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Abstract of the Dissertation

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We consider several exactly solvable models of strongly correlated electrons in one dimension, such as the Heisenberg XXX model, the supersymmetric t-J model and the Hubbard model. These models can be solved by using the method of graded algebraic Bethe ansatz. We use it to design graded tensor networks which can be contracted approximately to obtain a Matrix Product State.

This overcomes a major shortcoming of current density matrix renormalization group (DMRG) methods which work well on the ground states, but have difficulty working with the excited states of such models.

In addition, observables such as correlation functions are important as they are experimentally measurable, but have been analytically described in the double scaling limit only. Moreover, these analytical results are mostly expressed in the form of determinants, which are numerically inefficient to compute. With the tensor network description of the spin models, we can efficiently compute any expectation value of the eigenstates on finite length lattices for direct comparison with laboratory results.

As a proof of principle, we calculate correlation functions of ground states and excited states of such models on finite lattices of lengths in an intermediate regime which are of experimental interest.

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Chapter 1

Introduction

Bose-Einstein condensation was first experimentally observed in 1995 [6, 23], a phenomenon predicted by Bose and Einstein more than 70 years earlier [25].

Since then, experimental advancements in the cooling and trapping of gases have provided a better understanding of ultracold atoms [22, 65, 93, 67, 38, 17, 16, 19]. Experimental studies of quantum systems confined to one dimension (1D) have also yielded insights into quantum statistical and dynamical effects in quantum many-body systems [128, 18].

In particular, following these experimental advancements, several exactly solvable models found their way into the laboratory. For instance, the Tonks-Girardeau gas [39] was first observed experimentally in 2004 [88, 56], and the Gaudin-Yang model [36] in 2009 [68].

Inspired by these developments, the study of integrable models has experienced renewed interest over the past decade. Their capability to reveal non-perturbative aspects of quantum many-body systems provides an unique and important window into the nature of low dimensional condensed matter physics.

1.1 Bethe ansatz

The study of exactly solvable models began when Bethe (1931) [15] introduced a clever guess for the form of the wavefunction – the (coordinate) Bethe ansatz – to obtain the energy eigenspectrum of the 1D Heisenberg spin chain.

The Bethe ansatz [46, 75, 59] reduces the complex problem of diagonaliz-

ing the Hamiltonian to finding the solutions of a set of algebraic equations, known as the Bethe ansatz equations. The energy eigenspectrum of the Hamiltonian can be obtained exactly in terms of the Bethe ansatz equations, from which its physical properties can be derived via further mathematical analysis.

The Bethe ansatz nevertheless laid in obscurity for several decades, before it was eventually discovered to underpin a diverse range of physical problems, from superconductivity to string theory [9].

From 1931 to the early 1960s, developments in the Bethe ansatz was relatively slow, with only a few papers dealing with the thermodynamic limit and the XXZ model [45, 84, 121, 24, 41]. Yang and Yang (1966) [127] made an important step forward in proving the Bethe's hypothesis that Bethe's solution is indeed the groundstate of the XXZ spin chain.

A crucial insight arose when Yang [126] observed that a generalised Bethe's hypothesis, subject to a set of cubic equations being satisfied, works for the Lieb-Liniger model (a quantum 1D Bose gas with delta-function interaction, solved by Lieb and Liniger [70, 69] in 1963). On the other hand, Baxter independently showed that in the construction of commuting transfer matrices in two dimensional classical models in statistical mechanics, the same set of cubic equations appear [13, 11]. As such, this equation has been termed the Yang-Baxter equation [31].

During the 1970s to 1990s, the Yang-Baxter equation was discovered to be the key to the solvability of a number of quantum mechanical models, including the XYZ spin chain [12], the supersymmetric t-J model [32, 33, 27] and the Hubbard model [71, 99, 108, 83, 101, 34, 35, 26]. From a mathematical point of view, it could also be viewed as a generator for new exactly solvable models as well.

Another significant development in the theory of quantum integrable systems was the formulation of the algebraic Bethe ansatz [31, 106, 105]. It is an integral part of the Quantum Inverse Scattering method (QISM), which combines the ideas of the coordinate Bethe ansatz and the classical inverse scattering method into one unified framework.

The first model to be solved by the QISM is the non-linear Schrodinger equation [129], which can be shown to be equivalent to the Lieb-Liniger model [70, 69]. Its Lax representation was first constructed in [98], which is the starting point for the QISM.

It has since become a well established method for solving a wide range of quantum models, quantum field theories and non-linear differential equa-

tions [29, 46, 113, 62, 100, 63, 58, 30, 111, 26]). It not only reproduces the results of the coordinate Bethe ansatz, but reveals more about the structure of the eigenstates and provides new analytical tools to obtain scalar products [107], norms [58] and correlations [58].

More recently, extensions of the Yang-Baxter equation to three dimensional classical models have seen some exciting progress. One promising extension known as the Zamolodchikov tetrahedron equation (ZTE) [130, 14] yielded its first nontrivial, non-negative solutions of ZTE in a vertex formulation on a three dimensional cubic lattice [73]. In addition, sufficiency conditions for the integrability of a class of three dimensional classical models was found recently [55], in which the integrability of the Kitaev model [57] (an exactly solvable two dimensional quantum spin model with topological order) can be reformulated in this context.

1.2 Tensor networks

The description of quantum states using tensor networks has been very successful in recent literature. For instance, they have been used to study different systems in different dimensions, of finite or infinite size [21, 85, 47, 86, 66, 131, 125, 72], with different boundary conditions [118, 91] and symmetries [76, 104, 102, 103, 10, 90, 94, 96, 97, 122].

In addition, the extremely successful density matrix renormalization group (DMRG) [123, 124] finds its roots in the one-dimensional Matrix Product States (MPS) [1, 2]. MPS have also been applied to the field of quantum information and condensed matter physics [116, 89, 115, 102].

To describe the ground state of higher-dimensional systems, the projected entangled pair states (PEPS) [114] were introduced and proved to be useful for the numerical study of ground states of two-dimensional systems [79, 80]. In addition, the Multiscale Entanglement Renormalization Ansatz (MERA) [119, 120] provides a framework for the description and numerical study of quantum systems at criticality.

1.3 Algebraic Bethe ansatz and tensor networks

Even though the coordinate Bethe ansatz [15] has been successful at solving certain 1D quantum models exactly, it remains computationally intractable to get interesting properties out of the states – like their entanglement characteristics or their correlations, as the structure of the eigenstates is not evident.

Moreover, even though the algebraic Bethe ansatz [58] provides new tools for further mathematical analysis, the analytical results obtained are usually numerically inefficient to compute, as they are mostly expressed in the form of determinants, for which there are no known ways of effective numerical evaluation.

To further study the structure of the eigenstates of the exactly solvable models in detail, we could formulate the algebraic Bethe ansatz in the language of tensor networks.

In fact, it can be easily seen from the tensor network description of the Bethe eigenstates that the eigenstates can be described as MPS [52]. The calculation of properties of the eigenstates (for instance, the correlation functions) then requires the contraction of a tensor network such as the one depicted in Fig. 2.1. Correlation functions are important as they are experimentally measurable, but have been analytically described in the double scaling limit only. With the tensor network description of the spin models, we can investigate correlation functions of eigenstates on finite length lattices for comparison with laboratory results.

The tensor network constructed could be contracted approximately using a similar method as for time evolution in the Density Matrix Renormalization Group (DMRG) [123, 124, 115, 78]. Eventually, a MPS [1, 2, 116, 89, 115, 102] is obtained, from which expectation values of observables can be calculated directly. For the case of the antiferromagnetic XXX and XXZ model with both periodic and open boundary conditions, it has been shown that correlations [77] can be obtained for 50 sites with good precision.

1.4 This thesis

This thesis begins with a recap of the formulation of the algebraic Bethe ansatz of the XXX model in tensor network language, and numerical re-

sults of the correlation functions are then calculated using the approximate contraction of the tensor networks, following Murg et al. [77].

The main work done in this thesis is then presented - the formulation of the algebraic Bethe ansatz of the supersymmetric t-J model in tensor network language and numerical calculations of its correlation functions up to 18 sites as a proof of concept [20]. The main novelty of this work is the formulation of the nesting and grading of the algebraic Bethe ansatz into tensor network language, and computation of correlation functions on finite lattices for arbitrary ground states and excited states of the t-J model.

New unpublished work on the interpretation of the Hubbard model in terms of tensor networks then follows.

Finally, a summary of this thesis is given in Chapter 5.

Chapter 2

Heisenberg XXX model

2.1 The Heisenberg XXX model

We consider a one-dimensional model of spin-$\frac{1}{2}$ particles that have a nearest neighbor spin-spin interaction.

The Hamiltonian of the Heisenberg XXX model is given by

$$\begin{aligned} H_{XXX} &= \sum_{j=1}^{L} h_{XXX}^{(j,j+1)} \,, \\ h_{XXX}^{(j,j+1)} &= \frac{1}{2}\left[\sigma_x^j \otimes \sigma_x^{j+1} + \sigma_y^j \otimes \sigma_y^{j+1} + \sigma_z^j \otimes \sigma_z^{j+1} - \mathbb{1}\right] \end{aligned} \tag{2.1}$$

where the Pauli matrices are defined as

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_x = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_x = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \tag{2.2}$$

and the superscripts indicate the lattice site where the respective Pauli matrices act on.

The XXX model was introduced by Heisenberg [43] in 1928 as a quantum generalization of the Ising model. The coordinate Bethe ansatz [15] was first proposed by Bethe to solve for its eigenvectors and eigenvalues. Following that, Hulthen [45], des Cloiseaux and Pearson [24], Orbach [84], Yang and Yang [127], Baxter [12], Gaudin [37], Takahashi [110], Ovchinnikov [87], Kulish and Reshetikhin [60] et al. obtained important results using Bethe's method. The algebraic Bethe ansatz for the XXX model was later developed in the late 70's [112, 31, 61, 30].

In addition, Katsura and Maruyama had also previously shown that the alternative formulation of the Bethe ansatz by Alcaraz and Lazo [3, 4, 5] is equivalent to the algebraic Bethe ansatz for the XXX model.

This chapter is devoted to the tensor network description and numerical calculation of observables of the eigenstates of the XXX model, using the algebraic Bethe ansatz, as a recap of [77].

2.2 Algebraic Bethe ansatz for XXX model

In this section, the algebraic Bethe ansatz for the XXX model [58, 40, 26, 133, 92] is resketched in the language of tensor networks.

2.2.1 The Yang-Baxter Algebra

The (ungraded) Yang-Baxter equation will be described in the subsection. At the center of the machinery for the algebraic Bethe ansatz is the $R(\lambda, \mu)$ tensor

$$R^{\alpha\beta}_{\alpha'\beta'}(\lambda, \mu), \tag{2.3}$$

with $\alpha, \beta, \alpha', \beta'$ ranging from 1 to some "auxiliary" dimension d and λ, μ being some complex parameters. This tensor would define the model under study, as would be shown later. After joining indices $(\alpha\beta)$ and $(\alpha'\beta')$, the tensor (2.3) can also be interpreted as a matrix $R(\lambda, \mu)$ acting on the vector space $V \otimes V$ (with $V = \mathbb{C}^d$).

The R matrix (2.3) fulfills the Yang-Baxter equation if the following holds on $V \otimes V \otimes V$:

$$R^{(23)}(\lambda, \mu)R^{(12)}(\lambda, \nu)R^{(23)}(\mu, \nu) = R^{(12)}(\mu, \nu)R^{(23)}(\lambda, \nu)R^{(12)}(\lambda, \mu). \tag{2.4}$$

where the indices $1, 2, 3$ indicate in which of the three tensored spaces the matrices act nontrivially. The tensor $R(\lambda, \mu)$ defines the Yang-Baxter algebra $T^{\alpha}_{\alpha'}(\lambda)$ $(\alpha, \alpha' = 1, \ldots, d)$ by the relation

$$R^{\alpha\beta}_{\alpha'\beta'}(\lambda, \mu)T^{\alpha'}_{\alpha''}(\lambda)T^{\beta'}_{\beta''}(\mu) = T^{\alpha}_{\alpha'}(\mu)T^{\beta}_{\beta'}(\lambda)R^{\alpha'\beta'}_{\alpha''\beta''}(\lambda, \mu) \tag{2.5}$$

The common indices are summed over by the Einstein summation convention. $T(\lambda)$ can be considered as a 4-index tensor: 2 "virtual" indices α, α' of

dimension d select the operator $T^{\alpha}_{\alpha'}(\mu)$ within the matrix, and two "physical" indices operate as input and output index of the operator.

The fundamental representation of the Yang-Baxter algebra is formed by the operators $L^{\alpha}_{\alpha'}(\lambda,\nu)$ acting on $\mathbb{C}^d$ defined as

$$[L^{\alpha}_{\alpha'}(\lambda,\nu)]^{k}_{l} = R^{k\alpha}_{\alpha' l}(\lambda,\nu). \tag{2.6}$$

These L operators satisfy a "distorted" version of the Yang-Baxter equation known as the intertwining relation:

$$R(\lambda,\mu)\,[L(\lambda,\nu)\check{\otimes}L(\mu,\nu)] = [L(\mu,\nu)\check{\otimes}L(\lambda,\nu)]\,R(\lambda,\mu) \tag{2.7}$$

The parameter ν in $L(\lambda,\nu)$ is arbitrary up till now, but it can be set to a constant, $\nu=\nu_0$, for convenience.

Once a representation $L(\lambda)$ is known, more complex representations can be constructed by concatenating the $L(\lambda)$'s. Here, a monodromy matrix $T^{\alpha}_{\alpha'}(\lambda)$ acting on $(\mathbb{C}^d)^{\otimes L}$ is constructed out of L simple operators $L^{\alpha}_{\alpha'}(\lambda)$ acting on $\mathbb{C}^d$ via

$$[T^{\alpha}_{\alpha'}(\lambda)]^{k_1\cdots k_L}_{l_1\cdots l_L} = \sum_{\alpha_2\cdots\alpha_L} [L^{\alpha}_{\alpha_2}(\lambda)]^{k_1}_{l_1}[L^{\alpha_2}_{\alpha_3}(\lambda)]^{k_2}_{l_2}\cdots[L^{\alpha_L}_{\alpha'}(\lambda)]^{k_L}_{l_L}. \tag{2.8}$$

The monodromy matrix as defined also fulfills (2.5), which can be proven by induction, as the R tensor would subsequently interchange the operators $L^{\alpha}_{\alpha'}(\lambda)$ from left to right. Defining the matrices $\mathcal{L}^{k}_{l}(\lambda)$ as $\langle\alpha|\,\mathcal{L}^{k}_{l}(\lambda)\,|\alpha'\rangle := [L^{\alpha}_{\alpha'}(\lambda)]^{k}_{l}$, the monodromy matrix $T^{\alpha}_{\alpha'}(\lambda)$ can be written in the form of a Matrix Product Operator (MPO) [118],

$$T^{\alpha}_{\alpha'}(\lambda) = \sum_{\substack{k_1\cdots k_L\\ l_1\cdots l_L}} \langle\alpha|\,\mathcal{L}^{k_1}_{l_1}(\lambda)\cdots\mathcal{L}^{k_L}_{l_L}(\lambda)\,|\alpha'\rangle \tag{2.9}$$

where

$$o^{k_1}_{l_1}\otimes\cdots\otimes o^{k_L}_{l_L}, \quad o^{k}_{l} = |k\rangle\langle l|\,. \tag{2.10}$$

Now, we can obtain the transfer matrix $\tau(\lambda)$, as the trace of the monodromy matrix $T^{\alpha}_{\alpha'}(\lambda)$,

$$\tau(\lambda) := \mathrm{tr}\,\{T(\lambda)\} \equiv \sum_{\alpha} T^{\alpha}_{\alpha}(\lambda). \tag{2.11}$$

In the language of tensor network, $\tau(\lambda)$ is represented by an MPO with periodic boundary conditions [118].

2.2.2 Trace identities

Due to (2.5), the transfer matrix satisfies $[\tau(\lambda), \tau(\mu)] = 0$ for all λ and μ [58]. This property makes any function of $\tau(\lambda)$ a generator of an infinite set of commuting observables. In particular, taking $\mathcal{F}(\lambda) = \log \tau(\lambda)$, its Taylor expansion reads

$$\mathcal{F}(\lambda) = \mathcal{F}(\lambda_0) + (\lambda - \lambda_0)\mathcal{F}'(\lambda_0) + O\left((\lambda - \lambda_0)^2\right). \tag{2.12}$$

As such, $\mathcal{F}'(\lambda_0)$ gives the trace identity

$$\mathcal{F}'(\lambda_0) \equiv \frac{d}{d\lambda} \log \tau(\lambda)\Big|_{\lambda=\lambda_0} = \sum_{i=1}^{L} h^{(i,i+1)} = H \tag{2.13}$$

which is the definition of an integrable local Hamiltonian that acts on neighboring sites. In particular,

$$h = \frac{\partial}{\partial\lambda} R(\lambda, \nu_0)\Big|_{\lambda=\lambda_0}. \tag{2.14}$$

Models that emerge in such a way from fundamental representations are called fundamental models.

2.2.3 Algebraic Bethe ansatz for the XXX model

The Heisenberg XXX model is one such fundamental model, in which $d = 2$ and the R and L matrix are of the form

$$R(\lambda) = b(\lambda)I + a(\lambda)\Pi \tag{2.15}$$

$$[L^{\alpha}_{\alpha'}(\lambda)]^{k}_{l} = R^{k\alpha}_{\alpha' l}(\lambda) = [b(\lambda)\Pi + a(\lambda)I]^{k\alpha}_{\alpha' l} \tag{2.16}$$

where

$$a(\lambda) = \frac{\lambda}{\lambda + 1}\ , \quad b(\lambda) = \frac{1}{\lambda + 1} \tag{2.17}$$

and the identity and permutation operators are defined as

$$I^{b_1 b_1}_{a_1 a_2} = \delta^{b_1}_{a_1}\delta^{b_2}_{a_2}\ , \quad \Pi^{b_1 b_2}_{a_1 a_2} = \delta^{b_2}_{a_1}\delta^{b_1}_{a_2}, \tag{2.18}$$

where Π satisfies $\Pi(v \otimes w) = (w \otimes v)$. In particular, the R matrix satisfies:

$$R(0) = \mathbb{1}\,(\lambda_0 = 0)\ , \quad R'(0) = h_{XXX}, \tag{2.19}$$

Explicitly, the matrices $\mathcal{L}_l^k(\lambda)$ are

$$\begin{aligned}\mathcal{L}_0^0(\lambda) &= \begin{pmatrix} 1 & 0 \\ 0 & a(\lambda) \end{pmatrix}, \quad \mathcal{L}_1^0(\lambda) = \begin{pmatrix} 0 & 0 \\ b(\lambda) & 0 \end{pmatrix} \\ \mathcal{L}_0^1(\lambda) &= \begin{pmatrix} 0 & b(\lambda) \\ 0 & 0 \end{pmatrix}, \quad \mathcal{L}_1^1(\lambda) = \begin{pmatrix} a(\lambda) & 0 \\ 0 & 1 \end{pmatrix}.\end{aligned} \tag{2.20}$$

The Yang-Baxter Algebra with the R matrix (2.15) is generated by 4 quantum operators, such that the monodromy matrix is of the form

$$T(\lambda) = \begin{pmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & D(\lambda) \end{pmatrix} \tag{2.21}$$

The Hilbert space at the kth site of the lattice is spanned by two vectors $e_1 = (10) \equiv |0\rangle \equiv |\uparrow\rangle$, $e_2 = (01) \equiv |1\rangle \equiv |\downarrow\rangle$. We choose the reference state to be a state with all up spins

$$|0\rangle = \otimes_{n=1}^{L} |0\rangle_n ,$$

and we make an ansatz for the eigenstate of the monodromy matrix:

$$|\Psi(\mu_1, \ldots, \mu_M)\rangle = B(\mu_1) \cdots B(\mu_M) |0\rangle \tag{2.22}$$

In tensor network language, the creation operator $B(\mu)$ can also be represented as an MPO [117] by terminating the monodromy matrix with boundary vectors:

$$\begin{aligned} B(\lambda) &= \sum_{\substack{k_1 \cdots k_L \\ l_1 \cdots l_L}} \langle 0| T(\mu)_{l_1 \cdots l_L}^{k_1 \cdots k_L} |1\rangle \, o_{l_1}^{k_1} \otimes \cdots \otimes o_{l_L}^{k_L} \\ &= \sum_{\substack{k_1 \cdots k_L \\ l_1 \cdots l_L}} \langle 0| \mathcal{L}_{l_1}^{k_1}(\mu) \cdots \mathcal{L}_{l_L}^{k_L}(\mu) |1\rangle \, o_{l_1}^{k_1} \otimes \cdots \otimes o_{l_L}^{k_L} \end{aligned}$$

with $k, l \in \{0, 1\}$, $o_l^k = |k\rangle \langle l|$. The product of operators $B(\mu_1) \cdots B(\mu_M)$ can be interpreted as the contraction of the set of 4-index tensors $[\mathcal{L}_l^k(\mu_j)]_{r'}^r$ arranged in a rectangular grid, as shown in Fig. 2.1. Thereby, r, r', k and l label the left, right, up and down-indices, respectively. The multiplication of the MPOs with the product state $|0\rangle$ would yield a MPS [115, 116] with

bond-dimension 2^M. Each MPO $B(\lambda)$ is a creation operator which creates one down-spin, thus the MPS contains exactly M down-spins.

By making use of fundamental commutation relations generated from (2.5), we can use the algebraic Bethe ansatz to diagonalize the transfer matrix, from which the energy and momentum of the XXX Hamiltonian can be obtained [58]. We briefly outline the algebraic Bethe ansatz for the XXX model as below.

To make use of the trace identity (2.13), we must first obtain the diagonal elements $A(\lambda)$ and $D(\lambda)$. To find the diagonal elements, we use the fundamental commutation relations generated from (2.5). In particular, the equations that are relevant (out of the 16 equations generated) are:

$$B(\lambda)B(\mu) = B(\mu)B(\lambda) \tag{2.23}$$
$$A(\lambda)B(\mu) = f(\lambda,\mu)B(\mu)A(\lambda) + g(\lambda,\mu)B(\lambda)A(\mu) \tag{2.24}$$
$$D(\lambda)B(\mu) = f(\mu,\lambda)B(\mu)D(\lambda) + g(\mu,\lambda)B(\lambda)D(\mu) \tag{2.25}$$

where, for the XXX model,

$$f(\lambda,\mu) = 1/a(\mu,\lambda), \quad g(\lambda,\mu) = -b(\mu,\lambda)/a(\mu,\lambda). \tag{2.26}$$

and $a(\mu,\lambda)$ and $b(\mu,\lambda)$ are as defined in (2.17).

We note that the action of the diagonal elements on the reference state $|0\rangle$ is

$$\mathcal{A}(\lambda)\,|0\rangle = |0\rangle \tag{2.27}$$
$$\mathcal{D}(\lambda)\,|0\rangle = a(\lambda)\,|0\rangle \tag{2.28}$$
$$\tag{2.29}$$

and $C(\lambda)$ annihilates the reference state:

$$\mathcal{C}(\lambda)\,|0\rangle = 0. \tag{2.30}$$

From (2.24), we see that each commutation of $A(\lambda)$ with a $B(\mu_k)$ yields 2 terms, thus $A(\lambda)$ applied to $|\Psi(\mu_1,\ldots,\mu_M)\rangle$ should yield 2^M terms in principle, since it takes M commutations to move $A(\lambda)$ from left to right. However, upon closer inspection, we can see that these terms are "exchange" operators: the f-term in (2.24) exchanges the operators A and B, *but not* their arguments; the g-term, on the other hand, exchanges the operators A and B *and* their arguments. As such, after M commutations, we must have:

- Every term must contain M B's and one A.
- The $M+1$ coefficients $(\lambda, \mu_1, \ldots, \mu_M)$ are distributed among the M B's and the one A.

Since all B's commute (2.23), there are only 2 cases: either λ is an argument of A, in which the term would be of the form

$$B(\mu_1) \cdots B(\mu_M) A(\lambda) \left|0\right\rangle , \tag{2.31}$$

or that λ is an argument of one of the B's, in which the term is of the form

$$B(\lambda) \prod_{j \neq n} B(\mu_j) A(\mu_n) \left|0\right\rangle \tag{2.32}$$

with $n \in \{1, \ldots, M\}$. Thus, the 2^M terms can be collected into $M+1$ linearly independent terms:

$$\begin{aligned} A(\lambda) \left|\Psi(\mu_1, \ldots, \mu_M)\right\rangle = \Lambda B(\mu_1) \cdots B(\mu_M) A(\lambda) \left|0\right\rangle \\ + \sum_{n=1}^{M} \Lambda_n B(\lambda) \prod_{j \neq n} B(\mu_j) A(\mu_n) \left|0\right\rangle \end{aligned} \tag{2.33}$$

What remains to be done is the calculation of the coefficients Λ and Λ_n.

The expression (2.31) can be obtained after M commutations using the f-term in (2.24). However, the g-term must not be used, as it introduces an unwanted $B(\lambda)$. Thus

$$\Lambda = \prod_{j=1}^{M} f(\lambda, \mu_j). \tag{2.34}$$

To obtain (2.32), it is convenient to rewrite the Bethe ansatz (2.22) as

$$\left|\Psi(\mu_1, \ldots, \mu_M)\right\rangle = B(\mu_n) \prod_{j \neq n} B(\mu_j) \left|0\right\rangle , \tag{2.35}$$

which can be done for all n's, since all B's commute (2.23).

Since the expression (2.32) must not contain $B(\mu_n)$, the first commutation with $A(\lambda)$ must be performed using the g-term in (2.24). We would then obtain the expression

$$g(\lambda, \mu_n) B(\lambda) A(\mu_n) \prod_{j \neq n} B(\mu_j) \left|0\right\rangle . \tag{2.36}$$

All further commutations must use the f-term, because another use of the g-term would introduce $B(\mu_n)$ in the expression again. Thus, the coefficients must be

$$\Lambda_n = g(\lambda, \mu_n) \prod_{j \neq n} f(\mu_n, \mu_j) \tag{2.37}$$

The application of $D(\lambda)$ to $|\Psi(\mu_1, \ldots, \mu_M)\rangle$ can be also treated in a similar way using (2.23) and (2.25). Again, this would yield $M + 1$ terms:

$$\begin{aligned} D(\lambda) \, |\Psi(\mu_1, \ldots, \mu_M)\rangle = \tilde{\Lambda} B(\mu_1) \cdots B(\mu_M) D(\lambda) \, |0\rangle \\ + \sum_{n=1}^{M} \tilde{\Lambda}_n B(\lambda) \prod_{j \neq n} B(\mu_j) D(\mu_n) \, |0\rangle \, , \end{aligned} \tag{2.38}$$

in which the coefficients are

$$\tilde{\Lambda} = \prod_{j=1}^{M} f(\mu_j, \lambda). \tag{2.39}$$

and

$$\tilde{\Lambda}_n = g(\mu_n, \lambda) \prod_{j \neq n} f(\mu_j, \mu_n). \tag{2.40}$$

We note that the wanted terms on the RHS of (2.33) and (2.38) are the terms proportional to $|0\rangle$, and the other two terms are unwanted terms.

For the unwanted terms to cancel, we would require that

$$\Lambda_n + a(\mu_n) \tilde{\Lambda}_n = 0, \quad n = 1, \ldots, M \tag{2.41}$$

which would make $|\Psi(\mu_1, \ldots, \mu_M)\rangle$ an eigenvector of $t(\lambda) = A(\lambda) + D(\lambda)$.

These criterion for the spectral parameters are the Bethe ansatz equations, which can be written as

$$a(\mu_n) = \prod_{\substack{j=1 \\ j \neq n}}^{M} \frac{c(\mu_n, \mu_j)}{c(\mu_j, \mu_n)} \tag{2.42}$$

under the assumption that $g(\lambda, \mu)$ is an odd function in the sense that $g(\lambda, \mu) = -g(\mu, \lambda)$, as is the case for the Heisenberg XXX model.

By introducing a transformation $\mu_j = \frac{z_j}{2i} - \frac{1}{2}$, it can be easily seen that these Bethe ansatz equations obtained by the algebraic Bethe ansatz are exactly equivalent to those obtained by the coordinate Bethe ansatz [15, 45, 51, 49, 50]:

$$\left(\frac{z_n - i}{z_n + i}\right)^L = \prod_{\substack{j=1 \\ j \neq n}}^{M} \frac{z_n - z_j - 2i}{z_n - z_j + 2i} \quad , \qquad n = 1, \ldots, M. \tag{2.43}$$

From the wanted terms, we obtain the eigenvalues of the transfer matrix $\tau(\lambda)$ as

$$\tau(\lambda) = \Lambda + a(\lambda)\tilde{\Lambda}, \tag{2.44}$$

which can be expressed as

$$\tau(\lambda) = \prod_{j=1}^{M} \frac{1}{c(\mu_j, \lambda)} + a(\lambda) \prod_{j=1}^{M} \frac{1}{c(\lambda, \mu_j)}. \tag{2.45}$$

Using the trace identity (2.13), the energy is obtained as

$$E = \frac{\tau'(0)}{\tau(0)} = -\sum_{j=1}^{M} \frac{4}{z_j^2 + 1}. \tag{2.46}$$

2.3 Approximate contraction of the tensor network

The tensor network shown in Fig. 2.1 represents the correlation function

$$\langle \Psi(\mu_1, \ldots, \mu_M)| \, \sigma_x^{(i)} \sigma_x^{(j)} \, |\Psi(\mu_1, \ldots, \mu_M)\rangle \, . \tag{2.47}$$

It is composed of the network for $|\Psi(\mu_1, \ldots, \mu_M)\rangle$ shown in Fig. 2.1, with the conjugated tensor network representing $\langle \Psi(\mu_1, \ldots, \mu_M)|$ and two operators $\sigma_x^{(i)}$ and $\sigma_x^{(j)}$ between at sites i and j. The complexity to contract this network scales exponentially with the number of rows M or columns L (depending on the direction of contraction) as the Hilbert space grows, which renders exact computations infeasible.

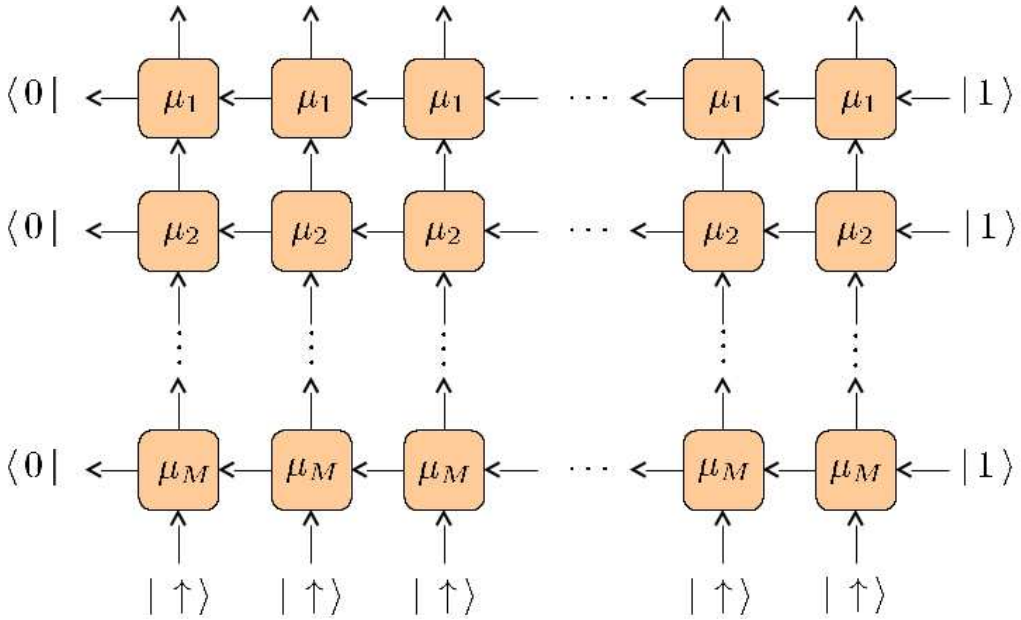


Figure 2.1: Tensor network representation of the Bethe eigenstate of the XXX model

To circumvent this problem, the contraction is done approximately. The network in Fig. 2.1 can be considered as the time evolution of the reference state $|0\rangle$ by the M creation operators $B(\mu_1), \ldots, B(\mu_M)$ to the final state $|\Psi(\mu_1, \ldots, \mu_M)\rangle$.

We denote the state obtained at each step as $|\Psi_m\rangle$, $m = 1, \cdots, M$. After each step, the product $|\Psi_m\rangle \equiv B(\mu_m) \left|\tilde{\Psi}_{m-1}\right\rangle$ remains an MPS, but its virtual dimension is increased by a factor of 2. To prevent exponential growth of the size of the tensor network, the MPS is approximated after each step by a simpler MPS $\left|\tilde{\Psi}_m\right\rangle$ that has maximal bond-dimension $\tilde{D}$ and is "closest" to $|\Psi_m\rangle$, in the sense that we try to minimize

$$K := \left\| |\Psi_m\rangle - \left|\tilde{\Psi}_m\right\rangle \right\|^2 \to \text{Min.} \tag{2.48}$$

by optimizing over all matrices of the MPS $\left|\tilde{\Psi}_m\right\rangle$.

This minimization problem also appears in the context of numerical calculation of expectation values with respect to PEPS [114, 79, 80], the cal-

culation of partition functions [78] and (imaginary) time-evolution of 1D quantum systems [115].

In this way, the MPS approximation of the tensor network is obtained for $m = M$. Thereby, $\{\mu_1, \ldots, \mu_M\}$ are the solutions of the Bethe ansatz equations. The error of the approximation is well controlled in the sense that the expectation value of the energy can always be calculated with respect to the approximated MPS $\left|\tilde{\Psi}_M\right\rangle$ and compared to the exact energy available from the Bethe ansatz.

In addition, for the XXX model with open boundary conditions and XXZ model with both periodic and open boundary conditions, the algebraic Bethe ansatz has been similarly formulated in tensor network language [77].

2.4 Numerical results for XXX model

Using the approximate contraction of the tensor network with cutoff $D = 500$, the structure factor of the ground state and selected two-spinon excited states with total spin $S = 1$ and total z-spin $S_z = 1$ of the XXX model have been investigated, where

$$S_z(q) = \frac{1}{L^2} \sum_{r,s} e^{iq(r-s)} \langle \sigma_z^r \sigma_z^s \rangle. \tag{2.49}$$

In Fig. 2.3, the structure factor at the point $q = \pi$, i.e. the squared staggered magnetization, is plotted as a function of the excitation energy and the momentum. Evidently, the excited states of the lowest branch show the highest staggered magnetization.

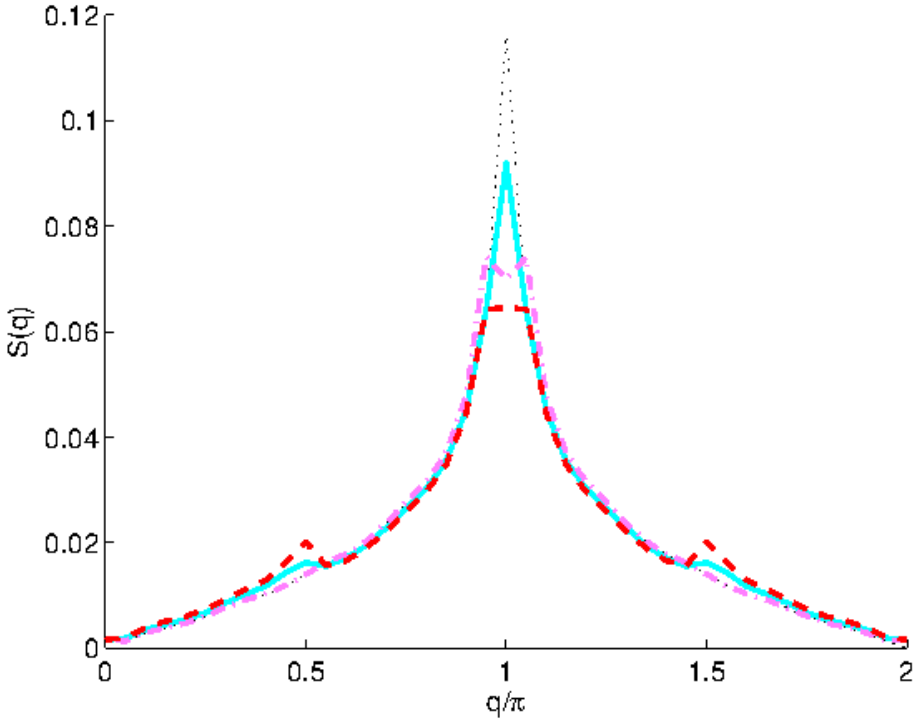


Figure 2.2: Structure factor for the ground state (dotted line) and three selected two-spinon excited states of the $L = 50$ XXX model as a function of the wave-vector q.

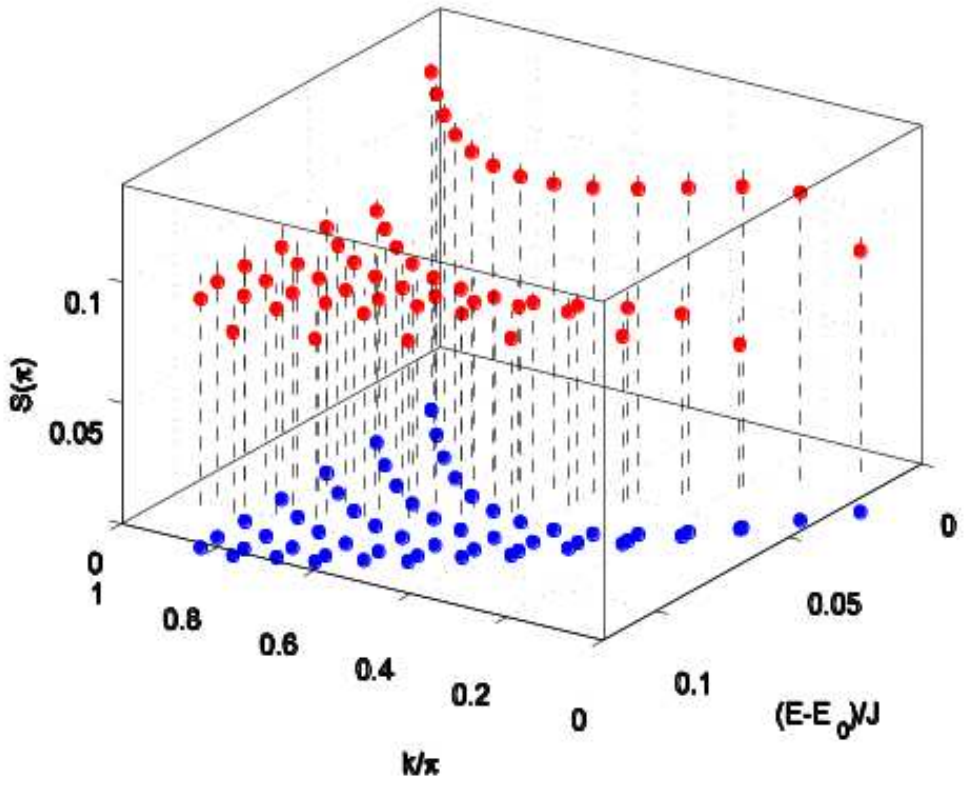


Figure 2.3: Structure factor $S(\pi)$ for selected two-spinon excited states of the $L = 30$ XXX model as a function of the momentum and the excitation energy.

Chapter 3

t-J model

3.1 The t-J model

Models of strongly correlated electrons in 1D, such as the Hubbard model and t-J model, can also be solved by the Bethe ansatz [32, 27]. In fact, the t-J model is an approximation of the strongly repulsive Hubbard model [26]. These models describe an important physical phenomena: spin and charge separation. The electron becomes unessential in this picture, and instead we have spin-waves and holons (holons carry electric charge with no spin).

This chapter is devoted to the tensor network description and numerical calculation of observables of the eigenstates of the t-J model, using the algebraic Bethe ansatz. We first describe the solution of the t-J model at supersymmetric coupling, then we proceed with the description of the tensor network and finally we would describe the numerical algorithm used and show the numerical results of the correlation functions.

In order to solve t-J model, the Bethe ansatz (and correspondingly, the tensor network) of the XXX model needs to be generalized by two steps: nesting and grading. Nesting means that when solving the Bethe ansatz, we find that a second Bethe ansatz nested within the first naturally appears. This is equivalent to diagonalizing the charge degrees and spin degrees of freedom in two separate steps [32, 27, 59]. Grading, on the other hand, is used to account for the fermionic nature of the electrons.

Using the tensor network description of the t-J model, computation of observables such as correlation functions can be done for both ground states and excited states at various fillings, overcoming a major hurdle of DMRG

methods, which can mainly deal with ground states only. Correlation functions for the t-J model have been described in the double scaling limit for the t-J model [54, 53] algebraically. Additionally, correlation functions have also been described using determinant representations [132], but they are highly difficult to evaluate numerically, since there is is no known algorithm for efficiently computing the exponential number of terms in the full expression of the determinants.

Since existing algebraic methods already suffice in the thermodynamic limit, we focus on the intermediate range of lattice lengths that are large enough to lie beyond the range of exact diagonalization, yet small enough to be qualitatively different from the thermodynamic limit. This regime is of major interest in current experiments with optical lattices and ion traps [42, 48]. As such, we have performed computations of the correlation functions of the eigenstates up to 18 lattice sites as a proof of principle.

3.2 Algebraic Bethe ansatz for the t-J model

In this section, we briefly outline the derivation of the algbraic Bethe ansatz for the t-J model, following Essler and Korepin[27].

3.2.1 Preliminaries

In the t-J model, electrons on a lattice of length L are described by operators $c_{j,\sigma}$, $j = 1, \cdots, L$, $\sigma = \pm 1$, which follow the anticommutation relations $\{c^\dagger_{i,\sigma}, c_{j,\tau}\} = \delta_{i,j}\delta_{\sigma,\tau}$. The state $|0\rangle$ (Fock vacuum) satisfies $c_{j,\sigma}|0\rangle = 0$. The Hilbert space of the Hamiltonian (3.3) is constrained to exclude double occupancy, thus there are three possible electronic states at a given lattice site i:

$$|0\rangle_i, |\uparrow\rangle_i = c^\dagger_{i,1}|0\rangle_i, |\downarrow\rangle_i = c^\dagger_{i,-1}|0\rangle. \tag{3.1}$$

We define the operators:

$$\begin{aligned} n_{i,\sigma} &= c^\dagger_{j,\sigma} c_{j,\sigma}, \quad n_i = n_{i,1} + n_{i,-1}, \quad N = \sum_{j=1}^{L} n_j \\ S_j &= c^\dagger_{j,1} c_{j,-1}, \quad S = \sum_{j=1}^{L} S_j \\ S^\dagger_j &= c^\dagger_{j,-1} c_{j,1}, \quad S^\dagger = \sum_{j=1}^{L} S^\dagger_j \\ S^z_j &= \tfrac{1}{2}(n_{j,1} - n_{j,-1}), \quad S^z = \sum_{j=1}^{L} S^z_j \end{aligned} \tag{3.2}$$

(We shall henceforth give local expressions O_j to imply the global ones are obtained as $O = \sum_{j=1}^{L} O_j$.)

The t-J Hamiltonian is given by

$$H = \sum_{j=1}^{L} \left\{ -t\mathcal{P} \sum_{\sigma=\pm 1} (c^\dagger_{j,\sigma} c_{j+1,\sigma} + H.c.)\mathcal{P} + J(\mathbf{S_j} \cdot \mathbf{S_{j+1}} - \tfrac{1}{4} n_j n_{j+1}) \right\} \tag{3.3}$$

where $\mathcal{P} = (1 - n_{j,-\sigma})$ is the projector which constrains the Hamiltonian to nondoubly occupied states. t represents nearest-neighbor hopping and J represents nearest-neighbor spin exchange and charge interactions.

Adding a term $2N - L$ to the Hamiltonian, and specializing to the value $J = 2t = 2$, the resultant Hamiltonian is supersymmetric and can be written as a graded permutation operator:

$$\begin{aligned} H_{\text{susy}} &= \mathcal{H} + 2N - L \\ &= -\sum_{j=1}^{L} \Pi^{j,j+1} \end{aligned} \tag{3.4}$$

The graded permutation operator permutes two adjacent lattice sites as follows (permuting two fermions gives a minus sign):

$$\begin{aligned} \Pi^{j,j+1} \left|0\right\rangle_j \left|0\right\rangle_{j+1} &= \left|0\right\rangle_j \left|0\right\rangle_{j+1} \\ \Pi^{j,j+1} \left|0\right\rangle_j \left|\sigma\right\rangle_{j+1} &= \left|\sigma\right\rangle_j \left|0\right\rangle_{j+1} \\ \Pi^{j,j+1} \left|\tau\right\rangle_j \left|\sigma\right\rangle_{j+1} &= -\left|\sigma\right\rangle_j \left|\tau\right\rangle_{j+1}, \qquad \sigma, \tau = \uparrow, \downarrow \end{aligned} \tag{3.5}$$

3.2.2 Grading

Consider the graded linear space $V^{(m|n)} = V^m \oplus V^n$, where m and n denote the dimensions of the "even" (V^m) and "odd" (V^n) parts, and $\oplus$ denotes the direct sum. Let $\{e_1, \cdots, e_{m+n}\}$ be a basis of $V^{(m+n)}$, such that $\{e_1, \cdots, e_m\}$ is a basis of V^m and $\{e_{m+1}, \cdots, e_n\}$ is a basis of V^n. The Grassmann parities of the basis vectors are given by $\{\epsilon_1 = \cdots = \epsilon_m = 0\}$ and $\{\epsilon_{m+1} = \cdots = \epsilon_{m+n} = 1\}$. Linear operators on $V^{(m|n)}$ can be represented in block form $[M \in \mathrm{End}(V^{(m|n)})]$:

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \quad \epsilon \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} = 0, \quad \epsilon \begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix} = 1 \tag{3.6}$$

The supertrace is defined as

$$\mathrm{str}(M) = \mathrm{tr}(A) - \mathrm{tr}(D), \tag{3.7}$$

where the traces on the rhs are the usual (non-graded) operator traces in V^m and V^n. We now define the graded tensor product of matrices in $V^{(m|n)} \otimes V^{(m|n)}$ as follows:

$$(F \otimes G)^{ab}_{cd} = F_{ab} G_{cd} (-1)^{\epsilon_c(\epsilon_a + \epsilon_b)} \tag{3.8}$$

The identity operator I and the permutation operator Π are defined as:

$$I^{a_1 b_1}_{a_2 b_2} = \delta_{a_1 b_1} \delta_{a_2 b_2} \tag{3.9}$$

$$\Pi(v \otimes w) = (w \otimes v),$$

$$(\Pi)^{a_1 b_1}_{a_2 b_2} = \delta_{a_1 b_2} \delta_{a_2 b_1} (-1)^{\epsilon_{b_1} \epsilon_{b_2}} \tag{3.10}$$

$V^{(m|n)}$ can be interpreted as the space of configurations at every site of a lattice gas of m species of bosons and n species of fermions. For the t-J model, we have $m = 1$, $n = 2$, and the three allowed configurations are given by (3.1).

3.2.3 Yang-Baxter equation

A matrix $R(\lambda)$ fulfills a graded Yang-Baxter equation if the following holds on $V^{(m|n)} \otimes V^{(m|n)} \otimes V^{(m|n)}$:

$$\begin{aligned} &[I \otimes R(\lambda - \mu)][R(\lambda) \otimes I][I \otimes R(\mu)] \\ &\quad = [R(\mu) \otimes I][I \otimes R(\lambda)][R(\lambda - \mu) \otimes I] \end{aligned} \tag{3.11}$$

The R matrix

$$\begin{aligned} R(\lambda) &= b(\lambda)I + a(\lambda)\Pi \\ a(\lambda) &= \frac{\lambda}{\lambda+i}, \quad b(\lambda) = \frac{i}{\lambda+i} \end{aligned} \tag{3.12}$$

is one such matrix that fulfills (3.11). We can rewrite (3.11) as

$$\begin{aligned} &R_{12}(\lambda-\mu)\{[\Pi_{13}R_{13}(\lambda)] \otimes [\Pi_{23}R_{23}(\mu)]\} \\ &\quad = \{[\Pi_{13}R_{13}(\mu)] \otimes [\Pi_{23}R_{23}(\lambda)]\}R_{12}(\lambda-\mu) \end{aligned} \tag{3.13}$$

where the indices $1, 2, 3$ indicate in which of the three tensored spaces the matrices act nontrivially. The tensor product in (3.13) is between spaces 1 and 2. We now call the third space "quantum space" and the first two spaces "matrix spaces". The quantum space and matrix space are usually called "physical space" and "auxiliary space" respectively in tensor network terms. The quantum space represents the Hilbert space of a single lattice site.

We now define the L operator on site k as a quantum operator valued linear operator on $\mathcal{H}_k \otimes V^{(m|n)}_{\text{matrix}}$ (where $\mathcal{H}_k \simeq V^{(m|n)}$ is the Hilbert space over the kth site, and $V^{(m|n)}_{\text{matrix}}$ is a matrix space):

$$L_k(\lambda)^{ab}_{\alpha\beta} = \Pi^{ac}_{\alpha\gamma}R(\lambda)^{cb}_{\gamma\beta} = [b(\lambda)\Pi + a(\lambda)I]^{ab}_{\alpha\beta}. \tag{3.14}$$

where the Greek (Roman) indices are the "quantum indices" ("matrix indices"). Rewriting (3.13) for the kth quantum space,

$$R(\lambda-\mu)[L_k(\lambda) \otimes L_k(\mu)] = [L_k(\mu) \otimes L_k(\lambda)]R(\lambda-\mu) \tag{3.15}$$

We shall now construct an integrable spin model based on the intertwining relation (3.15). We first define the monodromy matrix $T_L(\lambda)$ as the product (in the matrix space) of the L operators over all of the lattice sites:

$$T_L(\lambda) = L_L(\lambda)L_{L-1}(\lambda)\cdots L_1(\lambda) \tag{3.16}$$

$T_L(\lambda)$ is a quantum operator valued $(m+n) \times (m+n)$ matrix that acts nontrivially in the graded tensor product of all quantum spaces of the lattice. It also fulfills the same intertwining relation as the L operators (as can be proven by induction over the length of the lattice):

$$R(\lambda-\mu)[T_L(\lambda) \otimes T_L(\mu)] = [T_L(\mu) \otimes T_L(\lambda)]R(\lambda-\mu) \tag{3.17}$$

Taking the supertrace of the monodromy matrix, we get the transfer matrix $\tau(\lambda)$ of the spin model:

$$\tau(\lambda) = \mathrm{str}[T_L(\lambda)] = \sum_{a=1}^{m+n} (-1)^{\epsilon_a} [T_L(\lambda)]^{aa} \tag{3.18}$$

As a consequence of (3.17), transfer matrices with different spectral parameters commute. This implies that the transfer matrix is the generating functional of the Hamiltonian.

3.2.4 Trace identities

The Hamiltonian (3.3) can be obtained from the transfer matrix by taking its first logarithmic derivative at zero spectral parameter and shifting it by a constant:

$$\begin{aligned} H_{\mathrm{susy}} &= -i \left. \frac{\partial \ln[\tau(\lambda)]}{\partial \lambda} \right|_{\lambda=0} - L \\ &= -\sum_{k=1}^{L} (\Pi^{k,k+1}) \end{aligned} \tag{3.19}$$

3.2.5 Algebraic Bethe ansatz with FFB grading (Lai representation)

Let the Hilbert space at the kth site of the lattice be spanned by the three vectors $e_1 = (100)$, $e_2 = (010)$, and $e_3 = (001)$. In this section we consider a grading such that e_1 and e_2 are fermionic and e_3 is bosonic, representing the spin-down and spin-up electrons and the empty site respectively. This means that their Grassmann parities are $\epsilon_1 = \epsilon_2 = 1$ and $\epsilon_3 = 0$. We choose the reference state in the kth quantum space $|0\rangle_k$ and the reference state $|0\rangle$ of the whole lattice to be purely bosonic, i.e.,

$$|0\rangle_n = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, |0\rangle = \otimes_{n=1}^{L} |0\rangle_n \tag{3.20}$$

This choice of grading implies that $R(\mu) = b(\mu)I + a(\mu)\Pi$ can be written explicitly as:

$$R(\lambda) = \begin{pmatrix} b(\lambda)-a(\lambda) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & b(\lambda) & 0 & -a(\lambda) & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & b(\lambda) & 0 & 0 & 0 & a(\lambda) & 0 & 0 \\ 0 & -a(\lambda) & 0 & b(\lambda) & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & b(\lambda)-a(\lambda) & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & b(\lambda) & 0 & a(\lambda) & 0 \\ 0 & 0 & a(\lambda) & 0 & 0 & 0 & b(\lambda) & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & a(\lambda) & 0 & b(\lambda) & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \tag{3.21}$$

The L operator is defined by (3.14) and is of the form

$$L_n(\lambda) = \begin{pmatrix} a(\lambda) - b(\lambda)e_n^{11} & -b(\lambda)e_n^{21} & b(\lambda)e_n^{31} \\ -b(\lambda)e_n^{12} & a(\lambda) - b(\lambda)e_n^{22} & b(\lambda)e_n^{32} \\ b(\lambda)e_n^{13} & b(\lambda)e_n^{23} & a(\lambda) + b(\lambda)e_n^{33} \end{pmatrix}, \tag{3.22}$$

where $(e_n^{ab})_{\alpha\beta} = \delta_{a\alpha}\delta_{b\beta}$ are quantum operators in the nth quantum space. The monodromy matrix (3.16) can be represented as

$$\begin{aligned} T_L(\lambda) &= L_L(\lambda)L_{L-1}(\lambda)\cdots L_1(\lambda) \\ &= \begin{pmatrix} A_{11}(\lambda) & A_{12}(\lambda) & B_1(\lambda) \\ A_{21}(\lambda) & A_{22}(\lambda) & B_2(\lambda) \\ C_1(\lambda) & C_2(\lambda) & D(\lambda) \end{pmatrix}, \end{aligned} \tag{3.23}$$

which is a quantum operator valued 3×3 matrix. For clarity, we write (3.23) explicitly in component form:

$$\begin{aligned} \{[T_L(\lambda)]^{ab}\}_{\substack{\alpha_1\cdots\alpha_L \\ \beta_1\cdots\beta_L}} &= L_L(\lambda)^{ac_L}_{\alpha_L\beta_L} L_{L-1}(\lambda)^{c_L c_{L-1}}_{\alpha_{L-1}\beta_{L-1}} \cdots \\ &\quad \cdots L_1(\lambda)^{c_2 c_1}_{\alpha_1\beta_1} (-1)^{\sum_{j=2}^{L}(\epsilon_{\alpha_j}+\epsilon_{\beta_j})\sum_{i=1}^{j-1}\epsilon_{\alpha_i}} \end{aligned} \tag{3.24}$$

Note that the physical (greek) indices are subjected to the minus signs from the graded tensor product, while the matrix (latin) indices are not, as they are summed over (and not tensored). The transfer matrix is then given as

$$\tau(\mu) = \mathrm{str}[T_L(\mu)] = -A_{11}(\mu) - A_{22}(\mu) + D(\mu) \tag{3.25}$$

The action of $L_k(\lambda)$ on $|0\rangle_k$ is

$$L_k(\lambda)\,|0\rangle_k = \begin{pmatrix} a(\lambda) & 0 & 0 \\ 0 & a(\lambda) & 0 \\ b(\lambda)e_n^{13} & b(\lambda)e_n^{23} & 1 \end{pmatrix} |0\rangle_k \tag{3.26}$$

Using (3.23) and (3.26), we determine the action of the monodromy matrix on $|0\rangle$ to be

$$T_L(\lambda)\,|0\rangle = \begin{pmatrix} [a(\lambda)]^L & 0 & 0 \\ 0 & [a(\lambda)]^L & 0 \\ C_1(\lambda) & C_2(\lambda) & 1 \end{pmatrix} |0\rangle \tag{3.27}$$

We will now solve for a set of eigenstates of the transfer matrix using the Nested Algebraic Bethe Ansatz (NABA). By inspecting (3.27), $C_1(\lambda)$ and $C_2(\lambda)$ can be interpreted as creation operators (of odd Grassmann parity). We now make the following ansatz for the eigenstates of $\tau(\mu)$:

$$|\lambda_1, \cdots, \lambda_n|F\rangle = C_{a_1}(\lambda_1)C_{a_2}(\lambda_2)\cdots C_{a_n}(\lambda_n)\,|0\rangle\, F^{a_n\cdots a_1}, \tag{3.28}$$

where $a_j = 1, 2$, and $F^{a_n\cdots a_1}$ is a function of the spectral parameters λ. The action of the transfer matrix on states of the form (3.28) is determined by (3.27) and (3.17). The fundamental commutation relations from (3.17) which are relevant for the NABA are

$$\begin{aligned} A_{ab}(\mu)C_c(\lambda) &= (-1)^{\epsilon_a\epsilon_p}\frac{r(\mu-\lambda)^{dc}_{pb}}{a(\mu-\lambda)}C_p(\lambda)A_{ad}(\mu) \\ &\quad + \frac{b(\mu-\lambda)}{a(\mu-\lambda)}C_b(\mu)A_{ac}(\lambda), \\ D(\mu)C_c(\lambda) &= \frac{1}{a(\lambda-\mu)}C_c(\lambda)D(\mu) - \frac{b(\lambda-\mu)}{a(\lambda-\mu)}C_c(\mu)D(\lambda), \\ C_{a_1}(\lambda_1)C_{a_2}(\lambda_2) &= r(\lambda_1-\lambda_2)^{b_1a_2}_{b_2a_1}C_{b_2}(\lambda_2)C_{b_1}(\lambda_1), \end{aligned} \tag{3.29}$$

where

$$\begin{aligned} r(\mu)^{ab}_{cd} &= b(\mu)\delta_{ab}\delta_{cd} - a(\mu)\delta_{ad}\delta_{bc} \\ &= b(\mu)I^{ab}_{cd} + a(\mu)[\Pi^{(1)}]^{ab}_{cd} \end{aligned} \tag{3.30}$$

Here $[\Pi^{(1)}]^{ab}_{cd} = -\delta_{ad}\delta_{bc}$, is the 4×4 permutation matrix corresponding to the grading $\epsilon_1 = \epsilon_2 = 1$. Using (3.29) we find that the diagonal elements of the monodromy matrix $\tau(\mu)$ act on the states (3.28) as follows:

$$D(\mu)\,|\lambda_1,\cdots,\lambda_n|F\rangle = \prod_{j=1}^{n}\frac{1}{a(\lambda_j-\mu)}\,|\lambda_1,\cdots,\lambda_n|F\rangle + \sum_{k=1}^{n}(\tilde{\Lambda}_k)^{b_1\cdots b_n}_{a_1\cdots a_n}\,C_{b_k}(\mu)\prod_{\substack{j=1\\ j\neq k}}^{n}C_{b_j}(\lambda_j)\,|0\rangle\,F^{a_n\cdots a_1}, \quad (3.31)$$

$$\begin{aligned}&[A_{11}(\mu)+A_{22}(\mu)]\,|\lambda_1,\cdots,\lambda_n|F\rangle\\ &\quad= -[a(\mu)]^L\prod_{j=1}^{n}\frac{1}{a(\mu-\lambda_j)}\prod_{l=1}^{n}C_{b_l}(\lambda_l)\,|0\rangle\,\tau^{(1)}(\mu)^{b_1\cdots b_n}_{a_1\cdots a_n}F^{a_n\cdots a_1}\\ &\quad\quad+\sum_{k=1}^{n}(\Lambda_k)^{b_1\cdots b_n}_{a_1\cdots a_n}\,C_{b_k}(\mu)\prod_{\substack{j=1\\ j\neq k}}^{n}C_{b_j}(\lambda_j)\,|0\rangle\,F^{a_n\cdots a_1},\end{aligned} \quad (3.32)$$

where we define:

$$\begin{aligned}L^{(1)}_k &= b(\lambda)\Pi^{(1)} + a(\lambda)I^{(1)}\\ &= \Pi^{(1)}r(\lambda)\\ &= \begin{pmatrix} a(\lambda)-b(\lambda)e^{11}_k & -b(\lambda)e^{21}_k\\ -b(\lambda)e^{12}_k & a(\lambda)-b(\lambda)e^{22}_k\end{pmatrix} \quad (3.33)\\ T^{(1)}_n(\mu) &= L^{(1)}_n(\mu-\lambda_n)L^{(1)}_{n-1}(\mu-\lambda_{n-1})\\ &\quad\cdots L^{(1)}_2(\mu-\lambda_2)L^{(1)}_1(\mu-\lambda_1) \quad (3.34)\\ &= \begin{pmatrix} A^{(1)}(\mu) & B^{(1)}(\mu)\\ C^{(1)}(\mu) & D^{(1)}(\mu)\end{pmatrix}, \quad (3.35)\\ \tau^{(1)}(\mu) &= \mathrm{str}[T^{(1)}_n(\mu)]\\ &= -A^{(1)}(\mu) - D^{(1)}(\mu), \quad (3.36)\end{aligned}$$

$r(\mu)$ satisfies a (graded) Yang-Baxter equation:

$$r(\lambda-\mu)^{a_2c_2}_{a_3c_3}r(\lambda)^{a_1b_1}_{c_2d_2}r(\mu)^{d_2b_2}_{c_3b_3} = r(\mu)^{a_1c_1}_{a_2c_2}r(\lambda)^{c_2d_2}_{a_3b_3}r(\lambda-\mu)^{c_1b_1}_{d_2b_2}. \quad (3.37)$$

$L^{(1)}$ and $r(\mu)$ can be interpreted as the L operator and R matrix of a fundamental spin model describing two species of fermions. $T_n^{(1)}(\mu)$ and $\tau^{(1)}(\mu)$ are the monodromy matrix and transfer matrix of the corresponding inhomogeneous model. Inspection of (3.31) and (3.32) together with (3.25) shows that the eigenvalue condition

$$\tau(\mu)\,|\lambda_1,\cdots,\lambda_n|F\rangle = \nu(\mu,\{\lambda_j\},F)\,|\lambda_1,\cdots,\lambda_n|F\rangle \tag{3.38}$$

leads to the requirements that F ought to be an eigenvector of the "nested" transfer matrix $\tau^{(1)}(\mu)$, and that the "unwanted terms" cancel, i.e.,

$$[-(\Lambda_k)^{b_1\cdots b_n}_{a_1\cdots a_n} + (\tilde{\Lambda}_k)^{b_1\cdots b_n}_{a_1\cdots a_n}]F^{a_n\cdots a_1} = 0 \tag{3.39}$$

The relative sign in (3.39) is due to the supertrace in (3.25) and (3.38). The explicit expressions of Λ_k and $\tilde{\Lambda}_k$ can be computed and upon substitution into (3.39), we obtain the following conditions on the spectral parameters λ, and coefficients F, which are necessary for (3.38) to hold:

$$\begin{aligned}[a(\lambda_k)]^{-L}\prod_{\substack{l=1\\ l\neq k}}^{n}\frac{a(\lambda_k-\lambda_l)}{a(\lambda_l-\lambda_k)}F^{b_n\cdots b_1}&\\ = \tau^{(1)}(\lambda_k)^{b_1\cdots b_n}_{a_1\cdots a_n}F^{a_n\cdots a_1},\quad k=1,\cdots,n&\end{aligned} \tag{3.40}$$

The first step of the NABA is completed, and we now solve the nesting. The condition that F is an eigenvector of $\tau^{(1)}(\mu)$ requires $\tau^{(1)}(\mu)$ to be diagonalized, which can be achieved by a second, "nested" Bethe ansatz. From (3.37), (3.33) and (3.34), the following intertwining relation can be derived:

$$r(\lambda-\mu)[T_L^{(1)}(\lambda)\otimes T_L^{(1)}(\mu)] = [T_L^{(1)}(\mu)\otimes T_L^{(1)}(\lambda)]r(\lambda-\mu) \tag{3.41}$$

Using (3.35), (3.41) and (3.30), we can obtain the nested fundamental commutation relations:

$$\begin{aligned}
D^{(1)}(\mu)C^{(1)}(\lambda) &= \frac{1}{a(\mu-\lambda)}C^{(1)}(\lambda)D^{(1)}(\mu)\\
&\quad - \frac{b(\lambda-\mu)}{a(\lambda-\mu)}C^{(1)}(\mu)D^{(1)}(\lambda),\\
A^{(1)}(\mu)C^{(1)}(\lambda) &= \frac{1}{a(\lambda-\mu)}C^{(1)}(\lambda)A^{(1)}(\mu)\\
&\quad + \frac{b(\mu-\lambda)}{a(\mu-\lambda)}C^{(1)}(\mu)A^{(1)}(\lambda),\\
C^{(1)}(\lambda)C^{(1)}(\mu) &= C^{(1)}(\mu)C^{(1)}(\lambda).
\end{aligned} \tag{3.42}$$

For the nested reference states, we choose:

$$|0\rangle_k^{(1)} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, |0\rangle = \otimes_{k=1}^{n} |0\rangle_k^{(1)} \tag{3.43}$$

The action of the nested monodromy matrix $T_n^{(1)}(\mu)$ on the $|0\rangle^{(1)}$ is determined by (3.33) and we find

$$\begin{aligned} A^{(1)}(\mu)\,|0\rangle^{(1)} &= \prod_{j=1}^{n} a(\mu-\lambda_j)\,|0\rangle^{(1)} \\ D^{(1)}(\mu)\,|0\rangle^{(1)} &= \prod_{j=1}^{n} [a(\mu-\lambda_j) - b(\mu-\lambda_j)]\,|0\rangle^{(1)} \\ &= \prod_{j=1}^{n} \frac{a(\mu-\lambda_j)}{a(\lambda_j-\mu)}\,|0\rangle^{(1)}\,. \end{aligned} \tag{3.44}$$

We now make the following ansatz for the eigenstates of $\tau^{(1)}(\mu)$

$$\left|\lambda_1^{(1)},\cdots,\lambda_{n_1}^{(1)}\right\rangle = C^{(1)}(\lambda_1^{(1)})C^{(1)}(\lambda_2^{(1)})\cdots C^{(1)}(\lambda_{n_1}^{(1)})\,|0\rangle^{(1)}\,, \tag{3.45}$$

In component form, this state can be written as $\left|\lambda_1^{(1)},\cdots,\lambda_{n_1}^{(1)}\right\rangle_{a_n\cdots a_1}$, which is directly identifiable with $F^{a_n\cdots a_1}$.
The action of $\tau^{(1)}(\mu)$ on the states (3.45) can be evaluated with the help of the nested fundamental commutation relations (3.42):

$$\begin{aligned} D^{(1)}(\mu)\left|\lambda_1^{(1)},\cdots,\lambda_{n_1}^{(1)}\right\rangle &= \prod_{j=1}^{n_1} \frac{1}{a(\mu-\lambda_j^{(1)})} \prod_{l=1}^{n} \frac{a(\mu-\lambda_l)}{a(\lambda_l-\mu)} \left|\lambda_1^{(1)},\cdots,\lambda_{n_1}^{(1)}\right\rangle \\ &\quad + \sum_{k=1}^{n_1} \tilde{\Lambda}_k^{(1)} C^{(1)}(\mu) \prod_{\substack{j=1 \\ j\neq k}}^{n} C^{(1)}(\lambda_j^{(1)})\,|0\rangle^{(1)}\,, \end{aligned} \tag{3.46}$$

$$\begin{aligned} A^{(1)}(\mu)\left|\lambda_1^{(1)},\cdots,\lambda_{n_1}^{(1)}\right\rangle &= \prod_{j=1}^{n_1} \frac{1}{a(\lambda_j^{(1)}-\mu)} \prod_{l=1}^{n} a(\mu-\lambda_l) \left|\lambda_1^{(1)},\cdots,\lambda_{n_1}^{(1)}\right\rangle \\ &\quad + \sum_{k=1}^{n_1} \Lambda_k^{(1)} C^{(1)}(\mu) \prod_{\substack{j=1 \\ j\neq k}}^{n} C^{(1)}(\lambda_j^{(1)})\,|0\rangle^{(1)}\,. \end{aligned} \tag{3.47}$$

From (3.47) and (3.46) one can read off the eigenvalues of $\tau^{(1)}(\mu)$:

$$\tau^{(1)}(\mu)\left|\lambda_1^{(1)},\cdots,\lambda_{n_1}^{(1)}\right\rangle = -\left[\prod_{j=1}^{n_1}\frac{1}{a(\mu-\lambda_j^{(1)})}\prod_{l=1}^{n}\frac{a(\mu-\lambda_l)}{a(\lambda_l-\mu)} + \prod_{j=1}^{n_1}\frac{1}{a(\lambda_j^{(1)}-\mu)}\prod_{l=1}^{n}a(\mu-\lambda_l)\right]\left|\lambda_1^{(1)},\cdots,\lambda_{n_1}^{(1)}\right\rangle. \tag{3.48}$$

Substituting (3.48) into (3.40) at $\mu=\lambda_k$, we obtain the first of Bethe ansatz equations

$$[a(\lambda_k)]^L = \prod_{i=1}^{n_1} a(\lambda_k-\lambda_i^{(1)}), k=1,\cdots,n. \tag{3.49}$$

The explicit expressions of Λ_k and $\tilde{\Lambda}_k$ can be computed and their cancellation [to ensure that the states (3.45) are eigenstates of the transfer matrix $\tau^{(1)}(\mu)$] leads to the following set of Bethe ansatz equations for the nesting:

$$\prod_{i=1}^{n} a(\lambda_i-\lambda_p^{(1)}) = \prod_{\substack{j=1\\ j\neq p}}^{n_1}\frac{a(\lambda_j^{(1)}-\lambda_p^{(1)})}{a(\lambda_p^{(1)}-\lambda_j^{(1)})},\ p=1,\cdots,n_1. \tag{3.50}$$

Due to our choice of grading, we find that $n=N_e=N_\uparrow+N_\downarrow$ and $n_1=N_\downarrow$. If we define the shifted spectral parameters $\tilde{\lambda}_k=\lambda_k+i/2$, we can rewrite the Bethe ansatz equations in their "generic" form:

$$\begin{aligned}
&\left[\frac{\tilde{\lambda}_k-i/2}{\tilde{\lambda}_k+i/2}\right]^L = \prod_{j=1}^{N_\downarrow}\frac{\tilde{\lambda}_k-\lambda_j^{(1)}-i/2}{\tilde{\lambda}_k-\lambda_j^{(1)}+i/2},\ k=1,\cdots,N_e\\
&\prod_{k=1}^{N_e}\frac{\tilde{\lambda}_k-\lambda_p^{(1)}-i/2}{\tilde{\lambda}_k-\lambda_p^{(1)}+i/2} = \prod_{\substack{j=1\\ j\neq p}}^{N_\downarrow}\frac{\lambda_j^{(1)}-\lambda_p^{(1)}-i}{\lambda_j^{(1)}-\lambda_p^{(1)}+i},\ p=1,\cdots,n_1
\end{aligned} \tag{3.51}$$

The eigenvalues of the transfer matrix (3.25) are given by

$$\nu(\mu,\{\lambda_j\},F) = [a(\mu)]^L \prod_{j=1}^{N_e} \frac{1}{a(\mu-\lambda_j)} \nu^{(1)}(\mu) + \prod_{j=1}^{N_e} \frac{1}{a(\lambda_j-\mu)}$$
$$\nu^{(1)}(\mu) = -\left(\prod_{i=1}^{N_\downarrow} \frac{1}{a(\mu-\lambda_i^{(1)})} \prod_{j=1}^{N_e} \frac{a(\mu-\lambda_j)}{a(\lambda_j-\mu)} + \prod_{i=1}^{N_h} \frac{1}{a(\lambda_i^{(1)}-\mu)} \prod_{j=1}^{N_e} a(\mu-\lambda_j) \right). \tag{3.52}$$

Using the trace identities (3.19), we can obtain the energy eigenvalues from the eigenvalues of the transfer matrix:

$$\begin{aligned} E_{\text{susy}} &= \sum_{j=1}^{N_e} \frac{1}{\tilde{\lambda}_j^2 + 1/4} - L \\ &= -2\sum_{j=1}^{N_e} \cos(k_j) + 2N_e - L, \end{aligned} \tag{3.53}$$

where we have reparameterized $\tilde{\lambda}_j = \frac{1}{2}\cot(k_j/2)$. The Bethe ansatz equations (3.51) and the energy (3.53) were also derived by Schlottmann[95] and Lai[64] independently.

3.2.6 Algebraic Bethe ansatz with BFF grading (Sutherland representation)

In this section we consider a grading such that e_2 and e_3 are fermionic and e_1 is bosonic, representing the spin-down and spin-up electrons and the empty site respectively. This means that their Grassmann parities are $\epsilon_2 = \epsilon_3 = 1$ (fermionic) and $\epsilon_1 = 0$ (bosonic). We choose the reference state in the kth quantum space $|0\rangle_k$ and the reference state $|0\rangle$ of the whole lattice to be fermionic with all spins up, i.e.,

$$|0\rangle_n = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, |0\rangle = \otimes_{n=1}^{L} |0\rangle_n \tag{3.54}$$

This choice of grading implies that R can be written as

$$R(\lambda) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & b(\lambda) & 0 & a(\lambda) & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & b(\lambda) & 0 & 0 & 0 & a(\lambda) & 0 & 0 \\ 0 & -a(\lambda) & 0 & b(\lambda) & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & b(\lambda)a(\lambda) & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & b(\lambda) & 0 & -a(\lambda) & 0 \\ 0 & 0 & a(\lambda) & 0 & 0 & 0 & b(\lambda) & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -a(\lambda) & 0 & b(\lambda) & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & b(\lambda) - a(\lambda) \end{pmatrix} \tag{3.55}$$

The L operator is

$$L_n(\lambda) = \begin{pmatrix} a(\lambda) + b(\lambda)e_n^{11} & b(\lambda)e_n^{21} & b(\lambda)e_n^{31} \\ b(\lambda)e_n^{12} & a(\lambda) - b(\lambda)e_n^{22} & -b(\lambda)e_n^{32} \\ b(\lambda)e_n^{13} & -b(\lambda)e_n^{23} & a(\lambda) - b(\lambda)e_n^{33} \end{pmatrix}, \tag{3.56}$$

The action of $L_k(\lambda)$ on $|0\rangle_k$ is

$$L_k(\lambda)\,|0\rangle_k = \begin{pmatrix} a(\lambda) & 0 & 0 \\ 0 & a(\lambda) & 0 \\ b(\lambda)e_n^{13} & -b(\lambda)e_n^{23} & a(\lambda) - b(\lambda) \end{pmatrix} |0\rangle_k \tag{3.57}$$

The monodromy matrix is partitioned as before in (3.23), which now gives the transfer matrix

$$\tau(\mu) = A_{11}(\mu) - A_{22}(\mu) - D(\mu) \tag{3.58}$$

The action of the monodromy matrix on $|0\rangle$ follows from (3.57):

$$T_L(\lambda)\,|0\rangle = \begin{pmatrix} [a(\lambda)]^L & 0 & 0 \\ 0 & [a(\lambda)]^L & 0 \\ C_1(\lambda) & C_2(\lambda) & [a(\lambda) - b(\lambda)]^L \end{pmatrix} |0\rangle \tag{3.59}$$

and by inspecting (3.59), $C_1(\lambda)$ and $C_2(\lambda)$ are found to be creation operators of odd and even Grasssmann parity respectively. We make the following ansatz for the eigenstates of $\tau(\mu)$:

$$|\lambda_1, \cdots, \lambda_n|F\rangle = C_{a_1}(\lambda_1)C_{a_2}(\lambda_2)\cdots C_{a_n}(\lambda_n)\,|0\rangle\,F^{a_n\cdots a_1}, \tag{3.60}$$

The fundamental commutation relations are found to be

$$\begin{aligned} A_{ab}(\mu)C_c(\lambda) &= (-1)^{\epsilon_a\epsilon_p+\epsilon_a+\epsilon_b}\frac{r(\mu-\lambda)^{dc}_{pb}}{a(\mu-\lambda)}C_p(\lambda)A_{ad}(\mu) \\ &\quad + (-1)^{(\epsilon_a+1)(\epsilon_b+1)}\frac{b(\mu-\lambda)}{a(\mu-\lambda)}C_b(\mu)A_{ac}(\lambda), \\ D(\mu)C_c(\lambda) &= \frac{1}{a(\lambda-\mu)}C_c(\lambda)D(\mu) - \frac{b(\lambda-\mu)}{a(\lambda-\mu)}C_c(\mu)D(\lambda), \\ C_{a_1}(\lambda_1)C_{a_2}(\lambda_2) &= r_{FB}(\lambda_1-\lambda_2)^{a_2b_1}_{a_1b_2}C_{b_2}(\lambda_2)C_{b_1}(\lambda_1), \end{aligned} \tag{3.61}$$

where

$$r(\mu)^{ab}_{cd} = b(\mu)I^{ab}_{cd} + a(\mu)(\Pi_{BF})^{ab}_{cd}, \quad r_{FB}(\mu)^{ab}_{cd} = b(\mu)I^{ab}_{cd} + a(\mu)(\Pi_{FB})^{ab}_{cd}, \tag{3.62}$$

and Π_{BF} and Π_{FB} are the permutation matrices for the gradings $\epsilon_1 = 0$, $\epsilon_2 = 1$ and $\epsilon_1 = 1, \epsilon_2 = 0$, respectively. Using (3.61) we find that the diagonal elements of the monodromy matrix act on the states (3.60) as follows:

$$\begin{aligned} D(\mu)\,|\lambda_1,\cdots,\lambda_n|F\rangle &= \prod_{j=1}^{n}\frac{1}{a(\lambda_j-\mu)}\left(\frac{a(\mu)}{a(-\mu)}\right)^L|\lambda_1,\cdots,\lambda_n|F\rangle \\ &\quad + \sum_{k=1}^{n}(\tilde{\Lambda}_k)^{b_1\cdots b_n}_{a_1\cdots a_n}C_{b_k}(\mu)\prod_{\substack{j=1\\ j\neq k}}^{n}C_{b_j}(\lambda_j)\,|0\rangle\,F^{a_n\cdots a_1}, \end{aligned} \tag{3.63}$$

$$\begin{aligned} &[A_{11}(\mu)-A_{22}(\mu)]\,|\lambda_1,\cdots,\lambda_n|F\rangle \\ &\quad = [a(\mu)]^L\prod_{j=1}^{n}\frac{1}{a(\mu-\lambda_j)}\prod_{l=1}^{n}C_{b_l}(\lambda_l)\,|0\rangle\,\tau^{(1)}(\mu)^{b_1\cdots b_n}_{a_1\cdots a_n}F^{a_n\cdots a_1} \\ &\qquad + \sum_{k=1}^{n}(\Lambda_k)^{b_1\cdots b_n}_{a_1\cdots a_n}C_{b_k}(\mu)\prod_{\substack{j=1\\ j\neq k}}^{n}C_{b_j}(\lambda_j)\,|0\rangle\,F^{a_n\cdots a_1}, \end{aligned} \tag{3.64}$$

where

$$\begin{aligned} \tau^{(1)}(\mu)^{b_1\cdots b_n}_{a_1\cdots a_n} &= (-1)^{\epsilon_c}L^{(1)}_n(\mu-\lambda_n)^{cc_{n-1}}_{b_na_n}L^{(1)}_{n-1}(\mu-\lambda_{n-1})^{c_{n-1}c_{n-2}}_{b_{n-1}a_{n-1}} \\ &\quad \cdots L^{(1)}_1(\mu-\lambda_1)^{c_1c}_{b_1a_1}(-1)^{\epsilon_c\sum_{i=1}^{n-1}(\epsilon_{b_i}+1)\sum_{i=1}^{n-1}\epsilon_{c_i}(\epsilon_{b_i}+1)}, \end{aligned} \tag{3.65}$$

Here all the indices c_i and c are summed over. $\tau^{(1)}(\mu)$ is the transfer matrix of an inhomogeneous spin model of a boson and fermion on a lattice of n sites. Our reference state $|0\rangle$ is now of fermionic nature and we have to define a graded tensor product reflecting this fact:

$$(F\overline{\otimes}G)^{ab}_{cd} = F_{ab}G_{cd}(-1)^{(\epsilon_c+1)(\epsilon_a+\epsilon_b)} \tag{3.66}$$

In terms of this tensor product, the transfer matrix $\tau^{(1)}(\mu)$ given by (3.72) can be obtained as

$$\begin{aligned}\tau^{(1)}(\mu)^{b_1\cdots b_n}_{a_1\cdots a_n} &= \mathrm{str}[T^{(1)}_n(\mu)] \\ &= \mathrm{str}[L^{(1)}_n(\mu-\lambda_n)\overline{\otimes}L^{(1)}_{n-1}(\mu-\lambda_{n-1})\overline{\otimes}\cdots\overline{\otimes}L^{(1)}_1(\mu-\lambda_1)],\end{aligned} \tag{3.67}$$

$$\begin{aligned}L^{(1)}_k &= b(\lambda)\Pi^{(1)}_{BF} + a(\lambda)I^{(1)} \\ &= \begin{pmatrix} a(\lambda)+b(\lambda)e^{11}_k & b(\lambda)e^{21}_k \\ b(\lambda)e^{12}_k & a(\lambda)-b(\lambda)e^{22}_k \end{pmatrix}\end{aligned} \tag{3.68}$$

In (3.67) we have explicitly written the tensor product $\overline{\otimes}$ between the quantum spaces over the sites of the inhomogeneous model (and the L operators are multiplied within the matrix space). As before, $F^{a_n\cdots a_1}$ must be an eigenvector of $\tau^{(1)}(\mu)$ if $|\lambda_1,\cdots,\lambda_n|F\rangle$ is to be an eigenstate of $\tau(\mu)$. The unwanted terms can be computed in a similar way to the ones described for the FFB grading. The condition of the cancellation of the unwanted terms,

$$[(\Lambda_k)^{b_1\cdots b_n}_{a_1\cdots a_n} - (\tilde{\Lambda}_k)^{b_1\cdots b_n}_{a_1\cdots a_n}]F^{a_n\cdots a_1} = 0, \tag{3.69}$$

leads to the conditions

$$F^{a_n\cdots a_1} = [a(-\lambda_k)]^L[\tau^{(1)}(\lambda_k)F]^{a_n\cdots a_1},\ k=1,\cdots,n. \tag{3.70}$$

To solve the nesting we first have to note that, due to our change of tensor product, the nested L operators $L^{(1)}(\lambda)$ are now interwined by the R matrix

$$\widehat{r}(\mu)^{ab}_{cd} = b(\mu)\delta_{ab}\delta_{cd} + a(\mu)\delta_{ad}\delta_{bc}(-1)^{\epsilon_a+\epsilon_c+\epsilon_a\epsilon_c}. \tag{3.71}$$

The intertwining relation

$$\widehat{r}(\lambda-\mu)[T^{(1)}_L(\lambda)\overline{\otimes}T^{(1)}_L(\mu)] = [T^{(1)}_L(\mu)\overline{\otimes}T^{(1)}_L(\lambda)]\widehat{r}(\lambda-\mu) \tag{3.72}$$

together with the choice of reference state,

$$|0\rangle_k^{(1)} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, |0\rangle = \overline{\otimes}_{k=1}^{n} |0\rangle_k^{(1)} \tag{3.73}$$

can be analyzed similar to what was done in previous section. It can be shown that they represent a model of the permutation type with BF grading. The resulting Bethe ansatz equations are

$$[a(-\lambda_l)]^L = \prod_{\substack{m=1 \\ m\neq l}}^{n} \frac{a(\lambda_m - \lambda_l)}{a(\lambda_l - \lambda_m)} \prod_{j=1}^{n_1} a(\lambda_k - \lambda_i^{(1)}), \quad l = 1, \cdots, n. \tag{3.74}$$

$$1 = \prod_{j=1}^{n} a(\lambda_j - \lambda_k^{(1)}), \ k = 1, \cdots, n_1. \tag{3.75}$$

Due to our choice of grading, we find that $n = N_h + N_\downarrow$ and $n_1 = N_h$ respectively, where $N_h = N - N_e$ is the number of holes. If we define the shifted spectral parameters

$$\tilde{\lambda}_j = \lambda_j - i/2, \ \tilde{\lambda}_j^{(1)} = \lambda_j^{(1)} - i, \tag{3.76}$$

we obtain Sutherland's [109] form of the periodic boundary conditions:

$$\begin{aligned} \left[\frac{\tilde{\lambda}_k - i/2}{\tilde{\lambda}_k + i/2}\right]^L &= \prod_{\substack{m=1 \\ m\neq l}}^{N_h+N_\downarrow} \frac{\tilde{\lambda}_l - \tilde{\lambda}_m - i}{\tilde{\lambda}_l - \tilde{\lambda}_m + i} \prod_{j=1}^{N_h} \frac{\tilde{\lambda}_l - \tilde{\lambda}_j^{(1)} - i/2}{\tilde{\lambda}_l - \tilde{\lambda}_j^{(1)} + i/2}, \quad l = 1, \cdots, N_h + N_\downarrow, \\ 1 &= \prod_{k=1}^{N_h+N_\downarrow} \frac{\tilde{\lambda}_j - \tilde{\lambda}_k^{(1)} - i/2}{\tilde{\lambda}_j - \tilde{\lambda}_k^{(1)} + i/2}, \quad k = 1, \cdots, N_h \end{aligned} \tag{3.77}$$

The eigenvalues of the transfer matrix are

$$\begin{aligned} \nu(\mu, \{\lambda_j\}, F) &= [a(\mu)]^L \prod_{j=1}^{N_h+N_\downarrow} \frac{1}{a(\mu - \lambda_j)} \nu^{(1)}(\mu) - \prod_{j=1}^{N_h+N_\downarrow} \frac{1}{a(\mu - \lambda_j)} \left(\frac{a(\mu)}{a(-\mu)}\right)^L \\ \nu^{(1)}(\mu) &= \prod_{l=1}^{N_h} \frac{1}{a(\mu - \lambda_j^{(1)})} \left(\prod_{j=1}^{N_h+N_\downarrow} a(\mu - \lambda_j) - \prod_{j=1}^{N_h+N_\downarrow} \frac{a(\mu - \lambda_j)}{a(\lambda_j - \mu)} \right) \end{aligned} \tag{3.78}$$

Using the trace identities (3.19), we can obtain the energy eigenvalues as:

$$\begin{aligned} E_{\text{susy}} &= L - \sum_{j=1}^{N_h+N_\downarrow} \frac{1}{\tilde{\lambda}_j^2 + 1/4} \\ &= L - 2(N_h + N_\downarrow) - 2\sum_{j=1}^{N_e} \cos(k_j), \end{aligned} \tag{3.79}$$

where we have reparameterized $\tilde{\lambda}_j = \frac{1}{2}\tan(k_j/2)$

3.3 Tensor network description of the Bethe ansatz

3.3.1 Tensor network form

We now represent the above NABA in tensor network form. If we leave the considerations for grading aside, the (abstract) form of the tensor network is the same for both Lai and Sutherland representation (only actual mathematical representation differs). We proceed below to consider the general form of the tensor network for both representations without considering the grading first, after which we then consider the grading in detail in Sec. 3.4.

We represent each L operator $L(\lambda)^{ab}_{\alpha\beta}$ (a tensor with four indices) as shown in Fig. 3.1a. We construct the transfer matrix $T_L(\lambda) = L_L(\lambda)L_{L-1}(\lambda)\cdots L_1(\lambda)$ as shown in Fig. 3.1b.

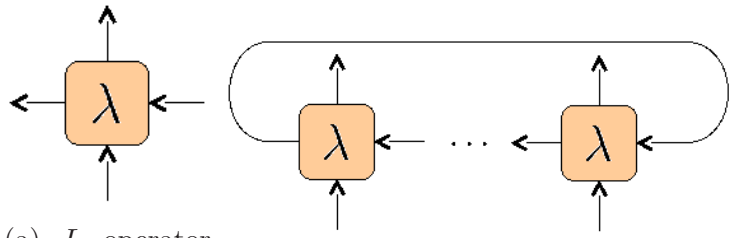
For the first level Bethe ansatz, the set of creation operators $\{C_1, C_2\}$ in (3.23) is constructed by terminating the ends of the transfer matrix by boundary vectors/matrices as shown in Fig. 3.2. The boundary row vector (001) on the left selects the third row of the transfer matrix $T(\lambda)$. The matrix K, which selects the first and second column of $T(\lambda)$, is defined as:

$$K = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \tag{3.80}$$

We call the matrix K the connector for it will be the bridge between the first level and nested Bethe ansatz.

For the nested Bethe ansatz, the creation operator $C^{(1)}(\lambda)$ in (3.35) is constructed by terminating the ends of the transfer matrix by boundary

vectors (0 1) on the left and $(1\ 0)^\intercal$ on the right (selecting the second row and first column respectively) as shown in Fig. 3.3.



(a) L operator $L(\lambda)$ (b) Monodromy matrix $T(\lambda)$

Figure 3.1: Tensor network representation of $L(\lambda)$ and $T(\lambda)$

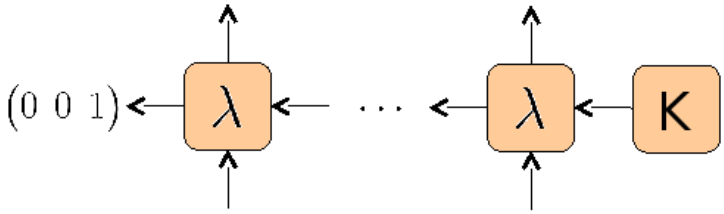


Figure 3.2: Creation operators $\{C_1(\lambda), C_2(\lambda)\}$

Now, we can construct the general tensor network form of the algebraic Bethe ansatz for both representations, as shown in Fig. 3.4, where we define:

$$\omega_{ab}^{(1)} = \lambda_a^{(1)} - \lambda_b \tag{3.81}$$

$$\{n, n_1\} = \begin{cases} \{N_e, N_\downarrow\}, & \text{Lai representation} \\ \{N_h + N_\downarrow, N_h\}, & \text{Sutherland representation} \end{cases} \tag{3.82}$$

The tensor network is split into two main parts: the first level Bethe ansatz and the nested Bethe ansatz. The first level and the nested level are connected by contracting the indices $a_1, \cdots, a_n$ of C_{a_i} of the creation operators in the first level with the wavefunction of the nested level, as shown in equation (3.28). The matrix K in Fig. 3.4 (as defined in (3.80)) selects the two first level creation operators $\{C_1, C_2\}$ and connects them to the corresponding index of the wavefunction in the nested Bethe ansatz.

The bond dimension of each bond in the tensor network for the first level Bethe ansatz is 3, while that for the nested level is 2.

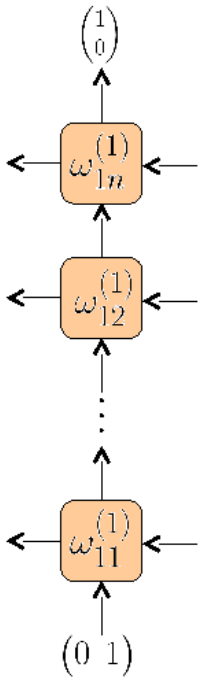


Figure 3.3: Nested creation operator $C^{(1)}(\lambda)$

3.4 Grading in terms of tensor networks

In this section, we explicitly consider the grading for both representations in detail. The tensor product is graded by assigning Grassmann parities to the basis vectors, which represents the fermionic nature of the t-J model. This introduces minus signs which are shown explicitly in (3.24) and (3.65). These minus signs are non-local at first glance, as the exponent of the minus sign of each element in the monodromy matrix depends on the parities of the indices to its right. However, in order to perform the approximate contraction of the tensor network (described in Sec. 3.5) in a sequential manner, we have to localize these minus signs. The graded Bethe ansatz can be mapped to a graded tensor network, which can be further mapped to a non-graded tensor network in which the virtual bond dimension is doubled to localize the minus signs. We describe two ways to perform the mapping in the following.

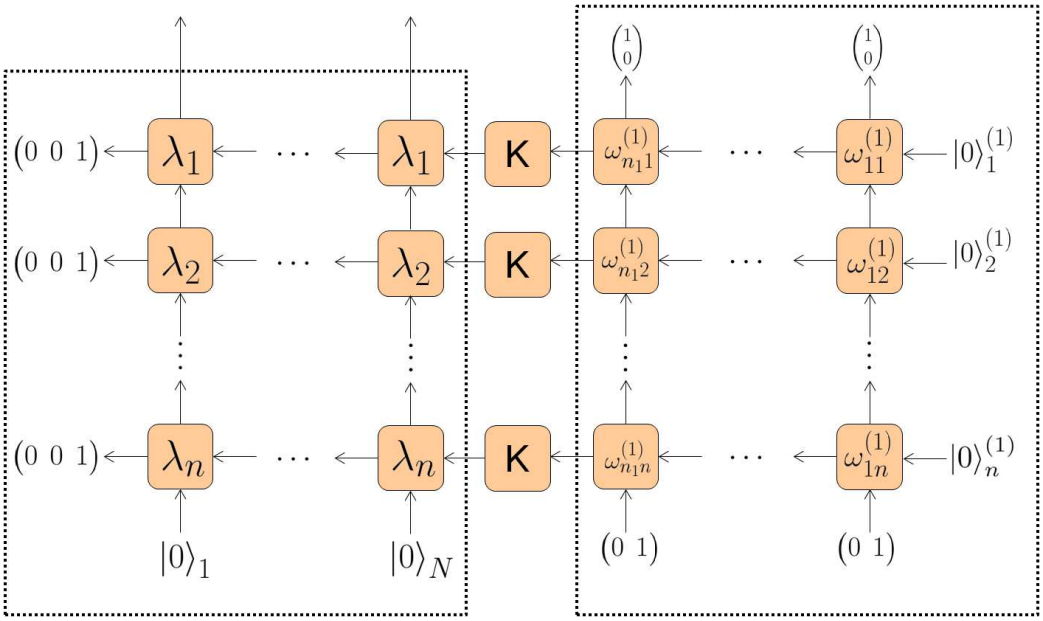


Figure 3.4: Tensor Network representation

3.4.1 Method 1

In this method we shall write the monodromy matrices in the form of a recurrence relation such that the minus signs are included locally in the L operators. Using such a representation in the form of matrices allows us to contract the tensor network efficiently, especially in languages like Matlab which matrix computations are designed for speed.

Lai representation

In Lai representation, the graded tensor products in the first level Bethe ansatz produce non-local minus signs as shown in equation (3.24). However, since the nested Bethe ansatz consist of a system of two fermions (in which the minus signs cancel), the graded tensor products do not produce any explicit (non-local) minus signs.

We introduce the following notation:

$$\varepsilon_k = \epsilon_{\alpha_k} + \epsilon_{\beta_k} \tag{3.83}$$

$$L_k(\lambda)^{ab}_{\alpha_k\beta_k}\big|_{\varepsilon_k=y} = L_k(\lambda)^{ab}_{\alpha_k\beta_k}\delta_{\varepsilon_k,y}, \qquad y = 0,1 \tag{3.84}$$

The delta function picks out only the quantum operators of the desired Grassmann parity ($\varepsilon_k = 0$ or 1). In Lai representation, the fermionic ($\varepsilon_k = 1$) operators are C_a and B_b in (3.23) ($a, b = 1, 2$), and the rest are bosonic ($\varepsilon_k = 0$). The original L operator is simply expressed by $L_k(\lambda) = L_k(\lambda)|_{\varepsilon_k=0} + L_k(\lambda)|_{\varepsilon_k=1}$. We define the following primed L operator and monodromy matrix:

$$L'_k(\lambda)^{ab}_{\alpha\beta} = L_k(\lambda)^{ab}_{\alpha\beta}(-1)^{\epsilon_\alpha} \tag{3.85}$$

$$\{[T'_L(\lambda)]^{ab}\}^{\alpha_1\cdots\alpha_L}_{\beta_1\cdots\beta_L} = L'_L(\lambda)^{ac_L}_{\alpha_L\beta_L} L'_{L-1}(\lambda)^{c_L c_{L-1}}_{\alpha_{L-1}\beta_{L-1}} \cdots L'_1(\lambda)^{c_2 c_1}_{\alpha_1\beta_1} (-1)^{\sum_{j=2}^{L}(\epsilon_{\alpha_j}+\epsilon_{\beta_j})\sum_{i=1}^{j-1}\epsilon_{\alpha_i}} \tag{3.86}$$

Now, we can write (3.24) in the form of a recurrence relation that allows the minus signs to be localized:

$$\begin{pmatrix} \{[T_{k+1}(\lambda)]^{ab}\}^{\alpha_1\cdots\alpha_{k+1}}_{\beta_1\cdots\beta_{k+1}} \\ \{[T'_{k+1}(\lambda)]^{ab}\}^{\alpha_1\cdots\alpha_{k+1}}_{\beta_1\cdots\beta_{k+1}} \end{pmatrix} = \begin{pmatrix} L_{k+1}(\lambda)^{ac_{k+1}}_{\alpha_{k+1}\beta_{k+1}}\Big|_{\varepsilon_{k+1}=0} & L_{k+1}(\lambda)^{ac_{k+1}}_{\alpha_{k+1}\beta_{k+1}}\Big|_{\varepsilon_{k+1}=1} \\ L'_{k+1}(\lambda)^{ac_{k+1}}_{\alpha_{k+1}\beta_{k+1}}\Big|_{\varepsilon_{k+1}=1} & L'_{k+1}(\lambda)^{ac_{k+1}}_{\alpha_{k+1}\beta_{k+1}}\Big|_{\varepsilon_{k+1}=0} \end{pmatrix} \times \begin{pmatrix} \{[T_k(\lambda)]^{c_{k+1}b}\}^{\alpha_1\cdots\alpha_k}_{\beta_1\cdots\beta_k} \\ \{[T'_k(\lambda)]^{c_{k+1}b}\}^{\alpha_1\cdots\alpha_k}_{\beta_1\cdots\beta_k} \end{pmatrix} \tag{3.87}$$

The minus signs are absorbed locally into the definition of $L'_k(\lambda)$. The L operators are now embedded in a larger matrix space, which we call the external matrix space. To use this construction to handle the grading, we would have to alter our tensor network so to include the external matrix space.

K' is defined as:

$$K' = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \tag{3.88}$$

The boundary vectors on the left of Fig. 3.5 and K' live in the space $V^{(0|2)} \otimes V^{(1|2)}$, where the first space $V^{(0|2)}$ is the external matrix space and the second space $V^{(1|2)}$ is the matrix space.

Sutherland representation

For Sutherland representation, the graded tensor products in both the first level and nested Bethe ansatz produce minus signs. The minus signs produced by the tensor product in the first level Bethe ansatz is exactly the

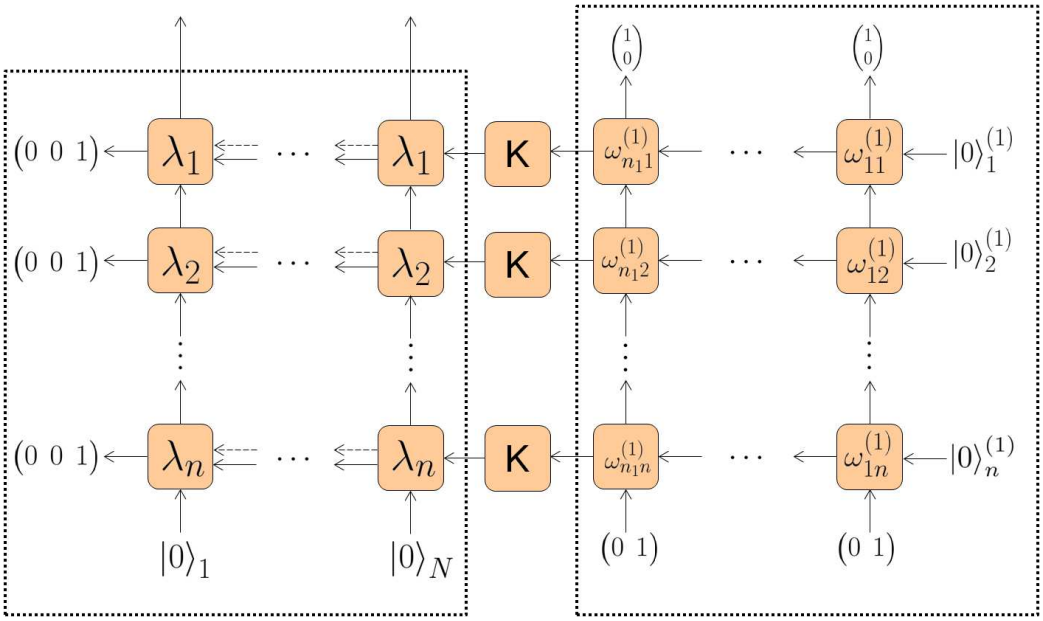


Figure 3.5: Graded Tensor Network for Lai representation

same as in Lai representation as shown in equation (3.24). However, due to the choice of grading in Sutherland representation, the fermionic ($\varepsilon_k = 1$) operators are B_1, C_1, A_{12} and A_{21} in (3.23), and the rest are bosonic ($\varepsilon_k = 0$). Nevertheless, the form of the recursion relation of the first level monodromy matrix for Sutherland representation is exactly the same as (3.87) in Lai representation.

Now, for the graded tensor product (3.65) in the nested Bethe ansatz, we introduce the following:

$$L_k^{(1)\prime}(\lambda)^{ab}_{\alpha\beta} = L_k^{(1)}(\lambda)^{ab}_{\alpha\beta}(-1)^{\epsilon_\alpha} \tag{3.89}$$

$$\begin{aligned}\{[T_L^{(1)\prime}(\lambda)]^{ab}\}^{\alpha_1\cdots\alpha_L}_{\beta_1\cdots\beta_L} &= L_L^{(1)\prime}(\lambda)^{ac_L}_{\alpha_L\beta_L} L_{L-1}^{(1)\prime}(\lambda)^{c_L c_{L-1}}_{\alpha_{L-1}\beta_{L-1}} \cdots L_1^{(1)\prime}(\lambda)^{c_2 c_1}_{\alpha_1\beta_1} \\ &\quad \times (-1)^{\sum_{j=2}^{L}(\epsilon_{\alpha_j}+\epsilon_{\beta_j})\sum_{i=1}^{j-1}(\epsilon_{\alpha_i}+1)}\end{aligned} \tag{3.90}$$

Now, we can write (3.65) in the form of a recurrence relation that allows

the minus signs to be localized:

$$\begin{pmatrix} \{[T^{(1)}_{k+1}(\lambda)]^{ab}\}^{\alpha_1\cdots\alpha_{k+1}}_{\beta_1\cdots\beta_{k+1}} \\ \{[T^{(1)\prime}_{k+1}(\lambda)]^{ab}\}^{\alpha_1\cdots\alpha_{k+1}}_{\beta_1\cdots\beta_{k+1}} \end{pmatrix} = \begin{pmatrix} L^{(1)}_{k+1}(\lambda)^{ac_{k+1}}_{\alpha_{k+1}\beta_{k+1}}\Big|_{\varepsilon_{k+1}=0} & L^{(1)}_{k+1}(\lambda)^{ac_{k+1}}_{\alpha_{k+1}\beta_{k+1}}\Big|_{\varepsilon_{k+1}=1} \\ L^{(1)\prime}_{k+1}(\lambda)^{ac_{k+1}}_{\alpha_{k+1}\beta_{k+1}}\Big|_{\varepsilon_{k+1}=1} & L^{(1)\prime}_{k+1}(\lambda)^{ac_{k+1}}_{\alpha_{k+1}\beta_{k+1}}\Big|_{\varepsilon_{k+1}=0} \end{pmatrix} \times \begin{pmatrix} \{[T^{(1)}_{k}(\lambda)]^{c_{k+1}b}\}^{\alpha_1\cdots\alpha_k}_{\beta_1\cdots\beta_k} \\ \{[T^{(1)\prime}_{k}(\lambda)]^{c_{k+1}b}\}^{\alpha_1\cdots\alpha_k}_{\beta_1\cdots\beta_k} \end{pmatrix} \tag{3.91}$$

The minus signs in the nested Bethe ansatz are absorbed locally into the definition of $L^{(1)\prime}_k(\lambda)$. To use this construction to handle the grading, we would have to alter our tensor network so to include the external matrix space (in both the first level and nested Bethe ansatz for Sutherland representation).

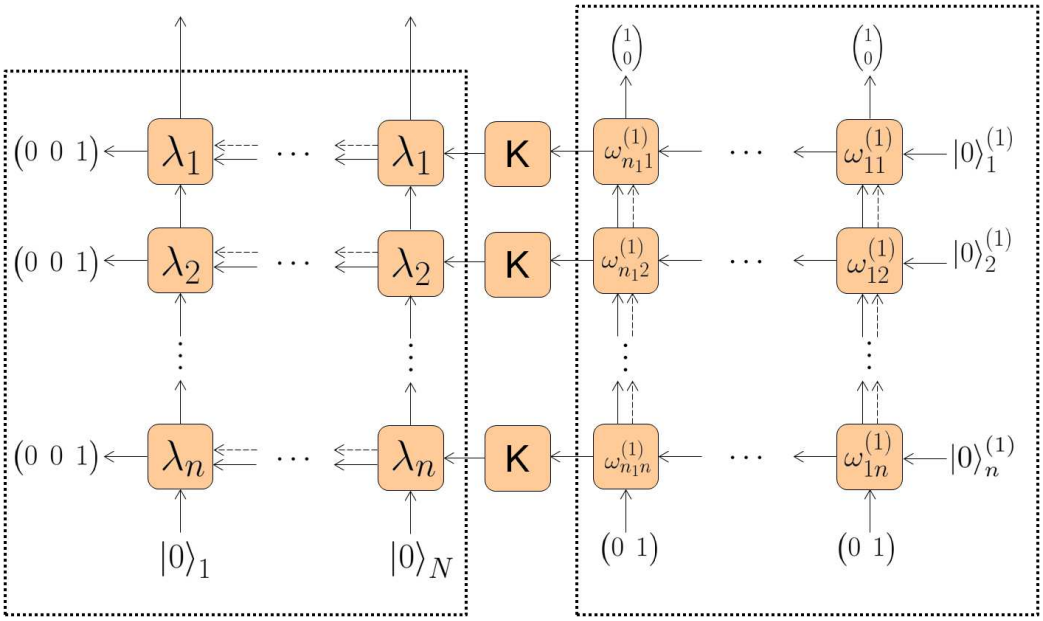


Figure 3.6: Graded Tensor Network for Sutherland representation

The boundary vectors on the left of Fig. 3.6 and K' live in the space $V^{(1|1)} \otimes V^{(1|2)}$, where the first space $V^{(1|1)}$ is the external matrix space and the second space $V^{(1|2)}$ is the matrix space, of the first level $L^{(1)}$ operators. The boundary vectors to the top and bottom of the nested Bethe ansatz live

similarly in the space $V^{(1|1)} \otimes V^{(1|1)}$, where the first space is the external matrix space and the second space is the matrix space, of the nested $L^{(1)}$ operators.

3.4.2 Method 2

Lai representation

In Lai representation, the grading of the first level Bethe network can also be handled by adding an extra bond that carries the parity information of the indices, denoted by the dotted lines in Fig. 3.5. The parity bond p_m at the m^{th} site satisfies the relation $p_m = p_{m-1} + \epsilon_{k_m} \pmod 2$, where $p_0 = 0$. In addition, these parity bonds, which store local information about the minus signs of (3.24), satisfy the recurrence relation

$$\begin{aligned}(-1)^{\sum_{j=2}^{m}(\epsilon_{k_j}+\epsilon_{l_j})\sum_{i=1}^{j-1}\epsilon_{k_i}} &= (-1)^{\sum_{j=2}^{m-1}(\epsilon_{k_j}+\epsilon_{l_j})\sum_{i=1}^{j-1}\epsilon_{k_i}} \\ &\quad \times (-1)^{(\epsilon_{k_m}+\epsilon_{l_m})p_m}\end{aligned} \tag{3.92}$$

As such, in the tensor network picture with grading, each L operator L_m becomes a tensor with 6 indices: 2 horizontal indices of dimension 3 describing the matrix space, 2 vertical indices k_m and l_m of dimension 3 describing the physical space and 2 parity indices p_{m-1} and p_m of dimension 2. Because of the recurrence relation (3.92), the nonlocal minus signs of (3.24) can be reproduced by multiplying each L operator with $(-1)^{(\epsilon_{k_m}+\epsilon_{l_m})p_m}$.

Sutherland representation

In Sutherland representation, both the first and the nested level Bethe network are graded, and they are handled by adding an extra bond that carries the parity information of the indices, denoted by the dotted lines in both levels of the Bethe ansatz in Fig. 3.6. As before, the parity bond p_m at the m^{th} site satisfies the relation $p_m = p_{m-1} + \epsilon_{k_m} \pmod 2$, where $p_0 = 0$, such that the minus signs of (3.24) can be localized.

3.4.3 Equivalence of the two methods

Upon joining the additional parity bonds (of dimension 2) in the second method with the bonds in the matrix space (of dimension 3) of the original tensor network, the L operators are now tensors of 6 by 6 in the matrix space

and 3 by 3 in the physical space, which has the same dimensions as that of the L operators of the first method. These two methods will then give rise to exactly the same tensor network, producing equivalent tensors (up to a unitary transformation). The first method can thus be simply considered as an explicit formulation of the joining of the parity bonds with the original bonds in the matrix space in the second method.

3.5 Approximate contraction of the tensor network

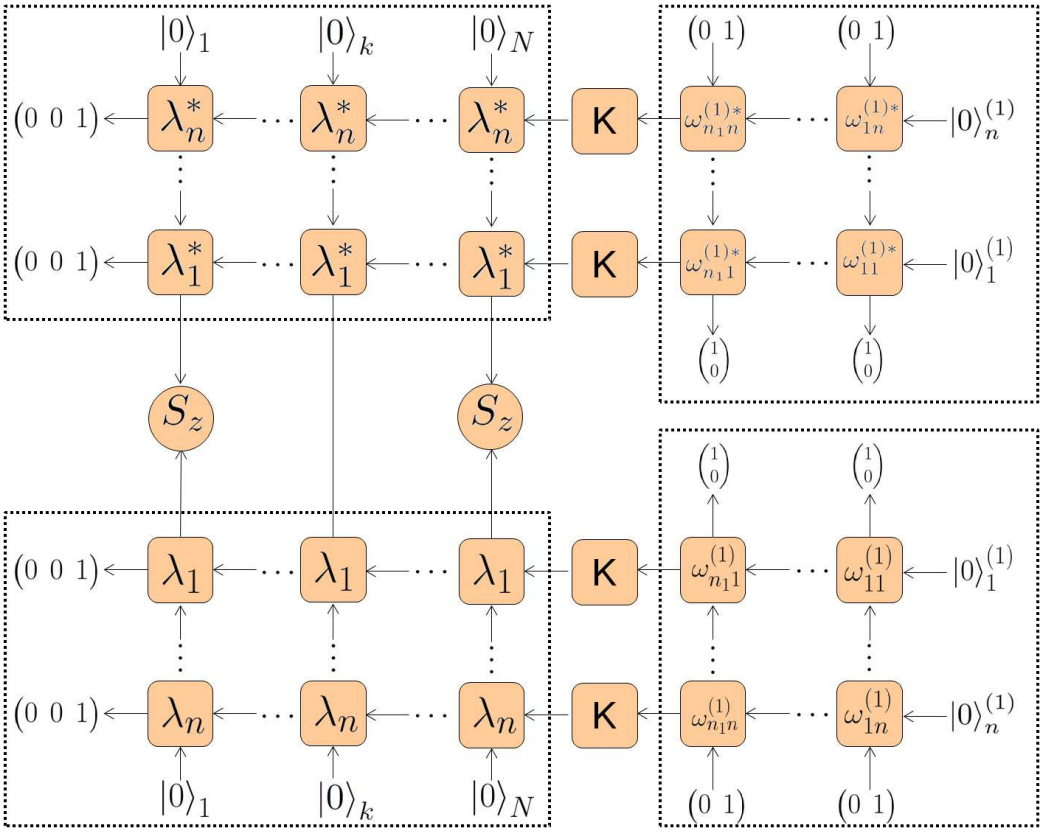


Figure 3.7: Tensor Network calculation of expectation values

The calculation of expectation values with respect to a Bethe eigenstate of the form of (3.28) is a considerably complex problem, because it requires

the contraction of the tensor network depicted in Fig. 3.7.

A tensor network with such a structure also appears in connection with the calculation of partition functions of two-dimensional classical systems and one-dimensional quantum systems and the calculation of expectation values with respect to PEPSs. The complexity of contracting this network scales exponentially with the number of rows M or columns N (depending on the direction of contraction), which renders exact calculations infeasible.

Following Murg et al.[77], to circumvent this problem, we attempt to perform the contraction in an approximative numerical way: the main idea is to consider the network in Fig. 3.4 as the time evolution of MPOs (L operators) acting on MPSs in a sequential order.

After each evolution step, the state remains an MPS, but the virtual dimension is increased, by a factor of 3 (first level) or 2 (nested level). Thus, we approximate the MPS after each evolution step by a MPS with smaller virtual dimension. Of course, we must exercise caution, as the creation operators are not unitary and the intermediate states of the evolution can be nonphysical (i.e., they might have to be represented by an MPS with high virtual dimension).

We choose the order of contraction to be such:

1. In the nested Bethe ansatz, act the n_1 nested creation operators $C^{(1)}(\lambda_{n_1}^{(1)}) \cdots C^{(1)}(\lambda_1^{(1)})$ on the initial MPS $|0\rangle^{(1)}$ sequentially, contracting approximately to get an MPS at each step, to produce a boundary MPS on the right of the first level Bethe ansatz.

2. Now, in the first level Bethe ansatz, act the n first level creation operators $C(\lambda_n) \cdots C(\lambda_1)$ on the initial MPS $|0\rangle$ sequentially, contracting approximately to get an MPS at each step, with the right end of the first level Bethe ansatz terminated by the boundary MPS produced in the first step.

At each step in the above contraction process, we let

$$|\Psi_m\rangle = C_{a_m}(\lambda_m) \left|\tilde{\Psi}_{m-1}\right\rangle, \ m = 1, \cdots, n \tag{3.93}$$

$$|\Psi_{m_1}\rangle^{(1)} = C^{(1)}(\lambda_{m_1}^{(1)}) \left|\tilde{\Psi}_{m_1-1}^{(1)}\right\rangle, \ m_1 = 1, \cdots, n_1 \tag{3.94}$$

where

$$\left|\tilde{\Psi}_0\right\rangle = |0\rangle, \quad \left|\tilde{\Psi}_0\right\rangle^{(1)} = |0\rangle^{(1)} \tag{3.95}$$

At each step of the first level Bethe ansatz, $|\Psi_m\rangle$ is approximated by the MPS $\left|\tilde{\Psi}_m\right\rangle$ that has maximal bond dimension D and is closest to $\left|\tilde{\Psi}_m\right\rangle$. In other words, we try solve the minimization problem

$$\min(M) := \min_{\left|\tilde{\Psi}_m\right\rangle \in \{MPS_D\}} \left\| |\Psi_m\rangle - \left|\tilde{\Psi}_m\right\rangle \right\|^2 \tag{3.96}$$

$$= \min\left(\left\langle \tilde{\Psi}_m | \tilde{\Psi}_m \right\rangle - 2 \left\langle \Psi_m | \tilde{\Psi}_m \right\rangle \right), \tag{3.97}$$

which is essentially a mimization problem of the form [115]

$$\min_{x^1, x^2, \cdots} \left[\sum_{k_1, k_2, \cdots} \left(x^1_{k_1} x^2_{k_2} x^3_{k_3} \cdots \right) \left(\bar{x}^1_{k_1} \bar{x}^2_{k_2} \bar{x}^3_{k_3} \cdots \right) \right.$$
$$\left. - \sum_{k_1, k_2, \cdots} \left(y^1_{k_1} y^2_{k_2} y^3_{k_3} \cdots \right) \left(\bar{x}^1_{k_1} \bar{x}^2_{k_2} \bar{x}^3_{k_3} \cdots \right) \right], \tag{3.98}$$

where $x^j_{k_j}$ and $y^j_{k_j}$ are the defining matrices of the MPS $\left|\tilde{\Psi}_m\right\rangle$ and $|\Psi_m\rangle$, respectively, and k_j ranges from 1 to 3 in the first level Bethe ansatz and from 1 to 2 in the nested level. The size of the matrices $x^j_{k_j}$ is constrained to $D \times D$ (except from the boundary matrices that are constrained to $1 \times D$ and $D \times 1$, respectively).

This minimization can be performed using the Alternating Least Squares (ALS) algorithm. The ALS is an iterative method that works as follows: after making an initial guess of the matrices $x^j_{k_j}$, all matrices are kept fixed except those on site 1, and optimization is done over $\{x^1_{k_1}\}$. Writing this set of matrices as a vector x^1 of dimension dD^2, this subproblem is of the form

$$\min_{x^1} \left(x^{1\dagger} N_1 x^1 - x^{1\dagger} \omega_1 \right), \tag{3.99}$$

which can be minimized by solving

$$\frac{\delta}{\delta x^{1\dagger}} \left(x^{1\dagger} N_1 x^1 - x^{1\dagger} \omega_1 \right) = 0 \implies N_1 x^1 = \omega_1. \tag{3.100}$$

This implies that the optimal x^1 can be obtained by solving a system of linear equations with coefficient matrix N_1 and inhomogenity ω_1, both of which can be obtained efficiently by contracting the appropriate tensor network [115].

For a MPS with open boundary conditions, a gauge condition can always be found that makes the coefficient matrix N_1 equal to the identity, thus making the solution of the system of linear equations numerically stable.

At the next step, all matrices are fixed except for those of site 2 (i.e. $\{x^2_{k_2}\}$) and the same optimization procedure is performed, and it continues optimizing for each site until the last site is reached. The sweep direction then changes from the last to the first site, and continues back and forth until convergence. In this way, the MPS approximation of the Bethe state is obtained for the whole tensor network.

The error of the approximation is well controlled in the sense that the expectation value of the energy can always be calculated with respect to the approximated MPS $\left|\tilde{\Psi}_m\right\rangle$ and compared to the exact energy available from the Bethe ansatz.

There is a (mathematical) degree of freedom that can be used to improve the approximation. This degree of freedom is due to the fact that the set of $\{\{\lambda\}, \{\lambda^{(1)}\}\}$ encode information about physical quantities and the ordering of the them should not change the final wavefunction produced. That is, permutation of order of applying the creation operators through permutation of the set of $\{\{\lambda\}, \{\lambda^{(1)}\}\}$ will not change the final wavefunction. However, the intermediate states are a priori not physical ground states; i.e., there is no reason for them to lie in the set of MPS with low bond dimension. Even so, similar to that which is noted in [77], there is always an ordering of the set of λ's such that the intermediate states contain as little entanglement as possible. We then use that ordering for doing the approximation.

3.6 Numerical solution of the Bethe ansatz equations

The spectral parameters satisfy their respective set of Bethe ansatz equations, which have to be solved and substituted into the tensor before the tensor network can be built and approximatively contracted.

However, the Bethe ansatz equations are a set of coupled non-linear equations which, in general, have no analytical means of solving. As such, we would require numerical methods to solve for them.

One method to solve the Bethe ansatz equations is to rewrite the equations into a suitable "log" form [51, 49], then approximate the spectral pa-

rameters using Newton's method for multivariate root finding, which works as follows:

An arbitrary function $f(\vec{x})$ can be approximated as

$$f(\vec{x}) \approx f(\vec{x}_0) + \mathrm{D}_f(\vec{x}_0)(\vec{x} - \vec{x}_0), \tag{3.101}$$

where $\mathrm{J}_f(\vec{x_0})$ represents the Jacobian matrix of the function f at the point $\vec{x_0}$, defined by

$$[\mathrm{J}_f(\vec{x})]_{ij} = \frac{\partial f_i(\vec{x})}{\partial x_j} \tag{3.102}$$

To find a root of $f(\vec{x})$, we set $f(\vec{x}) = 0$. Rearranging (3.101), we can obtain

$$\vec{x}_{n+1} \approx \vec{x}_n - [\mathrm{J}_f(\vec{x}_n)]^{-1} f(\vec{x}_n), \tag{3.103}$$

which can be iterated until the roots $\vec{x}_{n+1}$ reach a desired convergence.

However, in practice, x_{n+1} is not usually obtained by explicitly computing $[\mathrm{J}_f(\vec{x_n})]^{-1}$ and then multiplying by $f(\vec{x}_n)$, as this is computationally inefficient. Instead, it is more practical to solve the system of linear equations $[\mathrm{J}_f(\vec{x}_n)]\,\vec{s}_n = -f(\vec{x}_n)$ for the unknown $\vec{s}_n$, using a method such as Gaussian elimination, and then setting $\vec{x}_{n+1} = \vec{x}_n + \vec{s}_n$.

To verify that the Bethe ansatz equations are solved correctly, the analytical results obtained by Bares et al. [8] are computed using the Fie package [7] and compared with the energies obtained from the spectral parameters using Newton's method.

3.7 Numerical Results

Using the previously described method, we have obtained numerical results for the t-J model with periodic boundary conditions. The Lai and Sutherland representation are algebraically equivalent as proven in Essler and Korepin [27]. However, for numerical computations, the Sutherland representation works better as its tensor network is smaller near half filling (which increases the maximum lattice length that we can work with numerically), and its Bethe ansatz equations are more well behaved numerically. In fact, the Bethe ansatz equations of Lai representation blow up numerically at half filling when there are no holes (the infinities cancel algebraically). As such, after doing consistency checks between the two representations, we

have decided to only present the computational results of the Sutherland representation in this section.

To implement grading, we chose to use the first method as the explicit construction of the matrices can be more easily checked for errors. As a proof of principle, we obtain the correlation functions of eigenstates on lattices of length 18 as presented below. Calculations for lattices of larger length can be achieved through consideration of symmetries as was done in Murg et al. [77], or using mathematical packages which which can circumvent the default machine precision limit.

3.7.1 Electron correlator

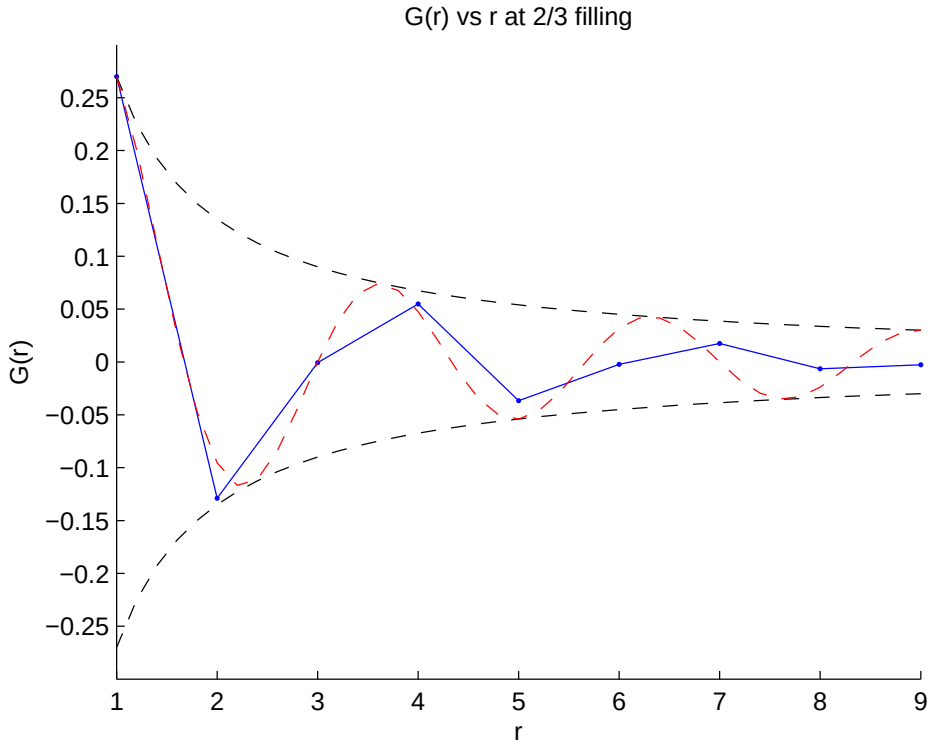


Figure 3.8: Spin-up correlator at 2/3 filling for ground state

The asymptotic behavior of the spin correlator is predicted by conformal

field theory to be

$$G_\sigma(r) = \left\langle c_\sigma^\dagger(r) c_\sigma(0) \right\rangle \propto r^{-\eta} cos(k_F r) \tag{3.104}$$

where η and k_F are as defined in [53]. This is strongly supported by our results, as can be gathered from Fig. 3.8. The deviations are caused by finite size effects, though the numerical results for spin-up correlator is clearly bounded by the theoretical results.

3.7.2 Singlet pair superconducting correlators

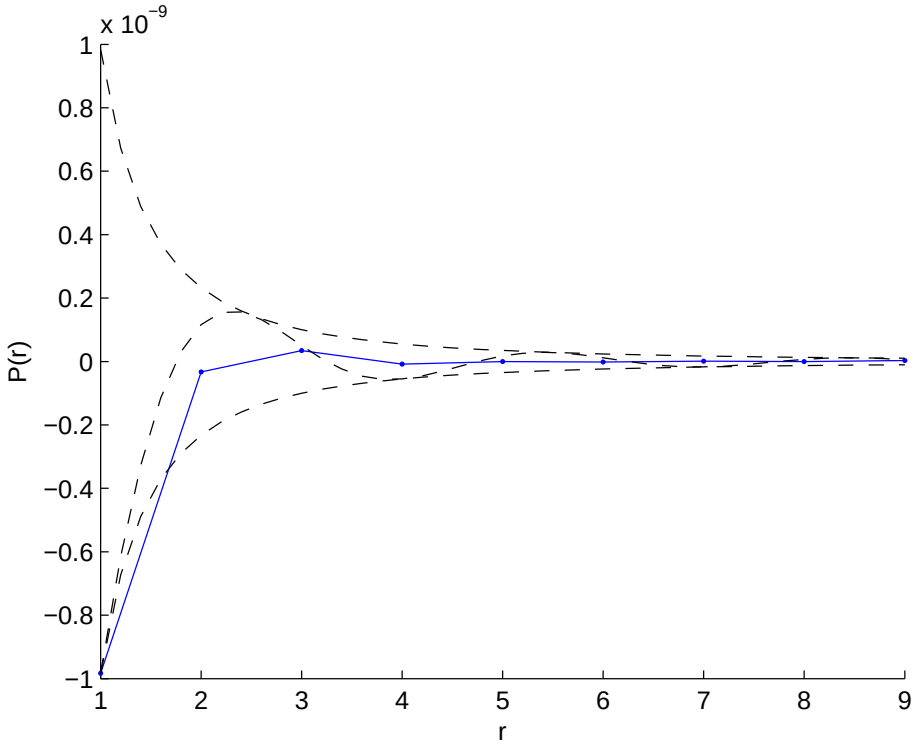


Figure 3.9: Singlet pair superconducting correlator at 2/3 filling for ground state

The asymptotic behavior of the singlet pair correlator is predicted by conformal field theory to be

$$P_s(r) \left\langle c_\uparrow^\dagger(r+1) c_\downarrow^\dagger(r) c_\uparrow(1) c_\downarrow(0) \right\rangle \propto r^{-\beta_s} cos(2k_F r) \tag{3.105}$$

where β_s and k_F are as defined in [53]. This is strongly supported by our results, as can be gathered from Fig. 3.9. The numerical result of the correlator is clearly bounded by the theoretical results, and the deviations are due to finite size effects.

3.7.3 Spin correlator

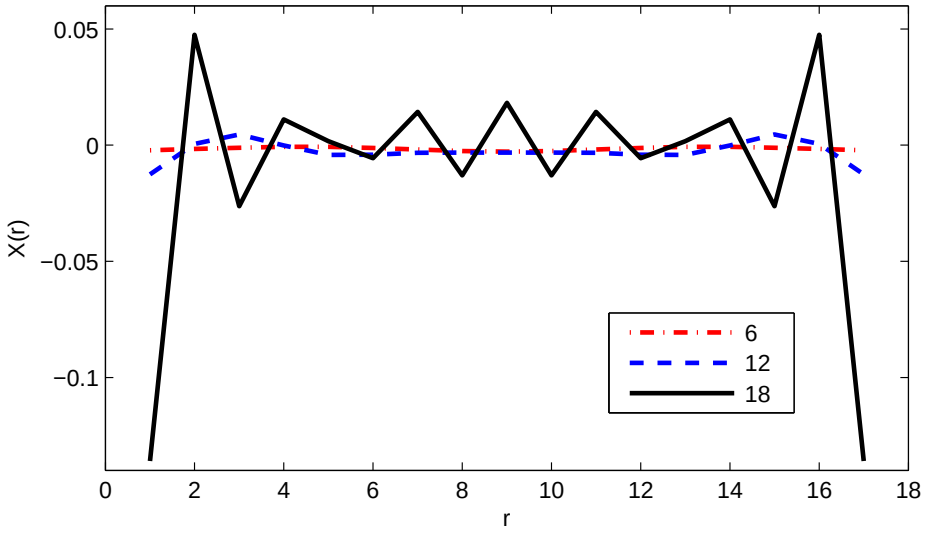


Figure 3.10: Spin correlator at various fillings for a charge triplet state - the number in the legend shows N, the total number of particles.

The spin correlator is defined as:

$$\chi(r) = \langle S_z(r) S_z(0) \rangle \quad , \quad S_z(r) = (n_\uparrow(r) - n_\downarrow(r)) \tag{3.106}$$

We consider the charge triplet state and calculate its spin correlator, as shown in Fig. 3.10. It shows that an interesting trend that as we decrease the number of holes in the lattice sites (i.e. increase the filling), the variation of the spin correlator increases, and that it tends toward a zigzag pattern that alternates between the even and odd lattice sites at a full lattice.

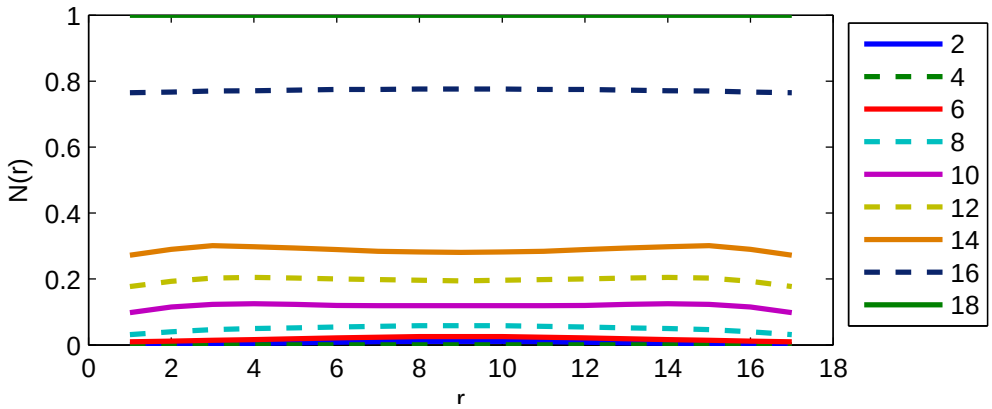


Figure 3.11: Charge density correlator at various fillings for a charge triplet state (color online) - the number in the legend shows N, the total number of particles

3.7.4 Charge density correlator

The charge density correlator is defined as:

$$N(r) = \langle n(r)n(0)\rangle \quad , \quad n(r) = (n_{\uparrow}(r) + n_{\downarrow}(r)) \tag{3.107}$$

We consider the charge triplet state and calculate its charge density correlator, as shown in Fig. 3.11. It does not fully show the trend of variation across the lattice sites, as it is dominated by the constant term in the correlator.

As such, we attempt to "normalize" the correlator by setting the correlator of the first site to be zero (by subtracting away the value of the correlator at the first site), as shown in Fig. 3.12. This clearly shows a trend that as N (total number of particles) increases, the magnitude of the variation of the correlator across the lattice sites increases, until $N = 14$ which reaches a peak, then decreases.

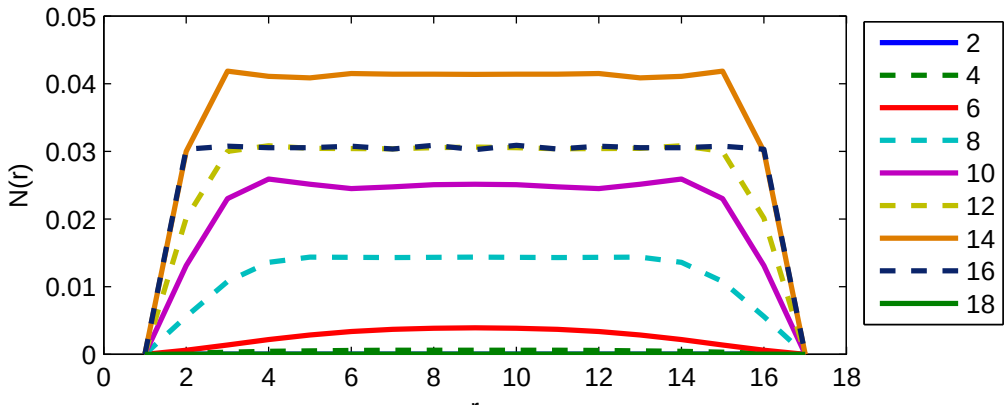


Figure 3.12: Normalized charge density correlator at various fillings for charge triplet state (color online) - the number in the legend shows N, the total number of particles.

Chapter 4

Hubbard model

4.1 The Hubbard model

Like the XXX model and the t-J model, the Hubbard model is a model of strongly correlated electrons. It was proposed in the 1960's by Hubbard [44] to understand the behavior of electrons in solids and has since become a candidate model for describing high-temperature superconductivity. More recently, the Bose-Hubbard model has been used to describe the behavior of ultracold atoms trapped in optical lattices [48].

The integrability of the Hubbard model in 1D was first proved by Shastry [99]. The algebraic Bethe ansatz of the Hubbard model is a generalized version of the algebraic Bethe ansatz used in solving the Heisenberg XXX model. There are several key ideas in the generalization of the Bethe ansatz: nesting, grading and recursion. Nesting means that we first diagonalize charge degrees of freedom and then spin degrees of freedom, thus the Bethe ansatz has two levels. Grading takes into account the fermionic nature of the electrons. Graded tensor network states have already been described in the literature [81]. In addition, recursion means that the Bethe wavefunction relies on the wavefunction of smaller length in a non-trivial recursion relation.

This chapter is devoted to the tensor network interpretation of the algebraic Bethe ansatz of the Hubbard model. From this tensor network interpretation, we can derive new insights into the structure of the algebraic Bethe ansatz.

4.2 Algebraic Bethe ansatz for the Hubbard model

In this section, we briefly outline the main ideas of the algebraic Bethe ansatz for the Hubbard model, following Martins and Ramos [74].

4.2.1 Preliminaries

In the Hubbard model, electrons on a lattice of length L are described by operators $c_{i,\sigma}$, $i = 1, \cdots, L$, $\sigma = \pm 1$, which follow the anticommutation relations $\{c^{\dagger}_{i,\sigma}, c_{j,\tau}\} = \delta_{i,j}\delta_{\sigma,\tau}$. The state $|0\rangle$ (Fock vacuum) satisfies $c_{i,\sigma}|0\rangle = 0$. The Hilbert space of the Hamiltonian (4.2) has four possible electronic states at a given lattice site i:

$$|0\rangle_i, |\uparrow\rangle_i = c^{\dagger}_{i,1}|0\rangle_i, |\downarrow\rangle_i = c^{\dagger}_{i,-1}|0\rangle, |\uparrow\downarrow\rangle_i = c^{\dagger}_{i,1}c^{\dagger}_{i,-1}|0\rangle. \tag{4.1}$$

The Hubbard Hamiltonian is given by

$$\begin{aligned} H = &-\sum_{i=1}^{L}\sum_{\sigma=\uparrow,\downarrow}[c^{\dagger}_{i\sigma}c_{i+1\sigma} + c^{\dagger}_{i+1\sigma}c_{i\sigma}] \\ &+ U\sum_{i=1}^{L}(n_{i\uparrow} - \frac{1}{2})(n_{i\downarrow} - \frac{1}{2}) \end{aligned} \tag{4.2}$$

where $n_{i\sigma} = c^{\dagger}_{i\sigma}c_{i\sigma}$ is the number operator. t represents the nearest-neighbor hopping and U represents the on-site Coulomb repulsion.

4.2.2 Grading

For the Hubbard model, the mathematical structure used for grading is the same as that of the t-J model 3.2.2, except that now we have $m = 2$, $n = 2$, and the four allowed configurations are given by (4.1).

4.2.3 Yang-Baxter equation

We provide a short reiteration of the Yang-Baxter equation here. The graded Yang-Baxter equation, as before, is

$$\begin{aligned} R_{12}(\lambda,\mu)\{[\Pi_{13}R_{13}(\lambda)]\otimes[\Pi_{23}R_{23}(\mu)]\} \\ = \{[\Pi_{13}R_{13}(\mu)]\otimes[\Pi_{23}R_{23}(\lambda)]\}R_{12}(\lambda,\mu) \end{aligned} \tag{4.3}$$

where the indices $1,2,3$ indicate in which of the three tensored spaces the matrices act nontrivially. The tensor product in (4.3) is between spaces 1 and 2. The third space is called "quantum space" and the first two spaces "matrix spaces". The L operator, $L_k(\lambda)^{ab}_{\alpha\beta}$, on site k is a quantum operator valued linear operator on $\mathcal{H}_k \otimes V^{(m|n)}_{\text{matrix}}$, where the Greek (Roman) indices are the "quantum indices" ("matrix indices"), and that $\mathcal{H}_k \simeq V^{(m|n)}$ is the Hilbert space over the kth site, and $V^{(m|n)}_{\text{matrix}}$ is a matrix space. Rewriting (4.3) for the kth quantum space,

$$R(\lambda,\mu)[L_k(\lambda)\otimes L_k(\mu)] = [L_k(\mu)\otimes L_k(\lambda)]R(\lambda,\mu) \tag{4.4}$$

Now, constructing an integrable spin model based on the intertwining relation (4.4), we first define the monodromy matrix $T_L(\lambda)$ as the product (in the matrix space) of the L operators over all of the lattice sites:

$$T_L(\lambda) = L_L(\lambda)L_{L-1}(\lambda)\cdots L_1(\lambda) \tag{4.5}$$

$T_L(\lambda)$ is a quantum operator valued $(m+n)\times(m+n)$ matrix that acts nontrivially in the graded tensor product of all quantum spaces of the lattice. It also fulfills the same intertwining relation as the L operators:

$$R(\lambda-\mu)[T_L(\lambda)\otimes T_L(\mu)] = [T_L(\mu)\otimes T_L(\lambda)]R(\lambda-\mu) \tag{4.6}$$

Taking the supertrace of the monodromy matrix, we get the transfer matrix $\tau(\lambda)$ of the spin model:

$$\tau(\lambda) = \text{str}[T_L(\lambda)] = \sum_{a=1}^{m+n}(-1)^{\epsilon_a}[T_L(\lambda)]^{aa} \tag{4.7}$$

As a consequence of (4.6), transfer matrices with different spectral parameters commute. This implies that the transfer matrix is the generating functional of the Hamiltonian.

For the Hubbard model, the (graded) R matrix is

$$\begin{pmatrix}
\alpha_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \alpha_5 & 0 & 0 & -i\alpha_9 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \alpha_5 & 0 & 0 & 0 & 0 & 0 & -i\alpha_9 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \alpha_4 & 0 & 0 & -i\alpha_{10} & 0 & 0 & i\alpha_{10} & 0 & 0 & \alpha_7 & 0 & 0 & 0 \\
0 & -i\alpha_8 & 0 & 0 & \alpha_5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \alpha_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & i\alpha_{10} & 0 & 0 & \alpha_3 & 0 & 0 & -\alpha_6 & 0 & 0 & -i\alpha_{10} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \alpha_5 & 0 & 0 & 0 & 0 & 0 & -i\alpha_8 & 0 & 0 \\
0 & 0 & -i\alpha_8 & 0 & 0 & 0 & 0 & 0 & \alpha_5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -i\alpha_{10} & 0 & 0 & -\alpha_6 & 0 & 0 & \alpha_3 & 0 & 0 & i\alpha_{10} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \alpha_1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \alpha_5 & 0 & 0 & -i\alpha_8 & 0 \\
0 & 0 & 0 & \alpha_7 & 0 & 0 & i\alpha_{10} & 0 & 0 & -i\alpha_{10} & 0 & 0 & \alpha_4 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -i\alpha_9 & 0 & 0 & 0 & 0 & 0 & \alpha_5 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -i\alpha_9 & 0 & 0 & \alpha_5 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \alpha_2
\end{pmatrix} \tag{4.8}$$

where $\alpha_i(\lambda,\mu)$, $i = 1 \ldots 10$ represent the Boltzmann weights [99]:

$$\alpha_1(\lambda,\mu) = \left\{ e^{[h(\mu)-h(\lambda)]} a(\lambda)a(\mu) + e^{-[h(\mu)-h(\lambda)]} b(\lambda)b(\mu) \right\} \tag{4.9}$$

$$\alpha_2(\lambda,\mu) = \left\{ e^{-[h(\mu)-h(\lambda)]} a(\lambda)a(\mu) + e^{[h(\mu)-h(\lambda)]} b(\lambda)b(\mu) \right\} \tag{4.10}$$

$$\alpha_3(\lambda,\mu) = \frac{e^{[h(\mu)+h(\lambda)]} a(\lambda)b(\mu) + e^{-[h(\mu)+h(\lambda)]} b(\lambda)a(\mu)}{a(\lambda)b(\lambda) + a(\mu)b(\mu)} \left\{ \frac{\cosh[h(\mu)-h(\lambda)]}{\cosh[h(\mu)+h(\lambda)]} \right\} \tag{4.11}$$

$$\alpha_4(\lambda,\mu) = \frac{e^{-[h(\mu)+h(\lambda)]} a(\lambda)b(\mu) + e^{[h(\mu)+h(\lambda)]} b(\lambda)a(\mu)}{a(\lambda)b(\lambda) + a(\mu)b(\mu)} \left\{ \frac{\cosh(h(\mu)-h(\lambda))}{\cosh(h(\mu)+h(\lambda))} \right\} \tag{4.12}$$

$$\alpha_5(\lambda,\mu) = 1 \tag{4.13}$$

$$\alpha_6(\lambda,\mu) = \left\{ \frac{e^{[h(\mu)+h(\lambda)]} a(\lambda)b(\mu) - e^{-[h(\mu)+h(\lambda)]} b(\lambda)a(\mu)}{a(\lambda)b(\lambda) + a(\mu)b(\mu)} \right\} \tag{4.14}$$

$$\times \left[b^2(\mu) - b^2(\lambda) \right] \frac{\cosh[h(\mu)-h(\lambda)]}{\cosh[h(\mu)+h(\lambda)]} \tag{4.15}$$

$$\alpha_7(\lambda,\mu) = \left\{ \frac{-e^{-[h(\mu)+h(\lambda)]} a(\lambda)b(\mu) + e^{[h(\mu)+h(\lambda)]} b(\lambda)a(\mu)}{a(\lambda)b(\lambda) + a(\mu)b(\mu)} \right\} \tag{4.16}$$

$$\times \left[b^2(\mu) - b^2(\lambda) \right] \frac{\cosh[h(\mu)-h(\lambda)]}{\cosh[h(\mu)+h(\lambda)]} \tag{4.17}$$

$$\alpha_8(\lambda,\mu) = \left\{ e^{[h(\mu)-h(\lambda)]} a(\lambda)b(\mu) - e^{-[h(\mu)-h(\lambda)]} b(\lambda)a(\mu) \right\} \tag{4.18}$$

$$\alpha_9(\lambda,\mu) = \left\{-e^{-[h(\mu)-h(\lambda)]}a(\lambda)b(\mu) + e^{[h(\mu)-h(\lambda)]}b(\lambda)a(\mu)\right\} \tag{4.19}$$

$$\alpha_{10}(\lambda,\mu) = \frac{b^2(\mu)-b^2(\lambda)}{a(\lambda)b(\lambda)+a(\mu)b(\mu)}\left\{\frac{\cosh[h(\mu)-h(\lambda)]}{\cosh[h(\mu)+h(\lambda)]}\right\} \tag{4.20}$$

4.2.4 Trace identities

The Hamiltonian (4.2) can be obtained from the transfer matrix by taking its first logarithmic derivative at zero spectral parameter:

$$H = \left.\frac{\partial \ln[T(\lambda)]}{\partial\lambda}\right|_{\lambda=0} \tag{4.21}$$

4.2.5 Algebraic Bethe ansatz of the Hubbard Model

Let the Hilbert space at the kth site of the lattice be spanned by the four vectors $e_1 = (1000)$, $e_2 = (0100)$, $e_3 = (0010)$, and $e_4 = (0001)$. We consider a grading such that e_2 and e_3 are fermionic, representing the spin-down and spin-up electrons respectively, and e_1 and e_4 are bosonic, representing the the empty site and doubly occupied site respectively. This means that their Grassmann parities are $\epsilon_2 = \epsilon_3 = 1$ and $\epsilon_1 = \epsilon_4 = 0$. We choose the reference state in the kth quantum space $|0\rangle_k$ as the doubly occupied state, and the reference state $|0\rangle$ of the whole lattice such that,

$$|0\rangle_n = \begin{pmatrix}1\\0\\0\\0\end{pmatrix}, |0\rangle = \otimes_{n=1}^{L}|0\rangle_n \tag{4.22}$$

The monodromy matrix (4.5) can be represented as

$$\begin{aligned} T_L(\lambda) &= L_L(\lambda)L_{L-1}(\lambda)\cdots L_1(\lambda) \\ &= \begin{pmatrix} B(\lambda) & \vec{B}(\lambda) & F(\lambda) \\ \vec{C}(\lambda) & \hat{A}(\lambda) & \vec{B}^*(\lambda) \\ C(\lambda) & \vec{C}^*(\lambda) & D(\lambda)\end{pmatrix}_{4\times 4} \end{aligned} \tag{4.23}$$

where $\vec{B}(\lambda)$, $\vec{C}^*(\lambda)$ and $\vec{B}^*(\lambda)$, $\vec{C}(\lambda)$ are two component vectors (dimensions 1×2 and 2×1 respectively), $\hat{A}(\lambda)$ is a 2×2 matrix (with elements $\hat{A}_{ab}(\lambda)$)

and $B(\lambda)$, $C(\lambda)$, $D(\lambda)$ and $F(\lambda)$ are scalars. The "dual" transformation is defined as:

$$(O^*(\lambda))^* \equiv O(\lambda) \ , \ A^*(\lambda) \equiv -A^t(\lambda), \tag{4.24}$$

$$B^*(\lambda) \equiv D(\lambda) \ , \ F^*(\lambda) = F(\lambda) \ , \ C^*(\lambda) = C(\lambda) \tag{4.25}$$

where A^t refers to the transpose of the 2×2 matrix A, and $\vec{B^*}(\lambda)$ corresponds to $U \to -U$ (ie $h \to -h$). For clarity, we write (4.23) explicitly in component form:

$$\begin{aligned}\{[T_L(\lambda)]^{ab}\}_{\substack{\alpha_1\cdots\alpha_L \\ \beta_1\cdots\beta_L}} = L_L(\lambda)^{ac_L}_{\alpha_L\beta_L} L_{L-1}(\lambda)^{c_L c_{L-1}}_{\alpha_{L-1}\beta_{L-1}} \cdots \\ \cdots L_1(\lambda)^{c_2 c_1}_{\alpha_1\beta_1} (-1)^{\sum_{j=2}^{L}(\epsilon_{\alpha_j}+\epsilon_{\beta_j})\sum_{i=1}^{j-1}\epsilon_{\alpha_i}}\end{aligned} \tag{4.26}$$

Note that the physical (greek) indices are subjected to the minus signs from the graded tensor product, while the matrix (latin) indices are not, as they are summed over (and not tensored). The transfer matrix is then given as

$$\tau(\mu) = \mathrm{str}[T_L(\mu)] = \left[B(\lambda) - \sum_{a=1}^{2} A_{aa}(\lambda) + D(\lambda)\right] \tag{4.27}$$

The action of $L_k(\lambda)$ on $|0\rangle_k$ is

$$L_k(\lambda)\,|0\rangle_k = \begin{pmatrix} \omega_1(\lambda) & \ddagger & \ddagger & \ddagger \\ 0 & \omega_2(\lambda) & 0 & \ddagger \\ 0 & 0 & \omega_2(\lambda) & \ddagger \\ 0 & 0 & 0 & \omega_3(\lambda) \end{pmatrix} |0\rangle_k \tag{4.28}$$

where $\ddagger$ represents arbitrary non-zero values, and

$$a(\lambda) = \cos(\lambda) \tag{4.29}$$

$$b(\lambda) = \sin(\lambda) \tag{4.30}$$

$$\sinh[2h(\lambda)] = \frac{U}{2} a(\lambda) b(\lambda) \tag{4.31}$$

$$\omega_1(\lambda) = [a(\lambda)]^2 e^{h(\lambda)} \tag{4.32}$$

$$\omega_2(\lambda) = a(\lambda) b(\lambda) e^{-h(\lambda)} \tag{4.33}$$

$$\omega_3(\lambda) = [b(\lambda)]^2 e^{h(\lambda)} \tag{4.34}$$

sUsing (4.23) and (4.28), we determine the action of the monodromy matrix on $|0\rangle$ to be

$$T_L(\lambda)\,|0\rangle = \begin{pmatrix} [\omega_1(\lambda)]^L & B_1(\lambda) & B_2(\lambda) & F(\lambda) \\ 0 & [\omega_2(\lambda)]^L & 0 & B_1^*(\lambda) \\ 0 & 0 & [\omega_2(\lambda)]^L & B_2^*(\lambda) \\ 0 & 0 & 0 & [\omega_3(\lambda)]^L \end{pmatrix} |0\rangle \tag{4.35}$$

In particular, we note that for the diagonal terms:

$$\begin{aligned} &B(\lambda)\,|0\rangle = [\omega_1(\lambda)]^L\,|0\rangle \quad , \\ &D(\lambda)\,|0\rangle = [\omega_3(\lambda)]^L\,|0\rangle \quad , \\ &\hat{A}_{aa}(\lambda)\,|0\rangle = [\omega_2(\lambda)]^L\,|0\rangle \ \text{ for } a = 1, 2. \end{aligned}$$

And, for the off-diagonal annihilation operators,

$$\begin{aligned} &\vec{C}(\lambda)\,|0\rangle = 0 \quad , \quad \vec{C}^*(\lambda)\,|0\rangle = 0 \quad , \\ &C(\lambda)\,|0\rangle = 0 \quad , \\ &\hat{A}_{ab}(\lambda)\,|0\rangle = 0 \text{ for } a \neq b \end{aligned}$$

Thus, by inspecting (4.35), $\vec{B}(\lambda)$ and $F(\lambda)$ can be interpreted as creation operators. With this insight, we will now describe the solution for a set of eigenstates of the transfer matrix using a generalized Nested Algebraic Bethe Ansatz (NABA).

From (4.6), the set of fundamental commutation relations which are relevant for the NABA can be derived. The first of them is

$$\begin{aligned} \vec{B}(\lambda) \otimes \vec{B}(\mu) = &\frac{\alpha_1(\lambda,\mu)}{\alpha_2(\lambda,\mu)} [\vec{B}(\mu) \otimes \vec{B}(\lambda)].\hat{r}(\lambda,\mu) \\ &- i\frac{\alpha_{10}(\lambda,\mu)}{\alpha_7(\lambda,\mu)} \{F(\lambda)B(\mu) - F(\mu)B(\lambda)\}\vec{\xi}, \end{aligned} \tag{4.36}$$

where

$$\vec{\xi} = (0 \quad 1 \quad -1 \quad 0) \quad ; \quad \hat{r}(\lambda,\mu) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \bar{a}(\lambda,\mu) & \bar{b}(\lambda,\mu) & 0 \\ 0 & \bar{b}(\lambda,\mu) & \bar{a}(\lambda,\mu) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{4.37}$$

and the auxiliary weights can be expressed as

$$\bar{a}(\tilde{\lambda},\tilde{\mu}) = \frac{U}{\tilde{\mu}-\tilde{\lambda}+U} \quad , \quad \bar{b}(\tilde{\lambda},\tilde{\mu}) = \frac{\tilde{\mu}-\tilde{\lambda}}{\tilde{\mu}-\tilde{\lambda}+U} \tag{4.38}$$

where

$$\tilde{\lambda} = \frac{a(\lambda)}{b(\lambda)}e^{2h(\lambda)} - \frac{b(\lambda)}{a(\lambda)}e^{-2h(\lambda)} - \frac{U}{2} \tag{4.39}$$

There is a physical interpretation of the terms in the fundamental commutation relation (4.36). $\vec{B}(\lambda)$ (of odd Grassmann parity) corresponds to single hole excitation at a site, and $F(\lambda)$ (of even Grassmann parity) corresponds the creation of a local hole pair with opposite spins, while $\vec{\xi}$ corresponds to imposing the Pauli exclusion principle on the spins to forbid doubly up or down spins.

The other relevant fundamental commutation relationships with $\vec{B}(\lambda)$ are

$$\begin{aligned}\hat{A}(\lambda)\otimes\vec{B}(\mu) = &-i\frac{\alpha_1(\lambda,\mu)}{\alpha_9(\lambda,\mu)}[\vec{B}(\mu)\otimes\hat{A}(\lambda)].\hat{r}(\lambda,\mu) + i\frac{\alpha_5(\lambda,\mu)}{\alpha_9(\lambda,\mu)}\vec{B}(\lambda)\otimes\hat{A}(\mu)\\ &- i\frac{\alpha_{10}(\lambda,\mu)}{\alpha_7(\lambda,\mu)}\left[\vec{B^*}(\lambda)B(\mu) + i\frac{\alpha_5(\lambda,\mu)}{\alpha_9(\lambda,\mu)}F(\lambda)\vec{C}(\mu)\right.\\ &\left.-i\frac{\alpha_2(\lambda,\mu)}{\alpha_9(\lambda,\mu)}F(\mu)\vec{C}(\lambda)\right]\otimes\vec{\xi}\end{aligned} \tag{4.40}$$

$$B(\lambda)\vec{B}(\mu) = i\frac{\alpha_2(\mu,\lambda)}{\alpha_9(\lambda,\mu)}\vec{B}(\mu)B(\lambda) - i\frac{\alpha_5(\mu,\lambda)}{\alpha_9(\lambda,\mu)}\vec{B}(\lambda)B(\mu) \tag{4.41}$$

$$\begin{aligned}D(\lambda)\vec{B}(\mu) = &-i\frac{\alpha_8(\lambda,\mu)}{\alpha_7(\lambda,\mu)}\vec{B}(\mu)D(\lambda) + \frac{\alpha_5(\lambda,\mu)}{\alpha_7(\lambda,\mu)}F(\mu)\vec{C^*}(\lambda)\\ &- \frac{\alpha_4(\lambda,\mu)}{\alpha_7(\lambda,\mu)}F(\lambda)\vec{C^*}(\mu) - i\frac{\alpha_{10}(\lambda,\mu)}{\alpha_7(\lambda,\mu)}\vec{\xi}.[\vec{B^*}(\lambda)\otimes\hat{A}(\mu)]\end{aligned} \tag{4.42}$$

The relevant fundamental commutation relationships with $\vec{F}(\lambda)$ are

$$\begin{aligned}\hat{A}_{ab}(\lambda)F(\mu) = &[1+\frac{\alpha_5^2(\lambda,\mu)}{\alpha_9(\lambda,\mu)\alpha_8(\lambda,\mu)}]F(\mu)\hat{A}_{ab}(\lambda)\\ &-\frac{\alpha_5^2(\lambda,\mu)}{\alpha_9(\lambda,\mu)\alpha_8(\lambda,\mu)}F(\lambda)\hat{A}_{ab}(\mu)\\ &+i\frac{\alpha_5(\lambda,\mu)}{\alpha_9(\lambda,\mu)}[\vec{B}(\lambda)\otimes\vec{B^*}(\mu)]_{ba}+i\frac{\alpha_5(\lambda,\mu)}{\alpha_8(\lambda,\mu)}[\vec{B^*}(\lambda)\otimes\vec{B}(\mu)]_{ab}\end{aligned} \tag{4.43}$$

$$B(\lambda)F(\mu) = \frac{\alpha_2(\mu,\lambda)}{\alpha_7(\mu,\lambda)}F(\mu)B(\lambda)-\frac{\alpha_4(\mu,\lambda)}{\alpha_7(\mu,\lambda)}F(\lambda)B(\mu) \tag{4.44}$$

$$+i\frac{\alpha_{10}(\mu,\lambda)}{\alpha_7(\mu,\lambda)}\{\vec{B}(\lambda)\otimes\vec{B}(\mu)\}.\vec{\xi}^t \tag{4.45}$$

$$D(\lambda)F(\mu) = \frac{\alpha_2(\lambda,\mu)}{\alpha_7(\lambda,\mu)}F(\mu)D(\lambda)-\frac{\alpha_4(\lambda,\mu)}{\alpha_7(\lambda,\mu)}F(\lambda)D(\mu) \tag{4.46}$$

$$-i\frac{\alpha_{10}(\lambda,\mu)}{\alpha_7(\lambda,\mu)}\vec{\xi}.\{\vec{B^*}(\lambda)\otimes\vec{B^*}(\mu)\} \tag{4.47}$$

where $\vec{\xi}^t$ stands for the transpose of $\vec{\xi}$.

The relevant fundamental commutation relationships between $\vec{B}(\lambda)$ and $\vec{F}(\lambda)$ are

$$[F(\lambda),F(\mu)] = 0 \tag{4.48}$$

$$F(\lambda)\vec{B}(\mu) = \frac{\alpha_5(\lambda,\mu)}{\alpha_2(\lambda,\mu)}F(\mu)\vec{B}(\lambda)-i\frac{\alpha_8(\lambda,\mu)}{\alpha_2(\lambda,\mu)}\vec{B}(\mu)F(\lambda) \tag{4.49}$$

$$\vec{B}(\lambda)F(\mu) = \frac{\alpha_5(\lambda,\mu)}{\alpha_2(\lambda,\mu)}\vec{B}(\mu)F(\lambda)-i\frac{\alpha_9(\lambda,\mu)}{\alpha_2(\lambda,\mu)}F(\mu)\vec{B}(\lambda) \tag{4.50}$$

There are a few additional commutation relationships required, as follows:

$$C_a(\lambda)B_b(\mu) = -\frac{\alpha_8(\lambda,\mu)}{\alpha_9(\lambda,\mu)}B_b(\mu)C_a(\lambda)+i\frac{\alpha_5(\lambda,\mu)}{\alpha_9(\lambda,\mu)}[B(\mu)A_{ab}(\lambda)-B(\lambda)A_{ab}(\mu)] \tag{4.51}$$

$$B_a^*(\lambda)B_b(\mu) = -\frac{\alpha_8(\lambda,\mu)}{\alpha_9(\lambda,\mu)}B_b(\mu)B_a^*(\lambda)+i\frac{\alpha_5(\lambda,\mu)}{\alpha_9(\lambda,\mu)}[F(\mu)A_{ab}(\lambda)-F(\lambda)A_{ab}(\mu)] \tag{4.52}$$

$$C_a^*(\lambda)B_b(\mu) = \frac{\alpha_3(\lambda,\mu)}{\alpha_7(\lambda,\mu)}B_a(\mu)C_b^*(\lambda) - \frac{\alpha_4(\lambda,\mu)}{\alpha_7(\lambda,\mu)}B_a(\lambda)C_b^*(\mu)$$
$$- \frac{\alpha_6(\lambda,\mu)}{\alpha_7(\lambda,\mu)}B_b(\mu)C_a^*(\lambda) + i\frac{\alpha_{10}(\lambda,\mu)}{\alpha_7(\lambda,\mu)}\xi_{lm}A_{la}(\lambda)A_{mb}(\mu)$$
$$+ i\frac{\alpha_{10}(\lambda,\mu)}{\alpha_7(\lambda,\mu)}\xi_{ab}[F(\mu)C(\lambda) - B(\mu)D(\lambda)] \tag{4.53}$$

We now make the following ansatz for the eigenstates of $\tau(\mu)$:

$$|\Phi_n(\lambda_1,\ldots,\lambda_n)\rangle = \vec{\Phi}_n(\lambda_1,\ldots,\lambda_n).\vec{\mathcal{F}}\,|0\rangle\,, \tag{4.54}$$

where $\vec{\Phi}_n(\lambda_1,\ldots,\lambda_n)$ are composed of creation fields $\vec{B}(\lambda)$ and $F(\lambda)$, for the "charge' degrees of freedom. $\vec{\mathcal{F}}$ represents a linear combination of "spin" degrees of freedom. We denote it as $\mathcal{F}^{a_n\cdots a_1}$, where $a_i = 1, 2$. The action of the transfer matrix on states of the form (4.54) is determined by (4.35) and (4.6).

We first solve for the wavefunction for a single particle excitation, then work towards a general wavefunction for n particle excitations.

We now make an ansatz for the one particle wavefunction:

$$|\Phi_1(\lambda_1)\rangle = \vec{B}(\lambda_1).\vec{\mathcal{F}}\,|0\rangle = B_a(\lambda_1)\mathcal{F}^a\,|0\rangle \tag{4.55}$$

This is consistent with the interpretation of the $\vec{B}(\lambda_1)$ as the creation operator for a single hole excitation.

Acting the diagonal operators on $|\Phi_1(\lambda_1)\rangle$, we find that

$$B(\lambda)\,|\Phi_1(\lambda_1)\rangle = i\frac{\alpha_2(\lambda_1,\lambda)}{\alpha_9(\lambda_1,\lambda)}[\omega_1(\lambda)]^L\,|\Phi_1(\lambda_1)\rangle - i\frac{\alpha_5(\lambda_1,\lambda)}{\alpha_9(\lambda_1,\lambda)}[\omega_1(\lambda_1)]^L\vec{B}(\lambda).\vec{\mathcal{F}}\,|0\rangle \tag{4.56}$$

$$D(\lambda)\,|\Phi_1(\lambda_1)\rangle = -i\frac{\alpha_8(\lambda,\lambda_1)}{\alpha_7(\lambda,\lambda_1)}[\omega_3(\lambda)]^L\,|\Phi_1(\lambda_1)\rangle - i\frac{\alpha_{10}(\lambda,\lambda_1)}{\alpha_7(\lambda,\lambda_1)}[\omega_2(\lambda_1)]^L[\vec{\xi}.(\vec{B^*}(\lambda)\otimes\hat{I})].\vec{\mathcal{F}}\,|0\rangle \tag{4.57}$$

$$\sum_{a=1}^{2}A_{aa}(\lambda)\,|\Phi_1(\lambda_1)\rangle = -i\frac{\alpha_1(\lambda,\lambda_1)}{\alpha_9(\lambda,\lambda_1)}\hat{r}^{a_1b_1}_{c_1a_1}(\lambda,\lambda_1)[\omega_2(\lambda)]^L B_{c_1}(\lambda_1)\mathcal{F}^{b_1}\,|0\rangle + i\frac{\alpha_5(\lambda,\lambda_1)}{\alpha_9(\lambda,\lambda_1)}[\omega_2(\lambda_1)]^L\vec{B}(\lambda).\vec{\mathcal{F}}\,|0\rangle - i\frac{\alpha_{10}(\lambda,\lambda_1)}{\alpha_7(\lambda,\lambda_1)}[\omega_1(\lambda_1)]^L[\vec{\xi}.(\vec{B^*}(\lambda)\otimes\hat{I})].\vec{\mathcal{F}}\,|0\rangle \tag{4.58}$$

The wanted terms are proportional to $|\Phi_1(\lambda_1)\rangle$ on the right hand side of the equations (4.56),(4.58) and (4.57), while the unwanted terms cancel out completely if we constrain λ_1 with the Bethe ansatz equation

$$\left[\frac{\omega_1(\lambda_1)}{\omega_2(\lambda_1)}\right]^L = 1 \tag{4.59}$$

Having solved the "charge" degrees of freedom for one particle excitation, we now define the define the auxiliary eigenvalue problem for the "spin" degrees of freedom:

$$T^{(1)}(\lambda,\lambda_1)^{a_1}_{b_1}\mathcal{F}^{a_1} = \hat{r}^{\alpha a_1}_{b_1\alpha}(\lambda,\lambda_1)\mathcal{F}^{a_1} = \Lambda^{(1)}(\lambda,\lambda_1)\mathcal{F}^{b_1}$$

This problem is the same as we encountered previously in the Heisenberg XXX model and the nested level of the supersymmetric t-J model. We will solve the auxiliary problem in full later, and we would obtain

$$\Lambda^{(1)}(\lambda,\lambda_1) = 1 + \bar{b}(\lambda,\lambda_1)$$

Now, summing up the eigenfunctions proportional to $|\Phi_1(\lambda_1)\rangle$ on the right hand side of the equations (4.56),(4.58) and (4.57) using (4.27), we obtain

the one-particle eigenvalue of the transfer matrix as

$$\begin{aligned}\Lambda(\lambda,\lambda_1) &= i\frac{\alpha_2(\lambda_1,\lambda)}{\alpha_9(\lambda_1,\lambda)}[\omega_1(\lambda)]^L - i\frac{\alpha_8(\lambda,\lambda_1)}{\alpha_7(\lambda,\lambda_1)}[\omega_3(\lambda)]^L \\ &\quad + i\frac{\alpha_1(\lambda,\lambda_1)}{\alpha_9(\lambda,\lambda_1)}\Lambda^{(1)}(\lambda,\lambda_1)[\omega_2(\lambda)]^L \\ &= i\frac{\alpha_2(\lambda_1,\lambda)}{\alpha_9(\lambda_1,\lambda)}[\omega_1(\lambda)]^L - i\frac{\alpha_8(\lambda,\lambda_1)}{\alpha_7(\lambda,\lambda_1)}[\omega_3(\lambda)]^L \\ &\quad + i\frac{\alpha_1(\lambda,\lambda_1)}{\alpha_9(\lambda,\lambda_1)}(1+\bar{b}(\lambda,\lambda_1))[\omega_2(\lambda)]^L\end{aligned} \tag{4.60}$$

Now, moving on the to the two particle excitation, we make an ansatz for its wavefunction of the "charge" degrees of freedom:

$$\vec{\Phi}_2(\lambda_1,\lambda_2) = \vec{B}(\lambda_1)\otimes\vec{B}(\lambda_2) + \vec{\xi}F(\lambda_1)B(\lambda_2)\hat{g}_0^{(2)}(\lambda_1,\lambda_2) \tag{4.61}$$

where $\hat{g}_0^{(2)}(\lambda_1,\lambda_2)$ is an arbitrary function to be determined later. This is the most general ansatz we can make given the interpretation of $\vec{B}(\lambda)$ as the creation operator for a single hole excitation and $\vec{F}(\lambda)$ as the creation operator for a local hole pair with opposite spins, with $\vec{\xi}$ imposing the Pauli exclusion principle. Note that the diagonal operator $B(\lambda)$ is also included here, as it turns out to be useful to include it in the following derivations.

We first proceed to find the action of $D(\lambda)$ on $\vec{\Phi}_2(\lambda_1,\lambda_2)$

$$D(\lambda)\vec{\Phi}_2(\lambda_1,\lambda_2) = D(\lambda)\vec{B}(\lambda_1)\otimes\vec{B}(\lambda_2) + D(\lambda)\vec{\xi}F(\lambda_1)B(\lambda_2)\hat{g}_0^{(2)}(\lambda_1,\lambda_2) \tag{4.62}$$

We first look at $D(\lambda)\vec{B}(\lambda_1)\otimes\vec{B}(\lambda_2)$ using (4.42), which gives

$$\begin{aligned}D(\lambda)\vec{B}(\lambda_1)\otimes\vec{B}(\lambda_2) = \Bigg[&-i\frac{\alpha_8(\lambda,\lambda_1)}{\alpha_7(\lambda,\lambda_1)}\vec{B}(\lambda_1)D(\lambda) + \frac{\alpha_5(\lambda,\lambda_1)}{\alpha_7(\lambda,\lambda_1)}F(\lambda_1)\vec{C}^*(\lambda) \\ &-\frac{\alpha_4(\lambda,\lambda_1)}{\alpha_7(\lambda,\lambda_1)}F(\lambda)\vec{C}^*(\lambda_1) - i\frac{\alpha_{10}(\lambda,\lambda_1)}{\alpha_7(\lambda,\lambda_1)}\vec{\xi}.(\vec{B}^*(\lambda)\otimes\hat{A}(\lambda_1))\Bigg] \\ &\otimes\vec{B}(\lambda_2)\end{aligned} \tag{4.63}$$

Now, using (4.42), (4.53) and (4.40), dropping the terms that have annihilation operators (C and C^*) on the rightmost, we have

$$
\begin{aligned}
&D(\lambda)\vec{B}(\lambda_1)\otimes\vec{B}(\lambda_2)\\
&=-i\frac{\alpha_8(\lambda,\lambda_1)}{\alpha_7(\lambda,\lambda_1)}\vec{B}(\lambda_1)\otimes\left[-i\frac{\alpha_8(\lambda,\lambda_2)}{\alpha_7(\lambda,\lambda_2)}\vec{B}(\lambda_2)D(\lambda)-i\frac{\alpha_{10}(\lambda,\lambda_2)}{\alpha_7(\lambda,\lambda_2)}\vec{\xi}.[\vec{B^*}(\lambda)\otimes\hat{A}(\lambda_2)]\right]\\
&\quad+\frac{\alpha_5(\lambda,\lambda_1)}{\alpha_7(\lambda,\lambda_1)}F(\lambda_1)\left[i\frac{\alpha_{10}(\lambda,\lambda_2)}{\alpha_7(\lambda,\lambda_2)}\vec{\xi}.[\hat{A}(\lambda)\otimes\hat{A}(\lambda_2)]-i\frac{\alpha_{10}(\lambda,\lambda_2)}{\alpha_7(\lambda,\lambda_2)}\vec{\xi}B(\lambda_2)D(\lambda)\right]\\
&\quad-\frac{\alpha_4(\lambda,\lambda_1)}{\alpha_7(\lambda,\lambda_1)}F(\lambda)\left[i\frac{\alpha_{10}(\lambda_1,\lambda_2)}{\alpha_7(\lambda_1,\lambda_2)}\vec{\xi}.[\hat{A}(\lambda_1)\otimes\hat{A}(\lambda_2)]-i\frac{\alpha_{10}(\lambda_1,\lambda_2)}{\alpha_7(\lambda_1,\lambda_2)}\vec{\xi}B(\lambda_2)D(\lambda_1)\right]\\
&\quad-i\frac{\alpha_{10}(\lambda,\lambda_1)}{\alpha_7(\lambda,\lambda_1)}\vec{\xi}.\left(\vec{B^*}(\lambda)\otimes\left[-i\frac{\alpha_1(\lambda_1,\lambda_2)}{\alpha_9(\lambda_1,\lambda_2)}[\vec{B}(\lambda_2)\otimes\hat{A}(\lambda_1)].\hat{r}(\lambda_1,\lambda_2)\right.\right.\\
&\quad\left.\left.+i\frac{\alpha_5(\lambda_1,\lambda_2)}{\alpha_9(\lambda_1,\lambda_2)}\vec{B}(\lambda_1)\otimes\hat{A}(\lambda_2)-i\frac{\alpha_{10}(\lambda_1,\lambda_2)}{\alpha_7(\lambda_1,\lambda_2)}\vec{B^*}(\lambda_1)B(\lambda_2)\otimes\vec{\xi}\right]\right)
\end{aligned}
\tag{4.64}
$$

Now, using (4.47), we get

$$
\begin{aligned}
D(\lambda)\vec{\xi}F(\lambda_1)B(\lambda_2)\hat{g}_0^{(2)}(\lambda_1,\lambda_2)=\vec{\xi}\Bigg[&\frac{\alpha_2(\lambda,\lambda_1)}{\alpha_7(\lambda,\lambda_1)}F(\lambda_1)D(\lambda)-\frac{\alpha_4(\lambda,\lambda_1)}{\alpha_7(\lambda,\lambda_1)}F(\lambda)D(\lambda_1)\\
&-i\frac{\alpha_{10}(\lambda,\lambda_1)}{\alpha_7(\lambda,\lambda_1)}\vec{\xi}.\{\vec{B^*}(\lambda)\otimes\vec{B^*}(\lambda_1)\}\Bigg]B(\lambda_2)\hat{g}_0^{(2)}(\lambda_1,\lambda_2)
\end{aligned}
\tag{4.65}
$$

Combining (4.64) and (4.65), we have

$$
\begin{aligned}
&D(\lambda)\vec{\Phi}_2(\lambda_1,\lambda_2)\\
&= -i\frac{\alpha_8(\lambda,\lambda_1)}{\alpha_7(\lambda,\lambda_1)}\vec{B}(\lambda_1)\otimes\left[-i\frac{\alpha_8(\lambda,\lambda_2)}{\alpha_7(\lambda,\lambda_2)}\vec{B}(\lambda_2)D(\lambda)-i\frac{\alpha_{10}(\lambda,\lambda_2)}{\alpha_7(\lambda,\lambda_2)}\vec{\xi}.[\vec{B^*}(\lambda)\otimes\hat{A}(\lambda_2)]\right]\\
&\quad+\frac{\alpha_5(\lambda,\lambda_1)}{\alpha_7(\lambda,\lambda_1)}F(\lambda_1)\left[i\frac{\alpha_{10}(\lambda,\lambda_2)}{\alpha_7(\lambda,\lambda_2)}\vec{\xi}.[\hat{A}(\lambda)\otimes\hat{A}(\lambda_2)]-i\frac{\alpha_{10}(\lambda,\lambda_2)}{\alpha_7(\lambda,\lambda_2)}\vec{\xi}B(\lambda_2)D(\lambda)\right]\\
&\quad-\frac{\alpha_4(\lambda,\lambda_1)}{\alpha_7(\lambda,\lambda_1)}F(\lambda)\left[i\frac{\alpha_{10}(\lambda_1,\lambda_2)}{\alpha_7(\lambda_1,\lambda_2)}\vec{\xi}.[\hat{A}(\lambda_1)\otimes\hat{A}(\lambda_2)]-i\frac{\alpha_{10}(\lambda_1,\lambda_2)}{\alpha_7(\lambda_1,\lambda_2)}\vec{\xi}B(\lambda_2)D(\lambda_1)\right]\\
&\quad-i\frac{\alpha_{10}(\lambda,\lambda_1)}{\alpha_7(\lambda,\lambda_1)}\vec{\xi}.\left(\vec{B^*}(\lambda)\otimes\left[-i\frac{\alpha_1(\lambda_1,\lambda_2)}{\alpha_9(\lambda_1,\lambda_2)}[\vec{B}(\lambda_2)\otimes\hat{A}(\lambda_1)].\hat{r}(\lambda_1,\lambda_2)\right.\right.\\
&\quad\left.\left.+i\frac{\alpha_5(\lambda_1,\lambda_2)}{\alpha_9(\lambda_1,\lambda_2)}\vec{B}(\lambda_1)\otimes\hat{A}(\lambda_2)-i\frac{\alpha_{10}(\lambda_1,\lambda_2)}{\alpha_7(\lambda_1,\lambda_2)}\vec{B^*}(\lambda_1)B(\lambda_2)\otimes\vec{\xi}\right]\right)\\
&\quad+\vec{\xi}\left[\frac{\alpha_2(\lambda,\lambda_1)}{\alpha_7(\lambda,\lambda_1)}F(\lambda_1)D(\lambda)-\frac{\alpha_4(\lambda,\lambda_1)}{\alpha_7(\lambda,\lambda_1)}F(\lambda)D(\lambda_1)\right.\\
&\quad\left.-i\frac{\alpha_{10}(\lambda,\lambda_1)}{\alpha_7(\lambda,\lambda_1)}\vec{\xi}.\{\vec{B^*}(\lambda)\otimes\vec{B^*}(\lambda_1)\}\right]B(\lambda_2)\hat{g}_0^{(2)}(\lambda_1,\lambda_2) \qquad (4.66)
\end{aligned}
$$

To eliminate the unwanted terms of the type

$$F(\lambda)D(\lambda_1)B(\lambda_2)\ \ ;\ \ \vec{\xi}.[\vec{B^*}(\lambda)\otimes\vec{B^*}(\lambda_1)]B(\lambda_2) \qquad (4.67)$$

we set $\hat{g}_0^{(2)}(\lambda_1,\lambda_2)=i\frac{\alpha_{10}(\lambda_1,\lambda_2)}{\alpha_7(\lambda_1,\lambda_2)}$ and we get:

$$
\begin{aligned}
&D(\lambda)\vec{\Phi}_2(\lambda_1,\lambda_2)\\
&= i\frac{\alpha_8(\lambda,\lambda_1)}{\alpha_7(\lambda,\lambda_1)}\vec{B}(\lambda_1)\otimes\left[i\frac{\alpha_8(\lambda,\lambda_2)}{\alpha_7(\lambda,\lambda_2)}\vec{B}(\lambda_2)D(\lambda)-i\frac{\alpha_{10}(\lambda,\lambda_2)}{\alpha_7(\lambda,\lambda_2)}\vec{\xi}.[\vec{B^*}(\lambda)\otimes\hat{A}(\lambda_2)]\right]\\
&\quad+\frac{\alpha_5(\lambda,\lambda_1)}{\alpha_7(\lambda,\lambda_1)}F(\lambda_1)\left[i\frac{\alpha_{10}(\lambda,\lambda_2)}{\alpha_7(\lambda,\lambda_2)}\vec{\xi}.[\hat{A}(\lambda)\otimes\hat{A}(\lambda_2)]-i\frac{\alpha_{10}(\lambda,\lambda_2)}{\alpha_7(\lambda,\lambda_2)}\vec{\xi}B(\lambda_2)D(\lambda)\right]\\
&\quad-\frac{\alpha_4(\lambda,\lambda_1)}{\alpha_7(\lambda,\lambda_1)}F(\lambda)\left[i\frac{\alpha_{10}(\lambda_1,\lambda_2)}{\alpha_7(\lambda_1,\lambda_2)}\vec{\xi}.[\hat{A}(\lambda_1)\otimes\hat{A}(\lambda_2)]\right]\\
&\quad-i\frac{\alpha_{10}(\lambda,\lambda_1)}{\alpha_7(\lambda,\lambda_1)}\vec{\xi}.\left(\vec{B^*}(\lambda)\otimes\left[-i\frac{\alpha_1(\lambda_1,\lambda_2)}{\alpha_9(\lambda_1,\lambda_2)}[\vec{B}(\lambda_2)\otimes\hat{A}(\lambda_1)].\hat{r}(\lambda_1,\lambda_2)\right.\right.\\
&\quad\left.\left.+i\frac{\alpha_5(\lambda_1,\lambda_2)}{\alpha_9(\lambda_1,\lambda_2)}\vec{B}(\lambda_1)\otimes\hat{A}(\lambda_2)\right]\right)\\
&\quad+\vec{\xi}\frac{\alpha_2(\lambda,\lambda_1)}{\alpha_7(\lambda,\lambda_1)}F(\lambda_1)D(\lambda)B(\lambda_2)i\frac{\alpha_{10}(\lambda_1,\lambda_2)}{\alpha_7(\lambda_1,\lambda_2)} \qquad (4.68)
\end{aligned}
$$

To make it easier to grouping terms together, we exchange $\lambda \leftrightarrow \lambda_1$ in the term $\vec{B}(\lambda_1) \otimes \vec{B^*}(\lambda) B(\lambda_2)$ using (4.52):

$$
\begin{aligned}
& D(\lambda)\vec{\Phi}_2(\lambda_1,\lambda_2) \\
& = -i\frac{\alpha_8(\lambda,\lambda_1)}{\alpha_7(\lambda,\lambda_1)} - i\frac{\alpha_8(\lambda,\lambda_2)}{\alpha_7(\lambda,\lambda_2)}\vec{B}(\lambda_1)\otimes\vec{B}(\lambda_2)D(\lambda) \\
& \quad + i\frac{\alpha_8(\lambda,\lambda_1)}{\alpha_7(\lambda,\lambda_1)} i\frac{\alpha_{10}(\lambda,\lambda_2)}{\alpha_7(\lambda,\lambda_2)} \left(-\frac{\alpha_8(\lambda_1,\lambda)}{\alpha_9(\lambda_1,\lambda)}\vec{B}(\lambda)\otimes\vec{\xi}.[\vec{B^*}(\lambda_1)\otimes\hat{A}(\lambda_2)] \right. \\
& \quad \left. + i\frac{\alpha_5(\lambda_1,\lambda)}{\alpha_9(\lambda_1,\lambda)}\vec{\xi}.[F(\lambda)\hat{A}(\lambda_1)\otimes\hat{A}(\lambda_2)] - F(\lambda_1)\hat{A}(\lambda)\otimes\hat{A}(\lambda_2)] \right) \\
& \quad + \frac{\alpha_5(\lambda,\lambda_1)}{\alpha_7(\lambda,\lambda_1)}F(\lambda_1)\left[i\frac{\alpha_{10}(\lambda,\lambda_2)}{\alpha_7(\lambda,\lambda_2)}\vec{\xi}.[\hat{A}(\lambda)\otimes\hat{A}(\lambda_2)] - i\frac{\alpha_{10}(\lambda,\lambda_2)}{\alpha_7(\lambda,\lambda_2)}\vec{\xi}B(\lambda_2)D(\lambda)\right] \\
& \quad - \frac{\alpha_4(\lambda,\lambda_1)}{\alpha_7(\lambda,\lambda_1)}F(\lambda)\left[i\frac{\alpha_{10}(\lambda_1,\lambda_2)}{\alpha_7(\lambda_1,\lambda_2)}\vec{\xi}.[\hat{A}(\lambda_1)\otimes\hat{A}(\lambda_2)]\right] - i\frac{\alpha_{10}(\lambda,\lambda_1)}{\alpha_7(\lambda,\lambda_1)}\vec{\xi}.\left(\vec{B^*}(\lambda)\right. \\
& \quad \left. \otimes\left[-i\frac{\alpha_1(\lambda_1,\lambda_2)}{\alpha_9(\lambda_1,\lambda_2)}[\vec{B}(\lambda_2)\otimes\hat{A}(\lambda_1)].\hat{r}(\lambda_1,\lambda_2) + i\frac{\alpha_5(\lambda_1,\lambda_2)}{\alpha_9(\lambda_1,\lambda_2)}\vec{B}(\lambda_1)\otimes\hat{A}(\lambda_2)\right]\right) \\
& \quad + \vec{\xi}\frac{\alpha_2(\lambda,\lambda_1)}{\alpha_7(\lambda,\lambda_1)}F(\lambda_1)D(\lambda)B(\lambda_2)i\frac{\alpha_{10}(\lambda_1,\lambda_2)}{\alpha_7(\lambda_1,\lambda_2)}
\end{aligned}
\tag{4.69}
$$

The long expression for $D(\lambda)$ will be simplified later. We now consider the action of the diagonal operator $B(\lambda)$, using (4.41) and (4.45), and we can obtain:

$$
B(\lambda)\vec{\Phi}_2(\lambda_1,\lambda_2) = B(\lambda)\vec{B}(\lambda_1)\otimes\vec{B}(\lambda_2) + B(\lambda)\vec{\xi}F(\lambda_1)B(\lambda_2)\hat{g}_0^{(2)}(\lambda_1,\lambda_2),
\tag{4.70}
$$

$$
\begin{aligned}
& B(\lambda)\vec{B}(\lambda_1)\otimes\vec{B}(\lambda_2) \\
& = \left[i\frac{\alpha_2(\lambda_1,\lambda)}{\alpha_9(\lambda,\lambda_1)}\vec{B}(\lambda_1)B(\lambda) - i\frac{\alpha_5(\lambda_1,\lambda)}{\alpha_9(\lambda,\lambda_1)}\vec{B}(\lambda)B(\lambda_1)\right]\otimes\vec{B}(\lambda_2) \\
& = i\frac{\alpha_2(\lambda_1,\lambda)}{\alpha_9(\lambda,\lambda_1)}\vec{B}(\lambda_1)\left(i\frac{\alpha_2(\lambda_2,\lambda)}{\alpha_9(\lambda,\lambda_2)}\vec{B}(\lambda_2)B(\lambda) - i\frac{\alpha_5(\lambda_2,\lambda)}{\alpha_9(\lambda,\lambda_2)}\vec{B}(\lambda)B(\lambda_2)\right) \\
& \quad - i\frac{\alpha_5(\lambda_1,\lambda)}{\alpha_9(\lambda,\lambda_1)}\vec{B}(\lambda)\left(i\frac{\alpha_2(\lambda_2,\lambda_1)}{\alpha_9(\lambda_1,\lambda_2)}\vec{B}(\lambda_2)B(\lambda_1) - i\frac{\alpha_5(\lambda_2,\lambda_1)}{\alpha_9(\lambda_1,\lambda_2)}\vec{B}(\lambda_1)B(\lambda_2)\right)
\end{aligned}
\tag{4.71}
$$

$$B(\lambda)\vec{\xi}F(\lambda_1)B(\lambda_2)\hat{g}_0^{(2)}(\lambda_1,\lambda_2) = \vec{\xi}\left(\frac{\alpha_2(\lambda_1,\lambda)}{\alpha_7(\lambda_1,\lambda)}F(\lambda_1)B(\lambda) - \frac{\alpha_4(\lambda_1,\lambda)}{\alpha_7(\lambda_1,\lambda)}F(\lambda)B(\lambda_1)\right.$$
$$\left. + i\frac{\alpha_{10}(\lambda_1,\lambda)}{\alpha_7(\lambda_1,\lambda)}\{\vec{B}(\lambda)\otimes\vec{B}(\lambda_1)\}.\vec{\xi}^t\right) B(\lambda_2)\hat{g}_0^{(2)}(\lambda_1,\lambda_2) \tag{4.72}$$

Combining (4.71) and (4.72), we get:

$$\begin{aligned}
B(\lambda)\vec{\Phi}_2(\lambda_1,\lambda_2) = {} & i\frac{\alpha_2(\lambda_1,\lambda)}{\alpha_9(\lambda,\lambda_1)}\vec{B}(\lambda_1)\left(i\frac{\alpha_2(\lambda_2,\lambda)}{\alpha_9(\lambda,\lambda_2)}\vec{B}(\lambda_2)B(\lambda) - i\frac{\alpha_5(\lambda_2,\lambda)}{\alpha_9(\lambda,\lambda_2)}\vec{B}(\lambda)B(\lambda_2)\right) \\
& - i\frac{\alpha_5(\lambda_1,\lambda)}{\alpha_9(\lambda,\lambda_1)}\vec{B}(\lambda)\left(i\frac{\alpha_2(\lambda_2,\lambda_1)}{\alpha_9(\lambda_1,\lambda_2)}\vec{B}(\lambda_2)B(\lambda_1) - i\frac{\alpha_5(\lambda_2,\lambda_1)}{\alpha_9(\lambda_1,\lambda_2)}\vec{B}(\lambda_1)B(\lambda_2)\right) \\
& + \vec{\xi}\left(\frac{\alpha_2(\lambda_1,\lambda)}{\alpha_7(\lambda_1,\lambda)}F(\lambda_1)B(\lambda) - \frac{\alpha_4(\lambda_1,\lambda)}{\alpha_7(\lambda_1,\lambda)}F(\lambda)B(\lambda_1)\right. \\
& \left. + i\frac{\alpha_{10}(\lambda_1,\lambda)}{\alpha_7(\lambda_1,\lambda)}\{\vec{B}(\lambda)\otimes\vec{B}(\lambda_1)\}.\vec{\xi}^t\right) B(\lambda_2)\hat{g}_0^{(2)}(\lambda_1,\lambda_2)
\end{aligned} \tag{4.73}$$

From (4.73), to rearrange the terms in a more convenient form, we exchange $\lambda \leftrightarrow \lambda_1$ in the term $\vec{B}(\lambda_1)\otimes\vec{B}(\lambda)B(\lambda_2)$ using (4.36) and we get:

$$\begin{aligned}
B(\lambda)\vec{\Phi}_2(\lambda_1,\lambda_2) = {} & i\frac{\alpha_2(\lambda_1,\lambda)}{\alpha_9(\lambda,\lambda_1)}i\frac{\alpha_2(\lambda_2,\lambda)}{\alpha_9(\lambda,\lambda_2)}\vec{B}(\lambda_1)\otimes\vec{B}(\lambda_2)B(\lambda) \\
& - i\frac{\alpha_2(\lambda_1,\lambda)}{\alpha_9(\lambda,\lambda_1)}i\frac{\alpha_5(\lambda_2,\lambda)}{\alpha_9(\lambda,\lambda_2)}\left[\frac{\alpha_1(\lambda_1,\lambda)}{\alpha_2(\lambda_1,\lambda)}[\vec{B}(\lambda)\otimes\vec{B}(\lambda_1)].\hat{r}(\lambda_1,\lambda)\right. \\
& \left. - i\frac{\alpha_{10}(\lambda_1,\lambda)}{\alpha_7(\lambda_1,\lambda)}\{F(\lambda_1)B(\lambda) - F(\lambda)B(\lambda_1)\}\vec{\xi}\right] B(\lambda_2) \\
& - i\frac{\alpha_5(\lambda_1,\lambda)}{\alpha_9(\lambda,\lambda_1)}\vec{B}(\lambda)\left(i\frac{\alpha_2(\lambda_2,\lambda_1)}{\alpha_9(\lambda_1,\lambda_2)}\vec{B}(\lambda_2)B(\lambda_1) - i\frac{\alpha_5(\lambda_2,\lambda_1)}{\alpha_9(\lambda_1,\lambda_2)}\vec{B}(\lambda_1)B(\lambda_2)\right) \\
& + \vec{\xi}\left(\frac{\alpha_2(\lambda_1,\lambda)}{\alpha_7(\lambda_1,\lambda)}F(\lambda_1)B(\lambda) - \frac{\alpha_4(\lambda_1,\lambda)}{\alpha_7(\lambda_1,\lambda)}F(\lambda)B(\lambda_1)\right. \\
& \left. + i\frac{\alpha_{10}(\lambda_1,\lambda)}{\alpha_7(\lambda_1,\lambda)}\{\vec{B}(\lambda)\otimes\vec{B}(\lambda_1)\}.\vec{\xi}^t\right) B(\lambda_2)\hat{g}_0^{(2)}(\lambda_1,\lambda_2)
\end{aligned} \tag{4.74}$$

The same bruteforce procedure can be done for $A(\lambda)$. At this point, we note that, as before, it is cumbersome and not very illuminating. However, Martins and Ramos have found a simplification of the expressions of $D(\lambda)$

and $B(\lambda)$, by simply grouping of relevant terms in (4.69) and (4.74) and using identities between the Boltzmann weights.

We can then obtain the action of the diagonal operators as follows:

$$\begin{aligned} B(\lambda)\left|\Phi_2(\lambda_1,\lambda_2)\right\rangle &= [\omega_1(\lambda)]^L \prod_{j=1}^{2} i\frac{\alpha_2(\lambda_j,\lambda)}{\alpha_9(\lambda_j,\lambda)}\left|\Phi_2(\lambda_1,\lambda_2)\right\rangle \\ &\quad - \sum_{j=1}^{2}[\omega_1(\lambda_j)]^L \left|\Psi_1^{(1)}(\lambda,\lambda_j;\{\lambda_l\})\right\rangle \\ &\quad + H_1(\lambda,\lambda_1,\lambda_2)[\omega_1(\lambda_1)\omega_1(\lambda_2)]^L \left|\Psi_0^{(3)}(\lambda,\lambda_j,\lambda_l;\{\lambda_k\})\right\rangle \end{aligned} \tag{4.75}$$

$$\begin{aligned} D(\lambda)\left|\Phi_2(\lambda_1,\lambda_2)\right\rangle &= [\omega_3(\lambda)]^L \prod_{j=1}^{2} -i\frac{\alpha_8(\lambda,\lambda_j)}{\alpha_7(\lambda,\lambda_j)}\left|\Phi_2(\lambda_1,\lambda_2)\right\rangle \\ &\quad - \sum_{j=1}^{2}[\omega_2(\lambda_j)]^L\Lambda^{(1)}(\lambda=\lambda_j,\{\lambda_l\}) \left|\Psi_1^{(2)}(\lambda,\lambda_j;\{\lambda_l\})\right\rangle \\ &\quad + H_2(\lambda,\lambda_1,\lambda_2)[\omega_2(\lambda_1)\omega_2(\lambda_2)]^L \left|\Psi_0^{(3)}(\lambda,\lambda_j,\lambda_l;\{\lambda_k\})\right\rangle \end{aligned}$$

$$\begin{aligned} \sum_{a=1}^{2}\hat{A}_{aa}(\lambda)\left|\Phi_2(\lambda_1,\lambda_2)\right\rangle &= [\omega_2(\lambda)]^L \prod_{j=1}^{2} -i\frac{\alpha_1(\lambda,\lambda_j)}{\alpha_9(\lambda,\lambda_j)}\Lambda^{(1)}(\lambda,\{\lambda_l\})\left|\Phi_2(\lambda_1,\lambda_2)\right\rangle \\ &\quad - \sum_{j=1}^{2}[\omega_2(\lambda_j)]^L\Lambda^{(1)}(\lambda=\lambda_j,\{\lambda_l\}) \left|\Psi_1^{(1)}(\lambda,\lambda_j;\{\lambda_l\})\right\rangle \\ &\quad - \sum_{j=1}^{2}[\omega_1(\lambda_j)]^L \left|\Psi_1^{(2)}(\lambda,\lambda_j;\{\lambda_l\})\right\rangle \\ &\quad + H_3(\lambda,\lambda_1,\lambda_2)[\omega_1(\lambda_1)\omega_2(\lambda_2)]^L \left|\Psi_0^{(3)}(\lambda,\lambda_j,\lambda_l;\{\lambda_k\})\right\rangle \\ &\quad + H_4(\lambda,\lambda_1,\lambda_2)[\omega_1(\lambda_2)\omega_2(\lambda_1)]^L \left|\Psi_0^{(3)}(\lambda,\lambda_j,\lambda_l;\{\lambda_k\})\right\rangle \end{aligned} \tag{4.76}$$

where the unwanted terms are of the form:

$$\left|\Psi_1^{(1)}(\lambda,\lambda_j;\{\lambda_l\})\right\rangle = i\frac{\alpha_5(\lambda_j,\lambda)}{\alpha_9(\lambda_j,\lambda)}\prod_{\substack{k=1\\k\neq j}}^{2} i\frac{\alpha_2(\lambda_k,\lambda_j)}{\alpha_9(\lambda_k,\lambda_j)} \times \vec{B}(\lambda)\otimes\vec{\Phi}_1(\lambda_k)\hat{O}_j^{(1)}(\lambda_j;\{\lambda_k\}).\vec{\mathcal{F}}\,|0\rangle \tag{4.77}$$

$$\left|\Psi_1^{(2)}(\lambda,\lambda_j;\{\lambda_l\})\right\rangle = i\frac{\alpha_{10}(\lambda,\lambda_j)}{\alpha_7(\lambda,\lambda_j)}\prod_{\substack{k=1\\k\neq j}}^{2} i\frac{\alpha_2(\lambda_k,\lambda_j)}{\alpha_9(\lambda_k,\lambda_j)} \times [\vec{\xi}.(\vec{B}^*(\lambda)\otimes\hat{I})]\otimes\vec{\Phi}_1(\lambda_k)\hat{O}_j^{(1)}(\lambda_j;\{\lambda_k\}).\vec{\mathcal{F}}\,|0\rangle \tag{4.78}$$

$$\left|\Psi_0^{(3)}(\lambda,\lambda_j,\lambda_l;\{\lambda_k\})\right\rangle = F(\lambda)\vec{\xi}.\vec{\mathcal{F}}\,|0\rangle \tag{4.79}$$

and the "ordering" factor $\hat{O}_j^{(1)}(\lambda_j;\{\lambda_k\})$ is defined as

$$\hat{O}_j^{(1)}(\lambda_j;\{\lambda_k\}) = \prod_{k=1}^{j-1}\frac{\alpha_1(\lambda_k,\lambda_j)}{\alpha_2(\lambda_k,\lambda_j)}\hat{r}_{k,k+1}(\lambda_k,\lambda_j) \tag{4.80}$$

and the functions $H_i(x,y,z)\ ;\ i=1,\ldots,4$ are:

$$\begin{aligned}
H_1(x,y,z) &= i\frac{\alpha_2(y,x)\alpha_5(z,x)\alpha_{10}(y,x)}{\alpha_9(y,x)\alpha_9(z,x)\alpha_7(y,x)} - i\frac{\alpha_4(y,x)\alpha_{10}(y,z)}{\alpha_7(y,x)\alpha_7(y,z)}\\
H_2(x,y,z) &= i\frac{\alpha_5(x,y)\alpha_{10}(x,z)}{\alpha_7(x,y)\alpha_7(x,z)} - i\frac{\alpha_4(x,y)\alpha_{10}(y,z)}{\alpha_7(x,y)\alpha_7(y,z)}\\
H_3(x,y,z) &= i\frac{\alpha_{10}(x,y)\alpha_5(x,y)\alpha_5(y,z)}{\alpha_7(x,y)\alpha_9(x,y)\alpha_9(y,z)} - i\frac{\alpha_2(x,y)\alpha_5(x,z)\alpha_{10}(x,y)}{\alpha_9(x,y)\alpha_9(x,z)\alpha_7(x,y)}\\
H_4(x,y,z) &= -i\frac{\alpha_{10}(x,y)\alpha_5(x,y)\alpha_5(y,z)}{\alpha_7(x,y)\alpha_9(x,y)\alpha_9(y,z)}\\
&\quad + i\frac{\alpha_1(x,y)\alpha_{10}(x,z)\alpha_5(x,y)[1+\bar{a}(x,y)]}{\alpha_9(x,y)\alpha_7(x,z)\alpha_8(x,y)}\\
&\quad - 2i\frac{\alpha_5^2(x,y)\alpha_{10}(y,z)}{\alpha_8(x,y)\alpha_9(x,y)\alpha_7(y,z)}
\end{aligned}$$

Having solved the "charge" degrees of freedom for the two particle excitation wavefunction, we now define the auxiliary problem for the "spin"

degrees of freedom:

$$T^{(1)}(\lambda,\{\lambda_l\})^{a_1a_2}_{b_1b_2}\mathcal{F}^{a_2a_1}=\hat{r}^{c_1a_1}_{b_1d_1}(\lambda,\lambda_1)\hat{r}^{d_1a_2}_{b_2c_1}(\lambda,\lambda_2)\mathcal{F}^{a_2a_1}=\Lambda^{(1)}(\lambda,\{\lambda_l\})\mathcal{F}^{b_2b_1} \tag{4.81}$$

To cancel the unwanted terms in the action of diagonal operators, we constrain the spectral parameters using the Bethe ansatz equation:

$$\left[\frac{\omega_1(\lambda_i)}{\omega_2(\lambda_i)}\right]^L=\Lambda^{(1)}(\lambda=\lambda_i,\{\lambda_j\}),\quad i=1,2 \tag{4.82}$$

And we would obtain the two-particle eigenvalue:

$$\begin{aligned}\Lambda(\lambda,\{\lambda_i\})=[\omega_1(\lambda)]^L\prod_{i=1}^{2}i\frac{\alpha_2(\lambda_i,\lambda)}{\alpha_9(\lambda_i,\lambda)}+[\omega_3(\lambda)]^L\prod_{i=1}^{2}-i\frac{\alpha_8(\lambda,\lambda_i)}{\alpha_7(\lambda,\lambda_i)}\\ -[\omega_2(\lambda)]^L\prod_{i=1}^{2}-i\frac{\alpha_1(\lambda,\lambda_i)}{\alpha_9(\lambda,\lambda_i)}\Lambda^{(1)}(\lambda,\{\lambda_j\})\end{aligned} \tag{4.83}$$

Again, we leave the solving of the auxiliary problem ($\Lambda^{(1)}(\lambda,\{\lambda_j\})$) to later.

Noting that

$$\vec{\xi}.\hat{r}(\lambda,\mu)=\frac{\alpha_{10}(\lambda,\mu)\alpha_7(\mu,\lambda)\alpha_2(\lambda,\mu)}{\alpha_7(\lambda,\mu)\alpha_{10}(\mu,\lambda)\alpha_1(\lambda,\mu)}\vec{\xi}, \tag{4.84}$$

we can verify that the two-particle vector satisfies:

$$\vec{\Phi}_2(\lambda_1,\lambda_2)=\frac{\alpha_1(\lambda_1,\lambda_2)}{\alpha_2(\lambda_1,\lambda_2)}\vec{\Phi}_2(\lambda_2,\lambda_1).\hat{r}(\lambda_1,\lambda_2), \tag{4.85}$$

which is a property of the two-particle vector under exchange of the spectral parameters. This is similar to the braiding statistics that arises in exchange of non-abelian anyons [82].

This exchange property can be generalized to the n-particle vector:

$$\begin{aligned}&\vec{\Phi}_n(\lambda_1,\ldots,\lambda_{j-1},\lambda_j,\ldots,\lambda_n)\\&=\frac{\alpha_1(\lambda_{j-1},\lambda_j)}{\alpha_2(\lambda_{j-1},\lambda_j)}\vec{\Phi}_n(\lambda_1,\ldots,\lambda_j,\lambda_{j-1},\ldots,\lambda_n).\hat{r}_{j-1,j}(\lambda_{j-1},\lambda_j)\end{aligned} \tag{4.86}$$

Now, for the three-particle excitation wavefunction, we make an ansatz:

$$\begin{aligned}\vec{\Phi}_3(\lambda_1,\lambda_2,\lambda_3) &= \vec{B}(\lambda_1)\otimes\vec{B}(\lambda_2)\otimes\vec{B}(\lambda_3)\\ &+[\vec{B}(\lambda_1)\otimes\vec{\xi}F(\lambda_2)B(\lambda_3)]\hat{g}_0^{(3)}(\lambda_1,\lambda_2,\lambda_3)\\ &+[\vec{\xi}\otimes F(\lambda_1)\vec{B}(\lambda_3)B(\lambda_2)]\hat{g}_1^{(3)}(\lambda_1,\lambda_2,\lambda_3)\\ &+[\vec{\xi}\otimes F(\lambda_1)\vec{B}(\lambda_2)B(\lambda_3)]\hat{g}_2^{(3)}(\lambda_1,\lambda_2,\lambda_3) \qquad (4.87)\end{aligned}$$

where the coefficients $\hat{g}_j^{(3)}(\lambda_1,\lambda_2,\lambda_3)$ are now determined by permutation properties of $\vec{\Phi}_3(\lambda_1,\lambda_3,\lambda_3)$

Imposing the exchange property (4.86) on exchanging $\lambda_2\leftrightarrow\lambda_3$ in $\vec{\Phi}_3(\lambda_1,\lambda_2,\lambda_3)$

$$\begin{aligned}&\vec{\Phi}_3(\lambda_1,\lambda_2,\lambda_3)\\ &=\frac{\alpha_1(\lambda_2,\lambda_3)}{\alpha_2(\lambda_2,\lambda_3)}\vec{\Phi}_3(\lambda_1,\lambda_3,\lambda_2).\hat{r}_{23}(\lambda_2,\lambda_3)\\ &=\vec{B}(\lambda_1)\otimes\vec{B}(\lambda_2)\otimes\vec{B}(\lambda_3)+i\frac{\alpha_{10}(\lambda_2,\lambda_3)}{\alpha_7(\lambda_2,\lambda_3)}[\vec{B}(\lambda_1)\otimes\vec{\xi}F(\lambda_2)B(\lambda_3)]\\ &+[\vec{B}(\lambda_1)\otimes\vec{\xi}F(\lambda_3)B(\lambda_2)][-i\frac{\alpha_{10}(\lambda_2,\lambda_3)}{\alpha_7(\lambda_2,\lambda_3)}+\frac{\alpha_1(\lambda_2,\lambda_3)}{\alpha_2(\lambda_2,\lambda_3)}\\ &\times\hat{g}_0^{(3)}(\lambda_1,\lambda_3,\lambda_2).\hat{r}_{23}(\lambda_2,\lambda_3)]\\ &+[\vec{\xi}\otimes F(\lambda_1)\vec{B}(\lambda_2)B(\lambda_3)]\frac{\alpha_1(\lambda_2,\lambda_3)}{\alpha_2(\lambda_2,\lambda_3)}\hat{g}_1^{(3)}(\lambda_1,\lambda_3,\lambda_2).\hat{r}_{23}(\lambda_2,\lambda_3)\\ &+[\vec{\xi}\otimes F(\lambda_1)\vec{B}(\lambda_3)B(\lambda_2)]\frac{\alpha_1(\lambda_2,\lambda_3)}{\alpha_2(\lambda_2,\lambda_3)}\hat{g}_2^{(3)}(\lambda_1,\lambda_3,\lambda_2).\hat{r}_{23}(\lambda_2,\lambda_3) \qquad (4.88)\end{aligned}$$

By comparing the above (4.88) to our original three particle ansatz (4.87), we obtain:

$$\hat{g}_0^{(3)}(\lambda_1,\lambda_2,\lambda_3)=i\frac{\alpha_{10}(\lambda_2,\lambda_3)}{\alpha_7(\lambda_2,\lambda_3)} \qquad (4.89)$$

$$\hat{g}_2^{(3)}(\lambda_1,\lambda_2,\lambda_3)=\frac{\alpha_1(\lambda_2,\lambda_3)}{\alpha_2(\lambda_2,\lambda_3)}\hat{g}_1^{(3)}(\lambda_1,\lambda_3,\lambda_2).\hat{r}_{23}(\lambda_2,\lambda_3) \qquad (4.90)$$

Now, to make use of the exchange property, we consider $\lambda_1\leftrightarrow\lambda_2$. We

first consider the following:

$$
\begin{aligned}
&\vec{\Phi}_3(\lambda_1,\lambda_2,\lambda_3)\\
&= \vec{B}(\lambda_1)\otimes\vec{B}(\lambda_2)\otimes\vec{B}(\lambda_3) + [\vec{B}(\lambda_1)\otimes\vec{\xi}F(\lambda_2)B(\lambda_3)]\hat{g}_0^{(3)}(\lambda_1,\lambda_2,\lambda_3)\\
&\quad + [\vec{\xi}\otimes F(\lambda_1)\vec{B}(\lambda_3)B(\lambda_2)]\hat{g}_1^{(3)}(\lambda_1,\lambda_2,\lambda_3) + [\vec{\xi}\otimes F(\lambda_1)\vec{B}(\lambda_2)B(\lambda_3)]\hat{g}_2^{(3)}(\lambda_1,\lambda_2,\lambda_3)\\
&= \frac{\alpha_1(\lambda_1,\lambda_2)}{\alpha_2(\lambda_1,\lambda_2)}\vec{B}(\lambda_2)\otimes\vec{B}(\lambda_1)\otimes\vec{B}(\lambda_3).\hat{r}_{12}(\lambda_1,\lambda_2) - i\frac{\alpha_{10}(\lambda_1,\lambda_2)}{\alpha_7(\lambda_1,\lambda_2)}[F(\lambda_1)B(\lambda_2)\vec{\xi}\otimes\vec{B}(\lambda_3)]\\
&\quad + i\frac{\alpha_{10}(\lambda_1,\lambda_2)}{\alpha_7(\lambda_1,\lambda_2)}[F(\lambda_2)B(\lambda_1)\vec{\xi}\otimes\vec{B}(\lambda_3)] + i\frac{\alpha_{10}(\lambda_2,\lambda_3)}{\alpha_7(\lambda_2,\lambda_3)}[\vec{B}(\lambda_1)\otimes\vec{\xi}F(\lambda_2)B(\lambda_3)]\\
&\quad + [\vec{\xi}\otimes F(\lambda_1)\vec{B}(\lambda_3)B(\lambda_2)]\hat{g}_1^{(3)}(\lambda_1,\lambda_2,\lambda_3) + [\vec{\xi}\otimes F(\lambda_1)\vec{B}(\lambda_2)B(\lambda_3)]\hat{g}_2^{(3)}(\lambda_1,\lambda_2,\lambda_3)\\
&= \frac{\alpha_1(\lambda_1,\lambda_2)}{\alpha_2(\lambda_1,\lambda_2)}\vec{B}(\lambda_2)\otimes\vec{B}(\lambda_1)\otimes\vec{B}(\lambda_3).\hat{r}_{12}(\lambda_1,\lambda_2) - i\frac{\alpha_{10}(\lambda_1,\lambda_2)}{\alpha_7(\lambda_1,\lambda_2)}[F(\lambda_1)B(\lambda_2)\vec{\xi}\otimes\vec{B}(\lambda_3)]\\
&\quad + i\frac{\alpha_{10}(\lambda_1,\lambda_2)}{\alpha_7(\lambda_1,\lambda_2)}[F(\lambda_1)\vec{\xi}\otimes(i\frac{\alpha_2(\lambda_2,\lambda_3)}{\alpha_9(\lambda_3,\lambda_2)}\vec{B}(\lambda_2)B(\lambda_3) - i\frac{\alpha_5(\lambda_2,\lambda_3)}{\alpha_9(\lambda_3,\lambda_2)}\vec{B}(\lambda_3)B(\lambda_2))]\\
&\quad + i\frac{\alpha_{10}(\lambda_2,\lambda_3)}{\alpha_7(\lambda_2,\lambda_3)}[\vec{B}(\lambda_1)\otimes\vec{\xi}F(\lambda_2)B(\lambda_3)]\\
&\quad + [\vec{\xi}\otimes F(\lambda_1)\vec{B}(\lambda_3)B(\lambda_2)]\hat{g}_1^{(3)}(\lambda_1,\lambda_2,\lambda_3) + [\vec{\xi}\otimes F(\lambda_1)\vec{B}(\lambda_2)B(\lambda_3)]\hat{g}_2^{(3)}(\lambda_1,\lambda_2,\lambda_3)
\end{aligned}
\tag{4.91}
$$

Comparing the coefficients of $[\vec{B}(\lambda_1)\otimes\vec{\xi}F(\lambda_2)B(\lambda_3)]$ in (4.91) and (4.86), with the help of (4.84), we have

$$
\hat{g}_1^{(3)}(\lambda_1,\lambda_2,\lambda_3) = i\frac{\alpha_{10}(\lambda_1,\lambda_2)}{\alpha_7(\lambda_1,\lambda_2)}i\frac{\alpha_2(\lambda_3,\lambda_2)}{\alpha_9(\lambda_3,\lambda_2)} \tag{4.92}
$$

Now, using (4.90), we have

$$
\hat{g}_2^{(3)}(\lambda_1,\lambda_2,\lambda_3) = i\frac{\alpha_{10}(\lambda_1,\lambda_3)}{\alpha_7(\lambda_1,\lambda_3)}i\frac{\alpha_1(\lambda_2,\lambda_3)}{\alpha_9(\lambda_2,\lambda_3)}\hat{r}_{23}(\lambda_2,\lambda_3) \tag{4.93}
$$

We note that, in particular, $\hat{g}_0^{(3)}(x,y,z) = \hat{g}_0^{(2)}(y,z)$, thus we can write $\vec{\Phi}_3(\lambda_1,\lambda_3,\lambda_2)$ as a recurrence relation:

$$
\begin{aligned}
\vec{\Phi}_3(\lambda_1,\lambda_2,\lambda_3) &= \vec{B}(\lambda_1)\otimes\vec{\Phi}_2(\lambda_2,\lambda_3)\\
&\quad + \sum_{j=2}^{3}\Big[\vec{\xi}\otimes F(\lambda_1)\vec{\Phi}_1(\lambda_2,\ldots,\lambda_{j-1},\lambda_{j+1},\ldots,\lambda_3)B(\lambda_j)\Big]\\
&\quad \times \hat{g}_{j-1}^{(3)}(\lambda_1,\lambda_2,\lambda_3)
\end{aligned}
\tag{4.94}
$$

Generalizing the three particle recurrence to n particles, we can make an ansatz:

$$\begin{aligned}\vec{\Phi}_n(\lambda_1,\ldots,\lambda_n) = \vec{B}(\lambda_1)\otimes\vec{\Phi}_{n-1}(\lambda_2,\ldots,\lambda_n) \\ + \sum_{j=2}^{n}\left[\vec{\xi}\otimes F(\lambda_1)\vec{\Phi}_{n-2}(\lambda_2,\ldots,\lambda_{j-1},\lambda_{j+1},\ldots,\lambda_n)B(\lambda_j)\right] \\ \times \hat{g}_{j-1}^{(n)}(\lambda_1,\ldots,\lambda_n) \qquad (4.95)\end{aligned}$$

where $\vec{\Phi}_0$ is identified with the unity vector.

To determine the constraints for $\hat{g}_j^{(n)}(\lambda_1,\ldots,\lambda_n)$, we first use the exchange property (4.86) and exchange $\lambda_j \leftrightarrow \lambda_{j-1}$ in $\vec{\Phi}_n(\lambda_1,\ldots,\lambda_n)$. We then use (4.95) on both sides of (4.86) and consider the coefficients of

$$[\vec{\xi}\otimes F(\lambda_1)\vec{\Phi}_{n-2}(\lambda_2,\ldots,\lambda_{j-2},\lambda_j,\ldots,\lambda_n)B(\lambda_{j-1})] \qquad (4.96)$$

and

$$[\vec{\xi}\otimes F(\lambda_1)\vec{\Phi}_{n-2}(\lambda_2,\ldots,\lambda_{j-1},\lambda_{j+1},\ldots,\lambda_n)B(\lambda_j)] \qquad (4.97)$$

on both sides of (4.86). We note that all the other terms will automatically satisfy (4.86).

As such, the $\hat{g}_j^{(n)}(\lambda_1,\ldots,\lambda_n)$ satisfy the recurrence:

$$\begin{aligned}\hat{g}_{j-1}^{(n)}(\lambda_1,\ldots,\lambda_{j-1},\lambda_j,\ldots,\lambda_n) = \frac{\alpha_1(\lambda_{j-1},\lambda_j)}{\alpha_2(\lambda_{j-1},\lambda_j)}\hat{g}_{j-2}^{(n)}(\lambda_1,\ldots,\lambda_j,\lambda_{j-1},\ldots,\lambda_n) \\ \times \hat{r}_{j-1,j}(\lambda_{j-1},\lambda_j) \qquad (4.98)\end{aligned}$$

We now consider $\lambda_1 \leftrightarrow \lambda_2$ as implemented for the three particle case, and we find that

$$\hat{g}_1^{(n)}(\lambda_1,\ldots,\lambda_n) = i\frac{\alpha_{10}(\lambda_1,\lambda_2)}{\alpha_7(\lambda_1,\lambda_2)}\prod_{k=3}^{n} i\frac{\alpha_2(\lambda_k,\lambda_2)}{\alpha_9(\lambda_k,\lambda_2)} \qquad (4.99)$$

Solving the $\hat{g}_j^{(n)}(\lambda_1,\ldots,\lambda_n)$ recursively from (4.98) and (4.99), we get

$$\begin{aligned}\vec{\Phi}_n(\lambda_1,\ldots,\lambda_n) = \vec{B}(\lambda_1)\otimes\vec{\Phi}_{n-1}(\lambda_2,\ldots,\lambda_n) + \sum_{j=2}^{n} i\frac{\alpha_{10}(\lambda_1,\lambda_j)}{\alpha_7(\lambda_1,\lambda_j)}\prod_{\substack{k=2\\k\neq j}}^{n} i\frac{\alpha_2(\lambda_k,\lambda_j)}{\alpha_9(\lambda_k,\lambda_j)}\\ \times\left[\vec{\xi}\otimes F(\lambda_1)\vec{\Phi}_{n-2}(\lambda_2,\ldots,\lambda_{j-1},\lambda_{j+1},\ldots,\lambda_n)B(\lambda_j)\right]\\ \times\prod_{k=2}^{j-1}\frac{\alpha_1(\lambda_k,\lambda_j)}{\alpha_2(\lambda_k,\lambda_j)}\hat{r}_{k,k+1}(\lambda_k,\lambda_j)\end{aligned} \tag{4.100}$$

Now, using the fundamental commutation relations, we can obtain the action of the diagonal operators on the n-particle vector:

$$\begin{aligned}B(\lambda)\left|\Phi_n(\lambda_1,\ldots,\lambda_n)\right\rangle = [\omega_1(\lambda)]^L\prod_{j=1}^{n} i\frac{\alpha_2(\lambda_j,\lambda)}{\alpha_9(\lambda_j,\lambda)}\left|\Phi_n(\lambda_1,\ldots,\lambda_n)\right\rangle\\ -\sum_{j=1}^{n}[\omega_1(\lambda_j)]^L\left|\Psi_{n-1}^{(1)}(\lambda,\lambda_j;\{\lambda_l\})\right\rangle\\ +\sum_{j=2}^{n}\sum_{l=1}^{j-1}H_1(\lambda,\lambda_l,\lambda_j)[\omega_1(\lambda_l)\omega_1(\lambda_j)]^L\left|\Psi_{n-2}^{(3)}(\lambda,\lambda_j,\lambda_l;\{\lambda_k\})\right\rangle\end{aligned} \tag{4.101}$$

$$\begin{aligned}D(\lambda)\left|\Phi_n(\lambda_1,\ldots,\lambda_n)\right\rangle = [\omega_3(\lambda)]^L\prod_{j=1}^{n} -i\frac{\alpha_8(\lambda,\lambda_j)}{\alpha_7(\lambda,\lambda_j)}\left|\Phi_n(\lambda_1,\ldots,\lambda_n)\right\rangle\\ -\sum_{j=1}^{n}[\omega_2(\lambda_j)]^L\Lambda^{(1)}(\lambda=\lambda_j,\{\lambda_l\})\left|\Psi_{n-1}^{(2)}(\lambda,\lambda_j;\{\lambda_l\})\right\rangle\\ +\sum_{j=2}^{n}\sum_{l=1}^{j-1}H_2(\lambda,\lambda_l,\lambda_j)[\omega_2(\lambda_1)\omega_2(\lambda_2)]^L\Lambda^{(1)}(\lambda=\lambda_j,\{\lambda_k\})\\ \times\Lambda^{(1)}(\lambda=\lambda_l,\{\lambda_k\})\left|\Psi_{n-2}^{(3)}(\lambda,\lambda_j,\lambda_l;\{\lambda_k\})\right\rangle\end{aligned} \tag{4.102}$$

$$
\begin{aligned}
&\sum_{a=1}^{2} \hat{A}_{aa}(\lambda) \left|\Phi_n(\lambda_1, \ldots, \lambda_n)\right\rangle \\
&= [\omega_2(\lambda)]^L \prod_{j=1}^{n} -i\frac{\alpha_1(\lambda, \lambda_j)}{\alpha_9(\lambda, \lambda_j)} \Lambda^{(1)}(\lambda, \{\lambda_l\}) \left|\Phi_n(\lambda_1, \ldots, \lambda_n)\right\rangle \\
&\quad - \sum_{j=1}^{n} [\omega_2(\lambda_j)]^L \Lambda^{(1)}(\lambda = \lambda_j, \{\lambda_l\}) \left|\Psi^{(1)}_{n-1}(\lambda, \lambda_j; \{\lambda_l\})\right\rangle \\
&\quad - \sum_{j=1}^{n} [\omega_1(\lambda_j)]^L \left|\Psi^{(2)}_{n-1}(\lambda, \lambda_j; \{\lambda_l\})\right\rangle \\
&\quad - \sum_{j=2}^{n} \sum_{l=1}^{j-1} H_3(\lambda, \lambda_l, \lambda_j)[\bar{a}(\lambda_l, \lambda_j) - \bar{b}(\lambda_l, \lambda_j)][\omega_1(\lambda_l)\omega_2(\lambda_j)]^L \\
&\quad \times \Lambda^{(1)}(\lambda = \lambda_j, \{\lambda_k\}) \left|\Psi^{(3)}_{n-2}(\lambda, \lambda_j, \lambda_l; \{\lambda_k\})\right\rangle \\
&\quad - \sum_{j=2}^{n} \sum_{l=1}^{j-1} H_3(\lambda, \lambda_j, \lambda_l) \frac{\alpha_1(\lambda_l, \lambda_j)}{\alpha_2(\lambda_l, \lambda_l)} [\omega_1(\lambda_j)\omega_2(\lambda_l)]^L \\
&\quad \times \Lambda^{(1)}(\lambda = \lambda_l, \{\lambda_k\}) \left|\Psi^{(3)}_{n-2}(\lambda, \lambda_j, \lambda_l; \{\lambda_k\})\right\rangle
\end{aligned}
\tag{4.103}
$$

There are three families of unwanted terms:

$$\left|\Psi^{(1)}_{n-1}(\lambda,\lambda_j;\{\lambda_l\})\right\rangle = i\frac{\alpha_5(\lambda_j,\lambda)}{\alpha_9(\lambda_j,\lambda)}\prod_{\substack{k=1\\k\neq j}}^{n} i\frac{\alpha_2(\lambda_k,\lambda_j)}{\alpha_9(\lambda_k,\lambda_j)}$$
$$\times \vec{B}(\lambda)\otimes\vec{\Phi}_{n-1}(\lambda_1,\dots\check{\lambda}_j,\dots,\lambda_n)$$
$$\times \hat{O}^{(1)}_j(\lambda_j;\{\lambda_k\}).\vec{\mathcal{F}}\,|0\rangle \tag{4.104}$$

$$\left|\Psi^{(2)}_{n-1}(\lambda,\lambda_j;\{\lambda_l\})\right\rangle = i\frac{\alpha_{10}(\lambda,\lambda_j)}{\alpha_7(\lambda,\lambda_j)}\prod_{\substack{k=1\\k\neq j}}^{n} i\frac{\alpha_2(\lambda_k,\lambda_j)}{\alpha_9(\lambda_k,\lambda_j)}$$
$$\times [\vec{\xi}.(\vec{B^*}(\lambda)\otimes\hat{I})]\otimes\vec{\Phi}_{n-1}(\lambda_1,\dots,\check{\lambda}_j,\dots,\lambda_n)$$
$$\times \hat{O}^{(1)}_j(\lambda_j;\{\lambda_k\}).\vec{\mathcal{F}}\,|0\rangle \tag{4.105}$$

$$\left|\Psi^{(3)}_{n-2}(\lambda,\lambda_j,\lambda_l;\{\lambda_k\})\right\rangle = \prod_{\substack{k=1\\\neq j,l}}^{n} i\frac{\alpha_2(\lambda_k,\lambda_j)}{\alpha_9(\lambda_k,\lambda_j)} i\frac{\alpha_2(\lambda_k,\lambda_l)}{\alpha_9(\lambda_k,\lambda_l)}$$
$$\times F(\lambda)\vec{\xi}\otimes\vec{\Phi}_{n-2}(\lambda_1,\dots,\check{\lambda}_l,\dots,\check{\lambda}_j,\dots,\lambda_n)$$
$$\times \hat{O}^{(2)}_{lj}(\lambda_l,\lambda_j;\{\lambda_k\}).\vec{\mathcal{F}}\,|0\rangle \tag{4.106}$$

where we have another "ordering" factor due to permutation properties of the spectral parameters $\{\lambda_j\}$:

$$\hat{O}^{(2)}_{lj}(\lambda_l,\lambda_j;\{\lambda_k\}) = \prod_{k=1}^{l-1}\frac{\alpha_1(\lambda_k,\lambda_j)}{\alpha_2(\lambda_k,\lambda_j)}\hat{r}_{k+1,k+2}(\lambda_k,\lambda_j)$$
$$\times\prod_{k=l+1}^{j-1}\frac{\alpha_1(\lambda_k,\lambda_j)}{\alpha_2(\lambda_k,\lambda_j)}\hat{r}_{k,k+1}(\lambda_k,\lambda_j)$$
$$\times\prod_{k=1}^{l-1}\frac{\alpha_1(\lambda_k,\lambda_l)}{\alpha_2(\lambda_k,\lambda_l)}\hat{r}_{k,k+1}(\lambda_k,\lambda_l) \tag{4.107}$$

where the symbol $\check{\lambda}_j$ means that the spectral parameter λ_j is absent from the set $\{\lambda_1,\dots,\lambda_n\}$.

Now, summing up the eigenfunctions proportional to $|\Phi_1(\lambda_1)\rangle$ on the right hand side of the equations (4.101),(4.102) and (4.103) using (4.27), we obtain

the eigenvalue of the transfer matrix as

$$\Lambda(\lambda,\{\lambda_j\}) = [\omega_1(\lambda)]^L \prod_{j=1}^{n} i\frac{\alpha_2(\lambda_j,\lambda)}{\alpha_9(\lambda_j,\lambda)} + [\omega_3(\lambda)]^L \prod_{j=1}^{n} -i\frac{\alpha_8(\lambda,\lambda_j)}{\alpha_7(\lambda,\lambda_j)} - [\omega_2(\lambda)]^L \prod_{j=1}^{n} -i\frac{\alpha_1(\lambda,\lambda_j)}{\alpha_9(\lambda,\lambda_j)}\Lambda^{(1)}(\lambda,\{\lambda_l\}) \tag{4.108}$$

where the spectral parameters satisfy the Bethe ansatz equation:

$$\left[\frac{\omega_1(\lambda_i)}{\omega_2(\lambda_i)}\right]^L = \Lambda^{(1)}(\lambda=\lambda_i,\{\lambda_j\}), \quad i=1,\ldots,n \tag{4.109}$$

Having solved the "charge" degrees of freedom for the n-particle excitation wavefunction, we now define the auxiliary problem for the "spin" degrees of freedom:

$$T^{(1)}(\lambda,\{\lambda_i\})^{b_1\cdots b_n}_{a_1\cdots a_n}\mathcal{F}^{b_n\cdots b_1} = \Lambda^{(1)}(\lambda,\{\lambda_i\})\mathcal{F}^{a_n\cdots a_1} \tag{4.110}$$

where the inhomogeneous transfer matrix $T^{(1)}(\lambda,\{\lambda_i\})$ is

$$T^{(1)}(\lambda,\{\lambda_i\})^{a_1\cdots a_n}_{b_1\cdots b_n} = \hat{r}^{c_1a_1}_{b_1d_1}(\lambda,\lambda_1)\hat{r}^{d_1a_2}_{b_2c_2}(\lambda,\lambda_2)\ldots\hat{r}^{d_{n-1}a_n}_{b_nc_1}(\lambda,\lambda_n) \tag{4.111}$$

We now solve the auxiliary problem following the same method as in the Heisenberg XXX model and the nested level of the supersymmetric t-J model.

The transfer matrix is

$$\mathcal{T}^{(1)}(\lambda,\{\lambda_j\}) = \mathcal{L}^{(1)}_{\mathcal{A}^{(1)}n}(\lambda,\lambda_n)\mathcal{L}^{(1)}_{\mathcal{A}^{(1)}n-1}(\lambda,\lambda_{n-1})\ldots\mathcal{L}^{(1)}_{\mathcal{A}^{(1)}1}(\lambda,\lambda_1) \tag{4.112}$$

where $\mathcal{A}^{(1)}$ is the two-dimensional "spin" auxiliary space. The Lax operator is

$$\mathcal{L}^{(1)}_{\mathcal{A}^{(1)}j}(\lambda,\lambda_j) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \bar{b}(\lambda,\lambda_j) & \bar{a}(\lambda,\lambda_j) & 0 \\ 0 & \bar{a}(\lambda,\lambda_j) & \bar{b}(\lambda,\lambda_j) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{4.113}$$

We write the monodromy matrix as

$$\mathcal{T}^{(1)}(\lambda,\{\lambda_j\}) = \begin{pmatrix} A^{(1)}(\lambda,\{\lambda_j\}) & B^{(1)}(\lambda,\{\lambda_j\}) \\ C^{(1)}(\lambda,\{\lambda_j\}) & D^{(1)}(\lambda,\{\lambda_j\}) \end{pmatrix} \tag{4.114}$$

And we define reference state as

$$\left|0^{(1)}\right\rangle = \prod_{j=1}^{n} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix}_j \tag{4.115}$$

Now, we can obtain the action of diagonal and lower triangular operators as

$$\begin{aligned} A^{(1)}(\lambda, \{\lambda_j\}) \left|0^{(1)}\right\rangle &= \left|0^{(1)}\right\rangle \\ D^{(1)}(\lambda, \{\lambda_j\}) \left|0^{(1)}\right\rangle &= \prod_{j=1}^{n} \bar{b}(\lambda, \lambda_j) \left|0^{(1)}\right\rangle \\ C^{(1)}(\lambda, \{\lambda_j\}) \left|0^{(1)}\right\rangle &= 0 \end{aligned}$$

We recall that the R matrix for the auxiliary problem is

$$\hat{r}(\lambda, \mu) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \bar{a}(\lambda, \mu) & \bar{b}(\lambda, \mu) & 0 \\ 0 & \bar{b}(\lambda, \mu) & \bar{a}(\lambda, \mu) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{4.116}$$

From the intertwining relationship

$$\hat{r}(\lambda, \mu)\mathcal{T}(\lambda) \otimes \mathcal{T}(\mu) = \mathcal{T}(\mu) \otimes \mathcal{T}(\lambda)\hat{r}(\lambda, \mu) \tag{4.117}$$

we can obtain the fundamental commutation relationships:

$$\begin{aligned} A^{(1)}(\lambda, \{\lambda_j\})B^{(1)}(\mu, \{\lambda_j\}) &= \frac{1}{\bar{b}(\mu, \lambda)} B^{(1)}(\mu, \{\lambda_j\})A^{(1)}(\lambda, \{\lambda_j\}) \\ &\quad - \frac{\bar{a}(\mu, \lambda)}{\bar{b}(\mu, \lambda)} B^{(1)}(\lambda, \{\lambda_j\})A^{(1)}(\mu, \{\lambda_j\}) \\ D^{(1)}(\lambda, \{\lambda_j\})B^{(1)}(\mu, \{\lambda_j\}) &= \frac{1}{\bar{b}(\lambda, \mu)} B^{(1)}(\mu, \{\lambda_j\})D^{(1)}(\lambda, \{\lambda_j\}) \\ &\quad - \frac{\bar{a}(\lambda, \mu)}{\bar{b}(\lambda, \mu)} B^{(1)}(\lambda, \{\lambda_j\})D^{(1)}(\mu, \{\lambda_j\}) \\ \left[B^{(1)}(\mu, \{\lambda_j\}), B^{(1)}(\lambda, \{\lambda_j\})\right] &= 0 \end{aligned}$$

We make an ansatz for nested level eigenstates:

$$|\phi\rangle = \prod_{l=1}^{m} B^{(1)}(\mu_l, \{\lambda_j\}) \left|0^{(1)}\right\rangle \tag{4.118}$$

In component form, this state can be written as $\left|\lambda_1^{(1)},\cdots,\lambda_{n_1}^{(1)}\right\rangle_{a_n\cdots a_1}$, which is directly identifiable with $\mathcal{F}^{a_n\cdots a_1}$.

Using the fundamental commutation relationships (4.118), we can obtain:

$$\begin{aligned}
A^{(1)}(\lambda,\{\lambda_j\})\left|\phi\right\rangle &= \prod_{l=1}^{m}\frac{1}{\bar{b}(\mu_l,\lambda)}\left|\phi\right\rangle \\
&\quad -\sum_{l=1}^{m}\frac{\bar{a}(\mu_l,\lambda)}{\bar{b}(\mu_l,\lambda)}\prod_{\substack{k=1\\k\neq l}}^{m}\frac{1}{\bar{b}(\mu_k,\mu_l)}B^{(1)}(\mu_k,\{\lambda_j\})\left|0^{(1)}\right\rangle \\
D^{(1)}(\lambda,\{\lambda_j\}) &= \prod_{j=1}^{n}\bar{b}(\lambda,\lambda_j)\prod_{l=1}^{m}\frac{1}{\bar{b}(\lambda,\mu_l)}\left|\phi\right\rangle \\
&\quad -\sum_{l=1}^{m}\frac{\bar{a}(\lambda,\mu_l)}{\bar{b}(\lambda,\mu_l)}\prod_{p=1}^{n}\bar{b}(\lambda_l,\lambda_p)\prod_{\substack{k=1\\k\neq l}}^{m}\frac{1}{\bar{b}(\mu_l,\mu_k)}B^{(1)}(\mu_k,\{\lambda_j\})\left|0^{(1)}\right\rangle
\end{aligned} \tag{4.119}$$

From the wanted terms, we find the eigenvalues of the auxiliary problem:

$$\Lambda^{(1)}(\lambda,\{\lambda_j\},\{\mu_l\}) = \prod_{l=1}^{m}\frac{1}{\bar{b}(\mu_l,\lambda)} + \prod_{j=1}^{n}\bar{b}(\lambda,\lambda_j)\prod_{l=1}^{m}\frac{1}{\bar{b}(\lambda,\mu_l)} \tag{4.120}$$

From the unwanted terms, we find the nested Bethe ansatz equation:

$$\prod_{j=1}^{n}\bar{b}(\mu_l,\lambda_j) = -\prod_{k=1}^{m}\frac{\bar{b}(\mu_l,\mu_k)}{\bar{b}(\mu_k,\mu_l)},\quad l=1,\ldots,m \tag{4.121}$$

We can express $\Lambda^{(1)}(\lambda,\{\lambda_j\},\{\mu_l\})$ as

$$\begin{aligned}
\Lambda^{(1)}(\lambda,\{\lambda_j\},\{\mu_l\}) &= [\omega_1(\lambda)]^L\prod_{j=1}^{n}i\frac{\alpha_2(\lambda_j,\lambda)}{\alpha_9(\lambda_j,\lambda)} + [\omega_3(\lambda)]^L\prod_{j=1}^{n}-i\frac{\alpha_8(\lambda,\lambda_j)}{\alpha_7(\lambda,\lambda_j)} \\
&\quad -[\omega_2(\lambda)]^L\left\{\prod_{j=1}^{n}-i\frac{\alpha_1(\lambda,\lambda_j)}{\alpha_9(\lambda,\lambda_j)}\prod_{l=1}^{m}\frac{1}{\bar{b}(\mu_l,\lambda)}\right. \\
&\quad \left.+\prod_{j=1}^{n}-i\frac{\alpha_8(\lambda,\lambda_j)}{\alpha_7(\lambda,\lambda_j)}\prod_{l=1}^{m}\frac{1}{\bar{b}(\lambda,\mu_l)}\right\},
\end{aligned}$$

and that the original Bethe ansatz equations for $\{\lambda_j\}$ satisfy:

$$\left[\frac{\omega_1(\lambda_j)}{\omega_2(\lambda_j)}\right]^L = \prod_{l=1}^{m} \frac{1}{\bar{b}(\mu_l, \lambda_j)} \tag{4.122}$$

We can introduce a new set of variables

$$z_-(\lambda_j) = \frac{a(\lambda_j)}{b(\lambda_j)} e^{2h(\lambda_j)} \quad ; \quad z_+(\lambda_j) = \frac{b(\lambda_j)}{a(\lambda_j)} e^{2h(\lambda_j)}, \tag{4.123}$$

and rewrite $\Lambda^{(1)}(\lambda, \{\lambda_j\}, \{\mu_l\})$ as

$$\begin{aligned}
&(-i)^n \Lambda(\lambda, \{z_\pm(\lambda_j)\}, \{\tilde{\mu}_l\}) \\
&= [\omega_1(\lambda)]^L \prod_{j=1}^{n} \frac{b(\lambda)}{a(\lambda)} \left[\frac{1 + z_-(\lambda_j)/z_+(\lambda)}{1 - z_-(\lambda_j)/z_-(\lambda)}\right] + [\omega_3(\lambda)]^L \prod_{j=1}^{n} \frac{b(\lambda)}{a(\lambda)} \left[\frac{1 + z_-(\lambda_j) z_-(\lambda)}{1 - z_-(\lambda_j) z_+(\lambda)}\right] \\
&\quad - [\omega_2(\lambda)]^L \left\{ \prod_{j=1}^{n} \frac{b(\lambda)}{a(\lambda)} \left[\frac{1 + z_-(\lambda_j)/z_+(\lambda)}{1 - z_-(\lambda_j)/z_-(\lambda)}\right] \times \prod_{l=1}^{m} \frac{z_-(\lambda) - 1/z_-(\lambda) - \tilde{\mu}_l + U/2}{z_-(\lambda) - 1/z_-(\lambda) - \tilde{\mu}_l - U/2} \right. \\
&\quad \left. + \prod_{j=1}^{n} \frac{b(\lambda)}{a(\lambda)} \left[\frac{1 + z_-(\lambda_j) z_-(\lambda)}{1 - z_-(\lambda_j) z_+(\lambda)}\right] \prod_{l=1}^{n} \frac{1/z_+(\lambda) - z_+(\lambda) - \tilde{\mu}_l - U/2}{1/z_+(\lambda) - z_+(\lambda) - \tilde{\mu}_l + U/2} \right\}
\end{aligned}$$

And the first level and nested Bethe ansatz equations can be rewritten as

$$[z_-(\lambda_j)]^L = \prod_{l=1}^{m} \frac{z_-(\lambda_j) - 1/z_-(\lambda_j) - \tilde{\mu}_l + U/2}{z_-(\lambda_j) - 1/z_-(\lambda_j) - \tilde{\mu}_l - U/2}, \tag{4.124}$$

$$\prod_{j=1}^{n} \frac{z_-(\lambda_j) - 1/z_-(\lambda_j) - \tilde{\mu}_l - U/2}{z_-(\lambda_j) - 1/z_-(\lambda_j) - \tilde{\mu}_l + U/2} = -\prod_{k=1}^{m} \frac{\tilde{\mu}_l - \tilde{\mu}_k + U}{\tilde{\mu}_l - \tilde{\mu}_k - U}; \quad l = 1, \ldots, m \tag{4.125}$$

Using the trace identity (4.21), and expressing $z_-(\lambda_j)$ in terms of hole momenta k_j:

$$z_-(\lambda_j) = e^{ik_j}, \tag{4.126}$$

we can finally obtain (first obtained by Lieb and Wu [71] using coordinate Bethe ansatz):

$$E_n(L) = \frac{U(L - 2n)}{4} + \sum_{j=1}^{n} 2\cos(k_j), \tag{4.127}$$

and the hole momenta k_j satisfy (with $\tilde{\mu}_l = 2i\bar{\mu}_l$):

$$e^{iLk_j} = \prod_{l=1}^{m} \frac{\sin(k_j) - \bar{\mu}_l - iU/4}{\sin(k_j) - \bar{\mu}_l + iU/4},$$
$$\prod_{j=1}^{n} \frac{\sin(k_j) - \bar{\mu}_l + iU/4}{\sin(k_j) - \bar{\mu}_l - iU/4} = -\prod_{k=1}^{m} \frac{\bar{\mu}_l - \bar{\mu}_k - iU/2}{\bar{\mu}_l - \bar{\mu}_k + iU/2}, \quad l = 1, \ldots, m$$

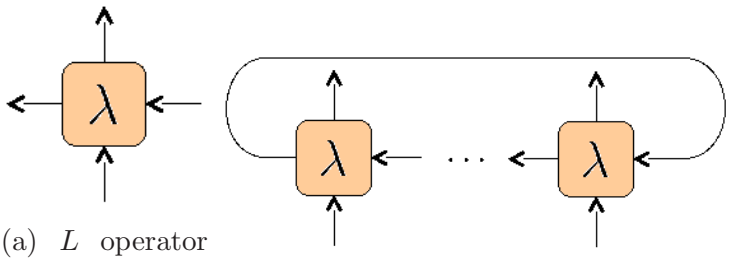
4.3 Tensor network description of the Bethe ansatz

4.3.1 Tensor network form

We now represent the above generalized NABA in tensor network form. The grading considerations are exactly the same as for the supersymmetric t-J model 3.4.

We first represent each L operator $L(\lambda)^{ab}_{\alpha\beta}$ (a tensor with four indices) as shown in Fig. 4.1a. We then construct the transfer matrix $T_L(\lambda) = L_L(\lambda)L_{L-1}(\lambda)\cdots L_1(\lambda)$ as shown in Fig. 4.1b.

For the nested Bethe ansatz, the creation operator $C^{(1)}(\lambda)$ in (4.114) is constructed by terminating the ends of the transfer matrix by boundary vectors $(1\ 0)$ on the left and $(0\ 1)^\intercal$ on the right (selecting the first row and second column respectively).



(a) L operator $L(\lambda)$

(b) Monodromy matrix $T(\lambda)$

Figure 4.1: Tensor network representation of $L(\lambda)$ and $T(\lambda)$

As such, if we define:

$$\omega^{(1)}_{ab} = \lambda^{(1)}_a - \lambda_b \tag{4.128}$$

the nested Bethe ansatz can be represented in tensor network form as depicted in Fig. 4.2.

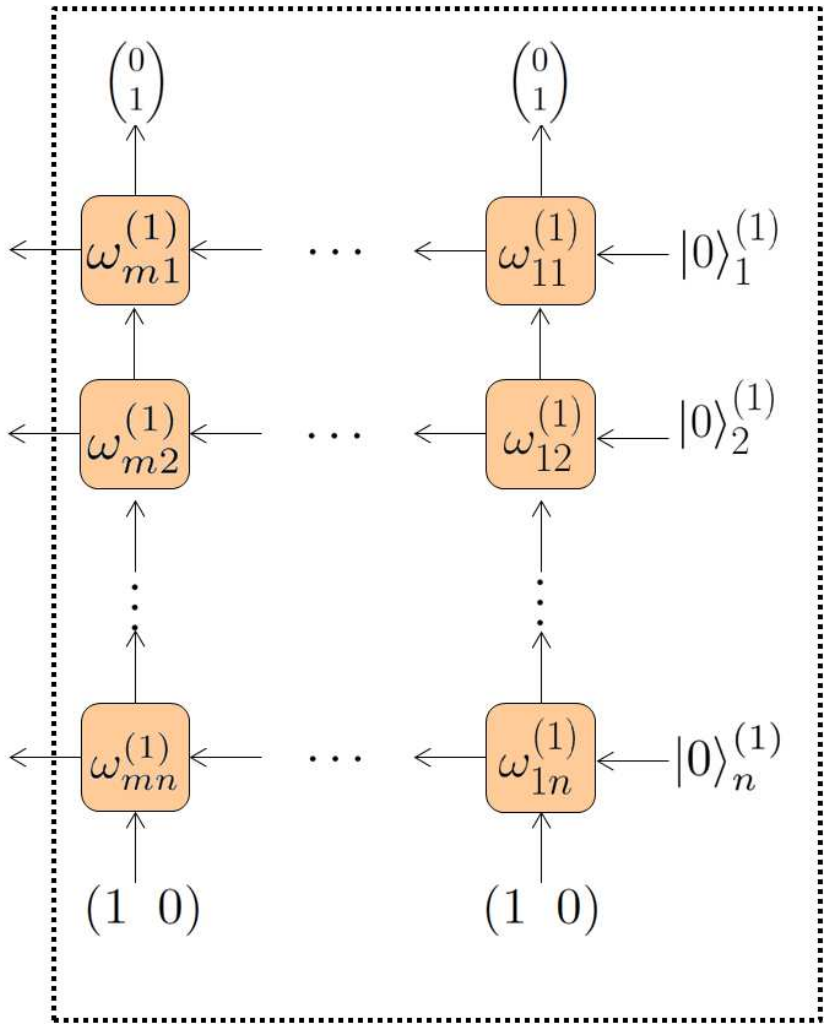


Figure 4.2: Tensor Network representation of nested level

Now we shall consider the first level Bethe ansatz. In the first term of the ansatz (4.100), $\vec{B}(\lambda_1) \otimes \vec{\Phi}_{n-1}(\lambda_2, \ldots, \lambda_n)$, the set of creation operators $\vec{B}$ is constructed by terminating the ends of the transfer matrix by appropriate boundary vectors/matrices. As such, the first term of the ansatz can be constructed as in Fig. 4.3, without connecting it to the nested level.

In each term of the sum in the second term of (4.100), the operators

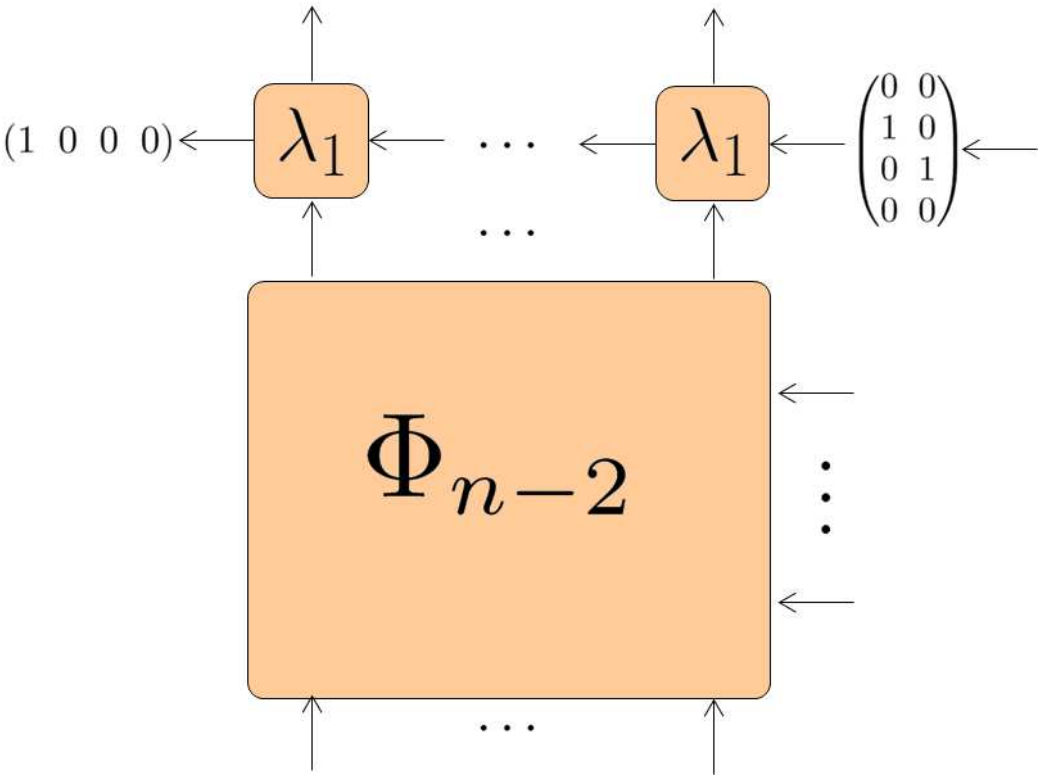


Figure 4.3: First term of the Bethe ansatz

B and F in (4.23) are constructed by terminating the ends of the transfer matrix by appropriate boundary vectors.

Figure 4.4: $\hat{r}$ operator

Now, to connect the nested Bethe ansatz with the first level Bethe ansatz, we need to understand the role of the set of operators $\hat{r}$ that trails each term in that sum. The $\hat{r}$ operator can be represented diagrammatically as shown in Fig. 4.4, and since for each term in the sum, the $\hat{r}$ sequentially "swaps" the adjacent indices from $k = j - 1$ to $k = 2$, the net effect is that it links the j^{th} index of the nested Bethe ansatz with the second index of the first

level Bethe ansatz, which, in turn, is the $\vec{\xi}$, that is also connected to the first index. This implies that the role of the $\vec{\xi}$ is to antisymmetrize the first and j^{th} index of the algebraic Bethe ansatz, which will form a pair of spins that will carry the spectral parameter λ_1 created by $F(\lambda_1)$.

We can now construct a tensor network form of the j^{th} term in the second term as shown in Fig. 4.5 and show diagrammatically how the $\hat{r}$'s form the connection to the nested level.

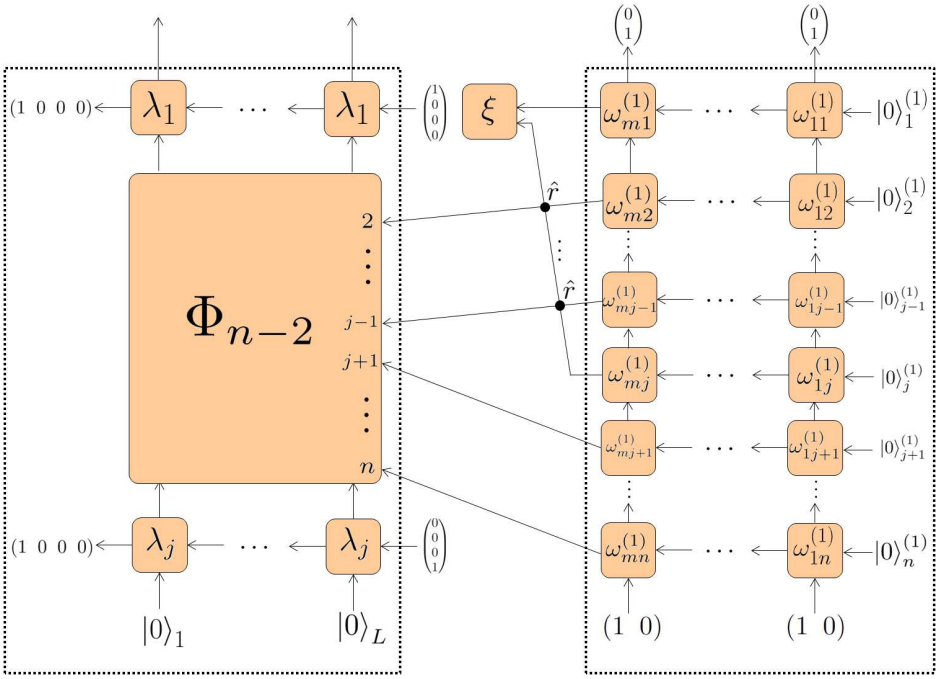


Figure 4.5: Second term of the Bethe ansatz

Having constructed both the first and second term of the Bethe ansatz (4.100) and their connection to the the nested level, the tensor network form of the algebraic Bethe ansatz for the Hubbard model can be constructed in full. However, the number of terms in the summation in (4.100) depends on the number of excitations, which means that even a pictorial description of its eigenstates would be very large, even for a particular ground state. Nevertheless, we note that the above provides sufficient information for full construction of the tensor network representation of the anastz Bethe ansatz for the Hubbard model.

Chapter 5

Summary

The Heisenberg XXX model, supersymmetric t-J model and the Hubbard model (all in 1D) are solvable by the algebraic Bethe ansatz, from which their respective tensor network representation were formulated, taking into account the generalizations of the ungraded Yang-Baxter formulalism of the XXX model through grading, nesting and recursion.

By making use of the fact that a Bethe eigenstate can be represented as a tensor network that consists of a series of MPOs applied to an MPS, the methods for approximative calculation of expectation values with respect to their Bethe eigenstates were then presented.

The virtual dimension can be systematically reduced after each multiplication and an MPS with small virtual dimension would eventually be obtained. This could then be used to calculate the expectation value of any observable, for arbitrary ground states and excited states of these models.

The work presented in this thesis therefore overcomes a major shortcoming of current DMRG methods, which work well mainly on the ground states only. In addition, with the tensor network description of these models, arbitrary expectation values of the eigenstates on finite length lattices within the regime of experimental interest can be efficiently computed.

As a proof of principle, we have obtained the correlation functions of eigenstates of the XXX model and supersymmetic t-J model on finite length lattices with our method.

Possible extensions to this work include actual computation of the correlation functions in the Hubbard model, and formulation of the algebraic Bethe ansatz of other integrable models, like the Essler-Korepin-Schoutens model [28], and continuous models such as the Lieb-Liniger model [70].

In addition, performing a tensor network interpretation of generalizations of the Yang-Baxter equation for two dimensional quantum models, such as the ZTE [130, 14] and the cubic equations for three dimensional classical models [55] (in which the integrability of the Kitaev model was reformulated) might potentially offer a better understanding of quantum many-body systems in higher dimensions.

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