B_2'(\mathbb{C}) \times \mathrm{GL}_2(\mathbb{C}), V_{\mathbb{C}})$ is defined to be

$$Z_{\mathrm{Shintani}}(s_1, s_2, w) = \sum_{A \in \left(B_2'(\mathbb{Z}) \times B_2'(\mathbb{Z}) \times \mathrm{SL}_2(\mathbb{Z})\right) \backslash V_{\mathbb{Z}}^{ss}} \frac{1}{|\mathrm{disc}(A)|^w |\det(A^F)|^{s_1} |\det(A^L)|^{s_2}}, \tag{3}$$

where $V_{\mathbb{Z}}^{ss}$ is the subset of semi-stable points of $V_{\mathbb{Z}}$ consisting of those orbits on which none of the three relative invariants vanishes. Denote by $\mathrm{Stab}(A)$ the stabilizer group in $B_2'(\mathbb{Z}) \times B_2'(\mathbb{Z}) \times \mathrm{SL}_2(\mathbb{Z})$ of the cube A. We need to show that the order $|\mathrm{Stab}(A)|$ is finite.

Proposition 3.1.2. *For any $A \in V_{\mathbb{Z}}^{ss}$, the stabilizer group* $\mathrm{Stab}(A)$ *in* $B_2'(\mathbb{Z}) \times B_2'(\mathbb{Z}) \times \mathrm{SL}_2(\mathbb{Z})$ *is:*

$$\{(I_2, I_2, I_2)\},$$

where I_2 is the 2×2 identity matrix. Therefore

$$|\mathrm{Stab}(A)| = 1$$

for any $A \in V_{\mathbb{Z}}^{ss}$.

Proof. For a given $2 \times 2 \times 2$ integer cube $A \in V_{\mathbb{Z}}^{ss}$, we write

$$Q_1(A) = mu^2 + xuv + sv^2,$$
$$Q_2(A) = nu^2 + yuv + tv^2$$

to be first two binary quadratic forms associated to it. Under the action of $B_2'(\mathbb{Z}) \times B_2'(\mathbb{Z}) \times \mathrm{SL}_2(\mathbb{Z})$ we can change A to another cube satisfying

$$c(A) = 0 \text{ and } 0 \leq x \leq 2|m| - 1 \text{ and } 0 \leq y \leq 2|n| - 1.$$

For such A, note that the entries $a(A), d(A), g(A) \neq 0$. Therefore the stabilizer group in $B_2'(\mathbb{Z}) \times B_2'(\mathbb{Z}) \times \mathrm{SL}_2(\mathbb{Z})$ must have the form

$$\left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{pmatrix} \right).$$

Now it becomes obvious that the diagonal elements in the last matrix have to be both positive. $\square$

We can rewrite the Shintani zeta function as

$$Z_{\text{Shintani}}(s_1, s_2, w) = \sum_{D \neq 0} \frac{1}{|D|^w} \sum_{m,n>0} \frac{B(D,m,n)}{m^{s_1} n^{s_2}}, \tag{4}$$

where

$$\begin{aligned} B(D,m,n) &= \#\{A \in V_{\mathbb{Z}}^{ss}/\sim : \mathrm{disc}(A) = D, |\det(A^F)| = m, |\det(A^L)| = n\} \\ &= 4 \cdot \#\{A \in V_{\mathbb{Z}}^{ss}/\sim : \mathrm{disc}(A) = D, \det(A^F) = m, \det(A^L) = n\}. \end{aligned}$$

3.2 Reduction theory

In this section, we will consider the geometry of integer orbits of the PHVS under the parabolic group action. This will help to reduce the sum over semi-stable orbits of $2 \times 2 \times 2$ integer cubes in the Shintani zeta function $Z_{\text{Shintani}}(s_1, s_2, w)$ to a sum over the tuples of integers (D, m, n) satisfying certain relations. First we need a lemma which establishes the existence of a $2 \times 2 \times 2$ integer cube with the required arithmetic invariants.

Lemma 3.2.1. *Let D, m and n be non-zero integers. For each solution (x, y) to the congruence equations*

$$\begin{aligned} x^2 &\equiv D \pmod{4m} \text{ for } 0 \leq x \leq 2|m| - 1, \\ y^2 &\equiv D \pmod{4n} \text{ for } 0 \leq y \leq 2|n| - 1, \end{aligned}$$

there exists a $2 \times 2 \times 2$ integer cube A such that

$$\begin{aligned} \mathrm{disc}(A) &= D, \\ Q_1(A)(u,v) &= mu^2 + xuv + sv^2, \\ Q_2(A)(u,v) &= nu^2 + yuv + tv^2. \end{aligned}$$

Moreover, the required $2 \times 2 \times 2$ integer cube A can be chosen such that in the bottom side A^D of A

$$c = 0 \text{ and } g.c.d.(d, g, h) = 1.$$

Proof. If there are solutions to the congruence equations, we have

$$D = x^2 - 4ms = y^2 - 4nt$$

for some integers s and t. It implies that D is congruent to 0 or 1 (mod 4), and the integers x, y have the same parity. Take

$$a = |g.c.d.(m, n, \frac{x+y}{2})|,$$

from

$$\frac{x-y}{2} \cdot \frac{x+y}{2} = ms - nt, \tag{5}$$

it follows that

$$g.c.d.(\frac{m}{a}, \frac{n}{a}) \mid \frac{x-y}{2}. \tag{6}$$

Set

$$d = \frac{m}{a},\ g = \frac{n}{a},\ \text{and } h = -\frac{x+y}{2a},$$

then

$$g.c.d.(d, g, h) = 1,$$

and (5) can be written as

$$\frac{x-y}{2} \cdot (-h) = ds - gt. \tag{7}$$

We claim that there is an integer f, such that

$$s + fg \equiv 0 \pmod{h},$$
$$t + fd \equiv 0 \pmod{h}.$$

This can be proved as follows: First if $h = 0$, then $ds = gt$. As in this case $g.c.d.(d, g) = 1$, we conclude that there exists such an integer f such that $s = -fg$ and $t = -fd$. If $h \neq 0$, for any prime divisor p of h, we have

$$p \mid ds - gt \text{ and } g.c.d.(d, g, p) = 1,$$

it follows that there is a unique solution (mod p) to the congruences

$$s + fg \equiv 0 \pmod{p},$$
$$t + fd \equiv 0 \pmod{p}.$$

Using the Chinese remainder theorem, we conclude that there are integers b and e such that

$$s = eh - fg,$$
$$t = bh - fd.$$

It follows that

$$ds - gt = (de - bg) \cdot h,$$

combined with (7), we have

$$\frac{x - y}{2} = bg - de.$$

If in the case of $h = 0$, from (6), we know that there always exist integers b and e such that the above equation holds.

Now we define a $2 \times 2 \times 2$ integer cube A by

$$(A^F, A^B) = \left(\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}, \begin{pmatrix} e & f \\ g & h \end{pmatrix} \right).$$

Then the two associated binary quadratic forms are

$$Q_1(A)(u, v) = adu^2 + (-ah + bg - de)uv + (eh - fg)v^2 = mu^2 + xuv + sv^2,$$
$$Q_2(A)(u, v) = agu^2 + (-ah - bg + de)uv + (bh - df\ )v^2 = nu^2 + yuv + tv^2.$$

So A is the cube required. In particular, we have shown that $g.c.d.(d, g, h) = 1$. $\square$

We next want to show that under the assumption that D is square-free, the data (D, m, n, x, y) uniquely determine a $B_2'(\mathbb{Z}) \times B_2'(\mathbb{Z}) \times \mathrm{SL}_2(\mathbb{Z})$-orbit.

Proposition 3.2.2. *Let D be a non-zero square-free integer and m, n be non-zero integers. Each solution to the congruence equations*

$$x^2 \equiv D \pmod{4m} \text{ for } 0 \leq x \leq 2|m| - 1,$$
$$y^2 \equiv D \pmod{4n\ } \text{ for } 0 \leq y \leq 2|n| - 1,$$

determines a unique orbit of integer cubes in $(B_2'(\mathbb{Z}) \times B_2'(\mathbb{Z}) \times \mathrm{SL}_2(\mathbb{Z})) \backslash V_{\mathbb{Z}}^{ss}$ such that

$$\mathrm{disc}(A) = D,$$
$$Q_1(A)(u, v) = mu^2 + xuv + sv^2,$$
$$Q_2(A)(u, v) = nu^2 + yuv + ty^2.$$

Moreover, two different solutions to the congruence equations correspond to two different orbits of integer cubes in $(B_2'(\mathbb{Z}) \times B_2'(\mathbb{Z}) \times \mathrm{SL}_2(\mathbb{Z}))\backslash V_{\mathbb{Z}}^{ss}$.

Proof. From the above lemma, we know that there always exists such a $2\times2\times2$ integer cube A. We want to show that such an $2\times2\times2$ integer cube A is uniquely determined by the data (D, m, n, x, y) satisfying the congruence equations.

Denote A by the pair of matrices

$$A = \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \begin{pmatrix} e & f \\ g & h \end{pmatrix} \right).$$

We first show that the (a, d, g, h) is uniquely determined up to the sign of a. As $\{1\} \times \{1\} \times \mathrm{SL}_2(\mathbb{Z}) \subset B_2'(\mathbb{Z}) \times B_2'(\mathbb{Z}) \times \mathrm{SL}_2(\mathbb{Z})$, we can assume that $c(A) = 0$. Then we have equations

$$\begin{aligned} bg - ah - ed &= x, \\ ed - ah - bg &= y, \\ ad &= m, \\ ag &= n. \end{aligned}$$

after adding the first two equations,

$$\begin{aligned} ah &= -(x+y)/2, \\ ad &= m, \\ ag &= n. \end{aligned}$$

Therefore we have

$$a \times g.c.d.(h, d, g) = g.c.d.((x+y)/2, m, n).$$

As

$$D = (bg - ah - ed)^2 - 4ad(eh - fg),$$

D is square-free implies that $g.c.d.(h, d, g) = 1$, therefore $a = |g.c.d.((x + y)/2, m, n)|$ as we can make a to be positive.

We next show that for two $2 \times 2 \times 2$ integer cubes with $c = 0$ and fixed discriminant D such that they have the same tuples (a, d, g, h) and $(Q_1(u, v), Q_2(u, v))$, then they are equal up to the action of $\{1\}\times\{1\}\times B_2(\mathbb{Z})$, the third one is an upper-triangular matrix in $\mathrm{SL}_2(\mathbb{Z})$. Here in this statement we again require D to be square-free.

We already showed that for a given A,

$$Q_1(A)(u,v) = mu^2 + xuv + sv^2 = adu^2 + (bg - ah - ed)uv + (eh - fg)v^2,$$
$$Q_2(A)(u,v) = nu^2 + yuv + tv^2 = agu^2 + (ed - ah - bg)uv + (bh - df)v^2.$$

As $(Q_1(A_i)(u,v), Q_2(A_i)(u,v))$ are the same for two integer cubes by assumption, we can make the following equations by setting the coefficients are equal

$$\begin{aligned} x(A_1) = x(A_2) :& b_1g - ah - e_1d = b_2g - ah - e_2d, \\ y(A_1) = y(A_2) :& e_1d - ah - b_1g = e_2d - ah - b_2g, \\ s(A_1) = s(A_2) :& e_1h - f_1g = e_2h - f_2g, \\ t(A_1) = t(A_2) :& b_1h - df_1 = b_2h - df_2. \end{aligned}$$

Taking the subtraction of right side from the left side on each equation, we have:

$$\begin{aligned} \Delta(b)g - \Delta(e)d = 0, \\ \Delta(e)h - \Delta(f)g = 0, \\ \Delta(b)h - \Delta(f)d = 0, \end{aligned}$$

here $\Delta(b) = b_1 - b_2$. Then we have solutions to the above equation system:

$$\Delta(e); \Delta(b) = \Delta(e)d/g; \Delta(f) = \Delta(e)h/g.$$

Under the assumption of D being square-free, we have shown that $g.c.d.(d,g,h) = 1$. Therefore $g|\Delta(e)$, i.e., the solution has form

$$\Delta(e) = gk; \Delta(b) = dk; \Delta(f) = hk.$$

This implies that we can make A_1 equivalent to A_2 under the action of $\{1\} \times \{1\} \times B_2(\mathbb{Z})$.

Now we prove the second part of the proposition by contradiction. Given a square-free integer D and two non-zero integers m and n, if (x_1, y_1) and (x_2, y_2) are two different solutions to the congruence equations in the proposition. Let A_1 and A_2 be the two corresponding $2 \times 2 \times 2$ integer cubes satisfying $\mathrm{disc}(A_i) = D$, and

$$Q_1(A_i)(u,v) = mu^2 + x_iuv + s_iv^2,$$
$$Q_2(A_i)(u,v) = nu^2 + y_iuv + t_iv^2.$$

Suppose that A_1 and A_2 are $B_2'(\mathbb{Z}) \times B_2'(\mathbb{Z}) \times \mathrm{SL}_2(\mathbb{Z})$ equivalent. We assume $c(A_1) = c(A_2) = 0$ as before. Since $0 \le x_i \le 2m-1$ and $0 \le y_i \le 2n-1$, the group action actually belongs to $\{1\} \times \{1\} \times B_2(\mathbb{Z})$, which implies that $(Q_1(A_i)(u,v), Q_2(A_i)(u,v))$ are the same. In particular $x_1 = x_2$ and $y_1 = y_2$, contradiction. So A_1 and A_2 are not $B_2'(\mathbb{Z}) \times B_2'(\mathbb{Z}) \times \mathrm{SL}_2(\mathbb{Z})$ equivalent. $\square$

Now we turn to the general case without assuming square-free discriminant. First we consider the non-zero integer $D = D_0 p^2$, where D_0 is square-free.

Proposition 3.2.3. *Let m and n be non-zero integers, and $D = D_0 p^2$ where D_0 is square-free and p is a prime integer. The coefficient $B(D,m,n)$ from (4) is given by:*

$$B(D,m,n) = A\left(D,4m\right)A\left(D,4n\right) + b(D_0, \frac{m}{p}, \frac{n}{p}),$$

where the second term

$$b(D_0, \frac{m}{p}, \frac{n}{p}) = \begin{cases} pA(D_0, \frac{4m}{p})A(D_0, \frac{4n}{p}) & \text{if } p \text{ divides } g.c.d.(D,m,n), \\ 0 & \text{otherwise.} \end{cases}$$

Proof. By the existence Lemma 3.2.1, we can find an $2 \times 2 \times 2$ integer cube A satisfying

$$c = 0 \text{ and } g.c.d.(d,g,h) = 1,$$

such that

$$\mathrm{disc}(A) = D,\ \det(A^F) = m \text{ and } \det(A^L) = n.$$

From the proof of the last proposition, the condition of $g.c.d.(d,g,h) = 1$ implies that the integer cube A determines a unique orbit in $(B_2'(\mathbb{Z}) \times B_2'(\mathbb{Z}) \times \mathrm{SL}_2(\mathbb{Z})) \backslash V_{\mathbb{Z}}^{ss}(D)$. We denote by $V_{\mathbb{Z}}^{ss}(D)$ the semi-stable subset of $V_{\mathbb{Z}}$ with discriminant D.

If $p | g.c.d.(D,m,n)$, we write $m_0 = \frac{m}{p}$, $n_0 = \frac{n}{p}$. Applying the existence Lemma 3.2.1 to the non-zero integers D_0, m_0, n_0, we can find an $2 \times 2 \times 2$ integer cube A_0 satisfying

$$c_0 = 0 \text{ and } g.c.d.(d_0, g_0, h_0) = 1,$$

such that

$$\mathrm{disc}(A_0) = D_0,\ \det(A_0^F) = m_0 \text{ and } \det(A_0^L) = n_0.$$

The condition $g.c.d.(d_0, g_0, h_0) = 1$ implies that A_0 determines a unique orbit in $(B_2'(\mathbb{Z}) \times B_2'(\mathbb{Z}) \times \mathrm{SL}_2(\mathbb{Z})) \backslash V_{\mathbb{Z}}^{ss}(D_0)$.

For each g in $\{(1) \times (1) \times \begin{pmatrix} 1 & j \\ 0 & p \end{pmatrix} : 0 \le j \le p-1\}$, the $2 \times 2 \times 2$ integer cube $g \cdot A_0$ determines uniquely the orbit in $(B_2'(\mathbb{Z}) \times B_2'(\mathbb{Z}) \times \mathrm{SL}_2(\mathbb{Z})) \backslash V_{\mathbb{Z}}^{ss}(D)$. □

Corollary 3.2.4. *If D is a fundamental discriminant, then we have*

$$B(D, m, n) = A(D, 4m)A(D, 4n).$$

Proof. If D is square-free then this follows from the above proposition we proved. Otherwise $D = 4D_0$ where D_0 is square-free, using the above proposition applied to the case $p = 2$, we have

$$B(D, m, n) = A\left(D, 4m\right) A\left(D, 4n\right) + b(D_0, \frac{m}{2}, \frac{n}{2}).$$

The second term is 0 as the discriminant of a quadratic form should be $D_0 = f^2 d_K$ where d_K is a fundamental discriminant, but D is already a fundamental discriminant. □

Proposition 3.2.5. *Let m and n be non-zero integers, and $D = D_0 p^{2k}$ where D_0 is square-free and p is a prime integer. We have*

$$\begin{aligned} B(D, m, n) =& A(D, 4m)A(D, 4n) + b(\frac{D}{p^2}, \frac{m}{p}, \frac{n}{p}) + \cdots \\ &+ b(\frac{D}{p^{2i}}, \frac{m}{p^i}, \frac{n}{p^i}) + \cdots + b(D_0, \frac{m}{p^k}, \frac{n}{p^k}), \end{aligned}$$

for some $1 \le i \le k$,

$$b(\frac{D}{p^{2i}}, \frac{m}{p^i}, \frac{n}{p^i}) = \begin{cases} p^i A(\frac{D}{p^{2i}}, \frac{4m}{p^i}) A(\frac{D}{p^{2i}}, \frac{4n}{p^i}) & \text{if } p^i \text{ divides } g.c.d.(D, m, n), \\ 0 & \text{otherwise.} \end{cases}$$

Proof. By the existence Lemma 3.2.1, we can find an $2 \times 2 \times 2$ integer cube A satisfying

$$c = 0 \text{ and } g.c.d.(d, g, h) = 1,$$

such that

$$\operatorname{disc}(A) = D,\ \det(A^F) = m \text{ and } \det(A^L) = n.$$

The condition of $g.c.d.(d, g, h) = 1$ implies that the integer cube A determines a unique orbit in $(B_2'(\mathbb{Z}) \times B_2'(\mathbb{Z}) \times \mathrm{SL}_2(\mathbb{Z})) \backslash V_{\mathbb{Z}}^{ss}(D)$.

If $p^i | g.c.d.(D, m, n)$, we write $m_i = \frac{m}{p^i}$, $n_i = \frac{n}{p^i}$. Applying the existence Lemma 3.2.1 to the non-zero integers $\frac{D}{p^{2i}}$, $\frac{m}{p^i}$, and $\frac{n}{p^i}$, we can find an $2 \times 2 \times 2$ integer cube A_i satisfying

$$c_i = 0 \text{ and } g.c.d.(d_i, g_i, h_i) = 1,$$

such that

$$\operatorname{disc}(A_i) = \frac{D}{p^{2i}},\ \det(A_i^F) = \frac{m}{p^i} \text{ and } \det(A_i^L) = \frac{n}{p^i}.$$

The condition $g.c.d.(d_i, g_i, h_i) = 1$ implies that A_i determines a unique orbit in $(B_2'(\mathbb{Z}) \times B_2'(\mathbb{Z}) \times \mathrm{SL}_2(\mathbb{Z})) \backslash V_{\mathbb{Z}}^{ss}(\frac{D}{p^{2i}})$.

For each g in $\{(1) \times (1) \times \begin{pmatrix} 1 & j \\ 0 & p^i \end{pmatrix} : 0 \leq j \leq p^i - 1\}$, the $2 \times 2 \times 2$ integer cube $g \cdot A_i$ determines uniquely the orbit in $(B_2'(\mathbb{Z}) \times B_2'(\mathbb{Z}) \times \mathrm{SL}_2(\mathbb{Z})) \backslash V_{\mathbb{Z}}^{ss}(D)$. $\square$

In summary we have the full general formula for the orbits counting number $B(D, m, n)$.

Proposition 3.2.6. *Let m and n be non-zero integers, and $D = D_0 D_1^2$ where D_0 is square-free. We have*

$$B(D, m, n) = \sum_{d | D_1} b(\frac{D}{d^2}, \frac{m}{d}, \frac{n}{d}), \tag{8}$$

where

$$b(\frac{D}{d^2}, \frac{m}{d}, \frac{n}{d}) = \begin{cases} d \cdot A(\frac{D}{d^2}, \frac{4m}{d}) \cdot A(\frac{D}{d^2}, \frac{4n}{d}) & \text{if } d \text{ divides } g.c.d.(D_1, m, n), \\ 0 & \text{otherwise.} \end{cases}$$

Proof. By the existence Lemma 3.2.1, we can find an $2 \times 2 \times 2$ integer cube A satisfying

$$c = 0 \text{ and } g.c.d.(d, g, h) = 1,$$

such that

$$\operatorname{disc}(A) = D,\ \det(A^F) = m \text{ and } \det(A^L) = n.$$

The condition of $g.c.d.(d, g, h) = 1$ implies that the integer cube A determines a unique orbit in $(B_2'(\mathbb{Z}) \times B_2'(\mathbb{Z}) \times \mathrm{SL}_2(\mathbb{Z})) \backslash V_{\mathbb{Z}}^{ss}(D)$.

If $d | g.c.d.(D_1, m, n)$, we write $m' = \frac{m}{d}$, $n' = \frac{n}{d}$. Applying the existence Lemma 3.2.1 to the non-zero integers $\frac{D}{d^2}$, $\frac{m}{d}$, and $\frac{n}{d}$, we can find an $2 \times 2 \times 2$ integer cube A' satisfying

$$c' = 0 \text{ and } g.c.d.(d', g', h') = 1,$$

such that

$$\operatorname{disc}(A') = \frac{D}{d^2},\ \det\left((A')^F\right) = \frac{m}{d} \text{ and } \det\left((A')^L\right) = \frac{n}{d}.$$

The condition $g.c.d.(d', g', h') = 1$ implies that A' determines a unique orbit in $(B_2'(\mathbb{Z}) \times B_2'(\mathbb{Z}) \times \mathrm{SL}_2(\mathbb{Z})) \backslash V_{\mathbb{Z}}^{ss}(\frac{D}{d^2})$.

For each g in $\{(1) \times (1) \times \begin{pmatrix} 1 & j \\ 0 & d \end{pmatrix} : 0 \le j \le d-1\}$, the $2 \times 2 \times 2$ integer cube $g \cdot A'$ determines uniquely the orbit in $(B_2'(\mathbb{Z}) \times B_2'(\mathbb{Z}) \times \mathrm{SL}_2(\mathbb{Z})) \backslash V_{\mathbb{Z}}^{ss}(D)$. $\square$

Finally we obtain the explicit formula for the Shintani zeta function associated to the PVS of $2 \times 2 \times 2$ cubes.

Theorem 3.2.7. *The Shintani zeta function $Z_{\mathrm{Shintani}}(s_1, s_2, w)$ can be expressed as*

$$\sum_{A \in \left(B_2'(\mathbb{Z}) \times B_2'(\mathbb{Z}) \times \mathrm{SL}_2(\mathbb{Z})\right) \backslash V_{\mathbb{Z}}^{ss}} \frac{1}{|\operatorname{disc}(A)|^w |\det(A^F)|^{s_1} |\det(A^L)|^{s_2}}$$
$$= \sum_{D = D_0 D_1^2} \frac{1}{|D|^w} \sum_{m,n>0} \frac{\sum_{\substack{d|D_1 \\ d|m, d|n}} d \cdot A(\frac{D}{d^2}, \frac{4m}{d}) \cdot A(\frac{D}{d^2}, \frac{4n}{d})}{m^{s_1} n^{s_2}}.$$

In particular, if D is an odd integer, then (8) becomes

$$B(D, m, n) = \sum_{d|D_1} d \cdot A(\frac{D}{d^2}, \frac{4m}{d}) \cdot A(\frac{D}{d^2}, \frac{4n}{d}), \tag{9}$$

as when d is an odd integer it is a divisor of m if and only if it is a divisor of $4m$. So we have the following

Corollary 3.2.8. *The partial Shintani zeta function* $Z^{\mathrm{odd}}_{\mathrm{Shintani}}(s_1, s_2, w)$ *defined by*

$$Z^{\mathrm{odd}}_{\mathrm{Shintani}}(s_1, s_2, w) = \sum_{\substack{A \in \left(B_2'(\mathbb{Z}) \times B_2'(\mathbb{Z}) \times \mathrm{SL}_2(\mathbb{Z})\right) \backslash V_{\mathbb{Z}}^{ss} \\ \mathrm{dics}(A) \ \mathrm{odd}}} \frac{1}{|\mathrm{disc}(A)|^w |\det(A^F)|^{s_1} |\det(A^L)|^{s_2}}$$

can be written as

$$\sum_{D = D_0 D_1^2} \frac{1}{|D|^w} \sum_{m,n>0} \frac{\sum_{d|D_1} d \cdot A(\frac{D}{d^2}, \frac{4m}{d}) \cdot A(\frac{D}{d^2}, \frac{4n}{d})}{m^{s_1} n^{s_2}}.$$

Chapter 4

A_3 Weyl Group Multiple Dirichlet Series

In this section, we will relate the Shintani zeta function $Z_{\text{Shintani}}(s_1, s_2, w)$ to the quadratic A_3-Weyl group multiple Dirichlet series. The idea is first to construct a multiple Dirichlet series $Z_{\text{WMDS}}(s_1, s_2, w)$ and then show its relation to the Shintani zeta function of PVS of $2 \times 2 \times 2$ cubes, using the results we did for the relation between $Z_{A_2}(s, w)$ and $Z_{\text{Shintani}}(s, w; B_2')$. Finally we show the multiple Dirichlet series $Z_{\text{WMDS}}(s_1, s_2, w)$ is the desired A_3 Weyl group multiple Dirichlet series by computing the generating function of its p-parts.

In the A_2-WMDS

$$Z_{A_2}(s, w) = \sum_{\substack{m>0, \\ D \text{ odd discriminant}}} \frac{\chi_D(\hat{m}) a(D, m)}{m^s |D|^w},$$

we let $\tilde{A}(D, m) = \chi_D(\hat{m}) a(D, m)$. Write $D = D_0 D_1^2$ where D_0 stands for the square-free part of D. Define

$$Z_{\text{WMDS}}(s_1, s_2, w) = \sum_{D \text{ odd discriminant}} \frac{1}{|D|^w} \sum_{m,n>0} \frac{\chi_D(\hat{m}) \chi_D(\hat{n})}{m^{s_1} n^{s_2}} a(D, m, n),$$

where

$$a(D, m, n) = \sum_{\substack{d | D_1 \\ d|m, \ d|n}} d \cdot a(\frac{D}{d^2}, \frac{m}{d}) \cdot a(\frac{D}{d^2}, \frac{n}{d}).$$

Lemma 4.0.1. *The multiple Dirichlet series $Z_{\mathrm{WMDS}}(s_1, s_2, w)$ can be expressed as*

$$Z_{\mathrm{WMDS}}(s_1, s_2, w) = \sum_{D \text{ odd discriminant}} \frac{1}{|D|^w} \sum_{m,n>0} \frac{\chi_D(\hat{m})\chi_D(\hat{n})}{m^{s_1} n^{s_2}} a(D, m, n)$$

$$= \sum_{\substack{D = D_0 D_1^2 \\ D \text{ odd discriminant}}} \frac{1}{|D|^w} \frac{\sum_{\substack{d|D_1 \\ d|m, d|n}} d \cdot \tilde{A}(\frac{D}{d^2}, \frac{m}{d}) \cdot \tilde{A}(\frac{D}{d^2}, \frac{n}{d})}{m^{s_1} n^{s_2}}.$$

Proof. This follows from the fact that the quadratic character $\chi_D(\cdot)$ is the same as $\chi_{D/d^2}(\cdot)$ by definition, and the factor m is prime to D if and only if the factor $\frac{m}{d}$ is prime to $\frac{D}{d^2}$. Therefore $\chi_D(\hat{m})\chi_D(\hat{n})$ is a common factor for fixed integers D, m and n. $\square$

As we did in the Theorem 2.3.1, we will show the relation between the inner sum of $Z^{\mathrm{odd}}_{\mathrm{Shintani}}(s_1, s_2, w)$ and $Z_{\mathrm{WMDS}}(s_1, s_2, w)$.

Proposition 4.0.2. *Let D be an odd integer. The inner sum of the Shintani zeta function $Z^{\mathrm{odd}}_{\mathrm{Shintani}}(s_1, s_2, w)$ can be expressed by*

$$\sum_{m,n>0} \frac{B(D, m, n)}{m^{s_1} n^{s_2}} = \tilde{P}_2(D, s_1, s_2) \frac{\zeta(s_1)}{\zeta(2s_1)} \frac{\zeta(s_2)}{\zeta(2s_2)} \sum_{m,n>0} \frac{\chi_D(\hat{m})\chi_D(\hat{n})}{m^{s_1} n^{s_2}} a(D, m, n),$$

where

$$\tilde{P}_2(D, s_1, s_2) = \tilde{P}_2(D, s_1)\tilde{P}_2(D, s_2)$$

and $\tilde{P}_2(D, s_i)$ is defined in Theorem 2.3.1.

Proof. Recall that in the proof of Theorem 2.3.1, we have shown that

$$\sum_{m>0} \frac{A(D, 4m)}{m^s} = \tilde{P}_2(D, s) \frac{\zeta(s)}{\zeta(2s)} \sum_{m>0} \frac{\chi_D(\hat{m}) a(D, m)}{m^s}.$$

Therefore,

$$\sum_{m,n>0} \frac{A(D, 4m)A(D, 4n)}{m^{s_1} n^{s_2}} = \tilde{P}_2(D, s_1)\tilde{P}_2(D, s_2) \frac{\zeta(s_1)}{\zeta(2s_1)} \frac{\zeta(s_2)}{\zeta(2s_2)} \cdot \sum_{m,n>0} \frac{\chi_D(\hat{m})\chi_D(\hat{n})}{m^{s_1} n^{s_2}} a(D, m)a(D, n).$$

In particular for any $d^2|D$, replace D by $\frac{D}{d^2}$, m by $\frac{m}{d}$ and n by $\frac{n}{d}$, we have

$$\sum_{m,n>0} \frac{A(\frac{D}{d^2},\frac{4m}{d})A(\frac{D}{d^2},\frac{4n}{d})}{m^{s_1}n^{s_2}} = \tilde{P}_2(\frac{D}{d^2},s_1)\tilde{P}_2(\frac{D}{d^2},s_2)\frac{\zeta(s_1)}{\zeta(2s_1)}\frac{\zeta(s_2)}{\zeta(2s_2)} \cdot \sum_{m,n>0} \frac{\chi_D(\hat{m})\chi_D(\hat{n})}{m^{s_1}n^{s_2}} a(\frac{D}{d^2},\frac{m}{d})a(\frac{D}{d^2},\frac{n}{d}).$$

As D is odd and $D \equiv D/d^2 \pmod 8$, by the definition of $\tilde{P}_2(D,s)$, it follows that

$$\tilde{P}_2(D,s) = \tilde{P}_2(\frac{D}{d^2},s).$$

Finally, taking the sum over all d with $d^2|D$, we have

$$\sum_{m,n>0}\sum_{d^2|D} \frac{d \cdot A(\frac{D}{d^2},\frac{4m}{d}) \cdot A(\frac{D}{d^2},\frac{4n}{d})}{m^{s_1}n^{s_2}} = \tilde{P}_2(D,s_1)\tilde{P}_2(D,s_2)\frac{\zeta(s_1)}{\zeta(2s_1)}\frac{\zeta(s_2)}{\zeta(2s_2)} \cdot \sum_{m,n>0} \frac{\chi_D(\hat{m})\chi_D(\hat{n})}{m^{s_1}n^{s_2}} a(D,m,n).$$

□

Recall the formula of $\tilde{P}_2(D,s)$,

$$\tilde{P}_2(D,s) = \begin{cases} 2 & D \equiv 1 \pmod 4, \\ 0 & \text{otherwise.} \end{cases}$$

Now we can give an explicit relation between $Z_{\text{Shintani}}(s_1,s_2,w)$ and $Z_{\text{WMDS}}(\cdot)$. We further relate the Shintani zeta function $Z_{\text{Shintani}}(s_1,s_2,w)$ to a Weyl group multiple Dirichlet series by showing that $Z_{\text{WMDS}}(s_1,s_2,w)$ is a quadratic A_3-WMDS.

Theorem 4.0.3. *The Shintani zeta function of* PVS *of* $2\times 2\times 2$ *cubes can be related to the multiple Dirichlet series* $Z_{\text{WMDS}}(s_1,s_2,w)$ *by*

$$Z^{\text{odd}}_{\text{Shintani}}(s_1,s_2,w) = 4\frac{\zeta(s_1)}{\zeta(2s_1)}\frac{\zeta(s_2)}{\zeta(2s_2)}Z_{\text{WMDS}}(s_1,s_2,w).$$

Theorem 4.0.4. $Z_{\text{WMDS}}(s_1,s_2,w)$ *is a quadratic* A_3 *Weyl group multiple Dirichlet series.*

Proof. Consider the p-parts of our $a(D, m, n)$ defined by

$$a_{klt}(p) = a(p^k, p^l, p^t).$$

Explicit from its definition,

$$a_{klt}(p) = a(p^k, p^l)a(p^k, p^t) + pa(p^{k-2}, p^{l-1})a(p^{k-2}, p^{t-1}) + \cdots .$$

In [CG07] and [CG10], the authors developed a systematical way to construct the Weyl group multiple Dirichlet series. The idea is to construct a rational function invariant under the Weyl group action. In the case of root system of A_3 type, the p-parts of the rational function is given by

$$f_{A_3}(x, y, z) = \frac{(1 - xy - yz + xyz + pxy^2z - px^2y^2z - pxy^2z^2 + px^2y^3z^2)}{(1 - x)(1 - y)(1 - z)(1 - py^2z^2)(1 - py^2x^2)(1 - p^2x^2y^2z^2)}.$$

Write the expansion

$$f_{A_3}(x, y, z) = \sum b_{klt}(p)x^l y^k z^t,$$

then we can compare our $\{a_{klt}\}$ with $\{b_{klt}\}$. They coincide with each other as follows: we know that for $|x|, |y|, |z| < 1/p$ there is

$$f_{A_3}(x, y, z) = \frac{1}{1 - pxy^2z} \int f_{A_2}(x, t) f_{A_2}(yt^{-1}, z) \frac{dt}{t},$$

where

$$f_{A_2}(x_1, x_2) = \sum_{k,l \geq 0} a(p^k, p^l)x_1^k x_2^l$$

and the integral is taken over the circle $|t| = 1/p$ ([CG07, Example 3.7]). Substituting the above expansion of f_{A_2} into f_{A_3}, note that $a(p^k, p^l) = a(p^l, p^k)$, we have

$$\begin{aligned} f_{A_3}(x, y, z) &= \frac{1}{1 - pxy^2z} \sum_{k,l,t \geq 0} a(p^k, p^l)a(p^k, p^t)x^l y^k z^t \\ &= \sum_{s=0}^{\infty} p^s x^s y^{2s} z^s \sum_{k,l,t \geq 0} a(p^k, p^l)a(p^k, p^t)x^l y^k z^t \\ &= \sum_{k,l,t \geq 0} (a(p^k, p^l)a(p^k, p^t) + pa(p^{k-2}, p^{l-1})a(p^{k-2}, p^{t-1}) + \cdots)x^l y^k z^t \\ &= \sum_{k.l,t \geq 0} a(p^k, p^l, p^t)x^l y^k z^t. \end{aligned}$$

Therefore, $a_{klt} = b_{klt}$. □

Chapter 5

Moduli parameterizes ideals of a quadratic ring

We first recall the classical results in the theory of binary quadratic forms [Cox89]. Given a primitive integral binary quadratic form

$$Q(x, y) = ax^2 + bxy + cy^2$$

with discriminant $D = b^2 - 4ac$ such that $K = \mathbb{Q}(\sqrt{D})$ a quadratic field, there is a canonical way to associate it an integral ideal I in the quadratic order $R = R(D)$. Let τ be one of the two roots of the quadratic function $Q(x, 1) = 0$, then

$$R = \langle 1, a\tau \rangle \text{ and } I = \langle a, a\tau \rangle.$$

If f is the conductor of the quadratic order R, then we can express $a\tau$ as:

$$a\tau = \frac{-b \mp f d_K}{2} \pm f w_K,$$

where $w_K = \frac{d_K + \sqrt{d_K}}{2}$, and d_K is the fundamental discriminant of the quadratic field K. It follows that R has a $\mathbb{Z}$-basis $[1, f w_K]$, which only depends on the discriminant $D = f^2 d_K$ of R. Notice that I is the ideal of R satisfying $R/I \cong \mathrm{N}(I)\mathbb{Z}$, where the norm $\mathrm{N}(I) = |a|$. The classical Gauss composition law can be stated as the correspondence

$$\mathrm{Cl}\left((\mathrm{Sym}^2\mathbb{Z}^2)^*; D\right) \to \mathrm{Cl}^+\left(R(D)\right)$$

is an isomorphism of groups, where $\mathrm{Cl}^+(R(D))$ ($= \mathrm{Cl}(R(D))$ as K is imaginary) is the narrow ideal class group.

In order to extend the above construction to an arbitrary integral binary quadratic form

$$Q(x, y) = ax^2 + bxy + cy^2 \tag{10}$$

with discriminant $D = b^2 - 4ac \neq 0$, we need to first recall the definition of oriented quadratic ring introduced in the paper [Bha04a]. A quadratic ring is the commutative ring with unity whose underlying additive group is $\mathbb{Z}^2$. There is a unique automorphism for a quadratic ring R. With the automorphism, we can define the trace of an element $x \in R$ by taking $\mathrm{Tr}(x) = x + x'$, where x' denotes the image of x under the automorphism. Alternatively, the trace function $\mathrm{Tr} : R \to \mathbb{Z}$ is defined as the trace of the endomorphism $R \xrightarrow{\times\alpha} R$. We also define the norm of an element $x \in R$ by taking $\mathrm{N}(x) = x \cdot x'$. The discriminant $\mathrm{disc}(R)$ of R is defined to be the determinant $\det(\mathrm{Tr}(\alpha_i \alpha_j))$ where $\{\alpha_i\}$ is any $\mathbb{Z}$-basis of R. As the $\mathbb{Z}$-basis of any quadratic R has the form $[1, \tau]$, where τ satisfies the equation $\tau^2 + r\tau + s = 0$, the discriminant of R is given explicitly by $\mathrm{disc}(R) = r^2 - 4s$. Conversely, given any integer $D \equiv 0$ or $1 \pmod 4$, there exists a unique quadratic ring $R(D)$ with discriminant D. Canonically, $R(D)$ has a $\mathbb{Z}$-basis $[1, \tau_D]$, where τ_D is determined by

$$\tau_D^2 = \frac{D}{4} \text{ or } \tau_D^2 = \frac{D-1}{4} + \tau_D, \tag{11}$$

in accordance to whether $D \equiv 0 \pmod 4$ or $D \equiv 1 \pmod 4$. We call R non-degenerate if $\mathrm{disc}(R) \neq 0$. From now on, we only consider the case of non-degenerate quadratic rings.

For an integer $D \neq 0$, the quadratic ring $R(D)$ has a unique non-trivial automorphism. The quadratic ring $R(D)$ is oriented if we specify the choice of τ_D. For an oriented quadratic ring $R(D)$, the specific choice of τ in any $\mathbb{Z}$-basis $[1, \tau]$ is made such that the change-of-basis matrix from the basis $[1, \tau]$ to the canonical basis $[1, \tau_D]$ has positive determinant. We call such a basis $[1, \tau]$ positively oriented. For the rest of the section, we always assume that the quadratic ring $R(D)$ is oriented with each $\mathbb{Z}$-basis $[1, \tau]$ positively oriented.

Finally, for a quadratic ring R with non-zero discriminant D, we define for it the narrow class group $\mathrm{Cl}^+(R)$, the group of oriented ideal classes. Recall that an oriented ideal is the pair (I, ϵ), where I is a (fractional) ideal of R in $K(R) = R \otimes \mathbb{Q}$, and $\epsilon = \pm 1$ gives the orientation of the ideal I. For an element $k \in K(R)$, the product $k \cdot (I, \epsilon)$ is defined to be the oriented ideal $(kI, \operatorname{sgn}(\mathrm{N}(k))\,\epsilon)$. Two oriented ideal (I_1, ϵ_1) and (I_2, ϵ_2) belong to the same oriented ideal class if they satisfy $(I_1, \epsilon_1) = k \cdot (I_2, \epsilon_2)$. We will suppress ϵ for the rest of the section and assume I always oriented. For an oriented ideal $I \subset R$, the unoriented norm of I is defined to be $\mathrm{N}(I) = |R/I|$; while the oriented norm of I is denoted by $\epsilon \cdot \mathrm{N}(I)$.

Now for the binary quadratic form (10), the oriented quadratic ring R is defined to be

$$R = \langle 1, \tau \rangle,$$

where the choice of τ is specified and it satisfies

$$\tau^2 + b\tau + ac = 0.$$

Further, the oriented ideal I is defined to be

$$I = \langle a, \tau \rangle$$

with the orientation given by the ordered basis $[a, \tau]$. It is easy to see that I is the ideal contained in R with the norm $\mathrm{N}(I) = |a|$. If the binary quadratic form is primitive, i.e., $g.c.d.(a, b, c) = 1$, then the ideal I is proper, which means it has an inverse in the quadratic algebra $K(R)$.

Let $B_2'(\mathbb{Z}) \subset \mathrm{SL}_2(\mathbb{Z})$ be the subgroup of lower-triangular integer matrices with positive diagonal elements. Then we will show the following result which says the set of pairs (R, I) with oriented ideal I with cyclic quotient in R can be parameterized by the integer orbits of the PVS of binary quadratic forms acted on by the Borel subgroup $B_2'(\mathbb{C})$.

Proposition 5.0.1. *The natural map*

$$\begin{aligned} &B_2'(\mathbb{Z})\backslash\{Q(u,v) = au^2 + buv + cv^2 : b^2 - 4ac \neq 0, a \neq 0\} \\ &\to \mathrm{Iso}\backslash\{(R, I) : R/I \cong \mathrm{N}(I)\mathbb{Z}\} \end{aligned}$$

defined above is a bijection. The isomorphism f from the pair (R_1, I_1) to another (R_2, I_2) is defined to be the orientation-preserving isomorphism from R_1 to R_2 and sending I_1 to I_2.

Proof. From the construction above, the map is well defined. We first prove the surjectivity. Given an oriented quadratic ring R with $\mathrm{disc}(R) = D$ and an oriented ideal $I \subset R$ defined by:

$$R = \langle 1, \tau_D \rangle \text{ and } I = \langle \alpha, \beta \rangle,$$

where τ_D is defined in (11) and the orientation of I is determined by the ordered basis $[\alpha, \beta]$, we can always assume that the norm $\mathrm{N}(\alpha) = \alpha \cdot \alpha' \neq 0$. This is trivial in the number field case. To prove it in the general case, we define a binary quadratic form by

$$(\alpha \cdot \alpha')u^2 - (\alpha' \cdot \beta + \alpha \cdot \beta')uv + (\beta \cdot \beta')v^2, \tag{12}$$

then it is easy to see that the discriminant of (12) is exactly the discriminant of I given by

$$\mathrm{disc}(I) = \left(\det \begin{pmatrix} \alpha & \beta \\ \alpha' & \beta' \end{pmatrix}\right)^2.$$

As

$$\mathrm{disc}(I) = (\mathrm{N}(I))^2 \cdot \mathrm{disc}(R),$$

and R is non-degenerate, it implies that $\mathrm{disc}(I) \neq 0$. So at least one of the coefficients of u^2 and v^2 is non-zero. We can assume $\alpha \cdot \alpha' \neq 0$ by changing the order of α and β. As $\alpha, \beta \in I$, $\mathrm{N}(I)|\mathrm{N}(\alpha)$ and $\mathrm{N}(I)|\mathrm{N}(\beta)$, it follows that $\mathrm{N}(I)$ is the common factor of all coefficients of (12). After canceling this common factor, we write it as

$$mu^2 + nuv + lv^2 \tag{13}$$

with $D = n^2 - 4ml$. So the quadratic ring $R(D)$ can be also written as

$$R(D) = [1, \tau_1],$$

where the choice of τ_1 is specified and it satisfies

$$\tau_1^2 + n\tau_1 + ml = 0. \tag{14}$$

We write

$$(\alpha, \beta) = (1, \tau_1)\begin{pmatrix} p & r \\ q & s \end{pmatrix}.$$

Substituting $u = \beta$ and $v = \alpha$ into (13), it becomes zero, by comparing it with the defining equation (14) of τ_1, then

$$p = ms + nq \text{ and } r = -lq; \quad \text{or} \quad p = -ms \text{ and } r = ns + lq.$$

As I is an ideal with cyclic quotient, we must have $g.c.d.(q, s) = 1$. Then by the elementary divisor theorem, we can transform the matrix $\begin{pmatrix} p & r \\ q & s \end{pmatrix}$

by a left multiplication in $B_2(\mathbb{Z})$ and a right multiplication in $\mathrm{SL}_2(\mathbb{Z})$ to the matrix has the form $\begin{pmatrix} a & * \\ 0 & 1 \end{pmatrix}$ with $a = \pm\mathrm{N}(I)$. Therefore under a certain basis, R and I can be written as

$$R(D) = \langle 1, \tau_2 \rangle \text{ and } I = \langle a, \tau_2 \rangle, \tag{15}$$

where the choice of τ_2 is made such that the basis $[1, \tau_2]$ is positively oriented. With α replaced by a and β replaced by τ_2, by (12) there is a binary quadratic form. From the discussion above, all coefficients of it are divisible by $|a| = \mathrm{N}(I)$. After canceling this common factor, we have

$$au^2 + buv + cv^2,$$

with $D = b^2 - 4ac$. This is the required binary quadratic form.

To prove the injectivity of the map, suppose that two binary quadratic forms

$$Q_i(u, v) = a_i u^2 + b_i uv + c_i v^2,$$

where $a_1 = a_2 = a \neq 0$ and $D = b_i^2 - 4a_i c_i \neq 0$ for $i = 1, 2$, have the same image. We want to prove that $b_2 = b_1 + 2na$ and $c_2 = n^2 a + b_1 n + c_1$ for some integer n. From the definition of the map, the oriented quadratic ring and the oriented ideal can be written as

$$R_1 = \langle 1, \tau_1 \rangle, I_1 = \langle a, \tau_1 \rangle,$$
$$R_2 = \langle 1, \tau_2 \rangle, I_2 = \langle a, \tau_2 \rangle$$

respectively, where the choice of $\tau_i' s$ are made such that both bases $[1, \tau_1]$ and $[1, \tau_2]$ are positively oriented, and they satisfy

$$\tau_i^2 + b_i \tau_i + ac_i = 0.$$

As they have the same image, there exists an isomorphism f from R_1 to R_2 preserving the orientation, so it has the form:

$$f(\tau_1) = \tau_2 + s,$$

As it also satisfies $f(I_1) = I_2 = \mathbb{Z}a + \mathbb{Z}\tau_2$, so we have

$$s = na,$$
$$b_2 = b_1 + 2na,$$
$$c_2 = n^2 a + b_1 n + c_1,$$

for some integer n. $\square$

Now we return to the case of $2\times 2\times 2$ integer cubes. Giver a $2\times 2\times 2$ integer cube A, suppose that the $D = \mathrm{disc}(A) \neq 0$, we consider the two binary quadratic forms $Q_1(A)(u,v)$ and $Q_2(A)(u,v)$ associated to A. Suppose that the coefficients a_i of u^2 are not zero, then applying the map in the last proposition to each $Q_i(A)(u,v)$, we get the pairs $(R; I_1, I_2)$ where I_i is the oriented ideal with $R/I_i \cong |a_i|\mathbb{Z}$. Explicitly, the map is given by

$$\begin{aligned}&\{A : Q_i(A)(u,v) = a_iu^2 + b_iuv + c_iv^2, i = 1,2\}\\ &\to \{(R; I_1, I_2) : I_i = \langle a_i, \tau_i\rangle, \tau_i^2 + b_i\tau + a_ic_i = 0\}.\end{aligned} \tag{16}$$

Theorem 5.0.2. *With the notation above. Then the natural map of* (16) *defines a surjective and finite morphism*

$$\begin{aligned}&(B_2'(\mathbb{Z}) \times B_2'(\mathbb{Z}) \times \mathrm{SL}_2(\mathbb{Z}))\backslash V_{\mathbb{Z}}^{ss}\\ &\to \mathrm{Iso}\backslash\{(R; I_1, I_2) : R/I_1 \cong \mathrm{N}(I_1)\mathbb{Z}, R/I_2 \cong \mathrm{N}(I_2)\mathbb{Z}\}.\end{aligned}$$

The cardinality $n(R; I_1, I_2)$ of the fiber is equal to

$$\sigma_1(D_1, a_1, a_2),$$

where $D = D_0D_1^2 = \mathrm{disc}(R)$, and D_0 is square-free. And it satisfies

$$\sum_{\substack{(R;I_1,I_2)/\sim\\ \mathrm{N}(I_i)=|a_i|}} n(R; I_1, I_2) = B(D, |a_1|, |a_2|).$$

Proof. The map is described above and it is easy to see well defined. We first prove the surjectivity of the map. Given a pair $(R; I_1, I_2)$ with I_i an oriented ideal of R and $R/I_i \cong \mathrm{N}(I_i)\mathbb{Z}$, by Proposition 5.1, we know that there are two binary quadratic forms

$$Q_i(u,v) = a_iu^2 + b_iuv + c_iv^2$$

for $i = 1,2$ with $D = \mathrm{disc}(R) = b_i^2 - 4a_ic_i$, such that R and I_i are determined by

$$\begin{aligned}R &= \langle 1, \tau_D\rangle = \langle 1, \tau_1\rangle = \langle 1, \tau_2\rangle,\\ I_1 &= \langle a_1, \tau_1\rangle \text{ and } I_2 = \langle a_2, \tau_2\rangle,\end{aligned}$$

where τ_D is defined in (11), and τ_i satisfies

$$\tau_i^2 + b_i\tau_i + a_ic_i = 0$$

for $i = 1, 2$. By the Lemma 3.3, we know that there exists a $2 \times 2 \times 2$ integer cube A such that

$$Q_i(A)(u, v) = Q_i(u, v).$$

Under the correspondence of (16), we conclude that the integer cube A maps to the given pair $(R; I_1, I_2)$.

To prove the second part of the theorem, let A and A' be the two $2 \times 2 \times 2$ integer cubes which are in the fiber of $(R; I_1, I_2)$ with $\mathrm{N}(I_i) = |a_i|$, then they have the following arithmetic property: $\mathrm{disc}(R) = \mathrm{disc}(A) = \mathrm{disc}(A') = D$, and the two quadratic forms associated to them are the same $Q_i(A)(u, v) = Q_i(A')(u, v) = a_i u^2 + b_i uv + c_i v^2$, where $0 \leq b_i \leq 2|a_i| - 1$ is assumed. Write $D = D_0 D_1^2$. From our general formula of $B(D, m, n)$ in the Shintani zeta function $Z_{\text{Shintani}}(s_1, s_2, w)$, we know that the fiber counting function is equal to

$$n(R; I_1, I_2) = \sum_{\substack{d|D_1 \\ d|a_1, d|a_2}} f(d), \text{where } f(d) = \begin{cases} d \text{ if } (\frac{b_i}{d})^2 \equiv \frac{D}{d^2} \pmod{\frac{4a_i}{d}}, \\ 0 \text{ otherwise.} \end{cases}$$

Note that as $b_i^2 \equiv D \pmod{4a_i}$ already holds, so it automatically implies $(\frac{b_i}{d})^2 \equiv \frac{D}{d^2} \pmod{\frac{4a_i}{d}}$ if $d | g.c.d.(D_1, a_1, a_2)$. It follows that

$$n(R; I_1, I_2) = \sigma_1 \left(g.c.d.(D_1, |a_1|, |a_2|) \right).$$

Furthermore, the sum of cardinalities over the fixed norms of $\mathrm{N}(I_1) = |a_1|$ and $\mathrm{N}(I_2) = |a_2|$ is exactly

$$\begin{aligned} & B(D, |a_1|, |a_2|) \\ & = \#\{A \in V_{\mathbb{Z}}^{ss}/\sim : \mathrm{disc}(A) = D, |\det(A^F)| = |a_1|, |\det(A^L)| = |a_2|\}. \end{aligned}$$

$\square$

Chapter 6

Shintani zeta functions and Periods of automorphic forms

Let Λ_d be the set of Heegner points of discriminant d. In [KS93], Katok-Sarnak proved that for a GL_2 Maass cusp form f of weight 0,

$$\sum_{z\in\Lambda_d} f(z) = 24\pi|d|^{3/4} \sum_{\mathrm{Shim}(F_i)=f} \rho_i(d)\overline{\rho_i(1)} \tag{17}$$

where the sum on the right side runs through an orthonormal basis $\{F_i\}$ for the space of half-integral weight Maass cusp forms whose Shimura lifting is f, and $\rho_i(d)$ denotes the d-th Fourier coefficient of F_i. This is the analogue of Maass's result for GL_2 Eisenstein series. Let

$$E(z,s) = \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} \Im(\gamma z)^s$$

where $\Gamma_\infty = \{\begin{pmatrix} \pm1 & n \\ 0 & \pm1 \end{pmatrix} : n\in\mathbb{Z}\}$ and

$$\tilde{E}(z,s) = \sum_{\gamma\in\Gamma_\infty\backslash\Gamma_0(4)} \left(\frac{c}{d}\right) \epsilon_d^{-1}(cz+d)^{-\frac{1}{2}}\Im(\gamma z)^s$$

where $\Gamma_\infty = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} : c\equiv 0 \bmod 4\}$ be the classical Eisenstein series of weight 0 and 1/2 respectively.

A classical result of Maass relating a weighted sum of $E(z,s)$ over CM points of discriminant $d<0$ with the ζ function of the imaginary extension $\mathbb{Q}(d)$ is the following:

$$\zeta_K(s) = \zeta(s)L(s,\chi_{d_K}) = \frac{2^{1+s}}{w_K|d_K|^{s/2}}\zeta(2s)\sum_z E(z,s). \tag{18}$$

On the other hand, the L-function $L(s,\chi_{d_K})$ shows up in the Fourier expansion of the metaplectic Eisenstein series:

$$\tilde{E}(z,s) = y^s = c_0(s)y^s\frac{\zeta(4s-1)}{\zeta(4s)} + \sum_{m\neq 0} b_m(s)K_m(s,y)e^{2\pi imx},$$

where for m square-free,

$$b_m(s) = c_m(s)\frac{L(2s,\chi_m)}{\zeta(4s+1)}.$$

Therefore, the expression (18) is analogous to the result (17) of Katok-Sarnak.

We want to study the compact orthogonal periods of automorphic forms of GL_3. In [CO12], the authors provide an explicit formula for an orthogonal period integral of Eisenstein series of GL_3, which is a finite sum of quadratic Weyl group double Dirichlet series. Using the approach of Shintani zeta functions associated with the prehomogeneous vector spaces, Li-Mei Lim in her thesis obtains the general formula for a finite sum of orthogonal period integrals of Eisenstein series of GL_3. In general, it is a conjecture of Jacquet ([Jac91]) relating the orthogonal periods of automorphic forms of G with the Whittaker-Fourier coefficients of metaplectic group of G.

We propose to use Shintani zeta function associated to a certain PVS to study period integrals of automorphic forms. To illustrate this approach, we work on GL_2 automorphic forms.

6.1 Shintani Zeta Functions of Higher rank

Denote by V the prehomogeneous vector space of binary quadratic forms under the action of $G^+ = \mathrm{GL}_2^+ \times \mathrm{GL}_1^+$. It has a relative invariant defined by $\det(g_1)^2a^2$ where $g=(g_1,a)$. Set $\chi(g)=\det(g_1)^2a^2$. The subgroup T is defined such that G^+/T acts freely on V. Let V' to be the G-invariant subset consisting of elements with non-square discriminant. The Eisenstein series of $\mathrm{SL}_2(\mathbb{Z})$ is defined by

$$E(g,z) = \frac{1}{2}(\det g)^{(1+z)/2}\sum\{(g_{11}m + g_{21}n)^2 + (g_{12}m + g_{22}n)^2\}^{-(1+z)/2},$$

where the summation is over all pairs consisting of mutually prime integers and $\mathrm{Re}(z) > 1$.

Assume f is a Maass form of $\mathrm{SL}_2(\mathbb{Z})$ with weight 0. Let Φ_∞ to be the Schwartz function of V. The Shintani zeta integral is defined by

$$Z_\infty(\Phi_\infty, f) = \int_{G^+(\mathbb{Z})\backslash G^+(\mathbb{R})/T(R)} \sum_{V'(\mathbb{Z})} \Phi_\infty(xg)\chi(g)^{s_2+s_1/2}E(g, 2s_1-1)f(g)dg.$$

Using the arguments of Katok-Sarnak in [KS93], $Z_\infty(\Phi_\infty, f)$ is related to the period integral of Maass form f by the formula:

Proposition 6.1.1. *The Shintani zeta integral $Z_\infty(\Phi_\infty, f)$ and the period integral of Esenstein series $E(g,z)$ with Maass form f can be related in the following way:*

$$Z_\infty(\Phi_\infty, f) = \sum_{\substack{x\in V'(\mathbb{Z})/G^+(\mathbb{Z}) \\ \mathrm{disc}(x)<0}} \frac{E(g_x, 2s_1-1)f(g_x)}{|\Gamma_x||\mathrm{disc}(x)|^{s_2+s_1/2}} \cdot \Psi_1(\Phi_\infty, f) \tag{19}$$
$$+ \sum_{\substack{x\in V'(\mathbb{Z})/G^+(\mathbb{Z}) \\ \mathrm{disc}(x)>0}} \frac{\int_{\Gamma_x\backslash G_x} E(gg_x, 2s_1-1)f(gg_x)dg}{|\mathrm{disc}(x)|^{s_2+s_1/2}} \cdot \Psi_2(\Phi_\infty, f).$$

Proof. We want to express the above integral as the sum of periods over stabilizer groups.

$$\begin{aligned}
&\int_{\Gamma\backslash G^+} E(g, 2s_1-1)f(g)\chi(g)^{s_2+s_1/2}\sum_{x\in L}\Phi_\infty(x\cdot g)dg \\
&= \sum_{x\in\Gamma\backslash L}\int_{\Gamma\backslash G^+} E(g, 2s_1-1)f(g)\chi(g)^{s_2+s_1/2}\sum_{\gamma\in\Gamma_x\backslash\Gamma}\Phi_\infty(x\cdot\gamma g)dg \\
&= \sum_{x\in\Gamma\backslash L}\int_{\Gamma_x\backslash G^+} E(g, 2s_1-1)f(g)\chi(g)^{s_2+s_1/2}\Phi_\infty(x\cdot g)dg.
\end{aligned}$$

Given $x \in V'(\mathbb{Z})$, suppose that $\mathrm{disc}(x) < 0$, Γ_x is a finite group. Moreover, $G_1^+ = \mathrm{SL}_2(\mathbb{R}) \times \{1\}$ acts transitively on the real set of $\{x \in V_{\mathbb{R}} | \mathrm{disc}(x) = n\}$, so we can find $g_x \in G_1^+$ such that

$$x \cdot g_x = (a, b, c) = (\frac{\sqrt{|n|}}{2}, 0, \frac{\sqrt{|n|}}{2}).$$

Therefore, we have

$$\begin{aligned}
&\int_{\Gamma_x \backslash G^+} E(g, 2s_1 - 1) f(g) \chi(g)^{s_2 + s_1/2} \Phi_\infty(x \cdot g) dg \\
&= \frac{1}{|\Gamma_x|} \int_{G^+} E(g, 2s_1 - 1) f(g) \chi(g)^{s_2 + s_1/2} \Phi_\infty(x \cdot g) dg \\
&= \frac{1}{|\Gamma_x|} \int_{G^+} E(g_x g, 2s_1 - 1) f(g_x g) \chi(g)^{s_2 + s_1/2} \Phi_\infty((\frac{\sqrt{|n|}}{2}, 0, \frac{\sqrt{|n|}}{2}) \cdot g) dg.
\end{aligned}$$

Now

$$\Phi_\infty((\frac{\sqrt{|n|}}{2}, 0, \frac{\sqrt{|n|}}{2}) \cdot g) = \Phi_\infty((\frac{\sqrt{|n|}}{2}, 0, \frac{\sqrt{|n|}}{2}) \cdot k_1 g k_2).$$

Using the Cartan decomposition $\mathrm{GL}_2^+ = KA^+K$, we have

$$\begin{aligned}
&\frac{1}{|\Gamma_x|} \int_{G^+} E(g_x g, 2s_1 - 1) f(g_x g) \chi(g)^{s_2 + s_1/2} \Phi_\infty((\frac{\sqrt{|n|}}{2}, 0, \frac{\sqrt{|n|}}{2}) \cdot g) dg \\
&= \frac{1}{|\Gamma_x|} \int_{a=1}^{\infty} \int_{t=0}^{\infty} t^{2s_2 + s_1} \Phi_\infty \left(t(a^2 \frac{\sqrt{|n|}}{2}, 0, a^{-2} \frac{\sqrt{|n|}}{2}) \right) \\
&\quad \cdot \left(\int_K \int_K E(g_x k_1 \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} k_2, 2s_1 - 1) f(g_x g) dk_1 dk_2 \right) \delta(a) \frac{dt}{t} \frac{da}{a}
\end{aligned}$$

where $\delta(a) = \frac{a^2 - a^{-2}}{2}$. As

$$\int_K \int_K E(g_x k_1 g k_2, 2s_1 - 1) f(g_x g) dk_1 dk_2$$

is a spherical function. So by the uniqueness, it is equal to

$$E(g_x, 2s_1 - 1) f(g_x) \omega_\lambda(g),$$

where $\omega_\lambda(g)$ is the standard spherical function with $\omega_\lambda(e) = 1$. Therefore, we arrive at

$$\sum_{\substack{x \in V'(\mathbb{Z})/\Gamma \\ \mathrm{disc}(x)<0}} \frac{E(g_x, 2s_1 - 1) f(g_x)}{|\Gamma_x||n|^{s_2+s_1/2}} \cdot \Psi_1(\Phi_\infty, f).$$

where $\Psi_1(\Phi_\infty, f)$ is independent of n and is equal to

$$\int_{a=1}^{\infty} \int_{t=0}^{\infty} \left(|n|t^2\right)^{s_2+s_1/2} f\left(t(a^2\frac{\sqrt{|n|}}{2}, 0, a^{-2}\frac{\sqrt{|n|}}{2})\right)$$
$$\cdot\, \omega_\lambda\left(\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}\right) \delta(a) \frac{dt}{t}\frac{da}{a}.$$

Next, given $x \in V'(\mathbb{Z})$, suppose that $\mathrm{disc}(x) = n > 0$, then Γ_x is either trivial or infinite. $G_1^+ = \mathrm{SL}_2(\mathbb{R}) \times \{1\}$ also acts transitively on the real set of $\{x \in V_{\mathbb{R}} | \mathrm{disc}(x) = n\}$. So we can find

$$x \cdot g_x = (a, b, c) = (0, \sqrt{n}, 0).$$

Therefore, we have

$$\int_{\Gamma_x \backslash G^+} E(g, 2s_1 - 1) f(g) \chi(g)^{s_2+s_1/2} \Phi_\infty(x \cdot g) dg$$
$$= \int_{\Gamma'_x \backslash G^+} E(g_x g, s_1) f(g_x g) \chi(g)^{s_2+s_1/2} \Phi_\infty((0, \sqrt{n}, 0) \cdot g) dg,$$

where

$$\Gamma'_x = g_x^{-1} \Gamma_x g_x = \{\pm \begin{pmatrix} \epsilon & 0 \\ 0 & \epsilon^{-1} \end{pmatrix}^m | m \in \mathbb{Z}\}$$

if Γ_x is infinite. Using the Iwasawa decomposition, write

$$g = \left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} y^{1/2} & 0 \\ 0 & y^{-1/2} \end{pmatrix} k, t\right)$$

where $t, y > 0$, then the fundamental domain for $\Gamma'_x \backslash G^+ / K$ is $\{t > 0, -\infty <$

$\xi < \infty, 1 \le y \le \epsilon^2\}$. Then the integral becomes

$$\int_{t=0}^{\infty}\int_{1}^{\epsilon^2}\int_{-\infty}^{\infty} t^{2s_2+s_1}\Phi_\infty\left(t\cdot(0,\sqrt{n},\sqrt{n}\xi/y)\right)$$
$$\cdot E\left(g_x\begin{pmatrix} y^{1/2} & \xi y^{-1/2} \\ 0 & y^{-1/2}\end{pmatrix}, 2s_1-1\right) f(g_x g) d\xi \frac{dy}{y^2}\frac{dt}{t}$$
$$= \int_{t=0}^{\infty}\int_{1}^{\epsilon^2}\int_{-\infty}^{\infty} t^{2s_2+s_1}\Phi_\infty\left(t\cdot(0,\sqrt{n},\sqrt{n}\xi)\right)$$
$$\cdot E\left(g_x\begin{pmatrix} y^{1/2} & 0 \\ 0 & y^{-1/2}\end{pmatrix}\begin{pmatrix} 1 & \xi \\ 0 & 1\end{pmatrix}, 2s_1-1\right) f(g_x g) d\xi \frac{dy}{y}\frac{dt}{t}.$$

As

$$\int_{1}^{\epsilon^2} E\left(g_x\begin{pmatrix} y^{1/2} & 0 \\ 0 & y^{-1/2}\end{pmatrix} g, 2s_1-1\right)\frac{dy}{y}$$
$$= \int_{1}^{\epsilon^2}\int_K E\left(g_x\begin{pmatrix} y^{1/2} & 0 \\ 0 & y^{-1/2}\end{pmatrix} gk, s_1\right)\frac{dy}{y}dk,$$

it is determined by the values on $\begin{pmatrix} 1 & \xi \\ 0 & 1\end{pmatrix}$ in the decomposition of g, so it can be written as

$$\int_{1}^{\epsilon^2} E\left(g_x\begin{pmatrix} y^{1/2} & 0 \\ 0 & y^{-1/2}\end{pmatrix}, 2s_1-1\right)\frac{dy}{y}\cdot V_\lambda\left(\begin{pmatrix} 1 & \xi \\ 0 & 1\end{pmatrix}\right) + U_\lambda\left(\begin{pmatrix} 1 & \xi \\ 0 & 1\end{pmatrix}\right),$$

where $V_\lambda(g)$ is even in ξ while $U_\lambda(g)$ is odd and $V_\lambda(e) = 1$. If we choose the Schwartz function Φ_∞ to be even in ξ and notice that

$$\int_{1}^{\epsilon^2} E\left(g_x\begin{pmatrix} y^{1/2} & 0 \\ 0 & y^{-1/2}\end{pmatrix}, 2s_1-1\right)\frac{dy}{y}$$
$$= \int_{\Gamma'_x\backslash G'_x} E(g_x g, 2s_1-1)dg$$
$$= \int_{\Gamma_x\backslash G_x} E(g g_x, 2s_1-1)dg,$$

where the measure on G_x is induced from that of G'_x, hence we obtain that

$$\sum_{\substack{x\in V'(\mathbb{Z})/\Gamma \\ \mathrm{disc}(x)>0}} \frac{\int_{\Gamma_x\backslash G_x} E(g g_x, 2s_1-1) f(g g_x)dg}{|n|^{s_2+s_1/2}}\cdot\Psi_2(\Phi_\infty, f).$$

where $\Psi_2(\Phi_\infty, f)$ is independent of n and is equal to

$$\int_{t=0}^{\infty}\int_{-\infty}^{\infty}\left(|n|t^2\right)^{s_2+s_1/2}\Phi_\infty\left(t\cdot(0,\sqrt{n},\sqrt{n}\xi)\right)\cdot V_\lambda\left(\begin{pmatrix}1 & \xi\\ 0 & 1\end{pmatrix}\right)d\xi\frac{dt}{t}.$$

We remark that in the case of Γ_x trivial the integral $\int_1^{\epsilon^2}$ is changed to $\int_0^\infty$. $\square$

On the other hand, if we unfold the Eisenstein series first, splitting the measure with respect to the Iwasawa decomposition $\mathrm{GL}_2^+ = B_2^+\cdot K$, we have

$$\begin{aligned}
&Z(\Phi_\infty, f)\\
&= \int_{\Gamma\backslash G^+/T} E(g, 2s_1-1)f(g)\chi(g)^{s_2+s_1/2}\sum_{x\in V'(\mathbb{Z})}\Phi_\infty(x\cdot g)dg\\
&= \int_{B_{\mathbb{Z}}^+\backslash G^+/T}\det(g)^{s_1}\{(g_{21})^2+(g_{22})^2\}^{-s_1}f(g)\chi(g)^{s_2+s_1/2}\sum_{x\in V'(\mathbb{Z})}\Phi_\infty(x\cdot g)dg\\
&= \int_{B_{\mathbb{Z}}^+\backslash B^+/T} f(b)\chi_1((b,t))\chi_2((b,t))\sum_{x\in V'(\mathbb{Z})}\Phi_\infty(x\cdot(b,t))\frac{dt}{t}db, \qquad (20)
\end{aligned}$$

where

$$\chi_1((b,a)) = b_{11}^{2s_1}a^{s_1};\chi_2((b,a)) = \det(b)^{2s_2}a^{2s_2}.$$

If we take the residue at the pole $s_1 = 1$ of Eisenstein series, from (19) we know that it leads to a Dirichlet series with coefficients the periods of automorphic form f. On the other hand, from (20), the residue at $s_1 = 1$ is expected to be an L-function of half-integral weight. To this end, we adopt the adelic language in the next section.

6.2 Adelic Shintani Zeta Integrals

Again denote by V the prehomogeneous vector space of binary quadratic forms under the action of upper triangular Borel subgroup $B_2^+\times\mathrm{GL}_1^+$. Let B_2^0 to be the connected component of B_2^+. Write V' the B_2^+-invariant subset consisting of elements with non-square discriminant. It is obvious that $V'\subset V^{ss}$, where V^{ss} stands for the semi-stable subset determined by the two relative invariants specified later. Note that the action $B^+ = B_2^+\times\mathrm{GL}_1^+$ on V has stabilizer group T which is isomorphic to the center of B_2^+.

The character determined by the relative invariants $P(xg) = \chi(g)P(x)$ is defined by

$$\chi_1(b,a) = b_{11}^{2s_1}a^{s_1}; \chi_2(b,a) = \det(b)^{2s_2}a^{2s_2}.$$

As

$$B_2^+(\mathbb{A}) = B_2^0(\mathbb{R})B_2^+(\mathbb{Z}_f)B_2^+(\mathbb{Q})$$

for $b_{\mathbb{A}} \in B_2^0(\mathbb{R}) \times B_2^+(\mathbb{Z}_f)$, we can write $b_{\mathbb{A}} = b_\infty b_f$, $b_{\mathbb{A}} = \begin{pmatrix} t & c \\ 0 & u \end{pmatrix}$ with $t_\infty, u_\infty > 0$. Then

$$\chi_1(b_{\mathbb{A}},a) = t_\infty^{2s_1}|a|_{\mathbb{A}}^{s_1}; \chi_2(b_{\mathbb{A}},a) = \det(b_\infty)^{2s_2}|a|_{\mathbb{A}}^{2s_2}.$$

Assume f is an weight 0 and type v Maass form of $\mathrm{SL}_2(\mathbb{Z})$. There is a Whittaker expansion

$$f\left(\begin{pmatrix} 1 & x_\infty \\ & 1 \end{pmatrix}\begin{pmatrix} y_\infty & \\ & 1 \end{pmatrix}\right) = \sum_{n\neq 0} A(n)\sqrt{y_\infty}K_{v-1/2}(2\pi n y_\infty)e_\infty(x_\infty).$$

We further assume f is an Hecke eigenform with Whittaker coefficient $W(1) = 1$. Denote also by f the adelic lift to the automorphic form of $\mathrm{GL}_2(\mathbb{A})$. We have for $b_{\mathbb{A}} = b_\infty b_f \in B_2^0(\mathbb{R}) \times B_2^+(\mathbb{Z}_f)$,

$$f(b_\infty b_f) = f(b_\infty).$$

The adelic global Shintani zeta integral is

$$\begin{aligned} Z(\Phi, f) &= \int_{B^+(\mathbb{Q})\backslash B^+(\mathbb{A})/T(\mathbb{A})} \sum_{V'(\mathbb{Q})} \Phi(xg_{\mathbb{A}})\chi_1(g_{\mathbb{A}})\chi_2(g_{\mathbb{A}})f(b_{\mathbb{A}})|dg|_{\mathbb{A}} \qquad (21) \\ &= \int_{B^+(\mathbb{Q})\backslash B^+(\mathbb{A})/T(\mathbb{A})} \sum_{V'(\mathbb{Q})} \Phi(xg_{\mathbb{A}})\chi_1(g_{\mathbb{A}})\chi_2(g_{\mathbb{A}}) \sum_{\alpha\in\mathbb{Q}^\times} W\left(\begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix} b_{\mathbb{A}}\right)|db|_{\mathbb{A}} \\ &= \sum_{x\in V'(\mathbb{Q}))/B^+(\mathbb{Q})} |G_x|^{-1}Z_{x,\alpha}(\Phi, f), \end{aligned}$$

where

$$Z_{x,\alpha}(\Phi, f) = \int_{B^+(\mathbb{A})/T(\mathbb{A})} \Phi(xg_{\mathbb{A}})\chi_1(g_{\mathbb{A}})\chi_2(g_{\mathbb{A}})W\left(\begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix} b_{\mathbb{A}}\right)|db|_{\mathbb{A}}.$$

Write the global Whittaker function W as the pure tensor product of

$$W_p\left(\begin{pmatrix}1 & x_p\\ & 1\end{pmatrix}\begin{pmatrix}y_p & \\ & 1\end{pmatrix}\begin{pmatrix}r_p & \\ & r_p\end{pmatrix}k_p\right)$$
$$=\begin{cases}|y_p|_p^{1/2}A(|y_p|_p^{-1})e_p(x_p) & \text{for } y_p\in\mathbb{Z}_p,\\ 0 & \text{otherwise.}\end{cases}$$

and

$$W_\infty\left(\begin{pmatrix}1 & x_\infty\\ & 1\end{pmatrix}\begin{pmatrix}y_\infty & \\ & 1\end{pmatrix}\begin{pmatrix}r_\infty & \\ & r_\infty\end{pmatrix}k_\infty\right)$$
$$=\sqrt{y_\infty}K_{v-1/2}(2\pi y_\infty)e_\infty(x_\infty).$$

Denote by Φ_∞ the Schwartz function of vector space $V(\mathbb{R})$. Recall the global Shintani zeta integral over $\mathbb{R}$ is by (20)

$$Z_\infty(\Phi_\infty,f)=\int_{B^+(\mathbb{Z})\backslash B^+(\mathbb{R})/T(R)}\sum_{V'(\mathbb{Z})}\Phi_\infty(xg)\chi_1(g)\chi_2(g)f(b)|dg|_\infty.$$

Then it is easy to show that

Proposition 6.2.1. *Suppose $\Phi=\Phi_\infty\otimes\Phi_0$ where Φ_0 is the characteristic function of $\prod_{v\in M_f}V_{o_v}$, then we have*

$$Z_\infty(\Phi_\infty,f)=Z(\Phi,f).$$

Proof. Note that for $b_{\mathbb{A}}=b_\infty b_f\in B_2^0(\mathbb{R})\times B_2^+(\mathbb{Z}_f)$ and $a_{\mathbb{A}}\in\mathbb{A}_\infty^\times=\mathbb{R}_+^\times\times\mathbb{A}_f^\times$, $\Phi(x(b_{\mathbb{A}},a_{\mathbb{A}}))=\Phi_\infty(x_\infty(b_\infty,a_\infty))$ for $x_0\in V_{o_v}$, therefore

$$\sum_{x\in V'(\mathbb{Q})}\Phi(x(b_{\mathbb{A}},a_{\mathbb{A}}))=\sum_{x\in V'(\mathbb{Z})}\Phi_\infty(x_\infty(b_\infty,a_\infty)).$$

Start from the definition of $Z(\Phi,f)$

$$\begin{aligned}Z(\Phi,f)=&\int_{B^+(\mathbb{Q})\backslash B^+(\mathbb{A})/T(\mathbb{A})}\sum_{V'(\mathbb{Q})}\Phi(xg_{\mathbb{A}})\chi_1(g_{\mathbb{A}})\chi_2(g_{\mathbb{A}})f(b_{\mathbb{A}})|dg|_{\mathbb{A}}\\ =&\int_{\left(\left(B_2^0(\mathbb{Z})\backslash B_2^0(\mathbb{R})\right)\times B_2^+(\mathbb{Z}_f),\mathbb{R}_+^\times\times\mathbb{A}_f^\times\right)/T(\mathbb{A})}\sum_{V'(\mathbb{Z})}\Phi(xg_{\mathbb{A}})\chi_1(g_{\mathbb{A}})\chi_2(g_{\mathbb{A}})f(b_{\mathbb{A}})|db|_{\mathbb{A}}\\ =&\int_{\left(\left(B_2^0(\mathbb{Z})\backslash B_2^0(\mathbb{R})\right),\mathbb{R}_+^\times\right)/T(\mathbb{R})}\sum_{V'(\mathbb{Z})}\Phi(xg)\chi_1(g)\chi_2(g)f(b)|dg|_\infty\\ =&Z_\infty(\Phi_\infty,f).\end{aligned}$$

□

From now on, we always assume $\Phi = \Phi_\infty \otimes \Phi_0$ where Φ_0 is the characteristic function of $\prod_{v\in M_f} V_{o_v}$. Furthermore Φ_0 is the pure tensor of characteristic functions of each V_{o_v}. Then we can write $Z(\Phi, \chi)$ as

Proposition 6.2.2. *Under the above assumption, we have*

$$Z(\Phi, f) = \sum_{x\in V'(\mathbb{Q})/G^0(\mathbb{Q})} |G_x(\mathbb{Q})|^{-1} \sum_{\alpha\in Q^\times} \prod_v Z^x_{v,\alpha},$$

where

$$Z^x_{v,\alpha} = \int_{B^+(\mathbb{Q}_v)/T(\mathbb{Q}_v)} \Phi_v(xg)\chi_1(g)\chi_2(g)W_v\left(\begin{pmatrix}\alpha & 0\\ 0 & 1\end{pmatrix} b\right)|dg|_v.$$

Proof. As

$$\begin{aligned}&Z(\Phi, f)\\ &= \sum_{x\in V'(\mathbb{Q})/B^+(\mathbb{Q})} |B_x(\mathbb{Q})|^{-1} \int_{B^+(\mathbb{A})/T(\mathbb{A})} \Phi(xg_{\mathbb{A}})\chi_1(g_{\mathbb{A}})\chi_2(g_{\mathbb{A}})f(b_{\mathbb{A}})|dg|_{\mathbb{A}},\end{aligned}$$

substituting the global Whittaker function as the pure tensor product, the result is obtained. ☐

Note that the orbits of $V'(\mathbb{Q})$ under $B^+(\mathbb{Q})$ are in one-to-one correspondence with the quadratic field extensions over $\mathbb{Q}$, hence we can also write

$$Z(\Phi, f) = \sum_{k:[k:\mathbb{Q}]=2} |G_k(\mathbb{Q})|^{-1} \sum_{\alpha\in Q^\times} \prod_v Z^k_{v,\alpha}.$$

Hence we have set up the adelic Shintani zeta integrals. To study their residues at $s_1 = 1$, we will use the method which was first introduced by T. Shintani in [Shi72], [Shi75] to obtain the average class number of cubic and quadratic field extensions respectively. His method relies on the study of zeta integrals of singular orbits. Particularly relevant to our adelic setting is the work of B. Datskovsky and D. Wright in [DW86], [DW88], which leads to the average density of cubic field extensions, and is independent of the earlier method used by Davenport-Heilbronn in [DH69] and [DH71]. We also mention more recent work of T. Taniguchi in [Tan08], along the same line of research, obtains the average of square class number of quadratic field extensions by considering the PVS for a pair of simple algebras. Our continued research will appear elsewhere and set forth the study of periods of automorphic forms in relation to the Shintani zeta functions of PVS.

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