Exploring the Space of Superconformal Field Theories

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Madalena Duarte de Almeida Lemos

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Madalena Duarte de Almeida Lemos

We, the dissertation committee for the above candidate for the

Doctor of Philosophy degree, hereby recommend

acceptance of this dissertation.

Leonardo Rastelli – Dissertation Advisor
Professor, Department of Physics and Astronomy

Peter van Nieuwenhuizen – Chairperson of Defense
Distinguished Professor, Department of Physics and Astronomy

Matthew Dawber
Associate Professor, Department of Physics and Astronomy

Martin Roček
Professor, Department of Physics and Astronomy

Michael Anderson
Professor, Department of Mathematics

This dissertation is accepted by the Graduate School.

Charles Taber
Dean of the Graduate School

Abstract of the Dissertation

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This dissertation focuses on the study of superconformal field theories (SCFTs) through the so-called (super)conformal bootstrap program.

The goals of this program are twofold: to chart the space of allowed SCFTs and to solve specific theories. The hope is that symmetries and a few simple physical assumptions, combined with basic consistency requirements (crossing symmetry and unitarity) are powerful enough to completely "solve" these theories. In doing so we never have to provide a Lagrangian description for the theories in question, and thus we can employ the bootstrap in studying SCFTs for which no such description is known. This is the case of many of the theories considered in this dissertation, making the bootstrap an ideal tool to explore them.

Most of this dissertation focuses on SCFTs in four-dimensions with $\mathcal{N} = 2$ supersymmetry. The large amount of symmetry of these theories makes them more tractable, and we find that the crossing symmetry equations admit a solvable subsector. This gives rise to the identification of a two-dimensional chiral algebra inside the four-dimensional SCFT, which allows for exact results to be obtained, including new unitarity bounds, constraining the space of allowed SCFTs. However, if we want to study the full theory, we must analyze the full set of crossing

equations by resorting to numerical techniques. In this work we begin such an analysis, obtaining various bounds on central charges and operator dimensions which are valid for any SCFT. In the last part of this dissertation we extend the numerical bootstrap analysis for SCFTs in six-dimensions with $\mathcal{N} = (2, 0)$ supersymmetry.

Chapters 2, 3 and 4 are essentially identical to Refs. [1–3], and the results of the final chapter will appear in Ref. [4].

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Chapter 1

Introduction

The conformal bootstrap is an old dream in theoretical physics, that the symmetries and a few simple physical assumptions, combined with basic consistency requirements (crossing symmetry and unitarity) are powerful enough to completely "solve" conformal field theories (CFTs). By solving we mean obtaining the spectrum of operators and all three-point functions, the CFT data, which is enough to compute any correlator.

Conformal field theories are of natural importance in theoretical physics. They arise in many physical applications, from statistical mechanics, where they describe the critical behavior of systems at second order phase transitions, to perturbative string theory in the worldsheet formulation. Through the AdS/CFT correspondence conformal field theories in the large N limit are dual to semiclassical quantum gravity in asymptotically anti- de Sitter spacetimes. They also arise as fixed points of the renormalization group flow of quantum field theories, therefore being of natural importance for the study of quantum field theories in general.

The great amount of symmetry of these theories also makes them more tractable, especially in two-dimensions, where the conformal algebra is extended to a infinite-dimensional one, the Virasoro algebra. There the bootstrap was very successful, leading to many exactly solvable "rational" CFTs. The techniques used, however, do not apply to higher dimensions, or even generically to non-rational CFTs in two-dimensions.

The recent revival of the bootstrap stemmed from the work of Rattazzi, Rychkov, Tonni and Vichi [6], who were able to use these constraints, combined with numerical techniques, to exclude the existence of theories in certain regions of the space of conformal dimensions of operators. The statement that a given CFT spectrum is not consistent with crossing symmetry is translated into the existence of a solution to a linear or semidefinite programming problem, which can be solved by resorting to known algorithms. Owing to the very general assumptions that go into this program, these methods allow one to constrain the space of conformal field theories by placing bounds on the operator dimensions and also on operator product expansion (OPE) coefficients (such as the central charge of the theory). It was also found that some physically interesting theories, such as the two and three-dimensional Ising model, sit at special "kinks" of the exclusion curves separating the region in parameter space for which there are solutions to crossing symmetry compatible with unitarity and the region where there are none. In cases such as these there are even indications that there is an actual theory living on the boundary of the allowed region, and this method has been used to obtain operator dimensions and OPE coefficients with very good precision. It is worth mentioning that this was accomplished by just studying one particular correlator, whereas for there to be a consistent CFT one would have to study all possible correlators, and so allowed theories are much more constrained.[1]

A big advantage of this method is that it does not rely on the existence of any Lagrangian, contrasting most currently available methods, and as such it can be applied to theories for which no such description is known. It is also non-perturbative in nature, and the dimensions and OPE coefficients found this way are the complete answer, thus applying to any theory irrespectively of whether a parameter exists in which one could do perturbation theory. Therefore it is particularly useful for strongly coupled theories which have no marginal deformations, as is the case of many of the theories considered in this dissertation.

The focus of this dissertation will be the bootstrap program for superconformal field theories (SCFTs), and our goals are twofold: to chart the space of allowed theories, and to use it to study specific models. In what follows we describe the outline of the thesis and summarize our main results.

1.1 $\mathcal{N} = 2$ SCFTs in four dimensions

Most of this dissertation is devoted to the study of four dimensional $\mathcal{N} = 2$ SCFTs. The large amount of symmetry makes these theories more tractable than the lower or non-supersymmetric theories, and has lead to many exact results (see, *e.g.*, [10] for a recent review). Nevertheless the space of $\mathcal{N} = 2$ SCFTs is vast, and much richer than the maximally supersymmetric case. There has been an explosion of results, with many new SCFTs found, and there is still no complete classification of the landscape of such theories. While theories that can be described by a Lagrangian can be classified [11], there is a large number of theories for which no Lagrangian description is known. Yet the current landscape of theories appears to have some structure, hinting that a classification might be possible, as will be discussed in detail in section 4.2. We might also hope that some of the non-Lagrangian theories also are uniquely specified by few discrete data. For example the $\mathfrak{e}_6$ theory of Minahan and Nemeschansky [12] discussed in subsection 4.6.2 could plausibly be the unique theory with $\mathfrak{e}_6$ flavor symmetry, and with the appropriate central charges.

Altogether this makes the (superconformal) bootstrap program a natural tool to chart out the space of theories, and to attempt to "solve" some of them. One could even expect

[1] Work along these lines has been done recently in [7–9], where the authors considered multiple correlators and found even stronger constraints from the bootstrap.

the bootstrap program to prove more constraining due to the large amount of supersymmetry. This indeed turns out to be the case, and we found that the bootstrap equations for a certain class of operators admit a solvable subsector, allowing for exact results to be obtained, including new analytic bounds on the allowed central charges of the theory. These results can then be complemented by numerical bounds obtained using the techniques already mentioned.

In chapter 2 we describe a new correspondence between four-dimensional SCFT with $\mathcal{N} \geq 2$ and two-dimensional chiral algebras. We show that these four-dimensional theories contain a protected subsector which is captured by a non-unitary two-dimensional chiral algebra. The existence of a stress tensor in the four-dimensional theory implies the chiral algebra will have a meromorphic stress tensor, although with a negative central charge, fixed in terms of the four-dimensional c central charge. Moreover, flavor symmetries of the four-dimensional theory give rise to affine Kac-Moody currents at a negative level (fixed in terms of the four-dimensional flavor central charge k) in the chiral algebra. The graded partition function of the chiral algebra is exactly captured by a four-dimensional quantity – the Schur limit of the superconformal index [13]. The superconformal index [14] counts (with signs) the protected spectrum of the theory, setting to zero any combination of multiplets which might recombine to form a long multiplet. Therefore it can be very useful in the study of the protected spectrum of SCFTs, and a vast literature exists on it (see, *e.g.*, [15] for a recent review). Combining the information extracted from the index with the aforementioned correspondence between four-dimensional SCFTs and two-dimensional chiral algebras can prove valuable in studying the four-dimensional theory.

This is explored in chapter 3, where we focused on the chiral algebras associated with the so-called T_N theories.[2] These theories, which are non-Lagrangian for $N \geqslant 3$,[3] are of natural interest for being one of the building blocks of class $\mathcal{S}$ theories [17, 18]. As such through well defined four-dimensional operations, which also have a clear meaning in the two-dimensional chiral algebra, one can obtain many non-Lagrangian theories starting from these ones. By analysing the Schur limit of the superconformal index and accentuating its two-dimensional interpretation as a graded partition function by making manifest the critical affine module structure, we were able to conjecture the full set of generators of the chiral algebras associated with these theories. This allowed us to bootstrap the chiral algebra of the T_4 theory, by imposing associativity of the operator product algebra of the conjectured set of generators. Associativity turns out to be constraining enough to completely fix all OPE coefficients, including the central charges. Armed with this explicit construction we computed null relations on the chiral algebra, which correspond to Higgs branch chiral ring relations on the four-dimensional side, some of which were previously unknown.

Another direct consequence of the existence of this protected subsector is that the crossing symmetry equations split into two sets, one only involving the exchange of protected

[2] See [16] for a recent review of these theories.

[3] The T_3 theory is nothing more than the $\mathfrak{e}_6$ theory of Minahan and Nemeschansky already mentioned.

operators (which is simply the crossing equation for the two-dimensional chiral algebra), while the other involves both protected and unprotected operators. The bootstrap program then becomes a two step process: in the first step we solve the crossing equation for this protected sector analytically, serving as input for the second step, where the full-fledged crossing equations, governing the unprotected part, are analyzed numerically. By studying one particular scalar four-point function (of the moment map operators) the first step gave us new analytic bounds, for theories with a given flavor symmetry, involving both central charges k and c . It also allowed us to fix completely (for interacting theories) the protected contributions to the full four-dimensional correlator of this type of operators.

By also resorting to numerical techniques, further constraints can be obtained, yielding information about the unprotected operators in the theory, such as their dimensions and OPE coefficients. It also provides additional bounds on the central charges of the theory, valid for any $\mathcal{N}=2$ SCFT. In chapter 4 we initiate this numerical analysis. We take an abstract operator-algebraic viewpoint in order to unify the description of Lagrangian and non-Lagrangian theories. While there is much to explore by employing the superconformal bootstrap program to study $\mathcal{N}=2$ SCFTs, we have taken a first step by approaching these theories from two different angles. On one we study correlators of moment map operators (which are related by supersymmetry to the flavor currents), focusing on theories with a given flavor symmetry (in particular $\mathfrak{su}(2)$ and $\mathfrak{e}_6$), while on the other we study correlators of chiral primaries (whose vacuum expectation values (vevs) parametrize the Coulomb branch). For the former operators the bootstrap program is the two-step process described above, however the latter are not captured by the chiral algebra, and as such we can only resort to numerical methods to study them. From the first correlator we find numerical bounds involving the c and k central charges, while from the latter we find central charge bounds for theories with a Coulomb branch operator of a given dimension. It is also interesting to bound other OPE coefficients. For example in the four-point function of chiral primaries some of these can tell us about relations on the Coulomb branch chiral ring.

1.2 Six-dimensional $\mathcal{N}=(2,0)$ SCFTs

The final chapter 5 of this dissertation applies the bootstrap method for the maximally supersymmetric theory in six dimensions, which will appear in [4].[4] This theory plays an important role in theoretical physics, as many lower dimensional theories can be obtained from its compactification. In particular a vast class of four-dimensional $\mathcal{N}=2$ SCFTs, the theories of class $\mathcal{S}$, can be obtained as the low-energy limit of twisted compactifications of $\mathcal{N}=(2,0)$ theories on a Riemann surface [17, 18]. Dualities among the four-dimensional theories can then be understood from such compactifications. However, despite its central

[4]Six dimensions is also the largest in which one can have a superconformal field theory [19–21].

role, very little is known about $\mathcal{N} = (2,0)$ theories beyond the free case. The main problem starts with the fact that these theories have no Lagrangian description, and thus the usual tools we have to study SCFTs are not available. These theories are isolated, having no marginal deformations that preserve the superconformal algebra $\mathfrak{osp}(8^\star|4)$, corresponding to non-trivial fixed points of the renormalization group in six dimensions [22]. Nevertheless, as we have been stressing throughout this dissertation, this is not an impediment from the bootstrap point of view, therefore making the superconformal bootstrap the ideal tool to tackle these theories.

Contrasting with the four-dimensional $\mathcal{N} = 2$ SCFTs considered in the previous chapters, maximally supersymmetric theories are much more constrained. Whereas before a large part of our objectives was to constrain the space of allowed theories, for the $\mathcal{N} = (2,0)$ there is a proposed classification arising from string theory. The symmetry group of these theories does not allow for flavor symmetries, and the known SCFTs are classified by simply-laced Lie algebras [23]. However we can still ask the question of whether there can exist "exotic" theories, that do not find a realization in string or M-theory. For the theories of type A_n and D_n there is a dual description in the large n limit through the AdS/CFT correspondence in terms of supergravity on $AdS_7 \times S^4$ and $AdS_7 \times \mathbb{RP}^4$ respectively [24, 25]. As such, in this limit, the numerical bootstrap results obtained in this dissertation should recover the known results from supergravity on $AdS_7 \times S^4$ [26, 27]. Beyond the large n limit, the only results for correlation functions have been obtained in [28], where the authors found a protected sector of operators and observables isomorphic to a two-dimensional chiral algebra, allowing for the calculation of correlators in that subsector. Similarly to Chapters 2 and 4, to make statements about operators not captured by this chiral algebra we must resort to numerical techniques to analyze the full crossing symmetry equations.

As the starting point in the superconformal bootstrap program for $\mathcal{N} = (2,0)$ SCFTs we want to consider an operator any local conformal field theory must have – the stress tensor. However we will not have to take the four-point function of the stress tensor itself, as we can consider instead that of the superconformal primary of the super multiplet the stress tensor belongs to. This has the advantage that the superconformal primary is a spin zero operator, avoiding the technically challenging bootstrap of a spin two operator.

We start by asking the question of what the lowest possible central charge (for an interacting theory) can be. Our results suggest that the lowest possible central charge corresponds precisely to that of the A_1 theory, therefore ruling out any "exotic" theories with smaller central charges. Moreover we find evidence that there is a unique $\mathcal{N} = (2,0)$ SCFT with this central charge, which must correspond exactly to the A_1 theory (in M-theory this corresponds to the low-energy theory on two M5 branes [29, 30]). In addition we obtain bounds on OPE coefficients of protected operators which were not captured by the chiral algebra of [28], as well as bounds on the low-lying spectrum of unprotected operators. We will argue that, at least for the A_1 case, these bounds must be saturated by the physical theories, paving the way for a possible bootstrap of this theory.

Chapter 2

Infinite Chiral Symmetry in Four Dimensions

The contents of this chapter appear in [1]: "Infinite Chiral Symmetry in Four Dimensions", C. Beem, M. Lemos, P. Liendo, W. Peelaers, L. Rastelli and B. C. van Rees, arXiv:1312.5344 [hep-th], Commun. Math. Phys. **336**, no. 3, 1359 (2015)
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2.1 Introduction

It has long been recognized that supersymmetric quantum field theories enjoy many special properties that make them particularly useful testing grounds for more general ideas about quantum field theory. This is largely a consequence of the fact that many observables in such theories are "protected", in the sense of being determined by a semiclassical calculation with a finite number of corrections taken into account, or alternatively by some related "finite-dimensional" problem that admits the type of closed-form solution that is uncharacteristic of interacting quantum field theories. In most circumstances, these techniques have a semi-classical flavor to them. For example, in cases where supersymmetric partition functions can be computed by localization, the calculation is generally performed starting from a weakly coupled Lagrangian description of the theory.

A notable omission from the currently available techniques is a way to directly access the interacting superconformal phases of theories that do not admit a Lagrangian formulation. By now, there exists a veritable menagerie of models in various dimensions that exhibit conformal phases with varying amounts of supersymmetry, but only in the nicest cases do such models belong to families that include free theories as special points, allowing for properties of the interacting theory to be studied semiclassically. Even for those Lagrangian models, the standard supersymmetric toolkit does not seem to exploit some of the most powerful structures of conformal field theory, such as the existence of a state/operator map and of a well-controlled and convergent operator product expansion.

Meanwhile, recent years have witnessed a surprising resurgence of progress centering around precisely these aspects of conformal field theory in the form of the conformal bootstrap [31, 32]. In large part, this progress has been inspired by the development of numerical techniques for extracting constraints on the defining data of a CFT using unitarity and crossing symmetry [6, 33]. Generally speaking, these techniques are equally applicable to theories with and without supersymmetry, and despite promising early results [34–37], it has not been entirely clear the extent to which supersymmetry improves the situation. Nevertheless, the possibility that supersymmetry may act as a crucible in which exact results can be forged even for strongly interacting CFTs is irresistible, and we are led to ask the question:

> *Do the conformal bootstrap equations in dimension $d > 2$ admit a solvable truncation in the case of superconformal field theories?*

Having formulated the question, it is worth pausing to consider in what sense the answer could be "yes". The most natural possibilities correspond to known situations in which bootstrap-type equations are rendered solvable. There are two primary scenarios in which the constraints of crossing symmetry are nontrivial, yet solvable:

(I) Meromorphic (and rational) conformal field theories in two dimensions.

(II) Topological quantum field theories.

The subject of this chapter is the realization of the first option in the context of $\mathcal{N} \geqslant 2$ superconformal field theories in four dimensions. The same option is in fact viable for $(2,0)$ superconformal theories in six dimensions. That subject is elaborated upon in a separate article [28]. Although we will not discuss the subject at any length in the present work, the second option can also be realized using similar techniques to those discussed herein.

The primary hint that such an embedding should be possible was already observed in [36, 38], building upon the work of [27, 39–43]. In a remarkable series of papers [27, 38–43], the constraints of superconformal symmetry on four-point functions of half-BPS operators in $\mathcal{N} = 2$ and $\mathcal{N} = 4$ superconformal field theories were studied in detail. This analysis revealed that the superconformal Ward identities obeyed by these correlators can be conveniently solved in terms of a set of arbitrary real-analytic functions of the two conformal cross ratios $(z, \bar{z})$, along with a set of *meromorphic* functions of z alone. In a decomposition of the four-point function as an infinite sum of conformal blocks, these meromorphic functions capture the contribution to the double operator product expansion of intermediate "protected" operators belonging to shortened representations. The real surprise arises when these results are combined with the constraints of crossing symmetry. One then finds [36, 38] that the meromorphic functions obey a *decoupled* set of crossing equations, whose general solution can be parametrized in terms of a finite number of coefficients. For example, in the important case of the four-point function of stress-tensor multiplets in an $\mathcal{N} = 4$ theory, there is a one-parameter family of solutions, where the parameter has a direct physical interpretation as the central charge (conformal anomaly) of the theory. The upshot is that the protected part of this correlator is entirely determined by abstract symmetry considerations, with no reference to a free-field description of the theory.

In this chapter we establish a conceptual framework that explains and vastly generalizes this observation. For a general $\mathcal{N}=2$ superconformal field theory, we define a protected subsector by passing to the cohomology of a certain nilpotent supercharge $\mathbb{Q}$. This is a familiar strategy – for example, the definition of the chiral ring in an $\mathcal{N}=1$ theory follows the same pattern – but our version of this maneuver will be slightly unconventional, in that we take $\mathbb{Q}=\mathcal{Q}+\mathcal{S}$ to be a linear combination of a Poincaré and a conformal supercharge. In order to be in the cohomology of $\mathbb{Q}$, local operators must lie in a fixed plane $\mathbb{R}^2\subset\mathbb{R}^4$. Crucially, their correlators can be shown to be non-trivial *meromorphic* functions of their positions. This is in contrast to correlators of $\mathcal{N}=1$ chiral operators, which are purely topological in a general $\mathcal{N}=1$ model, and strictly vanish in an $\mathcal{N}=1$ *conformal* theory due to R-charge conservation.

The meromorphic correlators identified by this cohomological construction are precisely the ingredients that define a two-dimensional *chiral algebra*.[1] Our main result is thus the definition of a map χ from the space of four-dimensional $\mathcal{N}=2$ superconformal field theories to the space of two-dimensional chiral algebras,

$$\chi\ :\ \text{4d}\ \mathcal{N}=2\ \text{SCFT}\ \longrightarrow\ \text{2d Chiral Algebra}.$$

In concrete terms, the chiral algebra computes correlation functions of certain operators in the four-dimensional theory, which are restricted to be coplanar and further given an *explicit* space-time dependence correlating their $SU(2)_R$ orientation with their positions, see (2.27). For the case of four-point functions of half-BPS operators, assigning the external operators this "twisted" space-time dependence accomplishes precisely the task of projecting the full correlator onto the meromorphic functions appearing in the solution to the superconformal Ward identities. To recapitulate, those mysterious meromorphic functions are given a direct interpretation as correlators in the associated chiral algebra, and turn out to be special instances of a much more general structure.

The explicit space-time dependence of the four-dimensional operators is instrumental in making sure that they are annihilated by a common supercharge $\mathbb{Q}$ for any insertion point on the plane. From this viewpoint, our construction is in the same general spirit as [44] (see also [45]). These authors considered particular examples of correlators in $\mathcal{N}=4$ super

[1]We have settled on the expression "chiral algebra" as it is the most common in the physics literature. We consider it to be synonymous with "vertex operator algebra", though in the mathematical literature some authors make a distinction between the two notions. We trust no confusion will arise with the overloading of the word "chiral" due to its unavoidable use in the four-dimensional context, *e.g.*, "chiral and anti-chiral $4d$ supercharges", "the $\mathcal{N}=1$ chiral ring", etc.

Yang-Mills theory that are invariant under supercharges of the same schematic form $\mathcal{Q}+\mathcal{S}$. Their choices of supercharges are inequivalent to ours, and do not lead to meromorphic correlators.

The operators captured by the chiral algebra are precisely the operators that contribute to the Schur limit of the superconformal index [13, 14, 46], and we will refer to them as *Schur operators*. Important examples are the half-BPS operators that are charged under $SU(2)_R$ but neutral under $U(1)_r$, whose vacuum expectation values parameterize the Higgs branch of the theory, and the $SU(2)_R$ Noether current. The class of Schur operators is much larger, though, and encompasses a variety of supermultiplets obeying less familiar semi-shortening conditions. Operators associated to the Coulomb branch of the theory (such as the half-BPS operators charged under $U(1)_r$ but neutral under $SU(2)_R$) are *not* of Schur type. In a pithy summary, the cohomology of $\mathbb{Q}$ provides a "categorification" of the Schur index. It is a surprising and useful fact that this vector space naturally possesses the additional structure of a chiral algebra.

Chiral algebras are rigid structures. Associativity of their operator algebra translates into strong constraints on the spectrum and OPE coefficients of Schur operators in the parent four-dimensional theory. We have already pointed out that this leads to a unique determination of the protected part of four-point function of stress-tensor multiplets in the $\mathcal{N}=4$ context [36]. Another canonical example is the four-point function of "moment map" operators in a general $\mathcal{N}=2$ superconformal field theory. The moment map M is the lowest component of the supermultiplet that contains the conserved flavor current of the theory, and as such it transforms in the adjoint representation of the flavor group G. We find that the associated two-dimensional meromorphic operator $J(z):=\chi[M]$ is the dimension-one generating current of an affine Lie algebra $\hat{\mathfrak{g}}_{k_{2d}}$, with level k_{2d} fixed in terms of the four-dimensional flavor central charge. As the four-point function of affine currents is uniquely fixed, this relation completely determines the protected part of the moment map four-point function. In turn, this information serves as essential input to the full-fledged bootstrap equations that govern the contributions from generic long multiplets in the conformal block decomposition of these four-point functions. These equations can be studied numerically to derive interesting bounds on non-protected quantities, following the approach of [36]. We present numerical bounds that arise for various choices of G in chapter 4. It is worth emphasizing that the protected part of the four-point function receives contributions from an *infinite* tower of intermediate shortened multiplets, and without knowledge of its precise form the numerical bootstrap program would never get off the ground. In theories that admit a Lagrangian description, one could appeal to non-renormalization theorems and derive the same protected information in the free field limit; the chiral algebra then just serves as a powerful organizing principle to help obtain the same result. However, the abstract chiral algebra approach seems indispensable for the analysis of non-Lagrangian theories – for example, when G is an exceptional group.

As a byproduct of a detailed study of the moment map four-point function, we are able to derive new unitarity bounds that must be obeyed by the central charges of any interacting $\mathcal{N} = 2$ superconformal field theory. By exploiting the relation between the two- and four-dimensional perspectives, we are able to express certain coefficients of the four-dimensional conformal block decomposition of the four-point function in terms of central charges; the new bounds arise because those coefficients must be non-negative in a unitary theory. Saturation of the bounds signals special properties of the Higgs branch chiral ring. This is a particular instance of a more general encoding of four-dimensional physics in the chiral algebra, the surface of which we have only barely scratched. One notable aspect of this correspondence is the interplay between the geometry of the Higgs branch and the representation theory of the chiral algebra; for example, null vectors that appear at special values of the affine level imply Higgs branch relations.

We describe several structural properties of the map χ. Two universal features are the affine enhancement of the global flavor symmetry, and the Virasoro enhancement of the global conformal symmetry. The affine level in the chiral algebra is related to the flavor central charge in four dimensions as $k_{2d} = -\frac{1}{2}k_{4d}$, while the Virasoro central charge is proportional to the four-dimensional conformal anomaly coefficient,[2] $c_{2d} = -12c_{4d}$. A perhaps surprising feature of these relations is that the two-dimensional central charges and affine levels must be negative. Another universal aspect of the correspondence is a general prescription to derive the chiral algebra associated to a gauge theory whenever the chiral algebra of the original theory whose global symmetry is being gauged is known.

Turning to concrete examples, we start with the SCFTs of free hypermultiplets and free vector multiplets, which are associated to free chiral algebras. With the help of the general gauging prescription, we can combine these ingredients to find the chiral algebra associated to an arbitrary Lagrangian SCFT. We also sketch the structure of the chiral algebras associated to SCFTs of class $\mathcal{S}$, which are generally non-Lagrangian. In several concrete examples, we present evidence that the chiral algebra has an economical presentation as a $\mathcal{W}$-algebra, *i.e.*, as a chiral algebra with a finite set of generators [49]. We do not know whether all chiral algebras associated to SCFTs are finitely generated, or how to identify the complete set of generators in the general case. Indeed, an important open problem is to give a more precise characterization of the class of chiral algebras that can arise from physical four-dimensional theories. Ideally the distinguishing features of this class could be codified in a set of additional axioms. Since chiral algebras are on sounder mathematical footing than four-dimensional quantum field theories, it is imaginable that this could lead to a well-defined algebraic classification problem. If successful, this approach would represent concrete progress towards the loftier goal of classifying all possible $\mathcal{N} = 2$ SCFTs.

On a more formal note, four-dimensional intuition leads us to formulate a number of new conjectures about chiral algebras that may be of interest in their own right. The conjectures

[2] There are two tensorial structures in the four-dimensional trace anomaly, whose coefficients are conventionally denoted a and c. It is the c anomaly that is relevant for us, in contrast to the better studied a anomaly, which decreases monotonically under RG flow [47, 48].

generally take the form of an ansatz for the cohomology of a BRST complex, and include new free-field realizations of affine Lie algebras at special values of the level and new examples of quantum Drinfeld-Sokolov reduction for nontrivial modules. We present evidence for our conjectures obtained from a low-brow, level-by-level analysis, but we suspect that more powerful algebraic tools may lead to rigorous proofs.

The organization of this chapter is as follows. In §2.2 we review the arguments behind the appearance of infinite-dimensional chiral symmetry algebras in the context of two-dimensional conformal field theories. We explain how the same structure can be recovered in the context of $\mathcal{N} = 2$ superconformal field theories in four dimensions by studying observables that are well-defined after passing to the cohomology of a particular nilpotent supercharge in the superconformal algebra. This leads to the immediate conclusion that chiral symmetry algebras will control the structure of this subclass of observables. In §2.3, we describe in greater detail the resulting correspondence between $\mathcal{N} = 2$ superconformal models in four dimensions and their associated two-dimensional chiral algebras. We outline some of the universal features of the correspondence. We further describe an algorithm that defines the chiral algebra for any four-dimensional SCFT with a Lagrangian description in terms of a BRST complex. In §2.4, we describe the immediate consequences of this structure for more conventional observables of the original theory. It turns out that superconformal Ward identities that have previously derived for four-point functions of BPS operators are a natural outcome from our point of view. We further derive new unitarity bounds for the anomaly coefficients of conformal and global symmetries, many of which are saturated by interesting superconformal models. We point out that the state space of the chiral algebra provides a categorification of the Schur limit of the superconformal index. In §2.5, we detail the construction and analysis of the chiral algebras associated to some simple Lagrangian SCFTs. We also make a number of conjectures as to how to describe these chiral algebras as $\mathcal{W}$-algebras. In §2.6 we provide a sketch of the class of chiral algebras that are associated to four-dimensional theories of class $\mathcal{S}$. We conclude in §2.7 by listing a number of interesting lines of inquiry that are opened up by the results reported here. Several appendices are included that review relevant material concerning the superconformal algebras and representation theory used in our constructions.

2.2 Chiral symmetry algebras in four dimensions

The purpose of this section is to establish the existence of infinite chiral symmetry algebras acting on a restricted class of observables in any $\mathcal{N} = 2$ superconformal field theory in four dimensions. This is accomplished in two steps. First, working purely in terms of the relevant spacetime symmetry algebras, we identify a particular two-dimensional conformal subalgebra of the four-dimensional superconformal algebra,[3]

$$\mathfrak{sl}(2) \times \widehat{\mathfrak{sl}(2)} \subset \mathfrak{sl}(4\,|\,2) ,$$

with the property that the holomorphic factor $\mathfrak{sl}(2)$ commutes with a nilpotent supercharge, $\mathbb{Q}$, while the antiholomorphic factor $\widehat{\mathfrak{sl}(2)}$ is exact with respect to the same supercharge. We then characterize the local operators that represent nontrivial $\mathbb{Q}$-cohomology classes. The only local operators for which this is the case are restricted to lie in a plane $\mathbb{R}^2 \subset \mathbb{R}^4$ that is singled out by the choice of conformal subalgebra. The correlation functions of these operators are meromorphic functions of the insertion points, and thereby define a chiral algebra. As a preliminary aside, we first recall the basic story of infinite chiral symmetry in two dimensions in order to distill the essential ingredients that need to be reproduced in four dimensions. The reader who is familiar with chiral algebras in two-dimensional conformal field theory may safely proceed directly to §2.2.2.

2.2.1 A brief review of chiral symmetry in two dimensions

Let us take as our starting point a two-dimensional quantum field theory that is invariant under the global conformal group $SL(2,\mathbb{C})$. The complexification of the Lie algebra of infinitesimal transformations factorizes into holomorphic and anti-holomorphic generators,

$$\begin{aligned} L_{-1} &= -\partial_z \ , & L_0 &= -z\partial_z \ , & L_{+1} &= -z^2\partial_z \ , \\ \bar{L}_{-1} &= -\partial_{\bar{z}} \ , & \bar{L}_0 &= -\bar{z}\partial_{\bar{z}} \ , & \bar{L}_{+1} &= -\bar{z}^2\partial_{\bar{z}} \ , \end{aligned} \tag{2.1}$$

which obey the usual $\mathfrak{sl}(2) \times \mathfrak{sl}(2)$ commutation relations,

$$\begin{aligned} [L_{+1}, L_{-1}] &= 2L_0 \ , & [L_0, L_{\pm 1}] &= \mp L_{\pm 1} \ , \\ [\bar{L}_{+1}, \bar{L}_{-1}] &= 2L_0 \ , & [\bar{L}_0, \bar{L}_{\pm 1}] &= \mp \bar{L}_{\pm 1} \ . \end{aligned} \tag{2.2}$$

We need not assume that the theory is unitary, but for simplicity we will assume that the space of local operators decomposes into a direct sum of irreducible highest weight representations of the global conformal group. Such representations are labelled by holomorphic and anti-holomorphic scaling dimensions h and $\bar{h}$ of the highest weight state,

$$L_0|\psi\rangle_{h.w.} = h|\psi\rangle_{h.w.} \ , \qquad \bar{L}_0|\psi\rangle_{h.w.} = \bar{h}|\psi\rangle_{h.w.} \ , \tag{2.3}$$

[3] In this section, we adopt the convention of specifying the complexified versions of symmetry algebras. This will turn out to be particularly natural in the discussion of §2.2.2. We generally attempt to select bases for the complexified algebras that are appropriate for a convenient real form. Our basic constructions are insensitive to the signature of spacetime, though in places we explicitly impose constraints that follow from unitarity in Lorentzian signature.

and we further assume that h and $\bar{h}$ are not equal to negative half-integers (in which case one would find finite-dimensional representations of $\mathfrak{sl}(2)$).

Chiral symmetry arises as a consequence of the existence of any local operator $\mathcal{O}(z,\bar{z})$ which obeys a meromorphicity condition of the form

$$\partial_{\bar{z}}\mathcal{O}(z,\bar{z}) = 0 \implies \mathcal{O}(z,\bar{z}) := \mathcal{O}(z) \ . \tag{2.4}$$

Under the present assumptions, such an operator will transform in the trivial representation of the anti-holomorphic part of the symmetry algebra and by locality will have h equal to an integer or half-integer. Meromorphicity implies the existence of infinitely many conserved charges (and their associated Ward identities) defined by integrating the meromorphic operator against an arbitrary monomial in z,

$$\mathcal{O}_n := \oint \frac{dz}{2\pi i}\, z^{n+h-1}\, \mathcal{O}(z) \ . \tag{2.5}$$

The operator product expansion of two meromorphic operators contains only meromorphic operators, and the singular terms determine the commutation relations among the associated charges, *cf.* [49]. This is the power of meromorphy in two dimensions: an infinite dimensional symmetry algebra organizes the space of local operators into much larger representations, and the associated Ward identities strongly constrain the correlation functions of the theory.

Some examples of this structure are ubiquitous in two-dimensional conformal field theory. The energy-momentum tensor in a two-dimensional CFT is conserved and traceless in flat space, $\partial^\mu T_{\mu\nu} = T_\mu{}^\mu = 0$, leading to two independent conservation equations

$$\begin{aligned} \partial_{\bar{z}} T_{zz}(z,\bar{z}) = 0 &\implies T_{zz}(z,\bar{z}) := T(z) \ , \\ \partial_z T_{\bar{z}\bar{z}}(z,\bar{z}) = 0 &\implies T_{\bar{z}\bar{z}}(z,\bar{z}) := \overline{T}(\bar{z}) \ . \end{aligned} \tag{2.6}$$

The holomorphic stress tensor $T(z)$ is a meromorphic operator with $(h,\bar{h}) = (2,0)$, and its self-OPE is fixed by conformal symmetry to take the form

$$T(z)T(w) \sim \frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{(z-w)} \ , \tag{2.7}$$

which implies that the associated conserved charges obey the commutation relations of the Virasoro algebra with central charge c,

$$L_n := \oint \frac{dz}{2\pi i} z^{n+1} T(z) \ , \qquad [L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12} m(m^2-1)\delta_{m+n,0} \ . \tag{2.8}$$

Similarly, global symmetries can give rise to conserved holomorphic currents $J^A_z(z,\bar{z}) =:$

$J^A(z)$ with $(h, \bar{h}) = (1, 0)$. The self-OPEs of such currents are fixed to take the form

$$J^A(z)J^B(w) \sim \frac{k\,\delta^{AB}}{(z-w)^2} + \sum_C if^{ABC}\frac{J^C(w)}{(z-w)}\ , \tag{2.9}$$

with the structure constants f^{ABC} those of the Lie algebra of the global symmetry. The conserved charges in this case obey the commutation relations of an affine Lie algebra at level k,

$$J^A_n := \oint \frac{dz}{2\pi i}\, z^n\, J^A(z)\ , \qquad [J^A_m, J^B_n] = \sum_c if^{ABC}J^C_{m+n} + mk\,\delta^{AB}\delta_{m+n,0}\ . \tag{2.10}$$

The algebra of all meromorphic operators, or alternatively the algebra of their corresponding charges, constitutes the *chiral algebra* of a two-dimensional conformal field theory.

In most physics applications, the spectrum of a CFT will include non-meromorphic operators that reside in modules of the chiral algebra of the theory. In the generic case in which the chiral algebra is the Virasoro algebra, this just means that there are Virasoro primary operators with $\bar{h} \neq 0$. Nevertheless, the correlation functions of the meromorphic operators can be taken in and of themselves to define a certain meromorphic theory. Such theories are referred to by various authors as chiral algebras, vertex operator algebras, or meromorphic conformal field theories. Though some of these names are occasionally assigned to structures that possess some extra nice properties, such as modular invariant partition functions, we will be discussing the most basic version. Henceforth, by chiral algebra we will mean the operator product algebra of a set of meromorphic operators in the plane.[4] So defined, a chiral algebra is strongly constrained by the requirements of crossing symmetry. In what follows, we show that any $\mathcal{N} = 2$ superconformal field theory in four dimensions possesses a class of observables that define a chiral algebra in this sense.

2.2.2 Twisted conformal subalgebras

Chiral algebras are ordinarily thought to be a special feature of conformal-invariant models in two dimensions. Indeed, the appearance of an infinite number of conserved charges as defined in (2.5) follows from the interaction of two different ingredients that are special to two dimensions. Firstly, the operators that give rise to the chiral symmetry charges are invariant under (say) the anti-holomorphic factor of the two-dimensional conformal algebra, while transforming in a nontrivial representation of the holomorphic factor, so they are nontrivial holomorphic operators on the plane. The powerful machinery of complex analysis in a single

[4] In a preview of later discussions, we mention that by $\mathcal{W}$-algebra we will mean a chiral algebra for which the space of local operators is generated by a finite number of operators via the operations of taking derivatives and normal-ordered multiplication.

variable then produces the infinity of conserved charges in (2.5).[5]

In dimension $d > 2$, it is the first of these conditions that fails the most dramatically, while the latter seems more superficial. Indeed, correlation functions in a conformal field theory in higher dimensions can be restricted so that all operators lie on a plane $\mathbb{R}^2 \subset \mathbb{R}^d$, and the resulting observables will transform covariantly under the subalgebra of the d-dimensional conformal algebra that leaves the $\mathbb{R}^2$ in question fixed,

$$\mathfrak{sl}(2) \times \overline{\mathfrak{sl}(2)} \subset \mathfrak{so}(d+2) \ . \tag{2.11}$$

These correlation functions will be largely indistinguishable from those of an authentic two-dimensional CFT, and if one could locate operators that were chiral with respect to this subalgebra, then the arguments of §2.2.1 would go through unhindered and a chiral symmetry algebra could be constructed that would act on $\mathbb{R}^2$-restricted correlation functions. However, a local operator that transforms in the trivial representation of either copy of $\mathfrak{sl}(2)$ in (2.11) will necessarily be trivial with respect to *all* of $\mathfrak{so}(d+2)$. As such, the only "meromorphic" operator on the plane in a higher dimensional theory is the identity operator, and no chiral symmetry algebra can be constructed. This is ultimately a consequence of the simple fact that the higher dimensional conformal algebras do not factorize into a holomorphic and anti-holomorphic part: any two $\mathfrak{sl}(2)$ subalgebras will be related by conjugation.

The brief arguments given above are common knowledge, and essentially spell the end to any hopes of recovering chiral symmetry algebras in a general higher-dimensional conformal field theory. We have reproduced them here to clarify the mechanism by which they will be evaded in the coming discussion. In particular, we will see that the additional tools at our disposal in the case of *super*conformal field theories are sufficient to give life to chiral algebras in four dimensions. Before describing the construction, let us recall a simple example which illustrates the mechanism that will be used.

Intermezzo: translation invariance from cohomology

In a quantum field theory with $\mathcal{N} = 1$ supersymmetry in four dimensions, there exists a special class of operators known as *chiral operators* (not to be confused with the meromorphic operators of §2.2.1, which are chiral in a different sense) that lie in short representations of the supersymmetry algebra and satisfy a shortening condition in terms of a chiral half of the supercharges,

$$\{Q_\alpha, \mathcal{O}(x)] = 0 \ , \qquad \alpha = \pm \, . \tag{2.12}$$

[5]From another point of view, one can hardly hope to find a meromorphic sector in a higher dimensional CFT due to Hartogs' theorem, which implies the absence of singularities of codimension greater than one in a meromorphic function of several variables. This has been overcome in, *e.g.*, [50, 51] by considering extended operators that intersect in codimension one. The problem, then, is that the meromorphic structure does not impose constraints on the natural objects in the original theory – the local operators.

The translation generators in $\mathbb{R}^4$ are exact with respect to the chiral supercharges,

$$P_{\alpha\dot{\alpha}} = \{Q_\alpha, \widetilde{Q}_{\dot{\alpha}}\} \;, \tag{2.13}$$

and consequently, via the Jacobi identity, the derivative of a chiral operator is also exact,

$$[P_{\alpha\dot{\alpha}}, \mathcal{O}(x)] = \{Q_\alpha, \mathcal{O}'(x)] \;. \tag{2.14}$$

Because the chiral supercharges are nilpotent and anti-commute, the cohomology classes of chiral operators with respect to the supercharges Q_α are well-defined and independent of the insertion point of the operator. Schematically, one can write

$$[\mathcal{O}_i(x)]_{Q_\alpha} := \mathcal{O}_i \;. \tag{2.15}$$

Products of chiral operators are then free of short distance singularities and form a ring at the level of cohomology. Correlation functions of chiral operators have the excellent property of being independent of the positions of the operators,

$$\langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\ldots\mathcal{O}_n(x_n)\rangle = \langle [\mathcal{O}_1(x_1)][\mathcal{O}_2(x_2)]\ldots[\mathcal{O}_n(x_n)]\rangle = \langle \mathcal{O}_1\mathcal{O}_2\ldots\mathcal{O}_n\rangle \;. \tag{2.16}$$

A suggestive way of phrasing this well-known feature of the chiral ring is that although chiral operators transform in a nontrivial representation of the four-dimensional translation group, their cohomology classes with respect to the chiral supercharges transform in the trivial representation. The passage from local operators to their cohomology classes modifies the transformation properties of these local operators under the spacetime symmetry algebra, in this case rendering them trivial.

Holomorphy from cohomology

To recover chiral algebras in four dimensions, we adopt the same philosophy just illustrated in the example of the chiral ring. We will find a nilpotent supercharge with the property that *cohomology classes* of local operators with respect to said supercharge transform in a chiral representation of an $\mathfrak{sl}(2)\times\widehat{\mathfrak{sl}(2)}$ subalgebra of the full superconformal algebra, and as such behave as meromorphic operators. Such local operators will then necessarily constitute a chiral algebra as described in §2.2.1.

The first task that presents itself is an algebraic one. To realize chiral symmetry at the level of cohomology classes, we identify a two-dimensional conformal subalgebra of the four-dimensional superconformal algebra,

$$\mathfrak{sl}(2)\times\widehat{\mathfrak{sl}(2)} \subset \mathfrak{sl}(4\,|\,2) \;,$$

along with a privileged supercharge $\mathbb{Q}$ for which the following criteria are satisfied:

- The supercharge is nilpotent: $\mathbb{Q}^2 = 0$.

- $\mathfrak{sl}(2)$ and $\widehat{\mathfrak{sl}(2)}$ act as the generators of holomorphic and anti-holomorphic Möbius transformations on a complex plane $\mathbb{C} \subset \mathbb{R}^4$.

- The holomorphic generators spanning $\mathfrak{sl}(2)$ commute with $\mathbb{Q}$.

- The anti-holomorphic generators spanning $\widehat{\mathfrak{sl}(2)}$ are $\mathbb{Q}$ commutators.

In searching for such a subalgebra, we can first restrict our attention to subalgebras of $\mathfrak{sl}(4|2)$ that keep the plane fixed set-wise. There are two inequivalent maximal subalgebras of this kind: $\mathfrak{sl}(2|1) \times \mathfrak{sl}(2|1)$, which is the symmetry algebra of an $\mathcal{N} = (2,2)$ SCFT in two dimensions, and $\mathfrak{sl}(2) \times \mathfrak{sl}(2|2)$, which is the symmetry algebra of an $\mathcal{N} = (0,4)$ SCFT in two dimensions. One easily determines that the first subalgebra cannot produce the desired structure; we proceed directly to consider the second case.

The four-dimensional $\mathcal{N} = 2$ superconformal algebra and the two-dimensional $\mathcal{N} = (0,4)$ superconformal algebra are summarized in Appendix A. In embedding the latter into the former, we take the fixed two-dimensional subspace to be the one that is fixed pointwise by the rotation generator

$$\mathcal{M}^{\perp} := \mathcal{M}_{+}{}^{+} - \mathcal{M}^{\dot{+}}{}_{\dot{+}} \ . \tag{2.17}$$

The generator of rotations acting within the fixed plane is the orthogonal combination,

$$\mathcal{M} := \mathcal{M}_{+}{}^{+} + \mathcal{M}^{\dot{+}}{}_{\dot{+}} \ . \tag{2.18}$$

In more conventional terms, we are picking out the plane with $x_1 = x_2 = 0$. Introducing complex coordinates $z := x_3 + ix_4$, $\bar{z} := x_3 - ix_4$, the two-dimensional conformal symmetry generators in $\mathfrak{sl}(2) \times \mathfrak{sl}(2|2)$ can be expressed in terms of the four-dimensional ones as

$$\begin{aligned} L_{-1} &= \mathcal{P}_{+\dot{+}} \ , & L_{+1} &= \mathcal{K}^{\dot{+}+} \ , & 2L_0 &= \mathcal{H} + \mathcal{M} \ , \\ \bar{L}_{-1} &= \mathcal{P}_{-\dot{-}} \ , & \bar{L}_{+1} &= \mathcal{K}^{\dot{-}-} \ , & 2\bar{L}_0 &= \mathcal{H} - \mathcal{M} \ . \end{aligned} \tag{2.19}$$

The fermionic generators of $\mathfrak{sl}(2) \times \mathfrak{sl}(2|2)$ are obviously all anti-holomorphic, and upon embedding are identified with four-dimensional supercharges according to

$$\mathcal{Q}^{\mathcal{I}} = \mathcal{Q}^{\mathcal{I}}_{-} \ , \qquad \widetilde{\mathcal{Q}}_{\mathcal{I}} = \widetilde{\mathcal{Q}}_{\mathcal{I}\dot{-}} \ , \qquad \mathcal{S}_{\mathcal{I}} = \mathcal{S}^{-}_{\mathcal{I}} \ , \qquad \widetilde{\mathcal{S}}^{\mathcal{I}} = \widetilde{\mathcal{S}}^{\mathcal{I}\dot{-}} \ , \tag{2.20}$$

where $\mathcal{I} = 1,2$ is an $\mathfrak{sl}(2)_R$ index. Finally, the $\mathfrak{sl}(2|2)$ superalgebra has a central element $\mathcal{Z}$, which upon embedding is given in terms of four-dimensional symmetry generators as

$$\mathcal{Z} = r + \mathcal{M}^{\perp} \ , \tag{2.21}$$

where r is the generator of $U(1)_r$.

Amongst the supercharges listed in (2.20), one finds a variety of nilpotent operators. Any such operator will necessarily commute with the generators $L_{\pm 1}$ and L_0 in (2.19) since all of the supercharges do so. The requirement of $\mathbb{Q}$-exact anti-holomorphic Möbius transformations is harder to satisfy. In fact, up to similarity transformation using generators of the bosonic symmetry algebra, there are only two possible choices:

$$
\begin{aligned}
\mathbb{Q}_1 &:= \mathcal{Q}^1 + \widetilde{\mathcal{S}}^2 \,, \quad & \mathbb{Q}_2 &:= \mathcal{S}_1 - \widetilde{\mathcal{Q}}_2 \,, \\
\mathbb{Q}_1^\dagger &:= \mathcal{S}_1 + \widetilde{\mathcal{Q}}_2 \,, \quad & \mathbb{Q}_2^\dagger &:= \mathcal{Q}^1 - \widetilde{\mathcal{S}}^2 \,.
\end{aligned}
\tag{2.22}
$$

Interestingly, $\mathbb{Q}_1$ and $\mathbb{Q}_2$ give rise to *the same* $\mathbb{Q}$-exact generators of an anti-holomorphic $\widehat{\mathfrak{sl}(2)}$ algebra,

$$
\begin{array}{rclclcl}
\{\mathbb{Q}_1\,, \widetilde{\mathcal{Q}}_1\} &=& \{\mathbb{Q}_2\,, -\mathcal{Q}^2\} &=& \bar{L}_{-1} + \mathcal{R}^- &=:& \widehat{L}_{-1}\,, \\
\{\mathbb{Q}_1\,, \mathcal{S}_2\} &=& \{\mathbb{Q}_2\,, \widetilde{\mathcal{S}}^1\} &=& \bar{L}_{+1} - \mathcal{R}^+ &=:& \widehat{L}_{+1}\,, \\
\{\mathbb{Q}_1\,, \mathbb{Q}_1^\dagger\} &=& \{\mathbb{Q}_2\,, \mathbb{Q}_2^\dagger\} &=& 2(\bar{L}_0 - \mathcal{R}) &=:& 2\widehat{L}_0\,.
\end{array}
\tag{2.23}
$$

In addition, the central element of $\mathfrak{sl}(2|2)$ is exact with respect to both supercharges,

$$
\{\mathbb{Q}_1\,, \mathbb{Q}_2\} = -\mathcal{Z}\,. \tag{2.24}
$$

Note that while $\widehat{\mathfrak{sl}(2)}$ *does* act on the plane by anti-holomorphic conformal transformations, it is not simply a subalgebra of the original global conformal algebra. Rather, it is an $\mathfrak{sl}(2)_R$ twist of $\overline{\mathfrak{sl}(2)}$.[6] Because the relevant real forms of the $\overline{\mathfrak{sl}(2)}$ conformal algebra and $\mathfrak{sl}(2)_R$ are different, the generators of $\widehat{\mathfrak{sl}}(2)$ do not enjoy any reasonable hermiticity properties when acting on the Hilbert space of the four-dimensional theory. In particular, we can immediately see that $\widehat{L}_{\pm 1}^\dagger \neq \widehat{L}_{\mp 1}$. This would complicate matters considerably if our intention was to study operators that transform in nontrivial representations of this twisted algebra. Fortunately, our plan is precisely the opposite: chiral algebras can appear after passing to $\mathbb{Q}$-cohomology, at which point all of the objects of interest will effectively be invariant under the action of $\widehat{\mathfrak{sl}(2)}$. Consequently, reality/hermiticity conditions will play no role in the structure of the "physical" operators/observables defined at the level of cohomology.

2.2.3 The cohomology classes of local operators

Our next task is to study the properties of operators that define non-trivial $\mathbb{Q}_i$-cohomology classes. For the purposes of the present work, we are restricting our attention to *local operators* in four dimensions; the inclusion of non-local operators, such as line or surface

[6]In light of this, we may understand the absence of a similar construction using the $\mathfrak{sl}(2|1) \times \mathfrak{sl}(2|1)$ algebra as a consequence of there being no $\mathfrak{sl}(2)_R$ with which to twist. Similarly, our construction does not extend to $\mathcal{N} = 1$ superconformal theories since they only have an abelian R-symmetry.

operators, is an interesting extension that will be addressed in future work.

We begin by identifying the requirements for an operator inserted at the origin to define a nontrivial $\mathbb{Q}_i$-cohomology class. In particular, we will derive the conditions under which an operator $\mathcal{O}(x)$ obeys

$$\{\mathbb{Q}_i, \mathcal{O}(0)] = 0 \ , \qquad \mathcal{O}(0) \neq \{\mathbb{Q}_i, \mathcal{O}'(0)] \ , \tag{2.25}$$

for $i = 1$ or $i = 2$. Because both $\mathbb{Q}_i$ commute with $\widehat{L}_0$ and $\mathcal{Z}$, we lose no generality in restricting to definite eigenspaces of these charges. A standard cohomological argument then implies that since $\widehat{L}_0$ and $\mathcal{Z}$ are actually $\mathbb{Q}_i$-*exact*, an operator satisfying (2.25) must lie in the zero eigenspace of both charges. In terms of four-dimensional quantum numbers, this means that such an operator must obey[7]

$$\tfrac{1}{2}(E - (j_1 + j_2)) - R = 0 \ , \quad r + (j_1 - j_2) = 0 \ , \tag{2.26}$$

where E is the conformal dimension/eigenvalue of $\mathcal{H}$, j_1 and j_2 are $\mathfrak{sl}(2)_1$ and $\mathfrak{sl}(2)_2$ Lorentz quantum numbers/eigenvalues of $\mathcal{M}_+{}^+$ and $\mathcal{M}^{\dot{+}}{}_{\dot{+}}$, and R is the $\mathfrak{sl}(2)_R$ charge/eigenvalue of $\mathcal{R}$. As long as the four-dimensional SCFT is unitary, the last line of (2.23) implies that any operator with zero eigenvalue under $\widehat{L}_0$ must be annihilated by $\mathbb{Q}_i$ and $\mathbb{Q}_i^\dagger$ for *both* $i = 1$ and $i = 2$. The relations in (2.26) therefore characterize the harmonic representatives of $\mathbb{Q}_i$-cohomology classes of operators at the origin, and we see that the two supercharges actually define the *same* cohomology. Notably, these relations are known to characterize the operators that contribute to the Schur (and Macdonald) limits of the superconformal index in four dimensions [13], suggesting that the cohomology will be non-empty in any nontrivial $\mathcal{N} = 2$ SCFT. We will refer to the class of local operators obeying (2.26) as the *Schur operators* of the SCFT. We will have more to say about the features of these operators in §2.3.

Note that in contrast to the case of ordinary chiral operators in a supersymmetric theory, which are annihilated by a given Poincaré supercharge regardless of the insertion point, for operators to be annihilated by the $\mathbb{Q}_i$ when inserted away from the origin requires that they acquire a more intricate dependence on their position in $\mathbb{R}^4$. This is a consequence of the fact that the translation generators do not commute with the superconformal charges $\mathcal{S}_1^-$ and $\widetilde{\mathcal{S}}^{2\dot{-}}$ appearing in the definitions of the $\mathbb{Q}_i$. Indeed, there is no way to define the translation of a Schur operator from the origin to a point outside of the $(z, \bar{z})$ plane so that it continues to represent a $\mathbb{Q}_i$-cohomology class. Within the plane, though, we can accomplish this task using the $\mathbb{Q}_i$-exact, twisted $\widehat{\mathfrak{sl}(2)}$ of the previous subsection. In particular, because the twisted anti-holomorphic translation generator $\widehat{L}_{-1}$ is a $\mathbb{Q}_i$ anti-commutator and the holomorphic

[7]In fact, the second relation in (2.26) follows from the first as a consequence of unitarity and the four-dimensional superconformal algebra (see §2.3.1). We list it separately here since it is an algebraically independent constraint at the level of the quantum numbers.

translation generator L_{-1} is $\mathbb{Q}_i$-closed, we can define the *twisted-translated* operators

$$\mathcal{O}(z,\bar{z}) = e^{zL_{-1}+\bar{z}\widehat{L}_{-1}}\,\mathcal{O}(0)\,e^{-zL_{-1}-\bar{z}\widehat{L}_{-1}}\ , \tag{2.27}$$

where $\mathcal{O}(0)$ is a Schur operator. One way of thinking about this prescription for the translation of local operators is as the consequence of introducing a constant, complex background gauge field for the $\mathfrak{sl}(2)_R$ symmetry that is proportional to the $\mathfrak{sl}(2)$ raising operator. By construction, the twisted-translated operator is $\mathbb{Q}_i$ closed for both $i=1,2$, and the cohomology class of this operator is well-defined and depends on the insertion point holomorphically,

$$[\mathcal{O}(z,\bar{z})]_{\mathbb{Q}} \quad \Longrightarrow \quad \mathcal{O}(z)\ . \tag{2.28}$$

What does such an operator look like in terms of a more standard basis of local operators at the point $(z,\bar{z})$? To answer this, we must first note that Schur operators at the origin occupy the highest-weight states of their respective $\mathfrak{sl}(2)_R$ representation (this fact will be explained in greater detail in §2.3). If we denote the whole spin k representation of $\mathfrak{sl}(2)_R$ as $\mathcal{O}^{\mathcal{I}_1\mathcal{I}_2\cdots\mathcal{I}_{2k}}$ with $\mathcal{I}_i=1,2$, then the Schur operator at the origin is $\mathcal{O}^{11\cdots 1}(0)$, and the twisted-translated operator at any other point is defined as

$$\mathcal{O}(z,\bar{z}) := u_{\mathcal{I}_1}(\bar{z})\,\cdots\,u_{\mathcal{I}_{2k}}(\bar{z})\,\mathcal{O}^{\mathcal{I}_1\ldots\mathcal{I}_{2k}}(z,\bar{z})\,, \qquad u_{\mathcal{I}}(\bar{z}) := (1,\bar{z})\,. \tag{2.29}$$

At any given point $(z,\bar{z})$, this is a particular complex-linear combination of the different elements of the $\mathfrak{sl}(2)_R$ representation of the corresponding Schur operator. The precise combination depends on the insertion point as indicated. What we have discovered is that the correlation functions of these operators are determined at the level of their $\mathbb{Q}_i$-cohomology classes, and are therefore meromorphic functions of the insertion points.[8]

2.2.4 A chiral operator product expansion

The most efficient language for describing chiral algebras is that of the operator product expansion. Let us therefore study the structure of the operator product expansion of the twisted-translated Schur operators in order to see the emergence of meromorphic OPEs befitting a chiral algebra.

Consider two operators: $\mathcal{O}_1(z,\bar{z})$ is the twisted translation of a Schur operator from the origin to $(z,\bar{z})$, and $\mathcal{O}_2(0,0)$ is a Schur operator inserted at the origin. Given the general expression for the twisted-translated operator given in (2.29), the OPE of these two operators

[8]For $\mathcal{N}=4$ SYM, a similar contraction of the $SU(4)_R$ indices with position-dependent vectors was studied in [44]. The twists considered in that paper are different, and do not give rise to meromorphic operators and chiral algebras.

should take the form

$$\mathcal{O}_1(z,\bar{z})\mathcal{O}_2(0) = \sum_k \lambda_{12k} \frac{\bar{z}^{R_1+R_2-R_k}}{z^{h_1+h_2-h_k}\,\bar{z}^{\bar{h}_1+\bar{h}_2-\bar{h}_k}} \mathcal{O}_k(0) \ , \tag{2.30}$$

where the $\bar{z}^{R_1+R_2-R_k}$ in the numerator comes from the explicit factors of $\bar{z}$ appearing in (2.29), and R_k is the R-charge of the operator $\mathcal{O}_k$. This form of the OPE is so far a consequence of two-dimensional conformal invariance and conservation of R-charge under multiplication. We have introduced the two-dimensional quantum numbers h and $\bar{h}$, which are expressible in terms of four-dimensional quantum numbers as

$$h = \frac{E+(j_1+j_2)}{2} \ , \qquad \bar{h} = \frac{E-(j_1+j_2)}{2} \ . \tag{2.31}$$

Though the OPE does not look meromorphic yet, we are already well on our way. The left hand side of (2.30) is $\mathtt{Q}_i$-closed for any $(z,\bar{z})$, with the $\bar{z}$ dependence being $\mathtt{Q}_i$-exact. As a result, each individual term on the right hand side must be $\mathtt{Q}_i$-closed, and the sum should be reorganized into two groups. The first group will consist of the terms in which the operator $\mathcal{O}_k(0)$ is a Schur operator, while the second will consist of the remaining terms, for which the operator $\mathcal{O}_k(0)$ is $\mathtt{Q}_i$-exact. Recalling that the quantum numbers of Schur operators obey $\bar{h} = R$, we immediately see that for those terms in the OPE the $\bar{z}$ dependence cancels between the denominator and the numerator, thus providing the desired meromorphicity result:

$$\mathcal{O}_1(z,\bar{z})\,\mathcal{O}_2(0,0) \ = \sum_{k_{\text{Schur}}} \frac{\lambda_{12k}}{z^{h_1+h_2-h_k}}\,\mathcal{O}_k(0) \ + \{\mathtt{Q},\dots] \ . \tag{2.32}$$

From the four-dimensional construction, we expect this OPE to be single-valued, which implies that $h_1+h_2-h_k$ should be an integer. Indeed, this integrality follows from the fact that h is a sum of $SU(2)$ Cartans after applying $SU(2)$ selection rules. Clearly, in passing to $\mathtt{Q}_i$-cohomology classes the OPE stays well-defined and the $\mathtt{Q}_i$-exact piece can be set to zero. Thus at the level of cohomology, the twisted-translated operators can be reinterpreted as two-dimensional meromorphic operators with interesting singular OPEs.

It may be instructive to see how this meromorphic OPE plays out in a simple example. An extremely simple case, to which we shall return in §2.3, is that of free hypermultiplets in four dimensions. The scalar squarks Q and $\tilde{Q}$ of the hypermultiplet are Schur operators, and the corresponding twisted-translated operators take the form

$$q(z) := [Q(z,\bar{z}) + \bar{z}\tilde{Q}^*(z,\bar{z})]_{\mathtt{Q}} \ , \qquad \tilde{q}(z) := [\tilde{Q}(z,\bar{z}) - \bar{z}Q^*(z,\bar{z})]_{\mathtt{Q}} \ . \tag{2.33}$$

The singular OPE of these twisted operators can be easily worked out using the free OPE in four dimensions; we have

$$\begin{aligned} q(z)q(w) &\sim \text{regular} , & \tilde{q}(z)\tilde{q}(w) &\sim \text{ regular} , \\ q(z)\tilde{q}(w) &\sim \frac{1}{z-w} , & \tilde{q}(z)q(w) &\sim -\frac{1}{z-w} . \end{aligned} \tag{2.34}$$

This is example is in some respects deceptively simple, in that the terms appearing in the singular part of the OPE are meromorphic on the nose. In more complicated theories, there will be cohomologically trivial terms appearing in the singular part of the OPE, and meromorphicity will depend on a more detailed knowledge of the action of the nilpotent supercharges.

Let us briefly point out one difference between the structure observed here and that of a more conventional cohomological subalgebra. The chiral ring in the free hypermultiplet theory is generated by the operators $q(x)$ and $\tilde{q}(x)$. Because these operators both have $R = 1/2$, there can be no nonzero correlation functions in the chiral ring. The existence of nontrivial correlation functions in the chiral algebra described here follows precisely from the presence of subleading terms in the $\bar{z}$ expansion (2.33) with $SU(2)_R$ quantum numbers of opposite sign relative to the leading term.

Having established existence of nontrivial $\mathbb{Q}$-cohomology classes with meromorphic OPEs and correlators, we now take some time to develop the dictionary between four-dimensional SCFT structures and their two-dimensional counterparts.

2.3 The SCFT/chiral algebra correspondence

For any four-dimensional $\mathcal{N} = 2$ superconformal field theory, we have identified a subsector of operators whose correlation functions are meromorphic when they are restricted to be coplanar. This sector thus defines a map from four-dimensional SCFTs to two-dimensional chiral algebras:

$$\chi \ : \ \text{4d SCFT} \ \longrightarrow \ \text{2d Chiral Algebra.}$$

The aim of this section is to elaborate on the structure of this correspondence, focusing primarily on its more universal aspects. We begin with a short preview of some of the more prominent features of the correspondence.

Our first main result is the generic enhancement of the global $\mathfrak{sl}(2)$ conformal symmetry algebra to a full fledged Virasoro algebra. In other words, for any SCFT $\mathcal{T}$, we find that $\chi[\mathcal{T}]$ contains a meromorphic stress tensor. The two-dimensional central charge turns out to have a simple relationship to the four-dimensional conformal anomaly coefficient,

$$c_{2d} = -12 c_{4d} \ .$$

In particular, this implies that when $\mathcal{T}$ is unitary (which we always take to be the case),

$\chi[\mathcal{T}]$ is necessarily non-unitary. In a similar vein, we find that global symmetries of $\mathcal{T}$ are always enhanced into affine symmetries of $\chi[\mathcal{T}]$, and the respective central charges of these flavor symmetries enjoy another simple relationship,

$$k_{2d} = -\frac{1}{2}k_{4d} \ .$$

It is often helpful to think of a chiral algebra in terms of its generators. In the chiral algebra sense of the word, generators are those operators that cannot be expressed as the conformally normal-ordered products of derivatives of other operators. While we do not find a complete characterization of the generators of our chiral algebras, we do identify certain operators in four dimensions whose corresponding chiral operator will necessarily be generators. In particular, operators that are $\mathcal{N}=1$ chiral *and* satisfy the Schur shortening condition form a ring which is a consistent truncation of the $\mathcal{N}=1$ chiral ring, to which we refer as the Hall-Littlewood (HL) chiral ring. We find that every generator of the HL chiral ring necessarily leads to a generator of the associated chiral algebra. There may be additional generators of the chiral algebra beyond the stress tensor and the operators associated to generators of the HL chiral ring. We will find such additional generators in the example of §2.5.4.

For the special case of free SCFTs we completely characterize the associated chiral algebras. Unsurprisingly, free SCFTs give rise to free chiral algebras. In particular, free hypermultiplets correspond to the chiral algebra of dimension $1/2$ symplectic bosons, while free vector multiplets correspond to the small algebra of a (b, c) ghost system of dimension $(1, 0)$.

Finally, we describe the two-dimensional counterpart of gauging a flavor symmetry G in some general SCFT $\mathcal{T}_G$. Assuming that the chiral algebra associated to the ungauged SCFT is known, the prescription to find the chiral algebra of the new theory is as follows. The direct product of the original chiral algebra $\chi[\mathcal{T}_G]$ with a (b, c) system in the adjoint representation of G admits a nilpotent BRST operator precisely when the beta function for the four-dimensional gauge coupling vanishes. The chiral algebra of the gauged theory is then obtained by restricting to the BRST coholomogy. We find that this BRST operator precisely captures the one-loop correction to a certain four-dimensional supercharge, so that restricting to its cohomology is equivalent to the requirement that one should only retain those states that remain in their original short representations once one-loop corrections are taken into account.

2.3.1 Schur operators

As a first order of business, we pursue a more concrete characterization of the four-dimensional operators whose correlation functions are captured by the chiral algebra. Let us first reiterate the basic facts about these operators that were derived in §2.2. The chiral algebra computes correlation functions of operators that define nontrivial cohomology classes of the nilpotent supercharges $\mathbb{Q}_i$. Such operators are obtained by *twisted translations* (2.29) of Schur operators from the origin to an arbitrary point $(z, \bar{z})$ on the plane. A Schur operator is any operator that satisfies

$$[\widehat{L}_0, \mathcal{O}] = 0 \qquad \Longleftrightarrow \qquad \tfrac{1}{2}\left(E - (j_1 + j_2)\right) - R = 0 \ , \tag{2.35}$$

$$[\mathcal{Z}, \mathcal{O}] = 0 \qquad \Longleftrightarrow \qquad r + j_1 - j_2 = 0 \ . \tag{2.36}$$

If $\mathcal{T}$ is unitary, then these conditions can be equivalently formulated as the requirement that when inserted *at the origin*, an operator is annihilated by the two Poincaré and the two conformal supercharges that enter in the definition of the $\mathbb{Q}_i$, *i.e.*,

$$[\mathcal{Q}^{\,1}_{\,-}, \mathcal{O}(0)] = [\widetilde{\mathcal{Q}}_{\,2\dot{-}}, \mathcal{O}(0)] = [\mathcal{S}_1^{-}, \mathcal{O}(0)] = [\widetilde{\mathcal{S}}^{2\dot{-}}, \mathcal{O}(0)] = 0 \ . \tag{2.37}$$

This follows from the hermiticity conditions $\mathcal{Q}_{-}^{1\dagger} := \mathcal{S}_1^{-}$ and $\mathcal{Q}_{2\dot{-}}^{\dagger} := \widetilde{\mathcal{S}}^{2\dot{-}}$ in conjunction with the relevant anticommutators from Appendix A,

$$\{\mathcal{Q}^1_{-}\,, \mathcal{Q}_{-}^{1\dagger}\} = \widehat{L}_0 - \frac{1}{2}\mathcal{Z}\,, \quad \{\widetilde{\mathcal{Q}}_{2\dot{-}}\,, \widetilde{\mathcal{Q}}^{\dagger}_{2\dot{-}}\} = \widehat{L}_0 + \frac{1}{2}\mathcal{Z}\,. \tag{2.38}$$

It follows immediately that the state $\mathcal{O}(0)|0\rangle$ is annihilated by all four supercharges if and only if its quantum numbers obey (2.35) and (2.36). Actually, (2.38) implies the additional inequality

$$\widehat{L}_0 \geqslant \frac{|\mathcal{Z}|}{2} \ , \tag{2.39}$$

from which we may conclude that imposing only (2.35) is a necessary and sufficient condition to define a Schur operator. We further note that Schur operators are necessarily the highest-weight states of their respective $SU(2)_R$ representations, and so carry the maximum eigenvalue R of the Cartan generator. If this were not the case, states with greater R would have negative $\widehat{L}_0$ eigenvalues, in contradiction with unitarity. Similarly, Schur operators are necessarily the highest weight states of their $SU(2)_1 \times SU(2)_2$ Lorentz symmetry representation, carrying the largest eigenvalues for j_1 and j_2. The index structure of a Schur operator is therefore of the form $\mathcal{O}^{1\cdots 1}_{+\cdots+\,\dot{+}\cdots\dot{+}}$.

From the definition of $\widehat{L}_0$ in (2.19) and (2.35) we find that the holomorphic dimension h of a Schur operator is non-zero and fixed in terms of its quantum numbers,

$$h = \tfrac{1}{2}\left(E + j_1 + j_2\right) = R + j_1 + j_2 \ . \tag{2.40}$$

This is always a half integer, since R, j_1 and j_2 are all $SU(2)$ Cartans. It follows from (2.36) and (2.40), in conjunction with the non-negativity of j_1 and j_2, that the holomorphic dimension of a Schur operator is bounded from below in terms of its four-dimensional

Multiplet	$\mathcal{O}_{\text{Schur}}$	h	r	Lagrangian "letters"
$\hat{\mathcal{B}}_R$	$\Psi^{11\ldots1}$	R	0	Q, $\tilde{Q}$
$\mathcal{D}_{R(0,j_2)}$	$\widetilde{\mathcal{Q}}^1_{\dot{+}}\Psi^{11\ldots1}_{\dot{+}\ldots\dot{+}}$	$R+j_2+1$	$j_2+\frac{1}{2}$	Q, $\tilde{Q}$, $\tilde{\lambda}^1_{\dot{+}}$
$\bar{\mathcal{D}}_{R(j_1,0)}$	$\mathcal{Q}^1_{+}\Psi^{11\ldots1}_{+\ldots+}$	$R+j_1+1$	$-j_1-\frac{1}{2}$	Q, $\tilde{Q}$, λ^1_+
$\hat{\mathcal{C}}_{R(j_1,j_2)}$	$\mathcal{Q}^1_+\widetilde{\mathcal{Q}}^1_{\dot{+}}\Psi^{11\ldots1}_{+\ldots+\,\dot{+}\ldots\dot{+}}$	$R+j_1+j_2+2$	j_2-j_1	$D^n_{+\dot{+}}Q$, $D^n_{+\dot{+}}\tilde{Q}$, $D^n_{+\dot{+}}\lambda^1_+$, $D^n_{+\dot{+}}\tilde{\lambda}^1_{\dot{+}}$

Table 2.1: This table summarizes the manner in which Schur operators fit into short multiplets of the $\mathcal{N}=2$ superconformal algebra. For each supermultiplet, we denote by Ψ the superconformal primary. There is then a single conformal primary Schur operator $\mathcal{O}_{\text{Schur}}$, which in general is obtained by the action of some Poincaré supercharges on Ψ. We list the holomorphic dimension h and $U(1)_r$ charge r of $\mathcal{O}_{\text{Schur}}$ in terms of the quantum numbers (R, j_1, j_2) that label the shortened multiplet (left-most column). We also indicate the schematic form that $\mathcal{O}_{\text{Schur}}$ can take in a Lagrangian theory by enumerating the elementary "letters" from which the operator may be built. We denote by Q and $\tilde{Q}$ the complex scalar fields of a hypermultiplet, by $\lambda^{\mathcal{I}}_\alpha$ and $\tilde{\lambda}^{\mathcal{I}}_{\dot{\alpha}}$ the left- and right-moving fermions of a vector multiplet, and by $D_{\alpha\dot{\alpha}}$ the gauge-covariant derivatives.

R-charges,

$$h = R + j_1 + j_2 \geqslant R + |j_1 - j_2| = R + |r| \ . \tag{2.41}$$

The inequality is saturated if and only if j_1 or j_2 is zero.

Supermultiplets of Schur type

Schur operators belong to shortened representations of the $\mathcal{N}=2$ superconformal algebra. The complete list of possible shortening conditions is reviewed in Appendix B. In the notations of [52], the superconformal multiplets that contain Schur operators are the following,

$$\hat{\mathcal{B}}_R \ , \quad \mathcal{D}_{R(0,j_2)} \ , \quad \bar{\mathcal{D}}_{R(j_1,0)} \ , \quad \hat{\mathcal{C}}_{R(j_1,j_2)} \ . \tag{2.42}$$

For the purpose of enumeration, it is sufficient to focus on those Schur operators that are conformal primaries. Given such a *primary Schur operator*, there is a tower of descendant Schur operators that are obtained by the action $L_{-1} = P_{+\dot{+}} = -\partial_{+\dot{+}}$. It turns out that each of the supermultiplets listed in (2.42) contains exactly one conformal primary Schur operator. In the case of $\hat{\mathcal{B}}_R$, this is also the *super*conformal primary of the multiplet, whereas in the other cases it is a superconformal descendant. This representation-theoretic information is summarized in Table 2.1, where we also provide the schematic form taken by each type of operator in a Lagrangian theory.

The shortening conditions obeyed by the Schur operators make crucial use of the extended $\mathcal{N}=2$ supersymmetry. Indeed, the hallmark of a Schur operator is that it is annihilated by two Poincaré supercharges of *opposite* chiralities ($\mathcal{Q}^1_-$ and $\widetilde{\mathcal{Q}}_{2\dot{-}}$ in our conventions). This defines a consistent shortening condition because the supercharges have the same $SU(2)_R$ weight, and thus anticommute with each other. No analogous shortening condition exists in an $\mathcal{N}=1$ supersymmetric theory, because the anticommutator of opposite-chirality supercharges necessarily yields a momentum operator, which annihilates only the identity.

Although the most general Schur operators, which are those belonging to $\hat{\mathcal{C}}_{R(j_1,j_2)}$ multiplets, may seem somewhat exotic, the Schur operators of type $\hat{\mathcal{B}}_R$, $\mathcal{D}_{R(0,j_2)}$ and $\bar{\mathcal{D}}_{R(j_1,0)}$ are relatively familiar. Indeed, they can be understood as special cases of conventional $\mathcal{N}=1$ chiral or anti-chiral operators. Let us focus for the moment on the $\mathcal{N}=1$ Poincaré subalgebra that contains the supercharges

$$\mathcal{Q}^2_\alpha\ , \quad \widetilde{\mathcal{Q}}_{2\dot{\alpha}}\ . \tag{2.43}$$

We then ask what subset of Schur operators are also elements of the chiral ring for this $\mathcal{N}=1$ subalgebra. In particular, such operators will be annihilated by *both* spinorial components of the anti-chiral supercharge $\widetilde{\mathcal{Q}}_{2\dot{\alpha}}$, $\dot{\alpha}=\dot{\pm}$. These operators have $j_2=0$, and a quick glance at Table 2.1 tells us that they are Schur operators of types $\hat{\mathcal{B}}_R$ and $\bar{\mathcal{D}}_{R(j_1,0)}$. These operators saturate the inequality (2.41), with $r=-j_1<0$ for $\bar{\mathcal{D}}_{R(j_1,0)}$ and $r=0$ for the $\hat{\mathcal{B}}_R$. As these are precisely the operators that contribute to the Hall-Littlewood (HL) limit of the superconformal index, we refer to them as *Hall-Littlewood operators*. They form a ring, the *Hall-Littlewood chiral ring*, which is a consistent truncation of the full $\mathcal{N}=1$ chiral ring.

In a Lagrangian theory, the $\hat{\mathcal{B}}_R$ type Schur operators are gauge-invariant combinations of Q and $\tilde{Q}$, the complex hypermultiplet scalars that are bottom components of $\mathcal{N}=1$ chiral superfields (we are suppressing color and flavor indices). Schur operators of type $\bar{\mathcal{D}}_{R(j_1,0)}$ are obtained by further allowing as possible letters the gauginos λ^1_+, which are the bottom components of the field strength chiral superfield W_+. In the full $\mathcal{N}=1$ chiral ring, one also has the other Lorentz component W_- of the field strength, as well as the $\mathcal{N}=1$ chiral superfield belonging to the $\mathcal{N}=2$ vector multiplet. Operators that contain those letters are, however, *not* a part of the HL chiral ring.

In complete analogy, we may also define a Hall-Littlewood anti-chiral ring, which contains the Schur operators of type $\hat{\mathcal{B}}_R$ and $\mathcal{D}_{R(0,j_2)}$. These operators are annihilated by chiral supercharges $\mathcal{Q}^1_\alpha$, $\alpha=\pm$, and are thus $\mathcal{N}=1$ anti-chiral with respect to the $\mathcal{N}=1$ subalgebra that is orthogonal to (2.43). Schur operators of type $\hat{\mathcal{B}}_R$ belong to both HL rings – these are half-BPS operators that are annihilated by both $\mathcal{Q}^1_\alpha$ and $\widetilde{\mathcal{Q}}_{2\dot{\alpha}}$. They form a further truncation of the $\mathcal{N}=1$ chiral ring to the *Higgs chiral ring*, and their vacuum expectation values parametrize the Higgs branch of the theory. We note that in Lagrangian theories that are represented by acyclic quiver diagrams, all $\mathcal{D}$-type multiplets recombine and are lifted from the $\mathcal{N}=1$ chiral ring at one-loop order [13]. In such cases, the HL chiral ring will coincide with the more restricted Higgs branch chiral ring.

Let us now look in greater detail at some Schur-type shortened multiplets of particular physical interest:

- $\hat{\mathcal{C}}_{0(0,0)}$: Stress-tensor multiplet. The superconformal primary is a scalar operator of dimension two that is a singlet under the $SU(2)_R \times U(1)_r$. The $SU(2)_R$ and $U(1)_r$ conserved currents, the supercurrents, and the stress tensor all lie in this multiplet. The Schur operator is the highest weight component of the $SU(2)_R$ current: J^{11}_{++} of the $SU(2)_R$.

- $\hat{\mathcal{C}}_{0(j_1,j_2)}$: Higher-spin currents multiplets. These generalize the stress-tensor multiplet and contain conserved currents of spin higher than two. If any such multiplets are present, the SCFT must contain a decoupled free sector [53]. Requiring the absence of these higher spin multiplets will lead to interesting unitarity bounds for the central charge of interacting SCFTs in §2.4.

- $\hat{\mathcal{B}}_{\frac{1}{2}}$: This is the superconformal multiplet of free hypermultiplets.

- $\hat{\mathcal{B}}_1$: Flavor-current multiplet. The superconformal primary is the "moment map" operator $M^{\mathcal{I}\mathcal{J}}$, which is a scalar operator of dimension two that is an $SU(2)_R$ triplet, is $U(1)_r$ neutral, and transforms in the adjoint representation of the flavor group G_F. The highest weight state of the moment map – M^{11} – is the Schur operator. The claim to fame of $\hat{\mathcal{B}}_1$ multiplets is that they harbor the conserved currents $J^F_{\alpha\dot{\alpha}}$ that generate the continuous "flavor" symmetry group G_F of the SCFT, that is, the symmetry group that commutes with the superconformal group. Because $\hat{\mathcal{B}}_1$ multiplets do not appear in any of the recombination rules for short multiplets listed in Appendix B, it is absolutely protected: $J^F_{\alpha\dot{\alpha}}$ remains conserved on the entire conformal manifold of the SCFT.[9]

- $\mathcal{D}_{0(0,0)} \oplus \bar{\mathcal{D}}_{0(0,0)}$: This is the superconformal multiplet of free $\mathcal{N}=2$ vector multiplets.

- $\mathcal{D}_{\frac{1}{2}(0,0)} \oplus \bar{\mathcal{D}}_{\frac{1}{2}(0,0)}$: "Extra" supercurrent multiplets. The top components of these multiplets are spin 3/2 conserved currents of dimension $\Delta = 7/2$ ($J_{\alpha\dot{\alpha}\dot{\beta}}$ and $J_{\alpha\beta\dot{\alpha}}$). They generate *additional* supersymmetry transformations beyond the $\mathcal{N}=2$ superalgebra in question. In particular, in the $\mathcal{N}=2$ description of an $\mathcal{N}=4$ SCFT, one finds two copies of each of these multiplets transforming as a doublet of the "flavor" $SU(2)_F \subset SU(4)_R$ that commutes with $SU(2)_R \times U(1)_r \subset SU(4)_R$. The Schur operators have $\Delta = 5/2$, and have index structure $\mathcal{O}^{11}_{\dot{+}}$ and $\mathcal{O}^{11}_{+}$. In $\mathcal{N}=4$ supersymmetric

[9]The only other supermultiplet that contains a global flavor symmetry current is $\hat{\mathcal{C}}_{0(\frac{1}{2},\frac{1}{2})}$. However, that multiplet also contains higher-spin currents, thus showing that the only points on a conformal manifold at which the flavor symmetry enhances are the points where the SCFT develops a free decoupled subsector.

Yang-Mills theory, these are the operators $\operatorname{Tr} q_i^1 \tilde{\lambda}^1_{\dot{+}}$ and $\operatorname{Tr} q_i^1 \lambda^1_{+}$, where $i = 1, 2$ is the $SU(2)_F$ index.

2.3.2 Notable elements of the chiral algebra

Armed with a working knowledge of the relevant four-dimensional operators, we now proceed to derive some universal entries in the $4d/2d$ dictionary. We first recall from §2.2.3 the process by which a meromorphic operator in two dimensions is obtained from an appropriate protected operator in four dimensions. Starting with a Schur operator in four dimensions, we obtain a two-dimensional chiral operator via the following series of specializations:

$$
\begin{array}{cc}
\mathcal{O}^{1\cdots 1}_{+\cdots+\dot{+}\cdots\dot{+}}(x) & \text{Schur operator} \\
\downarrow & \\
\mathcal{O}(z,\bar{z}) \cong u_{\mathcal{I}_1}(\bar{z})\cdots u_{\mathcal{I}_{2R}}(\bar{z})\mathcal{O}^{(\mathcal{I}_1\cdots\mathcal{I}_{2R})}(z,\bar{z}) & \text{Twisted-translated Schur operator} \\
\downarrow & \\
[\mathcal{O}(z,\bar{z})]_{\mathbb{Q}} & \text{Chiral cohomology class} \\
\downarrow & \\
\mathcal{O}(z) & \text{Two-dimensional chiral operator}
\end{array}
$$

In general we will refer to this associated chiral operator via the following notation:

$$
\mathcal{O}(z) = \chi[\mathcal{O}^{1\cdots 1}_{+\cdots+\dot{+}\cdots\dot{+}}] ,
$$

where sometimes we will be lax about the argument of the χ map and allow $\mathcal{O}^{1\cdots 1}_{+\cdots+\dot{+}\cdots\dot{+}}$ to be replaced by the more generic form of the operator $\mathcal{O}^{\mathcal{I}_1\cdots\mathcal{I}_{2R}}_{\alpha_1\cdots\alpha_{2j_1}\dot{\alpha}_1\cdots\dot{\alpha}_{2j_2}}$. Our first task will be to understand the chiral operators that are related to certain characteristic Schur operators of a four-dimensional theory. In doing so we will discover some interesting and generic features of this correspondence.

Virasoro enhancement of the $\mathfrak{sl}(2)$ symmetry

The holomorphic $\mathfrak{sl}(2)$ algebra generated by $\{L_{-1}, L_0, L_1\}$ is a manifest symmetry of the chiral algebra. Remarkably, this global conformal symmetry is enhanced to the full Virasoro algebra. The Virasoro algebra is generated by the modes L_n, $n \in \mathbb{Z}$, of a holomorphic stress tensor of dimension two $T(z)$. Surveying Table 2.1, we find a suitable candidate that is present in any theory $\mathcal{T}$: the Schur operator belonging to stress tensor multiplet $\hat{\mathcal{C}}_{0(0,0)}$. One should note that the Schur operator in this multiplet is *not* the four-dimensional stress tensor, but rather the component $J^{11}_{+\dot{+}}$ of the $SU(2)_R$ current $J^{\mathcal{I}\mathcal{J}}_{\alpha\dot{\alpha}}$.

The corresponding twisted-translated operator is defined as follows,

$$\mathcal{J}_R(z,\bar{z}) := u_{\mathcal{I}}(\bar{z})\ u_{\mathcal{J}}(\bar{z})\ J^{\mathcal{I}\mathcal{J}}_{+\dot{+}}(z,\bar{z})\ . \tag{2.44}$$

Per the discussion of §2.2, we identify the cohomology class $[\mathcal{J}_R(z,\bar{z})]_{\mathbb{Q}_i}$ with a dimension two meromorphic operator in the chiral algebra $\chi[\mathcal{T}]$,

$$T_{\mathcal{J}}(z) := \kappa\ [\mathcal{J}_R(z,\bar{z})]_{\mathbb{Q}_i}\ . \tag{2.45}$$

We provisionally include the subscript $\mathcal{J}$ as a reminder of the definition (2.45); we still need to establish that the OPEs of $T_{\mathcal{J}}(z)$ with itself and with other operators in the chiral algebra take the standard forms appropriate to a two-dimensional stress tensor. With this in mind, we have also included a normalization factor κ, to be fixed momentarily in order to recover the canonical TT OPE.

The two- and three-point functions of the R-symmetry current with itself are fixed by $\mathcal{N}=2$ superconformal invariance in terms of a single parameter c_{4d}, which is one of the two conformal anomaly coefficients (the other being a_{4d}). Starting from the OPE of two $SU(2)_R$ currents [54],

$$J^{\mathcal{I}\mathcal{J}}_{\mu}(x)J^{\mathcal{K}\mathcal{L}}_{\nu}(0) \sim \frac{3c_{4d}}{4\pi^4}\epsilon^{\mathcal{K}(\mathcal{I}}\epsilon^{\mathcal{J})\mathcal{L}}\frac{x^2 g_{\mu\nu}-2x_\mu x_\nu}{x^8} + \frac{2i}{\pi^2}\frac{x_\mu x_\nu x\cdot J^{(\mathcal{K}(\mathcal{I}}\epsilon^{\mathcal{J})\mathcal{L})}}{x^6} + \cdots\ , \tag{2.46}$$

we find the following OPE of twisted-translated Schur operators,

$$\begin{aligned}\mathcal{J}_R(z,\bar{z})\mathcal{J}_R(0,0) \quad\sim\quad & -\frac{3c_{4d}}{2\pi^4 z^4} - \frac{1}{\pi^2}\frac{\mathcal{J}_R(0,0)}{z^2} \\ & -\frac{1}{\pi^2}\bar{z}\frac{u_{\mathcal{I}}u_{\mathcal{J}}J^{\mathcal{I}\mathcal{J}}_{-\dot{-}}(0)}{z^3} + \frac{i}{\pi^2}\bar{z}\frac{J^{21}_{+\dot{+}}(0)}{z^2} + \frac{i}{\pi^2}\bar{z}^2\frac{J^{21}_{-\dot{-}}(0)}{z^3} + \cdots\ .\end{aligned} \tag{2.47}$$

Because the last three terms have non-zero $\widehat{L}_0$ eigenvalue, they are guaranteed to be $\mathbb{Q}_i$-exact. Upon setting $\kappa=-2\pi^2$, we find the following meromorphic OPE for $T_{\mathcal{J}}$,[10]

$$T_{\mathcal{J}}(z)\ T_{\mathcal{J}}(0) \sim \frac{-6\,c_{4d}}{z^4} + \frac{2\ T_{\mathcal{J}}(0)}{z^2} + \frac{\partial T_{\mathcal{J}}(0)}{z}\ . \tag{2.48}$$

Happily, we recognize in (2.48) the familiar two-dimensional TT OPE with central charge c_{2d} given by

$$c_{2d} = -12\,c_{4d}\ . \tag{2.49}$$

[10] The term corresponding to the simple pole does not immediately follow from the OPE given in (2.47). In particular, though the presence of $\partial T_{\mathcal{J}}(0)$ is guaranteed as a consequence of the double pole, we may worry that an additional quasiprimary (in the two-dimensional sense) may also appear. Such a quasiprimary $\mathcal{O}$ would have to be a boson of holomorphic dimension $h=3$ and have nonzero three point function $\langle T_{\mathcal{J}}T_{\mathcal{J}}\mathcal{O}\rangle$. This is forbidden by Bose symmetry.

This is the first major entry in our dictionary. Note that unitarity of the four-dimensional theory requires $c_{4d} > 0$, so the chiral algebra will have negative central charge and will therefore necessarily be non-unitary.

It is not immediately clear from the arguments presented thus far that $T_{\mathcal{J}}(z)$ will have the correct OPE with operators of the chiral algebra. In other words, the assertion that $T_{\mathcal{J}}$ acts as the stress tensor of the chiral algebra means that the "geometric" $\mathfrak{sl}(2)$ generators $\{L_{-1}, L_0, L_{+1}\}$ defined by the embedding (2.19) of the two-dimensional conformal algebra into the four-dimensional one should coincide in cohomology with the generators $\{L^{\mathcal{J}}_{-1}, L^{\mathcal{J}}_0, L^{\mathcal{J}}_{+1}\}$ defined by the mode expansion of $T_{\mathcal{J}}(z)$. It would be sufficient to verify that this is the case for quasiprimary operators, by which we mean operators $\mathcal{O}(z)$ that, when inserted at the origin, are annihilated by the holomorphic special conformal generator

$$[L_{+1}, \mathcal{O}(0)] = 0 \ . \tag{2.50}$$

In our construction, such an $\mathcal{O}(z)$ arises as the cohomology class of a twisted-translated primary Schur operator. The assertion is then that in the chiral algebra (*i.e.*, up to $\mathbb{Q}_i$-exact terms), the $T_{\mathcal{J}}$ OPEs take the form

$$T_{\mathcal{J}}(z)\mathcal{O}(0) \sim \cdots + \frac{0}{z^3} + \frac{h\ \mathcal{O}(0)}{z^2} + \frac{\partial\mathcal{O}(0)}{z} \ , \tag{2.51}$$

where h is the holomorphic dimension of $\mathcal{O}$ and the dots indicate possible poles of order four or higher. Though we have not been able to find a general proof, we believe (2.51) to be a universal consequence of superconformal Ward identities. It is thanks to the relation for the conformal dimension $h = R + j_1 + j_2$ that the $SU(2)_R$ current can reproduce the appropriate scaling dimension, and the absence of additional operators should be excluded by selection rules for three-point functions of Schur-type superconformal multiplets. In practice, we have been able to give an abstract argument that this OPE holds only for the case where $\mathcal{O}$ is a scalar operator. For non-scalar operators in the abstract setting, we leave the structure of these OPEs as a conjecture. Later in this section, the OPE (2.51) will be shown to hold in full generality in the theories of free hypermultiplets and free vector multiplets. The abstract claim would follow if the most general solution of the requisite Ward identity is expressible as a linear combination of structures corresponding to free field models, which is empirically the case in all analogous situations with which the authors are familiar.

Affine enhancement of the flavor symmetry

We next turn to the role played by the flavor symmetries of $\mathcal{T}$ in the associated chiral algebra. When $\mathcal{T}$ enjoys a flavor symmetry G_F, the corresponding conserved current $J_{\alpha\dot{\alpha}}$ is an element of a $\hat{\mathcal{B}}_1$ supermultiplet, which additionally contains as its Schur primary the moment map operator M^{11} described in the list at the end of §2.3.1. We expect the presence of G_F symmetry to make itself known via the chiral operator associated to the moment map. Following the now-familiar procedure, we define a $\mathtt{Q}_i$-closed operator $M(z,\bar{z})$ via twisted translations of the Schur moment-map operator from the origin, and identify the corresponding cohomology class as a meromorphic operator in the chiral algebra,

$$M(z,\bar{z}) := u_{\mathcal{I}}(\bar{z})u_{\mathcal{J}}(\bar{z})\, M^{\mathcal{I}\mathcal{J}}(z,\bar{z}) \ , \qquad J(z) := \kappa[M(z,\bar{z})]_{\mathtt{Q}_i} \ . \tag{2.52}$$

The normalization constant κ will be determined momentarily. The meromorphic operator $J(z)$ has holomorphic dimension $h = 1$. We have suppressed flavor indices up to this point, but these operators all transform in the adjoint representation of the flavor symmetry group, and so we actually find $\dim G_F$ dimension one currents $J^A(z)$ in the chiral algebra. It is natural to suspect that these operators will behave as affine currents for the flavor symmetry. Indeed, a little calculation bears out this expectation. First, recall that the central charge k_{4d} of the flavor symmetry is defined in terms of the self-OPE of the conserved flavor symmetry current as follows,

$$J^A_\mu(x)\ J^B_\nu(0) \sim \frac{3k_{4d}}{4\pi^4}\delta^{AB}\frac{x^2 g_{\mu\nu} - 2x_\mu x_\nu}{x^8} + \frac{2}{\pi^2}\frac{x_\mu x_\nu f^{ABC} x\cdot J^C(0)}{x^6} + \cdots \ . \tag{2.53}$$

Here $A, B, C = 1, \ldots, \dim G_F$ are adjoint flavor indices, and we are using normalizations such that long roots of a Lie algebra have length $\sqrt{2}$ as in [54]. In the same conventions, the OPE of two moment maps reads

$$M^{A\,\mathcal{I}\mathcal{J}}(x)M^{B\,\mathcal{K}\mathcal{L}}(0) \sim -\frac{3k_{4d}}{48\pi^4}\frac{\epsilon^{\mathcal{K}(\mathcal{I}}\epsilon^{\mathcal{J})\mathcal{L}}\delta^{AB}}{x^4} - \frac{\sqrt{2}}{4\pi^2}\ \frac{f^{ABC}M^{C\,(\mathcal{I}(\mathcal{K}}\epsilon^{\mathcal{L})\mathcal{J})}}{x^2} + \cdots \ . \tag{2.54}$$

The OPE for the corresponding twisted-translated operators follows directly,

$$M^A(z,\bar{z})M^B(0,0) \sim -\frac{3k_{4d}}{48\pi^4}\frac{\delta^{AB}}{z^2} + \frac{\sqrt{2}}{4\pi^2}\ i\ \frac{f^{ABC}M^C(0,0)}{z} + \frac{\sqrt{2}}{4\pi^2}\ f^{ABC}M^{C\,21}(0)\frac{\bar{z}}{z} + \cdots \ , \tag{2.55}$$

where the last term is $\mathtt{Q}_i$-exact. Setting $\kappa = 2\sqrt{2}\pi^2$, we recognize the canonical current algebra OPE,[11]

$$J^A(z)J^B(w) \sim k_{2d}\frac{\delta^{AB}}{(z-w)^2} + \sum_C i f^{ABC}\frac{J^C(w)}{z-w} \ , \tag{2.56}$$

where the two-dimensional affine level k_{2d} is related to the four-dimensional flavor central charge k_{4d} by

$$k_{2d} = -\frac{k_{4d}}{2} \ . \tag{2.57}$$

[11] In two dimensions it is standard to define a convention-independent affine level k_{2d} as $k_{2d} := \frac{2\tilde{k}_{2d}}{\theta^2}$, where $\tilde{k}_{2d}$ is the level when the length of the long roots are normalized to be θ. In our conventions $\theta^2 = 2$ and so $\tilde{k}_{2d} = k_{2d}$.

This is the second important entry in the dictionary.

The Hall-Littlewood chiral ring and chiral algebra generators

An interesting problem that will be of particular concern in §2.5 is that of giving a simple description of the chiral algebra $\chi[\mathcal{T}]$ associated to a given $\mathcal{T}$ in terms of a set of generating currents. Generators of a chiral algebra are by definition those $\mathfrak{sl}(2)$ primary operators $\{\mathcal{O}_j\}$ for which the normal ordered products of their descendants, *i.e.*, operators of the form $\partial^{n_1}\mathcal{O}_1\partial^{n_2}\mathcal{O}_2\ldots\partial^{n_k}\mathcal{O}_k$, span the whole algebra.[12] When the chiral algebra has only a finite number of generators, it is customary to refer to it as a $\mathcal{W}$-algebra.

While we have given a clear set of rules that identifies the spectrum of the chiral algebra given the spectrum of the four-dimensional theory $\mathcal{T}$, these rules have little to say about the question of what operators are *generators* of $\chi[\mathcal{T}]$. There turns out to be a subset of generators that is always relatively easy to identify. Recall from §2.3.1 that the HL chiral and anti-chiral rings are consistent truncations of the $\mathcal{N}=1$ chiral and anti-chiral rings of $\mathcal{T}$, respectively. As such, they are commutative rings, and it is often possible to give them presentations in terms of generators and relations. What we show now is that the meromorphic operators associated to the generators of the HL chiral and antichiral rings are in fact generators of $\chi[\mathcal{T}]$ in the chiral algebra sense.

Given the shortening conditions they obey, one finds that the chiral algebra operators associated to HL operators have holomorphic dimension $h=R+|r|$. In order to establish the claim made above, we will show that an HL operator can never arise as a normal ordered product of other operators that are not themselves of HL type. Let $\mathcal{O}_1(z,\bar{z})$ and $\mathcal{O}_2(z,\bar{z})$ be two generic twisted-translated Schur operators, and let us assume that their OPE contains an HL operator $\mathcal{O}_3^{\rm HL}$,

$$\mathcal{O}_1(z,\bar{z})\,\mathcal{O}_2(0,0)\sim\frac{1}{z^{h_1+h_2-h_3}}\mathcal{O}_3^{\rm HL}(0,0)+\ldots \tag{2.59}$$

By assumption, $h_3=R_3+|r_3|$, while (2.41) implies that $h_1\geqslant R_1+|r_1|$, $h_2\geqslant R_2+|r_2|$. The $U(1)_r$ charge is conserved, so $r_3=r_1+r_2$ and $|r_3|\leqslant|r_1|+|r_2|$. Furthermore, $SU(2)_R$ selection rules imply the triangular inequality $R_3\leqslant R_1+R_2$. Combining these (in)equalities, we find that $h_3\leqslant h_1+h_2$, which implies that an HL operator may only appear on the right hand side as a singular term (if $h_3<h_1+h_2$) or as the leading non-singular term (if $h_3=h_1+h_2$). The latter possibility requires that $\mathcal{O}_1$ and $\mathcal{O}_2$ saturate the respective bounds (2.41) for h_1 and h_2, which is to say that they themselves must be HL operators. This argument establishes

[12] We are adopting the normal ordering conventions of [55], in which a sequence of chiral operators represents left-nesting of conformally normal-ordered products:

$$\mathcal{O}_1\mathcal{O}_2\cdots\mathcal{O}_{n-1}\mathcal{O}_n:=(\mathcal{O}_1(\mathcal{O}_2(\cdots(\mathcal{O}_{n-1}\mathcal{O}_n))))\ . \tag{2.58}$$

The algebra of operators so-defined is non-commutative and non-associative.

that HL operators cannot be generated as normal ordered products of non-HL operators, and so the generators of the HL chiral and antichiral rings must necessarily be generators of the chiral algebra.

The Hall-Littlewood chiral ring and Virasoro primaries

A further interesting feature of the HL chiral ring operators is that their corresponding meromorphic operators are always Virasoro primaries. For the generators of the HL chiral ring, this is already clear since the generators of any chiral algebra that includes a stress tensor are necessarily primaries of the Virasoro subalgebra. For other HL operators, though, this is a useful result that will help organize our thinking about some of the examples studied in §2.5.

The statement follows from a relatively straightforward analysis of the OPE of the meromorphic stress tensor with an arbitrary HL operator. In particular, let $\mathcal{O}_1(z)$ be the meromorphic operator associated to an HL operator in four dimensions. The quantum numbers of $\mathcal{O}_1$ obey the HL relation

$$h_1 = R_1 + |r_1| \ . \tag{2.60}$$

Now the crucial observation from which our result follows is this: from a four-dimensional perspective, the meromorphic stress tensor is a $\bar{z}$-dependent linear combination of operators with $r = 0$ and $R = 0, \pm 1$. Consequently, in the OPE of the meromorphic stress tensor with $\mathcal{O}_1(0)$, the only operators that may appear will have $R = R_1 \pm 1$ or $R = R_1$ and $r = r_1$. With what power of z can such an operator appear in the OPE? A Schur operator $\mathcal{O}_\gamma(0)$ with $R = R_1 + \gamma$ and $\mathcal{M} = |r_1| + 2\min(j_1, j_2)$ will appear in the OPE as

$$T(z)\mathcal{O}_1(0) \supset \frac{\mathcal{O}_\gamma(0)}{z^{2+R_1+|r_1|-R-\mathcal{M}}} = \frac{\mathcal{O}_\gamma(0)}{z^{2-\gamma-2\min(j_1,j_2)}} \ . \tag{2.61}$$

This is at most a pole of order three (when $\gamma = -1$ and $j_1 = 0$ or $j_2 = 0$), but such a pole cannot appear because HL operators are always $\mathfrak{sl}(2)$ primaries – thus the most singular term possible is a pole of order two. This is precisely the property that characterizes Virasoro primary operators, and so we have our result.

2.3.3 The chiral algebras of free theories

The simplest $\mathcal{N} = 2$ SCFTs are the theories of a free hypermultiplet and that of a free vector multiplet. For these special cases, we give a complete description of the associated chiral algebras. These chiral algebras are useful as the building blocks of interacting Lagrangian theories, some of which are discussed in §2.4. We describe in turn the cases of hypermultiplets and vector multiplets.

Free hypermultiplets

Let us consider the field theory of a single free hypermultiplet. The hypermultiplet itself lies in the short supermultiplet $\mathcal{B}_{\frac{1}{2}}$, in which the primary Schur operators are the scalars Q and $\tilde{Q}$. These are the highest weight states in a pair of $SU(2)_R$ doublets,

$$Q^{\mathcal{I}} = \left(\begin{array}{c} Q \\ \tilde{Q}^* \end{array} \right) , \qquad \tilde{Q}^{\mathcal{I}} = \left(\begin{array}{c} \tilde{Q} \\ -Q^* \end{array} \right) . \tag{2.62}$$

The single free hypermultiplet enjoys an $SU(2)_F$ flavor symmetry, under which $Q^{\mathcal{I}}$ and $\tilde{Q}^{\mathcal{I}}$ transform as a doublet. To work covariantly in terms of this $SU(2)_F$, we can introduce the following tensor,

$$Q^{\mathcal{I}}_{\hat{\mathcal{I}}} := \left(\begin{array}{cc} Q & \tilde{Q} \\ \tilde{Q}^* & -Q^* \end{array} \right) , \tag{2.63}$$

where $\hat{\mathcal{I}} = 1, 2$ is the newly minted $SU(2)_F$ index.

The Schur operators in this free theory are all the "words" that can be constructed out of the "letters" $\{Q, \tilde{Q}, \partial_{+\dot{+}}\}$. As there are no singularities in the products of ($\partial_{+\dot{+}}$ derivatives of) Q and $\tilde{Q}$, the operator associated to any given word is well-defined and the Schur operators in this theory form a commutative ring. The set of all meromorphic operators in the free hypermultiplet chiral algebra are therefore precisely the $\mathbb{Q}_i$ cohomology classes of the twisted-translated versions of these words. This chiral algebra is itself a free chiral theory in two dimensions. Let us see how this works.

The twisted-translated operators and the associated meromorphic operators for the hypermultiplet scalars themselves are defined as follows,

$$Q_{\hat{\mathcal{I}}}(z,\bar{z}) := u_{\mathcal{I}}(\bar{z})\, Q^{\mathcal{I}}_{\hat{\mathcal{I}}}(z,\bar{z}) , \qquad q_{\hat{\mathcal{I}}}(z) := [Q_{\hat{\mathcal{I}}}(z,\bar{z})]_{\mathbb{Q}_i} . \tag{2.64}$$

The relation to the operators defined in §2.2.4 is $q_{\hat{\mathcal{I}}}(z) = (q(z), \tilde{q}(z))$. This is an $SU(2)_F$ doublet of dimension $1/2$ meromorphic fields, the OPE of which can be computed using the free-field OPE in four dimensions and the definition of the twisted translated operators in (2.64),

$$q_{\hat{\mathcal{I}}}(z)\, q_{\hat{\mathcal{J}}}(w) \sim \frac{\varepsilon_{\hat{\mathcal{I}}\hat{\mathcal{J}}}}{z-w} . \tag{2.65}$$

It is reasonably easy to see that the entire spectrum of the chiral algebra of four-dimensional hypermultiplets is obtained by taking normal ordered products of the $q_{\hat{\mathcal{I}}}(z)$ and their descendants. In particular, one can show that the following diagram commutes,[13]

$$
\begin{tikzcd}
\{\mathcal{O}_i\ ,\mathcal{O}_j\} \arrow[r, "\times_{4d}"] \arrow[d, "\mathbb{Q}_i"'] & \mathcal{O}_i\mathcal{O}_j \arrow[d, "\mathbb{Q}_i"] \\
\{[\mathcal{O}_i]\ ,[\mathcal{O}_j]\} \arrow[r, "\times_{::}"] & :[\mathcal{O}_i][\mathcal{O}_j]:
\end{tikzcd}\ , \tag{2.66}
$$

where the top row represents multiplication in the ring of Schur operators, the bottom row represents *creation/annihilation* normal ordered products of chiral vertex operators, and the vertical arrows represent the identification of a Schur operator with its meromorphic counterpart in the chiral algebra. It follows that the meromorphic operator associated to any given word in ($\partial_{+\dot{+}}$derivatives of) Q and $\tilde{Q}$ is simply the corresponding creation/annihilation normal ordered product of (holomorphic derivatives of) q and $\tilde{q}$.

The chiral algebra of the free hypermultiplet is thus none other than the free symplectic boson algebra (*cf.* [56]). This simple example serves to illustrate some of the general points made in the previous subsections. The symplectic boson theory has a canonical stress tensor,

$$T(z) = \frac{1}{2}\varepsilon^{\hat{\mathcal{I}}\hat{\mathcal{J}}} q_{\hat{\mathcal{I}}}\partial q_{\hat{\mathcal{J}}}(z)\ , \tag{2.67}$$

and it is easy to check that the modes $\{L_{+1}, L_0, L_{-1}\}$ appearing in Laurent expansion of (2.67) reproduce the action of the holomorphic $\mathfrak{sl}(2)$ symmetry inherited from four dimensions. Thus the holomorphic $\mathfrak{sl}(2)$ is indeed enhanced to Virasoro symmetry. Moreover, we observe that given the form of the $SU(2)_R$ current in four dimensions

$$\mathcal{J}_\mu^{\mathcal{I}\mathcal{J}}(x) \sim \varepsilon^{\hat{\mathcal{I}}\hat{\mathcal{J}}} Q^{(\mathcal{I}}_{\hat{\mathcal{I}}}\partial_\mu Q^{\mathcal{J})}_{\hat{\mathcal{J}}}(x)\ , \tag{2.68}$$

The corresponding meromorphic operator $T_{\mathcal{J}}(z)$ will be equivalent to the canonical stress tensor,

$$T(z) = T_{\mathcal{J}}(z)\ . \tag{2.69}$$

From the TT OPE we read off the central charge $c_{2d} = -1$. Recalling that the conformal anomaly coefficient of a free hypermultiplet is $c_{4d} = 1/12$, this result is in agreement the universal relation $c_{2d} = -12c_{4d}$. The symplectic boson theory is like the theory of a complex free fermion (which of course has $c_{2d} = 1$), but with opposite statistics, hence the opposite value of the central charge.

Finally we mention a minor generalization of the above story for hypermultiplets. Gauge theories with $\mathcal{N} = 2$ supersymmetry are often described in terms of half-hypermultiplets instead of whole hypermultiplets. The generalization of the chiral algebra to the half-

[13]We will see when we come to consider interacting theories in §2.5 that product structures on Schur operators do not always translate so simply into those of the chiral algebra.

hypermultiplet conventions is straightforward. Let us consider half-hypermultiplets transforming in a pseudo-real representation R of some symmetry group G (at the moment we are working at zero coupling, so G is just a global symmetry group). The corresponding chiral algebra will be generated by $\dim R$ meromorphic fields,

$$q_i \ , \quad i = 1, \ldots, \dim R \ , \tag{2.70}$$

and the singular OPE of these operators will be given by

$$q_i(z) q_j(w) \sim \frac{\Omega_{ij}}{z-w} \ . \tag{2.71}$$

Here Ω_{ij} is the anti-linear involution that maps the representation R to its conjugate and squares to minus one. The description of the single full hypermultiplet in (2.65) actually fits into this framework with $G = SU(2)_F$.

Free vector multiplet

The other key ingredient in Lagrangian SCFTs is the theory of free vector multiplets. Free vectors lie in the short supermultiplet $\bar{\mathcal{D}}_{0(0,0)}$ and its conjugate $\mathcal{D}_{0(0,0)}$, whose superconformal primaries are the complex scalar ϕ and its conjugate $\bar{\phi}$, respectively. The primary Schur operators in these multiplets are the fermions λ^1_+ and $\tilde{\lambda}^1_{\dot{+}}$, and as in the case of hypermultiplets, the entire set of Schur operators in this theory is comprised of the words built out of the letters λ^1_+, $\tilde{\lambda}^1_{\dot{+}}$, and $\partial_{+\dot{+}}$.

The twisted-translated operators associated to the vector multiplet fermions are defined as follows,

$$\lambda(z,\bar{z}) := u_{\mathcal{I}}(\bar{z})\, \lambda^{\mathcal{I}}_+(z,\bar{z}) \ , \quad \tilde{\lambda}(z,\bar{z}) := u_{\mathcal{I}}(\bar{z}) \tilde{\lambda}^{\mathcal{I}}_{\dot{+}}(z,\bar{z}) \ , \tag{2.72}$$

and the $\mathbb{Q}_i$-cohomology classes of these operators are Grassmann-odd, holomorphic fields of dimension $h = 1$,

$$\lambda(z) := [\lambda(z,\bar{z})]_{\mathbb{Q}_i} \ , \quad \tilde{\lambda}(z) := [\tilde{\lambda}(z,\bar{z})]_{\mathbb{Q}_i} \ . \tag{2.73}$$

Using the four-dimensional free field OPEs, it is easy to derive the OPEs of these holomorphic fields. They are again the OPEs of a free chiral algebra:

$$\tilde{\lambda}(z)\lambda(0) \sim \frac{1}{z^2} \ , \quad \lambda(z)\tilde{\lambda}(0) \sim -\frac{1}{z^2} \ . \tag{2.74}$$

Indeed, the free-field form of these OPEs leads to an analogous commutative diagram to (2.66), which ensures that all the meromorphic operators in this theory are generated by $\lambda(z)$ and $\tilde{\lambda}(z)$ in the chiral algebra sense. We can recognize this chiral algebra as the (b, c)

ghost system of weight $(1,0)$,[14]

$$\tilde{\lambda} := b(z) \ , \quad \lambda(z) := \partial c(z) \ . \tag{2.75}$$

In making this identification, we have introduced an extra spurious mode – the zero mode c_0 of $c(z)$ – which is of absent in the algebra generated by $\lambda(z)$ and $\tilde{\lambda}(z)$. Thus, the more precise statement is that the chiral algebra associated to the vector multiplet is the so-called "small algebra" of the (b,c) system, which is by definition the algebra generated by $b(z)$ and $\partial c(z)$ (*cf.* [57, 58]). In other words, the Fock space of the small algebra is the subspace of the (b,c) Fock space that does *not* contain c_0, or equivalently, the subspace annihilated by b_0,

$$\mathcal{F}_{\text{small}} := \{\psi \in \mathcal{F}_{bc} \mid b_0\psi = 0\} \ . \tag{2.76}$$

The small algebra enjoys a global $SL(2,\mathbb{R})$ symmetry under which $\lambda(z)$ and $\tilde{\lambda}(z)$ transform as a doublet. We can make this symmetry manifest by introducing the notation ρ^α with $\alpha = \pm$, where $\rho^+ := \tilde{\lambda}$ and $\rho^- := \lambda$. Note that the Cartan generator of this symmetry acts as the $U(1)_r$ charge. In the language of the small algebra, the OPE can be put in a covariant form,

$$\rho^\alpha(z)\,\rho^\beta(0) \sim \frac{\varepsilon^{\alpha\beta}}{z^2} \ . \tag{2.77}$$

As in the hypermultiplet case, the action of the $\{L_{+1}, L_0, L_{-1}\}$ modes of the canonical ghost stress tensor can easily be seen to match the action of the geometric $\mathfrak{sl}(2)$ action inherited from the four-dimensional conformal algebra. Furthermore, given the $SU(2)_R$ current of the free vector theory,

$$\mathcal{J}^{\mathcal{I}\mathcal{J}}_{\alpha\dot{\alpha}}(x) \sim \lambda^{(\mathcal{I}}_{\alpha}\tilde{\lambda}^{\mathcal{J})}_{\dot{\alpha}}(x) \ , \tag{2.78}$$

we see that the canonical stress tensor coincides precisely with the dimension two current $T_{\mathcal{J}}$ obtained from the R-symmetry current by the usual map,

$$T(z) = -\frac{1}{2}\varepsilon_{\alpha\beta}\rho^\alpha\rho^\beta(z) = T_{\mathcal{J}}(z) \ . \tag{2.79}$$

The central charge of the (b,c) ghost system/small algebra is $c_{2d} = -2$, which can be seen to agree with the relation (2.49) upon recalling that $c_{4d} = \frac{1}{6}$ for a free vector multiplet.

2.3.4 Gauging prescription

The natural next step is to consider interacting SCFTs. Lagrangian $\mathcal{N} = 2$ SCFTs can be described using hypermultiplets and vector multiplets as elementary building blocks (see

[14]Recall that the derivative of a dimension zero conformal primary field – $c(z)$ in this case – is again a conformal primary.

[11] for a recent classification of all possibilities). In particular, such an SCFT consists of vector multiplets transforming in the adjoint representation of a semisimple gauge group $G = G_1 \times G_2 \cdots \times G_k$, along with a collection of (half)hypermultiplets transforming in some representation R of the gauge group such that the one-loop beta functions for all gauge couplings vanish. Supersymmetry ensures that the theory remains conformal at the full quantum level. The building blocks of the corresponding chiral algebra are a collection of symplectic bosons $\{q\,,\tilde{q}\}$ in the representation R, and a collection of $(b\,,c)$ ghost small algebras in the adjoint representation of G. When the gauge couplings are strictly zero, the chiral algebra is simply obtained by imposing the Gauss law constraint, *i.e.*, by restricting to the gauge-invariant operators of the free chiral algebra of symplectic bosons and ghosts. Our next step will be to determine what happens as we turn on the gauge couplings.

In fact, as Lagrangian theories are a small subset of all possible $\mathcal{N} = 2$ SCFTs, it is worthwhile to put the discussion in a more general context. Given a general superconformal field theory $\mathcal{T}$ with G_F flavor symmetry, a new SCFT is obtained by gauging a subgroup $G \subset G_F$ provided the gauge coupling beta function vanishes. We will denote the gauged theory with a nonzero gauge coupling g as $\mathcal{T}_G$.[15] Though $\mathcal{T}$ may be strongly coupled, the gauging procedure can be described in semi-Lagrangian language. By assumption, $\mathcal{T}$ possesses a conserved flavor symmetry current $J^A_{\alpha\dot{\alpha}}$, where $A = 1, \ldots \dim G$, which by $\mathcal{N} = 2$ supersymmetry is the top component of the moment map supermultiplet $\hat{\mathcal{B}}_1$. The gauged theory $\mathcal{T}_G$ is described by minimally coupling an $\mathcal{N} = 2$ vector multiplet to $\hat{\mathcal{B}}_1$. Of particular importance is the addition to the action, in $\mathcal{N} = 1$ notation, of the superpotential coupling

$$g \int d^2\theta\, \Phi^A\, M^{11,A} + \, h.c.\,, \tag{2.80}$$

where Φ is the $\mathcal{N} = 1$ chiral superfield in the $\mathcal{N} = 2$ vector multiplet, and M^{11} is the $\mathcal{N} = 1$ chiral superfield whose bottom component is the complex moment map M^{11}; both transform in the adjoint representation of G.

Let us assume that the chiral algebra $\chi[\mathcal{T}]$ is known. It will suffice to work abstractly, in the sense that the only features of $\chi[\mathcal{T}]$ that we will use follow directly from the existence of the global G symmetry. In particular, there will be an affine current $J^A(z)$ at level $k_{2d} = -\frac{1}{2}k_{4d}$ (*cf.* §2.3.2). As we mentioned above, at zero gauge coupling the chiral algebra of the gauged theory is obtained by imposing the Gauss law constraint on the tensor product algebra of $\chi[\mathcal{T}]$ with the G-ghost small algebra (ρ^+, ρ^-). In fact, it will be more useful to introduce the full (b, c) system and restrict to the small algebra by imposing the auxiliary condition $b^A_0\psi = 0$ for any state ψ.

[15] More precisely, there is one independent gauge coupling for each simple factor of the gauge group. To avoid clutter we focus on the procedure for gauging one simple factor at the time, so G will taken to be a simple group in what follows.

The affine current associated to the G symmetry in the ghost sector is

$$J^A_{\mathrm{gh}} := -i\, f^{ABC}\,(c^B b^C)\ . \tag{2.81}$$

The Gauss law, or gauge-invariance, constraint requires that all physical states should have vanishing total gauge charge, which is measured by the zero mode of the total gauge symmetry current,

$$J^A_{\mathrm{tot}}(z) := J^A(z) + J^A_{\mathrm{gh}}(z)\ . \tag{2.82}$$

Symbolically, we can therefore define the chiral algebra at zero gauge coupling as follows:

$$\chi[\mathcal{T}^{(0)}_G] = \{\psi \in \chi[\mathcal{T}] \otimes (b^A, c^A)\,|\, b^A_0\psi = J^A_{\mathrm{tot}\,0}\psi = 0\}\,. \tag{2.83}$$

We are now ready to address the problem of identifying the chiral algebra for $\mathcal{T}_G$ with $g \neq 0$.

BRST reduction of the chiral algebra

On general grounds, we expect that the chiral algebra of the interacting gauge theory will contain fewer operators than the non-interacting version, because some of the short multiplets containing Schur operators that are present at zero coupling will recombine into long multiplets and acquire anomalous dimensions. Ideally, we would like to describe this phenomenon using only the general algebraic ingredients that we have introduced so far. A crucial hint comes from phrasing the condition of conformal invariance of the gauge theory more abstractly. The vanishing of the one-loop beta function amounts to the requirement that in the ungauged theory, the flavor symmetry central charge is given by

$$k_{4d} = 4h^\vee\ , \tag{2.84}$$

where $h^\vee$ is the dual Coxeter number of the gauge group. This means that in two-dimensional language, the corresponding symmetry in $\chi[\mathcal{T}]$ must have its affine level given by

$$k_{2d} = -2h^\vee\ . \tag{2.85}$$

The affine level of the ghost-sector flavor currents J_{gh} is easily calculated to be $2h^\vee$, so the requirement of conformal invariance translates into the condition that the level of the total affine current J^A_{tot} be zero. Precisely in this case, it is possible to construct a nilpotent BRST operator in the chiral algebra. Imitating a construction familiar from coset conformal field theory [59], we define

$$Q_{\mathrm{BRST}} := \oint \frac{dz}{2\pi i}\, j_{\mathrm{BRST}}(z)\,, \quad j_{\mathrm{BRST}} := c_A\left[J^A + \frac{1}{2}J^A_{\mathrm{gh}}\right]\,. \tag{2.86}$$

Our contention is that the chiral algebra corresponding to the gauged theory at finite coupling is obtained by passing to the cohomology of Q_{BRST} *relative to* the ghost zero modes b_0^A,[16]

$$\chi[\mathcal{T}_G] = \mathcal{H}^*_{\text{BRST}}[\psi \in \chi[\mathcal{T}] \otimes (b^A, c^A) \bigm| b_0^A \psi = 0] \,. \tag{2.87}$$

Apart from its elegance, there are compelling physical arguments behind this claim. We will show that states of the chiral algebra that define nontrivial cohomology classes of Q_{BRST} correspond to the four-dimensional Schur states that survive in the interacting theory. By construction, all states of $\chi[\mathcal{T}_G^{(0)}]$ are annihilated by the four supercharges in (2.37). As we turn on the gauge coupling, those supercharges receive quantum corrections, and only a subset of states remains supersymmetric. We will see that Q_{BRST} precisely implements the $O(g)$ correction to one of the Poincaré supercharges, which will justify our conjecture under the assumption that higher order corrections do not remove any additional states.

A preliminary remark is that the Gauss law constraint is imposed automatically. Because

$$\{b_0^A \,, Q_{\text{BRST}}\} = J^A_{\text{tot}\, 0} \ , \tag{2.88}$$

states in the small algebra that are Q_{BRST}-closed are automatically gauge invariant. Consequently, we have the simpler expression,

$$\chi[\mathcal{T}_G] = \mathcal{H}^*_{\text{BRST}}[\chi[\mathcal{T}_G^{(0)}]] \ . \tag{2.89}$$

We can rewrite Q_{BRST} and separate out the ghost zero modes,

$$Q_{\text{BRST}} = c_0^A J^A_{\text{tot}\, 0} + b_0^A X^A + Q^- \ , \tag{2.90}$$

where we have defined

$$X^A := -\frac{i}{2} f^{ABC} \left(\sum_{n \neq 0} : c^B_{-n} c^C_n : - c^B_0 c^C_0 \right) \ , \tag{2.91}$$

while Q^- anticommutes with both c_0^A and b_0^A and can thus be expressed purely in terms of (ρ^{+A}, ρ^{-A}),

$$Q^- := \sum_{n \neq 0} \frac{1}{n} : \rho^{-A}_{-n} J^A_n : + \frac{i}{2} f^{ABC} \sum_{\substack{n \neq 0 \\ m \neq 0 \\ m \neq n}} \frac{1}{nm} : \rho^{-A}_{-n} \rho^{-B}_m \rho^{+C}_{n-m} : \ . \tag{2.92}$$

The operator Q^- fails to be nilpotent by a term proportional to $J^A_{tot\, 0}$, so it *is* nilpotent when

[16] In other terms, the BRST cohomology is being defined entirely in the small algebra: two Q_{BRST}-closed states belong to the same cohomology class if and only if they differ by an exact state $Q_{\text{BRST}}\lambda$, where λ is also in the small algebra.

acting on gauge-invariant states. It follows that (2.88) can be equivalently written as

$$\chi[\mathcal{T}_G] = \mathcal{H}^*_{Q^-}[\psi \in \chi[\mathcal{T}] \otimes (\rho^{+A}, \rho^{-A})\,, \text{with } J^A_{\text{tot}\,0}\psi = 0]\,. \tag{2.93}$$

This is the form of our conjecture that makes more immediate contact with four-dimensional physics. We will show that the action of Q^- precisely matches to the action of $\widetilde{\mathcal{Q}}^{(1)}_{2\dot{-}}$, the $O(g)$ term in the expansion of the supercharge $\widetilde{\mathcal{Q}}_{2\dot{-}}$,

$$\widetilde{\mathcal{Q}}_{2\dot{-}} = \widetilde{\mathcal{Q}}^{(0)}_{2\dot{-}} + g\,\widetilde{\mathcal{Q}}^{(1)}_{2\dot{-}} + O(g^2)\,. \tag{2.94}$$

In fact, Q^- is the lowest component of an $SL(2,\mathbb{R})$ doublet of operators Q^α, with

$$Q^+ := \sum_{n\neq 0} \frac{1}{n} :\rho^{+A}_{-n} J^A_n: + \frac{i}{2} f^{ABC} \sum_{\substack{n\neq 0\\ m\neq 0\\ m\neq n}} \frac{1}{mn} :\rho^{+A}_{-n}\rho^{+B}_m \rho^{-C}_{n-m}: \,. \tag{2.95}$$

In complete analogy, the action of Q^+ will be shown to be isomorphic to that of $\mathcal{Q}^{1(1)}_{-}$, the $O(g)$ term in the expansion of $\mathcal{Q}^1_-$. The two Poincaré supercharges $\mathcal{Q}^1_-$ and $\widetilde{\mathcal{Q}}_{2\dot{-}}$ play a completely symmetric role in the definition of Schur operators. The fact that Q_{BRST} contains Q^- rather than Q^+ is a consequence of our choice (2.75), which treated λ and $\tilde{\lambda}$ in a slightly asymmetric fashion.

Fortunately, to leading order in the gauge coupling the action of the relevant supercharges takes a universal form in the subspace of operators that obey the tree-level Schur condition. Such operators are obtained by forming gauge-invariant combinations of more elementary building blocks, namely the conformal primaries of the "matter" SCFT $\mathcal{T}$, the gauge-covariant derivative $D_{+\dot{+}}$, and the gauginos $\tilde{\lambda}^1_{\dot{+}}$ and λ^1_+. The supersymmetry variation of a gauge-invariant "word" is found by using the Leibniz rule to act on each elementary "letter".[17] It is then sufficient to specify the SUSY variations of the letters:

1. $\mathcal{Q}^1_-$ and $\widetilde{\mathcal{Q}}_{2\dot{-}}$ (anti)commute with the conformal primary operators in the matter sector $\mathcal{T}$.

2. For the gauge-covariant derivative $D_{+\dot{+}} := \partial_{+\dot{+}} + gA_{+\dot{+}}$,

$$[\mathcal{Q}^1_-, D_{+\dot{+}}] = g\tilde{\lambda}^1_{\dot{+}}\,, \quad [\widetilde{\mathcal{Q}}_{2\dot{-}}, D_{+\dot{+}}] = g\lambda^1_+\,, \tag{2.96}$$

   where we have just used the tree-level variation of the gauge field, times the explicit factor of g.

[17] For the special case of $\mathcal{N}=2$ superconformal QCD, a very explicit description of the action of $\mathcal{Q}^{1(1)}_{-}$ in the subsector of tree-level Schur operators can be found in Section 5 of [60].

3. Finally the variations of the gauginos can be deduced from the non-linear classical equations of motions of the vector multiplet, minimally coupled to the moment map supermultiplet $\hat{\mathcal{B}}_1$,

$$\begin{aligned} \{\widetilde{\mathcal{Q}}_{2\dot{-}}, \tilde{\lambda}^1_{\dot{+}}\} &= \{\mathcal{Q}^1_{-}, \lambda^1_{+}\} = F^{11} = gM^{11} \\ \{\widetilde{\mathcal{Q}}_{2\dot{-}}, \lambda^1_{+}\} &= \{\mathcal{Q}^1_{-}, \tilde{\lambda}^1_{\dot{+}}\} = 0\,, \end{aligned} \tag{2.97}$$

where F^{11} is the highest-weight of the $SU(2)_R$ triplet of auxiliary fields in the $\mathcal{N}=2$ vector multiplet.[18]

If a Schur operator in the free theory is to retain its Schur status at $O(g)$, then when inserted at the origin it must be annihilated by the one-loop corrections to the four relevant supercharges, $\{\widetilde{\mathcal{Q}}^{(1)}_{2\dot{-}}, (\widetilde{\mathcal{Q}}^{(1)}_{2\dot{-}})^\dagger, \mathcal{Q}^{1(1)}_{-}, (\mathcal{Q}^{1(1)}_{-})^\dagger\}$. Equivalently, it must define a nontrivial cohomology class with respect to $\widetilde{\mathcal{Q}}^{(1)}_{2\dot{-}}$ and $\mathcal{Q}^{1(1)}_{-}$. Conveniently, the recombination rules for shortened multiplets of Schur type (*cf.* Appendix B) are such that in any such recombination, the Schur operators of $\mathcal{T}^{(0)}$ are lifted in quartets that are related by the action of these two supercharges in the manner indicated in the following diagram:

$$\begin{tikzcd} & \hat{\mathcal{C}}_{R+\frac{1}{2}(j_1-\frac{1}{2},j_2)} \arrow[dr, "\widetilde{\mathcal{Q}}^{(1)}_{2\dot{-}}"] & \\ \hat{\mathcal{C}}_{R(j_1,j_2)} \arrow[ur, "\mathcal{Q}^{1(1)}_{-}"] \arrow[dr, "\widetilde{\mathcal{Q}}^{(1)}_{2\dot{-}}"'] & & \hat{\mathcal{C}}_{R+1(j_1-\frac{1}{2},j_2-\frac{1}{2})} \\ & \hat{\mathcal{C}}_{R+\frac{1}{2}(j_1,j_2-\frac{1}{2})} \arrow[ur, "\mathcal{Q}^{1(1)}_{-}"'] & \end{tikzcd} \tag{2.98}$$

In the diagram, we are labeling Schur operators by the name of the supermultiplet to which they belong.[19] Consequently, if an operator remains in the cohomology of *either* supercharge, it necessarily remains in the cohomology of both, and so stays a Schur operator at one-loop order. For example, if an operator becomes $\mathcal{Q}^{1(1)}_{-}$ exact then it is either at the right or at the top of the diagram and it follows that it is either $\widetilde{\mathcal{Q}}^{(1)}_{2\dot{-}}$ exact or not $\widetilde{\mathcal{Q}}^{(1)}_{2\dot{-}}$ closed, respectively. The other cases can be treated analogously.

Under the $4d/2d$ identifications

$$\widetilde{\mathcal{Q}}^{(1)}_{2\dot{-}} \to Q^- \,, \quad \mathcal{Q}^{1(1)}_{-} \to Q^+ \,, \quad D_{+\dot{+}} \to \partial \,, \quad \lambda^1_{+} \to \rho^- \,, \quad \tilde{\lambda}^1_{\dot{+}} \to \rho^+ \,, \tag{2.99}$$

[18] In an $\mathcal{N}=1$ description of the $\mathcal{N}=2$ vector multiplet, $F^{11} = \bar{F}$, where F is the top component of chiral superfield ϕ, whose superpotential coupling with the moment map is given in (2.80).

[19] To include all possible recombinations, we must formally allow j_1 and j_2 to take the value $-\frac{1}{2}$ as well, and re-interpret a $\hat{\mathcal{C}}$ multiplet with negative spins as a $\hat{\mathcal{B}}$, $\mathcal{D}$ or $\bar{\mathcal{D}}$ multiplet, according to the rules: $\hat{\mathcal{C}}_{R(j_1,-\frac{1}{2})} := \bar{\mathcal{D}}_{R+\frac{1}{2}(j_1,0)}$, $\hat{\mathcal{C}}_{R(-\frac{1}{2},j_2)} := \mathcal{D}_{R+\frac{1}{2}(0,j_2)}$, $\hat{\mathcal{C}}_{R(-\frac{1}{2},-\frac{1}{2})} := \hat{\mathcal{B}}_{R+1}$.

one easily checks that (2.92) and (2.95) have precisely the right form to reproduce the action of the $O(g)$ correction to the four-dimensional supercharges. Thus, the BRST cohomology specified in (2.87) is just the right thing to project out states whose corresponding Schur operators are lifted at one-loop order.

It is of some interest to note that this story of one-loop corrections to the spectrum of Schur operators admits a simple truncation to the case of HL chiral ring operators. The tree-level HL operators will be gauge-invariant combinations of the HL operators of $\mathcal{T}$ and the gaugino λ^1_+. The operators that are lifted from the spectrum at one-loop will be those that are related by the corrected supercharge $\widetilde{\mathcal{Q}}^{(1)}_{2\dot{-}}$, whose action in this sector is completely determined by (2.97). The problem of finding the HL operators in the spectrum of the interacting theory thus becomes a miniature "HL-cohomology" problem. In examples, it is sometimes useful to solve this problem as a first step in order to determine some important operators that will necessarily make an appearance in the chiral algebra.

Finally, a caveat is in order. We have assumed that the Schur operators that persist at infinitesimal coupling will remain protected at any finite value of the coupling. In some concrete cases, it can be demonstrated that no further recombination of shortened multiplets is possible. Moreover, in the examples of §2.5 we will propose simple economical descriptions for the chiral algebras defined by this cohomological recipe, and demonstrate that they have the symmetries expected at finite coupling from S-duality, giving strong evidence for our proposal, at least in those examples.

Non-renormalization of three-point couplings

So far, we have studied how the spectrum of operators is modified when the coupling is turned on, but we have said nothing about the OPE coefficients of the remaining physical operators in the gauged theory. Our implicit assumption has been that the OPE coefficients of operators that remain protected at finite coupling are actually independent of the coupling. From a two-dimensional perspective, it seems unlikely that the OPE coefficients could change due to the extremely rigid structure of chiral algebras, and we expect a corresponding non-renormalization statement to hold in four dimensions. Indeed, such a non-renormalization theorem directly follows from the methods and results of [61]. Let us consider the four-point function of three Schur-type operators and of the exactly marginal operator $\mathcal{O}_\tau$ responsible for changing the complexified gauge coupling,

$$\langle\, \mathcal{O}_1^{\mathcal{I}_1}(x_1)\, \mathcal{O}_2^{\mathcal{I}_2}(x_2)\, \mathcal{O}_3^{\mathcal{I}_3}(x_3)\, \mathcal{O}_\tau(x_4)\,\rangle\,, \tag{2.100}$$

where $\mathcal{I} = (\mathcal{I}^{(1)} \ldots \mathcal{I}^{(k)})$ with $\mathcal{I}^{(i)} = 1, 2$ are $SU(2)_R$ multi-indices and we have suppressed Lorentz indices. Non-renormalization of the appropriate three-point function of Schur-type operators will follow at once if we can argue that the above four-point function vanishes for any x_4 when $x_{1,2,3}$ all lie on the plane. By a conformal transformation, we can always take the fourth operator to lie on the same plane, and then focus on the $SU(1,1|2)$ subalgebra of

$SU(2,2|2)$ defined by the embedding (2.20). The Schur-type operators are chiral primaries of this subalgebra. The marginal operator $\mathcal{O}_\tau$, being the top component of an $\bar{\mathcal{E}}_2$ multiplet of $SU(2,2|2)$, is of the form $\mathcal{O}_\tau = \{\mathcal{Q}^1, [\mathcal{Q}^2, \dots]\}$ where $\mathcal{Q}^{\mathcal{I}} := \mathcal{Q}^{\mathcal{I}}_-$ are supercharges of $SU(1,1|2)$.[20] All the properties exploited in [61] to show the vanishing of the four-point function (2.100) are satisfied. The authors of [61] interpreted this result as a non-renormalization theorem for three-point functions of chiral primaries of two-dimensional $(0,4)$ theories, but exactly the same argument applies to our case as well.

We close this section by pointing out a curious aspect of the gauging prescription given here. Given a chiral CFT $\chi[\mathcal{T}]$ with affine G symmetry, one can introduce a two-dimensional vector field $A_{\bar{z}}$ and gauge G. Following standard arguments (for example, see [59, 62]), a change of variables in the path integral eliminates the gauge field in favor of an extra G current algebra at level $-(2h^\vee + k_{2d})$ and an adjoint-valued (b,c) ghost system. One must also impose invariance under the standard BRST operator associated to the gauge symmetry. In our case, $2h^\vee + k_{2d} = 0$ so the extra current algebra is trivial, and the BRST operator associated to the two-dimensional gauging takes precisely the form of (2.86). In some sense, we have found that "$4d$ gauging $= 2d$ gauging". We find it plausible that a localization-style argument may shed light on this correspondence.

2.4 Consequences for four-dimensional physics

The chiral symmetry algebras that we have uncovered have extensive consequences for the spectrum and structure constants of any $\mathcal{N}=2$ SCFT. To give a simple example, Virasoro symmetry implies that any Higgs branch half-BPS supermultiplet $\hat{\mathcal{B}}_R$ is accompanied by an entire module of semi-short $\hat{\mathcal{C}}_{R'(j,j)}$ multiplets with $R' = R-1, R, R+1$. In the four-dimensional theory, the descendant operators arise by taking repeated normal ordered products with certain components of the $SU(2)_R$ current, but the chiral algebra perspective makes this structure much more transparent.

In this section we elaborate on the relationship between the observables associated to the chiral algebra (*i.e.*, its correlation functions and torus partition function) and those of the parent four-dimensional theory. We first point out that the superconformal Ward identities for four-point functions of $\hat{\mathcal{B}}_R$ operators [42, 43] are a simple consequence of our cohomological construction. This new perspective makes it clear that analogous Ward identities must hold for four-point functions of general Schur operators. The presence of meromorphic functions in the solution of the Ward identities of [39, 42, 43] was one of the initial clues that led to our work. We now have a neat conceptual interpretation for them: they are nothing but the correlation functions of the associated chiral algebra. By exploiting the relationship between the two-dimensional and four-dimensional perspectives we are able to derive new unitarity

[20]Similarly, the conjugate operator $\bar{\mathcal{O}}_\tau$ is the top component of an $\mathcal{E}_2$ and can be written as $\{\widetilde{\mathcal{Q}}_1, [\widetilde{\mathcal{Q}}_2, \dots]\}$. An entirely analogous argument holds for the four-point function containing $\bar{\mathcal{O}}_\tau$.

bounds that must be satisfied by the conformal and flavor anomalies of a general *interacting* $\mathcal{N}=2$ SCFT. Finally, we delineate the relationship between the torus partition function of the chiral algebra and the superconformal index of the parent four-dimensional theory.

2.4.1 Conformal twisting and superconformal Ward identities

By construction, for a given SCFT $\mathcal{T}$, the correlation functions of $\chi[\mathcal{T}]$ are equal to certain correlation functions of physical operators in $\mathcal{T}$ restricted to lie on the plane. From the four-dimensional point of view these are somewhat unnatural correlators to study, as they have explicit space-time dependence built into the operators. On the other hand, each correlation function of $\chi[\mathcal{T}]$ is canonically associated to a family of more natural correlation functions of $\mathcal{T}$ that are obtained by replacing the twisted-translated operators with the corresponding untwisted operators at the same points in $\mathbb{R}^2$.

Let us consider such a correlator now. For simplicity, we specialize to a four-point function, in which case there is actually no loss of generality in restricting the operators to be coplanar. We denote the untwisted operators as $\mathcal{O}^{\mathcal{I}}(z,\bar{z})$, with $SU(2)_R$ multi-indices $\mathcal{I}=(\mathcal{I}^{(1)},\ldots,\mathcal{I}^{(k)})$ where $\mathcal{I}^{(i)}=1,2$. The components of the multi-index are symmetrized; the operator transforms in the spin $k/2$ representation of $SU(2)_R$. Recall that in our conventions, the Schur operator in this $SU(2)_R$ multiplet is the highest-weight state $\mathcal{O}^{1\cdots 1}(z,\bar{z})$. We represent the four-point function of such operators as

$$\mathcal{F}^{\mathcal{I}_1\mathcal{I}_2\mathcal{I}_3\mathcal{I}_4}(z_i,\bar{z}_i)=\langle\, \mathcal{O}_1^{\mathcal{I}_1}(z_1,\bar{z}_1)\,\mathcal{O}_2^{\mathcal{I}_2}(z_2,\bar{z}_2)\,\mathcal{O}_3^{\mathcal{I}_3}(z_3,\bar{z}_3)\,\mathcal{O}_4^{\mathcal{I}_4}(z_4,\bar{z}_4)\,\rangle\ . \tag{2.101}$$

This is actually a collection of four-point functions labelled by the different possible assignments for the R-symmetry indices. The full collection of four-point functions can be conveniently packaged by introducing two-component $SU(2)_R$ vectors $u(y_i)=(1,y_i)$ and defining contracted operators that depend on the auxiliary variable y as follows [42, 43]

$$\mathcal{O}_i(z_i,\bar{z}_i;y_i)=u_{I_1}(y_i)\cdots u_{\mathcal{I}_{k_i}}(y_i)\mathcal{O}_i^{(\mathcal{I}_1\cdots\mathcal{I}_{k_i})}(z_i,\bar{z}_i)\ . \tag{2.102}$$

A single function of x_i and y_i can be defined that encodes the full content of the collection of correlation functions in (2.101),

$$\mathcal{F}(z_i,\bar{z}_i;y_i)=\langle\, \mathcal{O}_1(z_1,\bar{z}_1;y_1)\,\mathcal{O}_2(z_2,\bar{z}_2;y_2)\,\mathcal{O}_3(z_3,\bar{z}_3;y_3)\,\mathcal{O}_4(z_4,\bar{z}_4;y_4)\,\rangle\ . \tag{2.103}$$

Charge conservation ensures that this function is homogeneous in the auxiliary y_i with weight $\frac{1}{2}\sum k_i$, and the correlation function for a given choice of external R-symmetry indices can be read off by selecting the coefficient of the appropriate monomial in the y_i variables.

This repackaging makes it simple to state the relationship with correlation functions of $\chi[\mathcal{T}]$. The twisted chiral operators defined in §2.2.2 are the specialization of the repackaged operators in (2.102) to $y_i=\bar{z}_i$. So if the related four-point function of meromorphic operators

$\mathcal{O}_i(z) = \chi[\mathcal{O}_i(z,\bar{z})]$ is defined as

$$f(z_1, z_2, z_3, z_4) = \langle \mathcal{O}_1(z_1)\mathcal{O}_2(z_2)\mathcal{O}_3(z_3)\mathcal{O}_4(z_4)\rangle \ , \tag{2.104}$$

then the correlation functions are related according to

$$f(z_i) = \mathcal{F}(z_i, \bar{z}_i; y_i)\big|_{y_i \to \bar{z}_i} \ . \tag{2.105}$$

The fact that the left-hand side of this equation is a meromorphic function of the operator insertion points is a consequence of the cohomological arguments of the previous sections, but it is also *precisely* the final form of the superconformal Ward identities for such a correlation function [27, 39–43].

This is a rather wonderful result: the entirety of the constraints imposed by superconformal Ward identities on the four-point function of half-BPS operators are captured by the existence of the twist of §2.2.2. It is worth noting that while the Ward identities of [42] were derived specifically for half-BPS operators in $\hat{\mathcal{B}}_R$ multiplets, here we see that the same type of Ward identities holds more generally for any Schur-type operators.

2.4.2 Four-dimensional unitarity and central charge bounds

The natural inner product on the Hilbert space of the radially quantized four-dimensional theory $\mathcal{T}$ does not survive the passage to $\mathbb{Q}$ cohomology. This is an immediate consequence of the fact that $\mathbb{Q}$ is not hermitian. Hence, unitarity in four dimensions does not imply unitarity in the chiral algebra. In fact, we have seen that a unitary theory $\mathcal{T}$ always gives rise to a chiral algebra $\chi[\mathcal{T}]$ with negative central charge, which is necessarily non-unitary. Nevertheless, there is an interesting interplay between the structure of the chiral algebra and four-dimensional unitarity. This leads to new unitarity bounds for the anomaly coefficients of any four-dimensional SCFT. In this section, we explore an elementary example that provides us with such bounds. It is possible that more extensive analysis could lead to further constraints; we leave such an analysis for future study.

The origin of nontrivial consistency conditions can be found in the fact that, as summarized in (2.105), the meromorphic correlator $f(z_i)$ can be computed in two different ways that must agree. The first computation is the two-dimensional one: once the singular OPEs of the meromorphic operators appearing in the correlator are known, the full correlation function is completely fixed by meromorphy. The meromorphic correlator further admits a unique decomposition into $\mathfrak{sl}(2)$ conformal blocks,[21] leading to an expression of the form

$$\begin{aligned} f(z_i) &= \left(\frac{z_{24}}{z_{14}}\right)^{h_{12}}\left(\frac{z_{14}}{z_{13}}\right)^{h_{34}}\frac{1}{z_{12}^{h_1+h_2}z_{34}^{h_3+h_4}}\sum_{\ell=0}^{\infty}(-1)^{\ell}\,a_\ell\,g_\ell(z)\ , \\ g_\ell(z) &:= (-\tfrac{1}{2}z)^{\ell-1}z\ {}_2F_1(\ell,\ell;2\ell;z)\ , \end{aligned} \tag{2.106}$$

where we have adopted the standard notation $z_{ij} := z_i - z_j$ and $z := \frac{z_{12}z_{34}}{z_{13}z_{24}}$. Additionally, h_i is the holomorphic scaling dimension of the i'th operator, and we have defined $h_{ij} = h_i - h_j$.

The second computation is the four-dimensional one. The correlator in (2.101) admits a decomposition into $\mathfrak{su}(2,2|2)$ superconformal blocks that each represent the contribution of a given superconformal multiplet to the four-point function. The contribution of each superconformal block to the meromorphic part of the amplitude defined by (2.105) is fixed up to the three-point coefficients. Thus for a given theory $\mathcal{T}$, the spectrum and three-point coefficients of BPS operators appearing in the conformal block expansion of a given correlation function can be determined directly from the correlation functions of $\chi[\mathcal{T}]$. Non-trivial constraints arise when we require that the three-point coefficients determined in this manner be consistent with unitarity.

Let us now turn to a specific example to study in detail. We consider the four-point function of superconformal primary operators in $\hat{\mathcal{B}}_1$ multiplets. As was explained in §2.3, these multiplets contain the spin one conserved currents that generate the global (non-R) symmetry of the theory, and the superconformal primaries are scalar moment map operators M^A. Consequently the results derived from this example will be relevant to any theory with non-trivial flavor symmetry. The moment map operators have dimension two and transform in the adjoint representations of both the flavor group G_F and $SU(2)_R$. The four-point function of such operators can be expanded in channels corresponding to each irreducible representation $\mathcal{R}$ of G_F in which the exchanged operators in the conformal block expansion may transform,

$$\langle M^A(z_1,\bar{z}_1;y_1)M^B(z_2,\bar{z}_2;y_2)M^C(z_3,\bar{z}_3;y_3)M^D(z_4,\bar{z}_4;y_4)\rangle = \sum_{\mathcal{R}\in\otimes^2\mathbf{adj}} P_{\mathcal{R}}^{ABCD}\mathcal{F}_{\mathcal{R}}(z_i,\bar{z}_i;y_i)\ , \tag{2.107}$$

where $P_{\mathcal{R}}^{ABCD}$ is the projector onto the irreducible representation denoted by $\mathcal{R}$. The projectors for the various groups can be obtained following the procedures described in [63].

Per the discussion of §2.3.2, the chiral operators $J^A = \chi[M^A]$ are affine currents, and the mermorphic correlators that emerge in the limit $y_i \to \bar{z}_i$ are equal to the four-point functions

[21]The result could also be expanded in Virasoro conformal blocks, but this is less natural for comparison to four-dimensional quantities.

in the corresponding chiral algebra,

$$z_{12}^2 z_{34}^2 \langle J^A(z_1) J^B(z_2) J^C(z_3) J^D(z_4)\rangle = f^{ABCD}(z) = \sum_{\mathcal{R}} P_{\mathcal{R}}^{ABCD} f_{\mathcal{R}}(z)\,. \tag{2.108}$$

Each such function can be examined independently as a potential source of nontrivial consistency conditions. In §2.3 we found that the level of the affine Lie algebra symmetry generated by these currents is $k_{2d} = -\frac{1}{2}k_{4d}$, so this meromorphic four-point function is completely fixed in terms of the structure constants of the associated non-affine Lie algebra and the flavor central charge,[22]

$$f^{ABCD}(z) = \delta^{AB}\delta^{CD} + z^2\delta^{AC}\delta^{BD} + \frac{z^2}{(1-z)^2}\delta^{AD}\delta^{CB} - \frac{z}{k_{2d}} f^{ACE}f^{BDE} - \frac{z}{k_{2d}(z-1)} f^{ADE}f^{BCE}\,. \tag{2.109}$$

This correlator can be decomposed into G_F channels, each of which can be expanded in $\mathfrak{sl}(2)$ conformal blocks as in (2.106). For example, for the singlet channel $\mathcal{R} = \mathbf{1}$, the above correlator gives

$$\begin{aligned} f_{\mathcal{R}=\mathbf{1}} &= \dim G_F + z^2\left(1 + \frac{1}{(1-z)^2}\right) + \frac{4z^2 h^\vee}{k_{2d}(z-1)} \\ &= \dim G_F - \sum_{\ell=0,2,\cdots} \frac{2^\ell(\ell+1)(\ell!)^2\,(2(\ell+1)(\ell+2)k_{2d} - 8\,h^\vee)}{k_{2d}(2\ell+1)!} g_{\ell+2}(z)\ , \end{aligned} \tag{2.110}$$

where $h^\vee$ is the dual Coxeter number.

This operator product expansion can be compared with that of the full four-point function in four dimensions. The superconformal block decomposition of such a four-point function has been worked out in [41]. In particular, operators that can potentially appear in the intermediate channel must belong to one of the following superconformal multiplets:

- $\mathcal{A}_{\Delta(j,j)}$: Long multiplets that are $SU(2)_R$ singlets with $j_1 = j_2 = j$.
- $\hat{\mathcal{C}}_{0(j,j)}$: Semishort multiplets with $j_1 = j_2 = j$ that contain conserved currents of spin $2j+2$.
- $\hat{\mathcal{C}}_{1(j,j)}$: Semishort multiplets with $j_1 = j_2 = j$.
- $\hat{\mathcal{B}}_1$: Half-BPS multiplets containing Higgs branch moment map operators.
- $\hat{\mathcal{B}}_2$: Half-BPS multiplets containing Higgs branch chiral ring operators of dimension four.

[22]Here we have rescaled the currents in such a way that the identity operator appears with unit normalization in the current-current OPE.

- $\mathcal{I}$: The identity operator.

The contribution of each such multiplet to the full four-point function is fixed up to a single coefficient corresponding to the three-point coupling (squared), and unitarity requires that this coefficient be real and positive. The contribution of each multiplet to the meromorphic functions $f_{\mathcal{R}}(z)$ appearing in the superconformal Ward identities has also been determined in [41]. The results are summarized as follows:

$$\begin{array}{lcr}
\mathcal{A}_{\Delta(\frac{\ell}{2},\frac{\ell}{2})} & : & 0 \ , \\
\hat{\mathcal{C}}_{0(\frac{\ell}{2},\frac{\ell}{2})} & : & \lambda^2_{\hat{\mathcal{C}}_{0(\frac{\ell}{2},\frac{\ell}{2})}} g_{\ell+2}(z) \ , \\
\hat{\mathcal{C}}_{1(\frac{\ell}{2},\frac{\ell}{2})} & : & -2\lambda^2_{\hat{\mathcal{C}}_{1(\frac{\ell}{2},\frac{\ell}{2})}} g_{\ell+3}(z) \ , \\
\hat{\mathcal{B}}_1 & : & \lambda^2_{\hat{\mathcal{B}}_1} g_1(z) \ , \\
\hat{\mathcal{B}}_2 & : & -2\lambda^2_{\hat{\mathcal{B}}_2} g_2(z) \ , \\
\mathrm{Id} & : & \lambda^2_{\mathrm{Id}} \ .
\end{array} \tag{2.111}$$

The coefficient $\lambda^2_{\bullet}$ of each contribution is required by unitarity to be non-negative.

Some of the coefficients appearing in (2.111) can be completely fixed by symmetry. For example, the identity operator can only appear in the singlet channel $f_{\mathcal{R}=\mathbf{1}}(z)$, where the corresponding coefficient is necessarily given by

$$\lambda^2_{\mathrm{Id}} = \dim G_F \ . \tag{2.112}$$

The multiplet $\hat{\mathcal{C}}_{0(0,0)}$ contains a spin two conserved current, *i.e.*, the stress tensor. There can only be one such multiplet, and it contributes to the meromorphic part of the four point function only in the singlet channel. The three-point coupling is fixed in terms of the four-dimensional central charge. In particular, one finds that in $f_{\mathcal{R}=\mathbf{1}}(z)$,

$$\lambda^2_{\hat{\mathcal{C}}_{0(0,0)}} = \frac{\dim G_F}{3c_{4d}} \ . \tag{2.113}$$

Finally, multiplets of type $\hat{\mathcal{B}}_1$ can contributes only to the adjoint channel, and the corresponding three-point coupling in $f_{\mathrm{adj}}(z)$ is fixed to be

$$\lambda^2_{\hat{\mathcal{B}}_1} = \frac{4h^\vee}{k_{4d}} \, . \tag{2.114}$$

As far as we know, these are the only contributions to this four-point function that are fixed by symmetry in terms of anomaly coefficients. Additionally, the multiplets $\hat{\mathcal{C}}_{0(\frac{\ell}{2},\frac{\ell}{2})}$ for $\ell \neq 0$ necessarily contain conserved currents of spin greater than two, and so are expected to be absent in interacting theories [53]. We will take this to be the case in the following analysis.

G_F	$h^\vee$	$\dim G_F$	G_F	$h^\vee$	$\dim G_F$
$\mathrm{SU}(N)$	N	N^2-1	E_6	12	78
$\mathrm{SO}(N)$	$N-2$	$\frac{N(N-1)}{2}$	E_7	18	133
$\mathrm{USp}(2N)$	$N+1$	$N(2N+1)$	E_8	30	248
G_2	4	14	F_4	9	52

Table 2.2: Dual Coxeter number and dimensions for simple Lie groups.

We can determine the three-point coefficients in, say, the $\mathcal{R} = \mathbf{1}$ channel by comparing with the expansion of the $\chi[\mathcal{T}]$ four-point function in (2.110). In particular, we find

$$\begin{aligned} \lambda^2_{\mathrm{Id}} &= \dim G_F \ , \\ \lambda^2_{\hat{\mathcal{C}}_{0(0,0)}} - 2\lambda^2_{\hat{\mathcal{B}}_2} &= \frac{8h^\vee}{k_{4d}} - 4 \ , \\ \lambda^2_{\hat{\mathcal{C}}_{1(\frac{\ell}{2},\frac{\ell}{2})}} &= \frac{2^{\ell+1}(\ell+2)((\ell+1)!)^2}{k_{4d}(2\ell+3)!}\left((\ell+2)(\ell+3)k_{4d} - 4h^\vee\right) \ , \end{aligned} \tag{2.115}$$

where in the last line only odd ℓ may appear. The second line of (2.115), after substituting the contribution of the stress tensor multiplet from (2.113), implies a nontrivial bound that must be satisfied in order for the contribution of the $\hat{\mathcal{B}}_2$ multiplet to be consistent with unitarity,

$$\frac{\dim G_F}{c_{4d}} \geqslant \frac{24h^\vee}{k_{4d}} - 12 \ . \tag{2.116}$$

For reference, the dimensions and dual Coxeter numbers of the semi-simple Lie algebras are displayed in Table 2.2. Similarly, the positivity of the last line in (2.115) for $\ell = 1$ implies the bound

$$k_{4d} \geqslant \frac{h^\vee}{3} \ . \tag{2.117}$$

The same analysis can be performed for the functions $f_{\mathcal{R}\neq\mathbf{1}}(z_i)$. In these channels there will be no contribution from the stress tensor multiplet, so the resulting bounds make reference only to the anomaly coefficient k_{4d}, as in (2.117). *A priori*, an independent bound may be obtained for each representation $\mathcal{R}$ appearing in the tensor product of two copies of the adjoint. For example, in the adjoint channel itself, there can be contributions from $\hat{\mathcal{B}}_1$ and $\hat{\mathcal{C}}_{1(\frac{\ell}{2},\frac{\ell}{2})}$ multiplets with even ℓ. Unitarity then imposes a bound on k_{4d} that turns out to be equivalent to that of (2.117). Stronger bounds can be found by considering other choices of $\mathcal{R}$, the possible values of which will depend on the particular choice of simple Lie algebra we

G_F		Bound	Representation
SU(N)	$N \geqslant 3$	$k_{4d} \geqslant N$	$\mathbf{N^2-1}_{\text{symm}}$
SO(N)	$N=4,\ldots,8$	$k_{4d} \geqslant 4$	$\mathbf{\frac{1}{24}N(N-1)(N-2)(N-3)}$
SO(N)	$N \geqslant 8$	$k_{4d} \geqslant N-4$	$\mathbf{\frac{1}{2}(N+2)(N-1)}$
USp($2N$)	$N \geqslant 3$	$k_{4d} \geqslant N+2$	$\mathbf{\frac{1}{2}(2N+1)(2N-2)}$
G_2		$k_{4d} \geqslant \frac{10}{3}$	$\mathbf{27}$
F_4		$k_{4d} \geqslant 5$	$\mathbf{324}$
E_6		$k_{4d} \geqslant 6$	$\mathbf{650}$
E_7		$k_{4d} \geqslant 8$	$\mathbf{1539}$
E_8		$k_{4d} \geqslant 12$	$\mathbf{3875}$

Table 2.3: Unitarity bounds for the anomaly coefficient k_{4d} arising from positivity of the $\hat{\mathcal{B}}_2$ three-point function in non-singlet channels.

consider. In general, we find that for a given choice of G_F, the strongest bound comes from requiring positivity of the contributions of $\hat{\mathcal{B}}_2$ multiplets in a single channel. The bounds from other channels are then automatically satisfied when the strongest bound is imposed. These strongest bounds are displayed in Table 2.3, where we also indicate the representation $\mathcal{R} \in \otimes^2\mathbf{adj}$ that leads to the bound in question. It should be noted that for the special case $G_F = SO(8)$, the same strongest bound is obtained from multiple channels. The representation appearing in the third line of Table 2.3 is in fact decomposable as $\mathbf{70} = \mathbf{35}_s \oplus \mathbf{35}_c$, and the degeneracy in the bounds can be understood as a consequence of $SO(8)$ triality. For $G_F = SU(2)$ one finds no additional bounds to the ones given in (2.116) and in (2.117). Finally, we can see that the bound (2.117) arising from positivity of the $\hat{\mathcal{C}}_{1(\frac{1}{2},\frac{1}{2})}$ multiplet in the singlet channel is made obsolete by bounds arising from other channels for all choices of G_F listed in the table.

2.4.3 Saturation of unitarity bounds

Given the existence of these unitarity bounds, it is incumbent upon us to consider the question of whether the bounds are saturated in any known superconformal models. To understand what sort of theory might saturate the bounds, it helps to identify any physical properties that a theory will necessarily possess if it saturates a bound. When the inequalities in (2.116) or Table 2.3 are saturated, it means precisely that there is no $\hat{\mathcal{B}}_2$ multiplet in the

G_F	A_1	A_2	D_4	E_6	E_7	E_8	F_4	G_2
$h^\vee$	2	3	6	12	18	30	9	4
k_{4d}	$\frac{8}{3}$	3	4	6	8	12	5	$\frac{10}{3}$
c_{4d}	$\frac{1}{2}$	$\frac{2}{3}$	$\frac{7}{6}$	$\frac{13}{6}$	$\frac{19}{6}$	$\frac{31}{6}$	$\frac{5}{3}$	$\frac{5}{6}$

Table 2.4: Central charges for $\mathcal{N}=2$ SCFTs with Higgs branches given by one-instanton moduli spaces for G_F instantons. Models corresponding to the right-most two columns are not known to exist, but must satisfy these conditions for their central charges if they do.

corresponding representation of G_F contributing to the four-point function in question. The absence of such an operator is intimately connected with a well-known feature of theories with $\mathcal{N}=2$ supersymmetry in four dimensions. Recalling that the Schur operators in the $\hat{\mathcal{B}}_R$ multiplets are Higgs branch chiral ring operators, the absence of a $\hat{\mathcal{B}}_2$ multiplet contributing to the four-point function of $\hat{\mathcal{B}}_1$ multiplets in the $\mathcal{R}$ channel amounts to a relation in the Higgs branch chiral ring of the form

$$(M\otimes M)\big|_{\mathcal{R}}=0\ , \tag{2.118}$$

where M is the moment map operator and the tensor product is taken in the chiral ring.

There exists an interesting set of theories for which precisely such relations are known to hold. These are the superconformal field theories that arise on a single $D3$ brane probing a codimension one singularity in F-theory on which the dilaton is constant [12, 64–68]. There are seven such singularities, labelled $H_0, H_1, H_2, D_4, E_6, E_7, E_8$, for which the corresponding SCFT has global symmetry given by the corresponding group (with $H_i\to A_i$). The Higgs branch of each such theory is isomorphic to the minimal nilpotent orbit of the flavor group G_F. These minimal nilpotent orbits admit a simple description: they are generated by a complex, adjoint-valued moment map M, subject to a set of relations that defined the so-called "Joseph ideal" (see [69] for a nice discussion),

$$(M\otimes M)\big|_{\mathcal{I}_2}=0\ ,\qquad \mathrm{Sym}^2(\mathbf{adj})=(2\,\mathbf{adj})\oplus\mathcal{I}_2\ , \tag{2.119}$$

where $(2\,\mathbf{adj})$ is the representation with Dynkin indices twice those of the adjoint representation.

This leads to an interesting set of conclusions. For one, these theories must saturate some of the $\hat{\mathcal{B}}_2$-type bounds listed above. In particular, this allows us to predict the value of c_{4d} and k_{4d} for these theories as a direct consequence of the Higgs branch relations. These predictions are listed in Table 2.4. Indeed, these anomaly coefficients have been computed by other means and the results agree [70]. On the other hand, an $\mathcal{N}=2$ superconformal theory with G_F symmetry can have as its Higgs branch the one-instanton moduli space of G_F instantons *only* if the $\hat{\mathcal{B}}_2$ bound for all representations in $\mathcal{I}_2$ can be simultaneously saturated. It is not hard to verify that the list of cases for which this can be true includes the cases described above in F-theory, along with $G_F = F_4$ and $G_F = G_2$. Theories with Higgs branches isomorphic to the one-instanton F_4 and G_2 moduli spaces appear to be absent from the literature, and it is tempting to speculate that such theories should nonetheless exist and have as their central charges the values listed in the right-most two columns of Table 2.4.

Finally, it is interesting to rephrase the above discussion purely in the language of the chiral algebra $\chi[\mathcal{T}]$. From this perspective, there is a marked difference between the bound (2.116) for the singlet sector and those of Table 2.3 for non-singlets. In a theory saturating the non-singlet bounds, the coefficient of a conformal block is actually set to zero in the OPE of 2.106. This should be considered in contrast to a theory that saturates the singlet bound, in which case all of the $\mathfrak{sl}(2)$ conformal blocks are present with nonzero coefficients. It follows that saturation of a non-singlet bound is equivalent to the presence of a null state in the chiral algebra. In particular, because the bounds in question appear in the $\hat{\mathcal{B}}_1$ four-point function, such null states can be understood entirely in terms of the affine Lie subalgebra of the chiral algebra. This interpretation can be verified directly by studying an affine Lie algebra with the level listed in Table 2.3.

The bound (2.116), on the other hand, does *not* imply the presence of a null state in the chiral algebra. Instead, a theory $\chi[\mathcal{T}]$ that saturates the singlet bound should have the property that the only $\mathfrak{sl}(2)$ primary of dimension two that appears in the OPE of two affine currents is identically equal to the chiral vertex operator that arises from the $\hat{\mathcal{C}}_{0(0,0)}$ multiplet in four dimensions, *i.e.*, it should be the two-dimensional stress tensor. We thus identify saturation of the singlet bound with the property that the Sugawara construction gives the true stress tensor of the chiral algebra,

$$T_{2d} = \frac{1}{k_{2d} + h^\vee}(J^a J^a) \ . \tag{2.120}$$

Sure enough, if the bound (2.116) is saturated, then we can rewrite the bound as an equation for the central charge

$$c_{2d} = \frac{k_{2d} \dim G_F}{k_{2d} + h^\vee} \ . \tag{2.121}$$

This is precisely the central charge associated with the Sugawara construction for the stress tensor of an affine Lie algebra.

Finally, we mention a number of additional theories that saturate some of the unitarity bounds derived here. In particular, though the rank one theory corresponding to the H_0 singularity has no flavor symmetry, it will have an extra $SU(2)$ symmetry for rank larger than one (as will all the other rank $\geqslant 1$ theories). In particular, for the case of rank two the flavor central charge corresponding to this extra $SU(2)$ is $\frac{17}{5}$ and the central charge is $c_{4d} = \frac{17}{12}$ [70]. This theory therefore saturates the bound (2.116). Additionally, we have found a number of theories that saturate bounds appearing in Table 2.3. In particular, the new rank one SCFTs found in [71] with flavor symmetry $USp(10)_7$ and $USp(6)_5 \times SU(2)_8$, where k_{4d} is indicated as a subscript for each group, saturate the bounds on k_{4d} for the USp factors. However for these theories the central charge bound is not saturated. The following theories described in [72] also saturate bounds on k_{4d}: S_5 with flavor symmetry $SU(10)_{10}$ (but not the rest of the S_N series), the $R_{0,N}$ series with flavor symmetry $SU(2)_6 \times SU(2N)_{2N}$, and the $R_{2,N}$ series with $SO(2N+4)_{2N} \times U(1)$ flavor symmetry.

2.4.4 Torus partition function and the superconformal index

Just as correlators of the chiral algebra are related to certain supersymmetric correlators of the parent four-dimensional theory, it will not come as a surprise that the torus partition function of the chiral algebra is related to a certain four-dimensional supersymmetric index – indeed, to the Schur limit of the superconformal index, as foreshadowed in our terminology.

We should first identify which quantum numbers can be meaningfully assigned to chiral algebra operators. Of the various Cartan generators of the four-dimensional superconformal algebra, only the holomorphic dimension L_0 and the transverse spin $M^{\perp} = j_1 - j_2$ (which is equal to $-r$ for Schur operators) survive as independent conserved charges of the chiral algebra. The torus partition function therefore takes the form[23]

$$Z(x,q) := \operatorname{Tr} x^{M^{\perp}} q^{L_0} \ . \tag{2.122}$$

As usual, the trace is over the Hilbert space in radial quantization, or equivalently over the local operators of the chiral algebra.

Specializing to $x = -1$, and noting that by the four-dimensional spin-statistics connection implies $(-1)^{j_1 - j_2} = (-1)^F$, where F is the fermion number, we find a weighted Witten index,

$$\mathcal{I}(q) := Z(-1,q) = \operatorname{Tr}\,(-1)^F\, q^{L_0} = \operatorname{Tr}\,(-1)^F q^{E-R} \ . \tag{2.123}$$

We recognize this as the trace formula that defines the Schur limit of the superconformal index [13], *cf.* Appendix B.[24] We should check that in the two-dimensional and four-dimensional interpretations of this formula the trace can be taken over the same space of states. Strictly speaking, in the four-dimensional interpretation the trace is over the entire Hilbert space of the radially quantized theory. However, the point of the Schur index is that only states obeying the Schur condition can conceivably contribute – the contributions of all other states cancel pairwise. As the states of the chiral algebra are in one-to-one correspondence with Schur states, the chiral algebra index (2.123) is indeed equivalent to the Schur index.

[23] To avoid clutter, we have omitted the obvious refinement by flavor fugacities. If the theory is invariant under some global symmetry group G_F, we may refine the trace formula by $\prod_i a_i^{f_i}$, where the f_i are Cartan generators of G_F and a_i the associated fugacities.

[24] It was observed in [73] that the Schur index has interesting modular properties under the action of $SL(2,\mathbb{Z})$ on the superconformal and flavor fugacities. The identification of the Schur index with a two-dimensional index may serve to shed some light on these observations.

The index is a cruder observable than the partition function, but because it is invariant under exactly marginal deformations, it is generally easier to evaluate. In practice, to evaluate the index of a Lagrangian SCFT, one enumerates all gauge-invariant states that can be formed by combining the elementary "letters" that obey the Schur condition, see Table 2.1. This combinatorial exercise is efficiently solved with the help of a matrix integral, where the integration over the gauge group enforces the projection onto gauge singlets. Examples of this prescription will be seen in the following section. By this procedure, one enumerates all gauge-invariant states that obey the tree-level Schur condition; there will be cancellations in the index corresponding to the recombinations of Schur multiplets into long multiplets that are *a priori* allowed by representation theory.

There is an entirely isomorphic computation in the associated chiral algebra. The "letters" obeying the tree-level Schur condition are nothing but the states of the symplectic bosons and the ghost small algebra (in the appropriate representations), and one is again instructed to project onto gauge singlets. To reiterate, to evaluate the index we do not really need to compute the cohomology of Q^-, which defines the states of the chiral algebra of the interacting gauge theory, *cf.* (2.93). We can simply let the trace run over the redundant set of states of the free theory. By contrast, the trace in the partition function (2.122) must be taken over only the states of the chiral algebra for the interacting theory, which are the cohomology classes of Q^-.

At the risk of being overly formal, we may point out that the physical state space of the chiral algebra (which for gauge theories is defined by the cohomological problem (2.93)), acts as a *categorification* of the Schur index. Once this vector space and the action of the charges are known, we can perform the more refined counting (2.122). In physical terms, the categorification contains extra information relative to the Schur index in that it knows about sets of short multiplets that are kinematically allowed to recombine but do not. In addition, there may be multiplets that cannot recombine but nonetheless make accidentally cancelling contributions to the index, and these are also seen in the categorification. Of course, the chiral algebra structure goes well beyond categorification – it is a rich algebraic system that also encodes the OPE coefficients of the Schur operators, and is subject to non-trivial associativity constraints.

It should be noted that as a graded vector space, we also have a categorification of the Macdonald limit of the superconformal index. Recall that the states contributing to the Macdonald index are really the same as the states that contribute to the Schur index, but their counting is refined by an extra fugacity t/q associated to the charge $r+R$ (for $t=q$ we recover the Schur index). Since each state in the vector space defined by the chiral algebra corresponds to a Schur operator, the additional grading by $r+R$ is perfectly well-defined. However, there is no obvious chiral algebra interpretation of the Macdonald limit of the superconformal index, because the additional grading is incompatible with the chiral algebra structure. More precisely, while L_0 and r are conserved charges for the twisted-translated operators (2.29), $r+R$ is not, since away from the origin the operators are linear combinations of operators with different R eigenvalues. In particular $r + R$ is *not* preserved by the OPE.

2.5 Examples and conjectures

In this section we consider a number of illustrative examples in which the four-dimensional superconformal field theory $\mathcal{T}$ admits a weakly coupled Lagrangian description. In such cases, the chiral algebra $\chi[\mathcal{T}]$ can be defined via the BRST procedure of §2.3, which at the very least allows for a level-by-level analysis of the physical states/operators in the algebra.

We can also consider the problem of giving an economical description of the chiral algebra in terms of a set of generators and their singular OPEs. A natural question is whether this set is finite, or in other words whether the chiral algebra is a $\mathcal{W}$-algebra. The results of §2.3.2 suggest a very general ansatz for a possible $\mathcal{W}$-algebra structure: the generators should be the operators associated to HL chiral ring generators in four dimensions, and possibly in addition the stress tensor. In each of the first three examples, our results are compatible with this guess, and we formulate concrete conjectures for the precise definition of each chiral algebra as a $\mathcal{W}$-algebra. In the final example, we find a counterexample to this simplistic picture. Namely, we find a theory for which the chiral algebra contains at least one additional generator beyond those included in our basic ansatz.

For the first example, we turn to perhaps the most familiar $\mathcal{N} = 2$ superconformal gauge theory.

2.5.1 $SU(2)$ superconformal QCD

The theory of interest is the $SU(2)$ gauge theory with four fundamental hypermultiplets. Many aspects of this theory that are relevant to the structure of the associated chiral algebra have been analyzed in, *e.g.*, [74]. The field content is an $SU(2)$ vector multiplet and four fundamental hypermultiplets. Because the fundamental representation of $SU(2)$ is pseudoreal, the obvious $U(4)$ global symmetry is enhanced to $SO(8)$, with the four fundamental hypermultiplets being reinterpreted as eight half-hypermultiplets. In $\mathcal{N} = 1$ notation we then have an adjoint-valued $\mathcal{N} = 1$ field strength superfield W^A_α, an adjoint-valued chiral multiplet Φ^B, and fundamental chiral multiplets Q^i_a transforming in the $\mathbf{8}_v$ of $SO(8)$. Here $a, b = 1, 2$ are vector color indices that can be raised and lowered with epsilon tensors, $A, B = 1, 2, 3$ are adjoint color indices, and $i = 1, \ldots, 8$ are $SO(8)$ vector indices. By a common abuse of notation, we use the same symbol for the scalar squarks in the matter chiral multiplets as for the superfields, whereas the gauginos in the vector multiplet are denoted λ^A_α and $\tilde{\lambda}_{A\dot{\alpha}}$. In terms of the $\mathcal{N} = 1$ superfields listed above, the Lagrangian density takes the form

$$\mathcal{L} = \mathrm{Im}\left[\tau \int d^2\,\theta d^2\bar{\theta}\ \mathrm{Tr}\left(\Phi^\dagger e^V \Phi + Q_i^\dagger e^V Q^i\right) + \tau \int d^2\theta\ \left(\tfrac{1}{2}\mathrm{Tr}\, W_\alpha W^\alpha + \sqrt{2} Q_a^i \Phi_b^a Q^{ib}\right)\right]\ , \tag{2.124}$$

Where $\tau = \theta/2\pi + 4\pi i/g_{\rm YM}^2$ is the complexified gauge coupling. The central charge of the $SU(2)$ color symmetry acting on the hypermultiplets is $k_{4d}^{SU(2)} = 8$, which satisfies condition (2.84) for τ to be an exactly marginal coupling. The central charge for the $SO(8)$ flavor symmetry and the conformal anomaly c_{4d} can also be read off directly from the field content,

$$k_{4d}^{SO(8)} = 4\ , \quad c_{4d} = \frac{7}{6}\ . \tag{2.125}$$

Although this description is sufficient to set up a BRST cohomology problem that defines the chiral algebra in the manner of §2.3, it is useful to first review some of the features of this theory that we expect to see reflected in the two-dimensional analysis. We have seen that a special role is played in the chiral algebra by the HL chiral ring, the elements of which are the superconformal primary operators in $\hat{\mathcal{B}}$ and $\mathcal{D}$-type multiplets. In this example, these are the lowest components of $\mathcal{N} = 1$ chiral superfields that are gauge-invariant polynomials in Q_a^i and W_α^A. As this theory is represented by an acyclic quiver diagram, all $\mathcal{D}$-type multiplets recombine and the HL chiral ring is identically the Higgs chiral ring.

In purely gauge invariant terms, the Higgs branch chiral ring is generated by a single dimension two operator in the adjoint of $SO(8)$,

$$M^{[ij]} = Q_a^i Q^{aj}\ . \tag{2.126}$$

This is the moment map for the action of $SO(8)$ on the Higgs branch.[25] There are additional relations that make the structure of the Higgs branch more interesting. Already at tree-level, there are relations that follow automatically from the underlying description in terms of squarks. When organized in representations of $SO(8)$, the of generators of these relations are as follows,

$$M \otimes M\big|_{\mathbf{35}_s} = 0\ , \qquad M \otimes M\big|_{\mathbf{35}_c} = 0\ . \tag{2.127}$$

On the other hand, there are F-term relations as a consequence of the superpotential in (2.124). They are absent in the theory with strictly zero gauge coupling, and encode the fact that certain operators that are present in the chiral ring of the free theory recombine and are lifted from the protected part of the spectrum when the coupling is turned on. The generators of F-term relations, again organized according to $SO(8)$ representation, are as

[25] It is a special feature of this theory (in contrast to, say, the $N_f = 2N_c$ theories with $N_c > 2$ that will be considered next) that the generators of the Higgs branch chiral ring all have dimension two. In general, there will be higher-dimensional baryonic generators that are not directly related to the global symmetry currents of the theory.

follows,

$$M \otimes M\big|_{\mathbf{35}_v} = 0\ , \qquad M \otimes M\big|_{\mathbf{1}} = 0\ . \tag{2.128}$$

One immediately recognizes the complete set of relations in (2.127) and (2.128) as defining the $SO(8)$ Joseph ideal described in §2.4. Indeed, for the particular case of $G_F = SO(8)$ we have $\mathcal{I}_2 = \mathbf{1} \oplus \mathbf{35}_v \oplus \mathbf{35}_s \oplus \mathbf{35}_c$. The Higgs branch of this theory is known to be isomorphic to the $SO(8)$ one-instanton moduli space, and the central charges (2.125) do in fact saturate the appropriate unitarity bounds outlined in §2.4.

As a final comment, let us recall that the gauge coupling appearing in the Lagrangian (2.124) is exactly marginal and parameterizes a one-complex-dimensional conformal manifold. S-duality acts by $SL(2,\mathbb{Z})$ transformations on τ, and the conformal manifold is identified with the familiar fundamental domain of $SL(2,\mathbb{Z})$ in the upper half plane. In the various weak-coupling limits the theory can always be described using the same $SU(2)$ gauge theory, but in comparing one such limit to another, the duality transformations act by triality on the $SO(8)$ flavor symmetry. Consequently, though a given Lagrangian description of this theory (and of the chiral algebra in the next subsection) singles out a certain triality frame, the protected spectrum of the theory, and so in particular the chiral algebra, should be triality invariant.

BRST construction of the associated chiral algebra

The chiral algebra can now be constructed using the procedure of §2.3. We first define the chiral algebra $\chi[\,\mathcal{T}_{\text{free}}\,]$ of the free theory. Each half-hypermultiplet gives rise to a pair of commuting, dimension 1/2 currents, whose OPE is that of symplectic bosons

$$q_a^i(z) := \chi[\,Q_a^i\,]\ , \qquad q_a^i(z)\, q_b^j(w) \sim \frac{\delta^{ij}\epsilon_{ab}}{z-w}\ . \tag{2.129}$$

Meanwhile, the vector multiplet contributes a set of adjoint-valued (b,c) ghosts of dimension $(1,0)$ with the standard OPE,

$$b^A(z) := \chi[\,\tilde{\lambda}^A]\ , \qquad \partial c^B(z) := \chi[\,\lambda^B]\ , \qquad b^A(z)c^B(w) \sim \frac{\delta^{AB}}{z-w}\ . \tag{2.130}$$

The generators of the $SU(2)$ gauge symmetry in the matter sector arise from the moment maps in the free theory, while in the ghost system they take the canonical form described in §2.3,

$$J^A(T^A)_a^{\ b} = q_a^i q^{ib}\ , \qquad J_{\text{gh}}^A = -if^{ABC}(c^B\, b^C)\ . \tag{2.131}$$

The chiral algebra of the free theory is then given by the gauge-invariant part of the tensor product of the symplectic boson and small algebra Fock spaces,

$$\chi[\mathcal{T}_{\text{free}}] = \{\psi \in \mathcal{F}(q_a^i, \rho_+^A, \rho_-^A)\,|\, J_{\text{tot},0}^A\psi = 0\}\ . \tag{2.132}$$

The current algebra generated by the $J^A_{\rm mat}$ has level $k^{SU(2)}_{2d} = -4 = -2h^\vee$, which ensures the existence of a nilpotent BRST differential. The BRST current and differential are then constructed in terms of these currents,

$$J_{\rm BRST} = c^A \left(J^A + \frac{1}{2} J^A_{\rm gh} \right) \ , \qquad Q_{\rm BRST} = \oint \frac{dz}{2\pi i} J_{\rm BRST}(z) \ . \tag{2.133}$$

The chiral algebra of the interacting theory is now the *BRST* cohomology

$$\chi[\,\mathcal{T}\,] = \mathcal{H}^*_{\rm BRST}\left[\chi[\mathcal{T}_{\rm free}]\right] \ . \tag{2.134}$$

We now perform a basic analysis of this cohomology. Already at this rudimentary level, we will find that a substantial amount of four-dimensional physics is packaged elegantly into the chiral algebra framework.

Enumerating physical states

It is a straightforward exercise to enumerate the physical operators up to any given dimension and to compute the singular terms in their OPEs. This is made easier with computer assistance – we have made extensive use of K. Thielemans' `Mathematica` package [55]. We now describe this enumeration in detail for operators of dimension one and two in the chiral algebra. In this example, the material we have reviewed above is already enough to predict the results of this enumeration. We will nevertheless find it instructive to explore in some detail how the inevitable spectrum comes about.

We begin at dimension one. Dimension one currents in the chiral algebra can only originate in $\mathcal{D}_{0(0,0)}$ and $\hat{\mathcal{B}}_1$ multiplets (*cf.* Table 2.1). The former contain free vector multiplets, and so are not gauge invariant. Thus the physical spectrum at dimension one should be isomorphic to the spectrum of $\hat{\mathcal{B}}_1$ multiplets. Sure enough, the complete list dimension-one operators in $\chi[\,\mathcal{T}_{\rm free}\,]$ is the following,

$$J^{[ij]} = q^i_a q^{ja} \ , \tag{2.135}$$

and these operators are the chiral counterparts of the $SO(8)$ moment maps, *i.e.*,

$$J^{[ij]} = \chi[M^{[ij]}] \ . \tag{2.136}$$

Direct computation further verifies that these operators exhaust the nontrivial BRST cohomology at dimension one. It is also straightforward to determine the singular terms in the OPEs of these currents,

$$J^{[ij]}(z) J^{[kl]}(0) \sim \frac{-2(\delta^{ik}\delta^{jl} - \delta^{il}\delta^{jk})}{z^2} + \frac{i f^{[ij][kl]}_{[mn]} J^{[mn]}(0)}{z} \ . \tag{2.137}$$

This is just an $\mathfrak{so}(8)$ affine Lie algebra at level $k_{2d} = -2$, which confirms the general prediction of §2.3 that flavor symmetries are affinized in the chiral algebra, subject to the relation $k_{2d} = -\frac{1}{2}k_{4d}$.

Moving on, the four-dimensional multiplets that can give rise to two-dimensional quasi-primary currents of dimension two are $\hat{\mathcal{C}}_{0(0,0)}$, $\hat{\mathcal{B}}_2$, $\mathcal{D}_{0(0,1)}$, and $\mathcal{D}_{\frac{1}{2}(0,\frac{1}{2})}$ multiplets (along with the conjugates of the last two). In addition, conformal descendants of dimension two can arise from holomorphic derivatives of the dimension one operators. Since no $\mathcal{D}$-type multiplets appear in this theory, the only quasi-primaries at dimension two will correspond to Higgs branch operators and the two-dimensional stress tensor.

The latter descends from the four-dimensional $SU(2)_R$ current. That current being bilinear in the free fields of the noninteracting theory, the corresponding two-dimensional operator can be obtained by simply replacing the four-dimensional fields with their chiral counterparts and conformally normal ordering,

$$T_{2d} = \tfrac{1}{2}q_a^i \partial q^{ia} - b^A \partial c_A \ . \tag{2.138}$$

Alternatively, this is just the canonical stress tensor for the combined system of free symplectic bosons and ghosts. Given the multiplicities of matter and ghost fields, the two-dimensional central charge is easily determined to be $c_{2d} = -14$.

The remaining BRST-invariant currents of dimension two can be constructed as normal ordered products and derivatives of the $\mathfrak{so}(8)$ affine currents,

$$\partial J^{[ij]} \ , \quad (J \otimes J)\big|_{\mathbf{1},\mathbf{35},\mathbf{35},\mathbf{35},\mathbf{300}} \ . \tag{2.139}$$

The singlet term in the tensor product above, once appropriately normalized, is the Sugawara stress tensor of the $\mathfrak{so}(8)$ affine Lie algebra,

$$T_{\text{sug}}^{\mathfrak{so}(8)} = \tfrac{1}{8}(J^{[ij]} J^{[ij]}) \ . \tag{2.140}$$

The Sugawara central charge is determined by the usual formula,

$$c_{\text{sug}} = \frac{k_{2d} \dim G_F}{k_{2d} + h^\vee} = -14 \ . \tag{2.141}$$

This matches the value for the canonical stress-tensor. This comes as no surprise, since the central charges of this theory saturate the unitarity bound (2.116), which implies that the canonical stress tensor should be equivalent to the Sugawara stress tensor. Indeed, (2.138) and (2.139) constitute an overcomplete list, and we in fact have the following relations,

$$
\begin{aligned}
J \otimes J\Big|_{\mathbf{1}} &= T_{2d} + \{Q_{\rm BRST}, q^i_a q^{ib} b^a_b\} \ , &\text{(2.142a)}\\
J \otimes J\Big|_{\mathbf{35}_v} &= \{Q_{\rm BRST}, q^{(i}_a q^{j)b} b^a_b\} \ , &\text{(2.142b)}\\
J \otimes J\Big|_{\mathbf{35}_c} &= 0 \ , &\text{(2.142c)}\\
J \otimes J\Big|_{\mathbf{35}_s} &= 0 \ , &\text{(2.142d)}
\end{aligned}
$$

The relations appearing here can be traced back to different aspects of the four-dimensional physics. Relations (2.142a) and (2.142b) are the two-dimensional avatars of the F-term relations in (2.128). Note that the first relation appears differently in this two-dimensional context due to the presence of the two-dimensional stress tensor on the right hand side. This is a remnant of the more complicated structure of normal ordering in the chiral algebra as compared to the chiral ring. Relations (2.142c) and (2.142d) are the tree-level relations. In the context of the chiral algebra, they can be seen as a simple consequence of Bose symmetry and normal ordering without making any reference to the BRST differential. This perfectly mirrors of the nature of tree-level relations in four dimensions.

A $\mathcal{W}$-algebra conjecture

Although the cohomological description of the chiral algebra is sufficient to compute the physical operators to any given level, it would be ideal to have a characterization entirely in terms of physical operators – for example, we may hope for a description as a $\mathcal{W}$ algebra. We have seen that the physical dimension two currents are all generated by the affine currents of dimension one, *i.e.*, the physical states enumerated so far all lie in the vacuum module of the $\mathfrak{so}(8)$ affine Lie algebra at level $k = -2$. What's more, these operators exhaust the list of operators that are guaranteed to be generators of the chiral algebra according to §2.3. We are thus led to a natural conjecture:

Conjecture 1. *When $\mathcal{T}$ is $\mathcal{N} = 2$ $SU(2)$ SQCD with four fundamental flavors, then $\chi[\mathcal{T}]$ is isomorphic to the $\mathfrak{so}(8)$ affine Lie algebra at level $k_{2d} = -2$.*

This is a mathematically well-posed conjecture, since the cohomological characterization of the chiral algebra is entirely concrete. It seems plausible that a more sophisticated approach to the cohomological problem could lead to a proof of the conjecture. We will be satisfied in the present work to test it indirectly.

level	$SO(8)$ representations and their multiplicities
0	$\mathbf{1}$
1	$\mathbf{28}$
2	$\mathbf{1},\ \mathbf{28},\ \mathbf{300}$
3	$\mathbf{1},\ 2\times\mathbf{28},\ \mathbf{300},\ \mathbf{350},\ \mathbf{1925}$
4	$2\times\mathbf{1},\ 3\times\mathbf{28},\ \mathbf{35}_v,\ \mathbf{35}_s,\ \mathbf{35}_c,\ 3\times\mathbf{300},\ \mathbf{350},\ \mathbf{1925},\ \mathbf{4096},\ \mathbf{8918}$
5	$2\times\mathbf{1},\ 6\times\mathbf{28},\ \mathbf{35}_v,\ \mathbf{35}_s,\ \mathbf{35}_c,\ 4\times\mathbf{300},\ 3\times\mathbf{350},\ \mathbf{567}_v,\ \mathbf{567}_s,\ \mathbf{567}_c,\ 3\times\mathbf{1925},$ $2\times\mathbf{4096},\ \mathbf{8918},\ \mathbf{25725},\ \mathbf{32928'}$

Table 2.5: The operator content of the chiral algebra up to level 5.

The superconformal index and affine characters

Conjecture 1 can be tested at the level of the indices of these theories. In particular, we have the following conjectural relationship

$$\mathcal{I}_{\mathrm{Schur}}(q;\vec{a}) = \mathrm{Tr}_{\chi[\mathcal{T}_{\mathrm{free}}]}(-1)^F q^{L_0}\prod_{i=1}^{4} a_i^{\mu_i} = \mathrm{Tr}_{\mathfrak{so}(8)_{-2}}(-1)^F q^{L_0}\prod_{i=1}^{4} a_i^{\mu_i}\ . \tag{2.143}$$

The shorthand $\vec{a} = (a_1, a_2, a_3, a_4)$ denotes the $SO(8)$ fugacities. Of course, the affine Lie algebra has only bosonic states, so the factor of $(-1)^F$ is immaterial. In particular this observation implies that if Conjecture 1 is correct, then all possible recombinations of tree-level Schur operators occur already at one loop.

On the one hand, the Schur limit of the superconformal index for this theory can be computed directly to fairly high orders in the q expansion by starting with the defining matrix integral,

$$\mathcal{I}_{\mathrm{Schur}}(q;\vec{a}) = \oint[db]\mathrm{P.E.}\left[\left(\frac{\sqrt{q}}{1-q}\right)\chi^{\mathbf{8}}_{SO(8)}(\vec{a})\chi^{\mathbf{2}}_{SU(2)}(b) + \left(\frac{-2q}{1-q}\right)\chi^{\mathbf{3}}_{SU(2)}(b)\right]\ , \tag{2.144}$$

and expanding the exponential. Here $\oint[db]$ denotes integration over the fugacity for the gauge group with the Haar measure.

On the other hand, the vacuum character of the $\mathfrak{so}(8)$ affine Lie algebra at level $k=-2$ can be computed once the spectrum of null primaries is known. Said spectrum can be determined with the aid of the Kazhdan-Lusztig polynomials, as we review in Appendix C.

Ultimately, both the character and the index are expanded in the form

$$1+\sum_{i=1}^{\infty} q^n \left(\sum_R d_{\mathcal{R}} \chi^{\mathcal{R}}(\vec{a}) \right) ,$$

where the $d_{\mathcal{R}}$ are positive integer multiplicities. At a given power of q, there are only a finite number of non-zero $d_{\mathcal{R}}$. Up to $O(q^5)$, the resulting degeneracies have been computed in both manners and agree. They are displayed in Table 2.5.

2.5.2 $SU(N)$ superconformal QCD with $N \geqslant 3$

We next consider the generalization of the previous example to the case of $SU(N)$ superconformal QCD with $N \geqslant 3$. In these theories, the Higgs branch has generators of dimension greater than two, thus guaranteeing the existence of nonlinear $\mathcal{W}$-symmetry generators in the chiral algebra. The cohomological construction of the corresponding chiral algebra is analogous to the $SU(2)$ case, *mutatis mutandi*. We will not repeat the description here in any detail. We first provide a brief outline of the relevant four-dimensional physics of these models, and then perform a systematic analysis of the physical operators of low dimension in the associated chiral algebra.

As in the $SU(2)$ theory, there is a Lagrangian description of these models in terms of the $\mathcal{N}=1$ chiral superfields

$$W_\alpha^A , \quad \Phi^B , \quad Q_a^i , \quad \widetilde{Q}_j^b , \tag{2.145}$$

where $a, b = 1, \ldots, N$ are vector color indices, $A, B = 1, \ldots, N^2-1$ are adjoint color indices, and $i, j = 1, \ldots, N_f$ with $N_f = 2N$ are vector flavor indices. The central charge is fixed by the field content to $c_{4d} = \frac{2N^2-1}{6}$.

For our purposes, the principal difference between the $N \geqslant 3$ theories and the $N=2$ case is in the structure of the Higgs branch chiral ring. In the higher rank theories, the hypermultiplets transform in a complex representation of the gauge group, so the global symmetry is not enhanced and we have $G_F = SU(N_f) \times U(1)$. The moment map operators for the global symmetry reside in mesonic $\hat{\mathcal{B}}_1$ multiplets, which can be separated into $SU(N_f)$ and $U(1)$ parts,

$$M_j{}^i := \widetilde{Q}_j^a Q_a^i \quad \Longrightarrow \qquad \mu := M_i{}^i , \quad \mu_j{}^i := M_j{}^i - \frac{1}{N_f} \mu \, \delta_j{}^i . \tag{2.146}$$

The level of the non-Abelian part of the global symmetry is $k_{4d}^{SU(N_f)} = 2N$. The baryons are of dimension N and no longer generate any additional global symmetries. Rather, they transform in the N-fold antisymmetric tensor representations of the flavor symmetry:

$$\begin{aligned} B^{i_1\ldots i_N} &:= Q^{i_1}_{a_1}\cdots Q^{i_N}_{a_N}\epsilon^{a_1\ldots a_N}\ , \\ \widetilde{B}_{i_1\ldots i_N} &:= \widetilde{Q}^{a_1}_{i_1}\cdots \widetilde{Q}^{a_N}_{i_N}\epsilon_{a_1\ldots a_N}\ . \end{aligned} \tag{2.147}$$

The mesons and baryons satisfy a set of polynomial relations. Following [74], we introduce notation where "$\cdot$" denotes contraction of an upper and a lower index and "$*$" denotes the contraction of flavor indices with the completely antisymmetric tensor in N_f indices. The relations are then given by

$$\begin{aligned} (*B)\widetilde{B} \ &= \quad *(M^N)\ , \qquad & M\cdot *B \ &= \ M\cdot *\widetilde{B}=0\ , \\ M'\cdot B \ &= \ \widetilde{B}\cdot M' \ = \ 0\ , & M\cdot M' \ &= \ 0\ , \end{aligned} \tag{2.148}$$

where $(M')_i{}^j := M_i{}^j - \frac{1}{N}\mu\delta_i^j = \mu_i{}^j - \frac{1}{2N}\mu\delta_i^j$. Additionally, all quantities antisymmetrized in more than N flavor indices must vanish.

This completes the description of the Hall-Littlewood chiral ring, since again this theory admits a linear quiver description, so there are no $\mathcal{D}$-type multiplets after turning on interactions. The final representation of canonical interest is the $\hat{\mathcal{C}}_{0(0,0)}$ multiplet, which again contributes an important Schur operator in the form of the $R=1$ component of the $SU(2)_R$ current:

$$\mathcal{J}^{R=1}_{+\dot{+}} \sim \frac{1}{2}\left(Q^i_a\partial_{+\dot{+}}\widetilde{Q}^a_i - \widetilde{Q}^a_i\partial_{+\dot{+}}Q^i_a\right) + \lambda^A_+\tilde{\lambda}_{\dot{+}A}\ . \tag{2.149}$$

Like the $SU(2)$ theory, these models all have one-complex-dimensional conformal manifolds with interesting behaviors at the boundary points, where S-dual descriptions become appropriate. In contrast to the $SU(2)$ theory, these S-dual descriptions are not the same as the original description, and rather involve intrinsically strongly-coupled non-Lagrangian sectors. While such dualities imply interesting structures for the associated chiral algebras, their dependence on non-Lagrangian theories takes us outside the scope of the current examples. This is discussed in much greater detail in [75].

Physical operators of low dimension

The nontrivial BRST cohomology classes of the chiral algebra can be computed by hand for small values of the dimension. The physical operators of dimension one again correspond to the moment map operators of the global symmetry, which in this case includes only the mesonic chiral ring operators,

$$J_i^j \ := \ q_{ai}\tilde{q}^{aj} - \frac{1}{N_f}\delta_i^j q_{ak}\tilde{q}^{ak} = \chi[\mu_i^j]\ , \tag{2.150}$$

$$J \ := \ q_{ak}\tilde{q}^{ak} = \chi[\mu]\ . \tag{2.151}$$

The singular OPEs of these currents are given by

$$\begin{aligned} J_i^j(z)J_k^l(0) &\sim -\frac{N(\delta_i^l\delta_k^j - \text{trace})}{z^2} + \frac{\delta_i^l J_k^j(z) - \delta_k^j J_i^l(z)}{z} \ , \\ J(z)J(0) &\sim -\frac{2N^2}{z^2} \ . \end{aligned} \tag{2.152}$$

This is an $\mathfrak{su}(N_f) \times \mathfrak{u}(1)$ affine Lie algebra at level $k_{2d} = -N$.

At dimension two, we first consider the operators that are invariant under the flavor symmetry. As expected, there is a canonical stress tensor,

$$T := \frac{1}{2}\left(q_{ai}\partial\tilde{q}^{ai} - \tilde{q}^{ai}\partial q_{ai}\right) - b_b^a\partial c_a^b = \chi[\mathcal{J}_{++}^1] \ , \tag{2.153}$$

whose self-OPE fixes the two-dimensional central charge,

$$c_{2d} = 2 - 4N^2 \ . \tag{2.154}$$

Additionally, the algebra generated by the affine $\mathfrak{su}(N_f) \times \mathfrak{u}(1)$ currents (2.150) contains a dimension two singlet that is the Sugawara stress tensor of the current algebra,

$$T_{\text{sug}} := \frac{1}{N_f}\left(J_i^j J_j^i - \frac{1}{N_f} JJ\right) \ . \tag{2.155}$$

The corresponding Sugawara central charge is also equal to $2 - 4N^2$, which suggests that the two stress tensors T and T^{sug} may be equivalent operators as they were in the $N = 2$ theory. Indeed, we expect this to be the case since the central charges in this theory again saturate the unitarity bound (2.116). A short computation verifies that their difference is BRST exact,

$$T - T_{\text{sug}} = \frac{1}{N_f}\{Q_{\text{BRST}}, q_{ai}\tilde{q}^{bj}b_b^a\} \ . \tag{2.156}$$

A complete basis for the physical flavor singlets of dimension two is given by T, JJ, and ∂J.

The remaining physical operators of dimension two are charged under $U(N_f)$. An overcomplete basis of such operators is given by flavored current bilinears $J_i^j J_k^l$ and $J_i^j J$, in addition to derivatives of the currents ∂J_i^j. These operators are not all independent. For example, the usual rules of conformal normal ordering imply that

$$J_i^j J_k^l - J_k^l J_i^j = \delta_i^l \partial J_k^j - \delta_k^j \partial J_i^l \ , \tag{2.157}$$

so the antisymmetric normal ordered product of two $SU(N_f)$ currents is a combination of descendants. For the symmetrized normal ordered product there exists another relation:

$$\frac{1}{2}(J_i^k J_k^j + J_k^j J_i^k) = \delta_i^j \left(\frac{1}{N_f^2} JJ + T\right) - \{Q_{\rm BRST}, q_{\alpha i}\tilde{q}^{\beta j} b_\beta^\alpha\}\,. \tag{2.158}$$

In group-theoretic terms, the relations amount to the statement that the parts of the symmetric product of two currents that transform in the singlet and adjoint representations do not correspond to independent operators.

It is worth jumping ahead to the case of dimension $N/2$, where we find operators that correspond to the baryonic chiral ring generators (2.147):

$$\begin{aligned} b_{i_1 i_2 \ldots i_{N_c}} &:= \varepsilon^{\alpha_1\alpha_2\ldots\alpha_{N_c}} q_{\alpha_1 i_1} q_{\alpha_2 i_2} \ldots q_{\alpha_{N_c} i_{N_c}} = \chi[B_{i_1 i_2 \cdots i_N}]\,, \\ \tilde{b}^{i_1 i_2 \ldots i_{N_c}} &:= \varepsilon_{\alpha_1\alpha_2\ldots\alpha_{N_c}} \tilde{q}^{\alpha_1 i_1} \tilde{q}^{\alpha_2 i_2} \ldots \tilde{q}^{\alpha_{N_c} i_{N_c}} = \chi[\tilde{B}^{i_1 i_2 \cdots i_N}]\,. \end{aligned} \tag{2.159}$$

These are Virasoro primaries of dimension $N_f/4$. The only non-trivial OPE that is not entirely fixed by symmetry is the $b \times \tilde{b}$ OPE. For $N_c = 3$, for example, it is given by

$$b_{i_1 i_2 i_3}(z)\tilde{b}^{j_1 j_2 j_3}(0) \sim \frac{36\,\delta_{[i_1}^{[j_1}\delta_{i_2}^{j_2}\delta_{i_3]}^{j_3]}}{z^3} - \frac{36\,\delta_{[i_1}^{[j_1}\delta_{i_2}^{j_2}J_{i_3]}^{j_3]}(0)}{z^2} + \frac{18\,\delta_{[i_1}^{[j_1}J_{i_2}^{j_2}J_{i_3]}^{j_3]}(0) - 18\,\delta_{[i_1}^{[j_1}\delta_{i_2}^{j_2}\partial J_{i_3]}^{j_3]}(0)}{z}\,, \tag{2.160}$$

where square brackets denote antisymmetrization with weight one.

Relation to the Higgs branch chiral ring

Again, certain features of the Higgs branch chiral ring arise organically from the chiral algebra. According to the general discussion in §2.3.2, the dimension two operators in the chiral algebra should in particular contain the image of the Schur operators in $\hat{\mathcal{B}}_2$ multiplets, which in the theories under consideration simply correspond to the product of two of the mesonic operators μ and μ_i^j subject to the final relation in (2.148). Furthermore, these Schur operators necessarily become Virasoro primary operators in the chiral algebra.

From amongst the BRST cohomology classes at level two – spanned by T, JJ, $J_i^j J$, the symmetrized combination $J_i^j J_k^l + J_k^l J_i^j$ modulo relation (2.158), and derivatives of level one currents – we find exactly three Virasoro primary operators:

$$\begin{aligned} \mathcal{X} &:= JJ - \frac{N_f^2}{N_f^2 - 2} T\,, \\ \mathcal{X}_i^j &:= J_i^j J\,, \\ \mathcal{X}_{ik}^{jl} &:= \frac{1}{2}(J_i^j J_k^l + J_k^l J_i^j) - \frac{N_f}{N_f^2 - 2}\left(\delta_i^l \delta_k^j - \frac{1}{N_f}\delta_i^j \delta_k^l\right) T\,, \end{aligned} \tag{2.161}$$

which are subject to the additional constraints,

$$\mathcal{X}_{ik}^{jl} = \mathcal{X}_{ki}^{lj}\,, \qquad \mathcal{X}_{ik}^{il} = 0\,, \qquad \mathcal{X}_{ij}^{jk} = \frac{1}{N_f^2}\delta_i^k \mathcal{X} + \{Q_{\mathrm{BRST}}, \ldots\}\ . \tag{2.162}$$

We see that we should identify $\mathcal{X} = \chi[\,\mu\mu\,]$, $\mathcal{X}_i^j = \chi[\,\mu\mu_i^j\,]$ and $\mathcal{X}_{ik}^{jl} = \chi[\,\mu_i^j\mu_k^l\,]$. The first two relations in (2.162) then reflect the natural symmetry properties of the original Schur operator, whilst the last equation precisely reproduces the final equation in (2.148).

We note that the definitions (2.161) somewhat obscure the relationship to four-dimensional physics because of the conformal normal ordering used to define the products of interacting fields. The same dimension two operators take a completely natural form in terms of creation/annihilation normal ordered products of symplectic bosons,

$$\begin{aligned} \mathcal{X} = &\quad : q_{\alpha i}\tilde{q}^{\alpha i} q_{\beta j}\tilde{q}^{\beta j} : \ , \\ \mathcal{X}_i^j = &\quad : q_{\alpha i}\tilde{q}^{\alpha j} q_{\beta k}\tilde{q}^{\beta k} : \ , \\ \mathcal{X}_{ik}^{jl} = &\quad : q_{\alpha i}\tilde{q}^{\alpha j} q_{\beta k}\tilde{q}^{\beta l} : \ , \end{aligned} \tag{2.163}$$

and this description also nicely illustrates the commutative diagram of §2.3.3.

Finally, at the level of Virasoro representations, the OPEs of the dimension one currents can now be summarized by the following fusion rules,

$$\begin{aligned} J_i^j \times J_k^l \quad &\to \quad -N(\delta_i^l\delta_k^j - \text{trace})\mathbb{1} + (\delta_i^l J_k^j - \delta_k^j J_i^l) + \mathcal{X}_{ik}^{jl} + \ldots\ , \\ J_i^j \times J \quad &\to \quad \mathcal{X}_i^j + \ldots\ , \\ J\ \times J \quad &\to \quad -2N^2\mathbb{1} + \mathcal{X} + \ldots\ , \end{aligned} \tag{2.164}$$

where we have omitted operators of dimension higher than two. We see that the product structure of the Higgs branch chiral ring is reproduced precisely by the $O(1)$ terms in these fusion rules.[26]

A $\mathcal{W}$-algebra conjecture

The chiral algebra is not as simple in this case as it was for the $SU(2)$ theory, since the generators b and $\tilde{b}$ are higher-spin $\mathcal{W}$-symmetry generators rather than simple affine currents. Nevertheless, there is a natural guess as to how to describe this more involved theory as a $\mathcal{W}$ algebra. It is useful to think of the operator content of the algebra in terms of representations of the affine $\mathfrak{u}(N_f)$ current algebra. From the analysis of levels one and two, we know that there is the vacuum representation – which in particular contains the affine currents and the stress tensor – and the "baryonic" representations, for which the highest weight state is

[26]We may similarly speculate that the Poisson bracket is encoded in the terms of the OPE that correspond to simple poles, but we have not checked this in detail.

given by the baryon or anti-baryon of (2.159). Other representations of the affine Lie algebra can only come from multi-baryon states or from new generators of dimension greater than two, where we have not performed a detailed analysis of the cohomology.

In four dimensions the mesons and the baryons are the complete set of generators for the Hall-Littlewood chiral ring. The most obvious conjecture is then that the corresponding two-dimensional operators generate the entire $\mathcal{W}$-algebra:

Conjecture 2. *When $\mathcal{T}$ is $\mathcal{N}=2$ $SU(N)$ superconformal QCD for with $2N$ flavors for $N>2$, then $\chi[\mathcal{T}]$ is isomorphic to the $\mathcal{W}$ algebra generated by affine $\mathfrak{u}(N_f)$ currents at level $k_{\mathfrak{su}(N_f)}=-N$ along with baryonic generators b and $\tilde{b}$ with the OPE (2.160) (or its generalizations to $N\geqslant 4$).*

Because no additional generators make an appearance in the singular OPEs of the affine currents and baryons, it is guaranteed to be the case that the $\mathcal{W}$ algebra we have just described forms a chiral subalgebra of $\chi[\mathcal{T}]$. Our conjecture is that this is in fact the whole thing. If true, this conjecture would imply that the Schur index for the $N_f=2N$ theories decomposes into characters of affine $\mathfrak{u}(2N)_{-N}$ with highest weights given by the vacuum or by one or more baryons.

Superconformal Index

We can provide support for this conjecture by comparing with the superconformal index. The Schur index of the theory is given by the following contour integral,

$$\begin{aligned}\mathcal{I}_{\rm Schur}(q;c,\vec{a}) = \int [d\vec{b}] P.E.\Bigg[&\frac{\sqrt{q}}{1-q}\left(c\,\chi^{\mathbf{N_f}}_{SU(N_f)}(\vec{a})\chi^{\mathbf{N}}_{SU(N)}(\vec{b})+c^{-1}\,\chi^{\mathbf{N_f}}_{SU(N_f)}(\vec{a}^{-1})\chi^{\mathbf{N}}_{SU(N)}(\vec{b}^{-1})\right)\\ &+\left(\frac{-2q}{1-q}\right)\chi^{\mathbf{N^2-1}}_{SU(N)}(\vec{b})\Bigg],\end{aligned} \tag{2.165}$$

where c is the $U(1)$ fugacity and $\vec{a}=(a_1,a_2,\ldots,a_{N_f-1})$ denotes $SU(N_f)$ fugacities. For $N=3$, the first few orders are given by

$$\begin{aligned}\mathcal{I}_{\rm Schur}(q;c,\vec{a}) =&1+\left(1+\chi^{\mathbf{35}}_{SU(6)}(\vec{a})\right)q+(c^3+c^{-3})\chi^{\mathbf{20}}_{SU(6)}(\vec{a})q^{3/2}\\ &+\left(\left(\chi^{sym^2(\mathbf{35})}_{SU(6)}(\vec{a})-\chi^{\mathbf{35}}_{SU(6)}(\vec{a})\right)+2\chi^{\mathbf{35}}_{SU(6)}(\vec{a})+2\right)q^2\\ &+(c^3+c^{-3})\Big(2\chi^{\mathbf{20}}_{SU(6)}(\vec{a})\\ &+\left(\chi^{\mathbf{35}\otimes\mathbf{20}}_{SU(6)}(\vec{a})-\chi^{\mathbf{20}}_{SU(6)}(\vec{a})-\chi^{\mathbf{70}}_{SU(6)}(\vec{a})-\chi^{\overline{\mathbf{70}}}_{SU(6)}(\vec{a})\right)\Big)q^{5/2}+\ldots,\end{aligned} \tag{2.166}$$

where we have explicitly indicated the presence of relations by listing them with a minus sign. The dimension two relations in the chiral algebra were elaborated upon in the previous subsection. At level 5/2, we can similarly determine the Virasoro primaries

$$Y_{ijk} = Jb_{ijk} + \partial b_{ijk}, \qquad \widetilde{Y}^{ijk} = J\widetilde{b}^{ijk} - \partial\widetilde{b}^{ijk} \tag{2.167}$$

$$Y^j_{i,klm} = \frac{1}{2}\left(J_i{}^j b_{klm} + b_{klm}J_i{}^j - \frac{1}{6}\delta^j_i\partial b_{klm} + \delta^j_{[k}\partial b_{|i|lm]}\right) \tag{2.168}$$

$$\widetilde{Y}^{j,klm}_i = \frac{1}{2}\left(J_i{}^j\widetilde{b}^{klm} + \widetilde{b}^{klm}J_i{}^j + \frac{1}{6}\delta^j_i\partial\widetilde{b}^{klm} - \delta^{[k}_i\partial\widetilde{b}^{|j|lm]}\right), \tag{2.169}$$

subject to the constraints

$$\epsilon^{iklmnp}\left(Y^j_{i,mnp} + \frac{1}{6}\delta^j_i Y_{mnp}\right) = 0, \qquad Y^j_{i,jlm} - \frac{1}{6}Y_{ilm} = \{Q_{\text{BRST}}, \ldots\} \tag{2.170}$$

$$\epsilon_{jklmnp}\left(\widetilde{Y}^{j,mnp}_i + \frac{1}{6}\delta^j_i\widetilde{Y}^{mnp}\right) = 0, \qquad \widetilde{Y}^{j,kli}_i - \frac{1}{6}\widetilde{Y}^{jkl} = \{Q_{\text{BRST}}, \ldots\}, \tag{2.171}$$

which again encode precisely the Higgs branch relations.

At level three, we have checked agreement between the Schur index and the cohomology generated by the $SU(6) \times U(1)$ currents and the baryons by explicitly computing the null states.

2.5.3 $\mathcal{N} = 4$ supersymmetric Yang-Mills theory

The theories considered in the previous two subsections all shared the special quality of admitting descriptions as linear quiver gauge theories, which meant that $\mathcal{D}$-type multiplets played no role in the analysis. We now turn to a case where this simplification no longer holds, and so there will necessarily be generators outside of the Higgs chiral ring. The theory in question is $\mathcal{N} = 4$ supersymmetric Yang-Mills theory with gauge group $SU(N)$. For our purposes, this is an $\mathcal{N} = 2$ theory with an $SU(N)$ vector multiplet and a single adjoint-valued hypermultiplet. In $\mathcal{N} = 1$ notation, we have the following chiral superfields,

$$W^A_\alpha\,, \qquad \Phi^A\,, \qquad Q^A_i\,, \tag{2.172}$$

where $A = 1, \ldots N^2 - 1$ an $SU(N)$ adjoint index and $i = 1, 2$ is an $SU(2)_F$ vector index. The flavor symmetry $SU(2)_F$ is the commutant of $SU(2)_R \times U(1)_r \subset SU(4)_R$, and so is an R-symmetry with respect to the full superalgebra. The central charges of the theory are given by

$$k^{SU(2)}_{4d} = N^2 - 1\,, \quad c_{4d} = \frac{(N^2-1)}{4}\,. \tag{2.173}$$

The Higgs branch chiral ring has $N - 1$ generators. In terms of the $N \times N$ matrices

$Q_i := Q_i^A t^A$, these are given by

$$\operatorname{Tr} Q_{(i_1} \cdots Q_{i_k)} \ , \qquad k = 1, \ldots, N-1 \ , \tag{2.174}$$

subject to trace relations. In this theory, the Hall-Littlewood chiral ring contains additional $\mathcal{D}$-type multiplets that are not described by the Higgs chiral ring. More specifically, for $SU(N)$ gauge group there are an additional $N-1$ HL generators given by

$$\operatorname{Tr} Q_{(i_1} \cdots Q_{i_k)} \tilde{\lambda}^1_{\dot{+}} \ , \qquad k = 1, \ldots, N-1 \ . \tag{2.175}$$

There are corresponding generators of the HL anti-chiral ring that lie in $\overline{\mathcal{D}}$ multiplets and take the same form with $\tilde{\lambda}^1_{\dot{+}}$ replaced by λ^1_{+}. Finally, the Schur component of the $SU(2)_R$ current, which will give rise to the stress tensor in two-dimensions, is given in terms of four-dimensional fields by

$$\mathcal{J}^{R=1}_{+\dot{+}} \sim \frac{1}{2} \operatorname{Tr} Q_i \partial_{+\dot{+}} Q_j \varepsilon^{ij} - \operatorname{Tr} \tilde{\lambda}_{\dot{+}} \lambda_{+} \ . \tag{2.176}$$

Cohomological description of the associated chiral algebra

The free chiral algebra follows the same pattern as the previous examples. The two dimensional counterparts of the hypermultiplet scalars and gauginos can be introduced as usual,

$$q_i^A(z) := \chi[Q_i^A] \ , \qquad b^A(z) := \chi[\tilde{\lambda}^A] \ , \qquad \partial c^A(z) := \chi[\lambda^A] \ . \tag{2.177}$$

The free chiral algebra has the free OPEs,

$$q_i^A(z) q_j^B(0) \sim \frac{\varepsilon_{ij} \delta^{AB}}{z} \ , \qquad b^A(z) c^B(0) \sim \frac{\delta^{AB}}{z} \ .$$

The stress tensor is given by the usual canonical expression

$$T = \frac{1}{2} q_i^A \partial q_j^B \ \varepsilon^{ij} - b^A \partial c^A \ , \tag{2.178}$$

which has a central charge of $c_{2d} = -3(N^2-1)$. The $SU(2)_F$ currents are given by

$$J_{ij} = -\frac{1}{2} q_i^A q_j^A \ , \tag{2.179}$$

and satisfy a current algebra at level $k_{2d} = -\frac{N^2-1}{2}$. The current algebra contains a Sugawara stress tensor of the usual form,

$$T_{\text{Sug}}(z) = \frac{1}{N^2-5} \ J_{ij} J_{kl} \ \varepsilon^{ik} \varepsilon^{jl} \ , \tag{2.180}$$

with central charge equal to $\frac{3(N^2-1)}{N^2-5}$. Note that precisely for $N=2$ and for no other value of N, the Sugawara central charge matches with the true central charge. As we will see, this is again a consequence of the two stress tensors being equivalent in BRST cohomology.

The $SU(N)$ currents for the matter and ghost sectors are given by

$$J^A = \frac{i}{2} f^{ABC} q_i^B q_j^C \ \varepsilon^{ij} \,, \qquad J_{\text{gh}}^A = -i f^{ABC} c^B b^C \,. \tag{2.181}$$

The levels for the corresponding current algebras are $-2N$ and $2N$, respectively. The BRST current is constructed as usual,

$$J_{\text{BRST}} = c^A \left(J^A_{SU(N)} + \frac{1}{2} J^A_{\text{gh}} \right) \,, \tag{2.182}$$

and its zero mode defines the nilpotent BRST operator Q_{BRST}.

Low-lying physical states

Let us first consider the case of $SU(2)$ gauge group. In this case the difference between the Sugawara stress tensor and the canonical stress tensor is BRST exact,

$$T - T_{\text{Sug}} \quad \sim \quad \{ Q_{\text{BRST}} \ , \ f^{ABC} q_i^A q_j^B b^C \ \varepsilon^{ij} \} \ . \tag{2.183}$$

Based on the description of the HL chiral ring generators, we expect that amongst the physical states should be an $SU(2)_F$ triplet of affine currents and an $SU(2)_F$ doublet of dimension $3/2$ fermionic generators. Up to dimension two, the cohomology is generated by precisely these operators,

$$\begin{aligned} J_{ij} = & \quad -\frac{1}{2}(q_i^A q_j^A) = \chi[\operatorname{Tr} Q_i Q_j] \ , \\ G_i := & \quad \sqrt{2}(q_i^A b^A) = \chi[\operatorname{Tr} Q_i \tilde{\lambda}_+] \ , \\ \tilde{G}_i := & -\sqrt{2}(q_i^A \partial c^A) = \chi[\operatorname{Tr} Q_i \lambda_+] \ . \end{aligned} \tag{2.184}$$

The OPEs of these generators can be computed directly,

$$J_{ij}(z)J_{kl}(w) \sim -\frac{N^2-1}{2}\frac{\varepsilon_{l(i}\varepsilon_{j)k}}{(z-w)^2} + \frac{2\varepsilon_{(k(i}J_{j)l)}}{z-w}\,, \tag{2.185}$$

$$J_{ij}(z)G_k(w) \sim \frac{\frac{1}{2}(\varepsilon_{ki}G_j(w)+\varepsilon_{kj}G_i(w))}{z-w}\,, \tag{2.186}$$

$$J_{ij}(z)\tilde{G}_k(w) \sim \frac{\frac{1}{2}(\varepsilon_{ki}\tilde{G}_j(w)+\varepsilon_{kj}\tilde{G}_i(w))}{z-w}\,, \tag{2.187}$$

$$G_i(z)G_j(w) \sim 0\,, \tag{2.188}$$

$$\tilde{G}_i(z)\tilde{G}_j(w) \sim 0\,, \tag{2.189}$$

$$G_i(z)\tilde{G}_j(w) \sim -\frac{2(N^2-1)\varepsilon_{ij}}{(z-w)^3} + \frac{4J_{ij}(w)}{(z-w)^2} + \frac{2\varepsilon_{ij}T(w)+2\partial J_{ij}(w)}{z-w}\,, \tag{2.190}$$

where $N=2$ and the symmetrization in the indices i,j and k,l has weight one. The value of N has been left unspecified in (2.185) because the OPEs will continue to hold for higher rank gauge groups. For the same reason, $T(z)$ has been included separately, though for $N=2$ it not a distinct generator, but rather is identified with the Sugawara stress tensor.

The operator product algebra in (2.185) can be immediately recognized to be the "small" $\mathcal{N}=4$ superconformal algebra with central charge $c_{2d}=-3(N^2-1)$ [76]. It is natural that there should be supersymmetry acting in the chiral algebra, since the holomorphic $\mathfrak{sl}(2)$ that commutes with the supercharges $\mathbb{Q}_i$ is in enhanced to a holomorphic $\mathfrak{sl}(2\,|\,2)$ when the four-dimensional theory is $\mathcal{N}=4$ supersymmetric. However, like the case of the global conformal algebra being generated not by the four-dimensional stress tensor but by the chiral operator associated to the $SU(2)_R$ current, here the enhanced supersymmetry in the chiral algebra is generated *not* by the four-dimensional supercurrents, but by the Schur operators that lie in the same $\mathcal{D}_{\frac{1}{2}(0,0)}$ and $\overline{\mathcal{D}}_{\frac{1}{2}(0,0)}$ multiplets with them. Those are the Schur operators that are transmuted into the two-dimensional supercurrents G_i and $\tilde{G}_i$.

In $SU(3)$ theory there are additional generators arising from the additional HL generators. Sure enough, direct computation produces the following list of new generators of dimension less than or equal to 5/2:

$$\begin{aligned}
B_{ijk} &:= \operatorname{Tr} q_i q_j q_k &&= \chi[\operatorname{Tr} Q_i Q_j Q_k]\ , \\
B_{ij} &:= \operatorname{Tr} q_i q_j b &&= \chi[\operatorname{Tr} Q_i Q_j \tilde{\lambda}_+]\ , \\
\tilde{B}_{ij} &:= \operatorname{Tr} q_i q_j \partial c &&= \chi[\operatorname{Tr} Q_i Q_j \lambda_{\dot{+}}]\ , \\
B_i &:= 3\operatorname{Tr} q_i b \partial c + \operatorname{Tr} \partial q_j q^j q_i &&= \chi[3\operatorname{Tr} Q_i \tilde{\lambda}_+ \lambda_{\dot{+}} + \operatorname{Tr} \partial_{+\dot{+}} Q_j Q^j Q_i]\ .
\end{aligned} \tag{2.191}$$

Precisely for the $SU(3)$ case, the operator B_i is in fact equivalent to a composite operator,

$$B_i \sim \varepsilon^{jj'}\varepsilon^{kk'}J_{jk}B_{ij'k'}\ . \tag{2.192}$$

This is a consequence of a chiral ring relation for this value of N which sets $\varepsilon^{jj'}\varepsilon^{kk'}\mathrm{Tr}\, Q_j Q_k$ $\mathrm{Tr}\, Q_i Q_{j'} Q_{k'}$ to zero. This will not be the case for higher rank gauge groups, and B_i will be an authentic generator of the algebra.

A super $\mathcal{W}$-algebra conjecture

Because the chiral algebras of $\mathcal{N}=4$ SYM theories are supersymmetric, we can introduce a more restrictive notion of generators for these algebras. More precisely, we would like to identify those operators that generate the chiral algebra under the operations of normal ordered products and *super*derivatives, or the action of $\mathfrak{sl}(2\,|\,2)$. In other words, we allow not just L_1 descendants, but also $G_{i,-\frac{1}{2}}$ and $\tilde{G}_{i,-\frac{1}{2}}$ descendants.

The last three generators in (2.191) are superdescendants of B_{ijk}, so we have really only found one additional super-generator in the $SU(3)$ theory. In general, HL operators will be grouped by $\mathcal{N}=4$ supersymmetry into multiplets comprised of a single $\hat{\mathcal{B}}$-type operator, an $SU(2)_F$ doublet of $\mathcal{D}$-type operators, and an $SU(2)_F$ doublet worth of $\bar{\mathcal{D}}$-type operators.

For a general simple gauge group, the natural guess is that the chiral algebra is generated by the small $\mathcal{N}=4$ superconformal algebra along with additional chiral primary operators arising from the Higgs chiral ring generators. Our conjecture is then the following:

Conjecture 3. *The chiral algebra for $\mathcal{N}=4$ SYM theory with gauge group G is isomorphic to an $\mathcal{N}=4$ super $\mathcal{W}$-algebra with* $\mathrm{rank}(G)$ *generators given by chiral primaries of dimensions $\frac{d_i}{2}$, where d_i are the degrees of the Casimir invariants of G.*

We now perform some tests of this conjecture at the level of the superconformal index.

The superconformal index

Conjecture 3 can be tested up to any given level by comparing the index of the chiral algebra defined in the conjecture with the superconformal index of $\mathcal{N}=4$ SYM in the Schur limit. For gauge group $SU(N)$, the Schur index is given by a contour integral,

$$\mathcal{I}_{\rm Schur}(q;a) = \oint [d\vec{b}]\mathrm{P.E.}\left[\left(\frac{\sqrt{q}}{1-q}\right)\chi^{\mathbf{2}}(a)\chi^{\mathbf{N^2-1}}(\vec{b}) + \left(\frac{-2q}{1-q}\right)\chi^{\mathbf{N^2-1}}(\vec{b})\right]\,, \tag{2.193}$$

where a is an $SU(2)_F$ flavor fugacity. For $SU(2)$ gauge group, expanding the integrand in powers of q and integrating gives the following result up to $O(q^4)$, where we have collected terms into $SU(2)_F$ characters $\chi^{\mathbf{R}}(a)$,

$$\begin{aligned}\mathcal{I}_{\rm Schur}(q;a) =& 1 + \chi^{\mathbf{3}}(a)q - 2\chi^{\mathbf{2}}(a)q^{3/2} + \left(\chi^{\mathbf{1}}(a)+\chi^{\mathbf{3}}(a)+\chi^{\mathbf{5}}(a)\right)q^2 \\ & - 2\left(\chi^{\mathbf{2}}(a)+\chi^{\mathbf{4}}(a)\right)q^{5/2} + \left(\chi^{\mathbf{1}}(a)+3\chi^{\mathbf{3}}(a)+\chi^{\mathbf{5}}(a)+\chi^{\mathbf{7}}(a)\right)q^3 \\ & - \left(4\chi^{\mathbf{2}}(a)+4\chi^{\mathbf{4}}(a)+2\chi^{\mathbf{6}}(a)\right)q^{7/2} \\ & + \left(3\chi^{\mathbf{1}}(a)+7\chi^{\mathbf{3}}(a)+4\chi^{\mathbf{5}}(a)+\chi^{\mathbf{7}}(a)+\chi^{\mathbf{9}}(a)\right)q^4 + \ldots\ .\end{aligned} \tag{2.194}$$

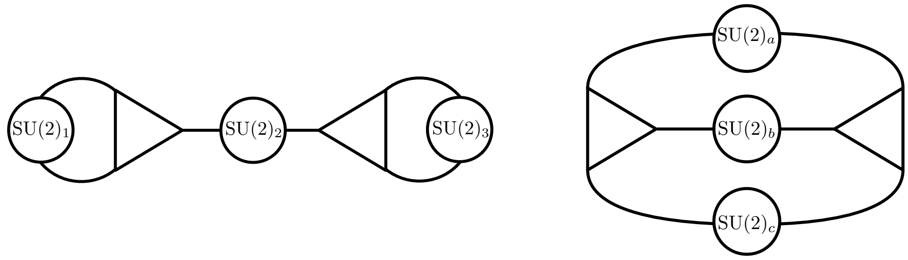

Figure 2.1: Weak coupling limits of the genus two class $\mathcal{S}$ theory.

We can compare this result with the index of the $\mathcal{W}$-algebra appearing in the conjecture (in this case, just the small superconformal algebra with the appropriate value of the central charge) by enumerating the states of the chiral algebra and then finding and subtracting the null states at each level. We have checked up to level four, and the results match exactly.

The same comparison can be done for the $SU(3)$ case, where the Schur index to $O(q^3)$ is given by

$$\begin{aligned}\mathcal{I}_{\text{Schur}}(q;a) =& 1+\chi^{\mathbf{3}}(a)q+\left(\chi^{\mathbf{4}}(a)-2\chi^{\mathbf{2}}(a)\right)q^{3/2}+\left(2\chi^{\mathbf{1}}(a)+\chi^{\mathbf{5}}(a)-\chi^{\mathbf{3}}(a)\right)q^2\\ &+\left(\chi^{\mathbf{6}}(a)-3\chi^{\mathbf{2}}(a)\right)q^{5/2}+\left(5\chi^{\mathbf{1}}(a)+\chi^{\mathbf{3}}(a)+2\chi^{\mathbf{7}}(a)-3\chi^{\mathbf{5}}(a)\right)q^3+\ldots\,.\end{aligned} \tag{2.195}$$

Up to level three the nulls were computed and they agree with the index. Note that in this case there are cancellations in the index of the chiral algebra, since there are bosonic and fermionic states appearing at the same level.

2.5.4 Class $\mathcal{S}$ at genus two

At this point, the reader may be starting to get the impression that the chiral algebra of any four-dimensional theory be entirely determined by the structure of its various chiral rings. The purpose of this next example is to show that such a simplistic picture is untenable.

Our example is the rank one class $\mathcal{S}$ theory associated to an unpunctured genus two Riemann surface [17, 18]. The theory admits two inequivalent weak-coupling limits, or S-duality frames, corresponding to the two generalized quiver constructions illustrated in Fig. 2.1. We will focus on the first case, which is sometimes called the dumbbell quiver. The gauge groups are denoted $SU(2)_1$ for the left loop, $SU(2)_2$ for central line, and $SU(2)_3$ for the right loop. The fields of the theory are two sets of half-hypermultiplets transforming in the trifundamental representation of $SU(2)^3$ and three $SU(2)$ vector multiplets. In $\mathcal{N}=1$ notation, we denote these by

$$Q_{a_1b_1a_2}\,,\qquad S_{a_3b_3a_2}\,,\qquad W^{(\nu)}_{\alpha\ A_\nu}\,,\qquad \Phi^{(\nu)}_{B_\nu}\,, \tag{2.196}$$

where $\nu = 1,2,3$ indexes the three $SU(2)$ gauge groups, a_ν, b_ν are fundamental indices of $SU(2)_\nu$, and A_ν, B_ν are adjoint indices of $SU(2)_\nu$. It is convenient to rearrange the fields $Q_{a_1b_1a_2}$ and $S_{a_3b_3a_2}$ in terms of irreducible representations of the gauge groups. In particular, we can define

$$\begin{aligned} Q_{A_1a_2} &:= -iQ_{a_1b_1a_2}(T_J)^{a_1b_1}\ , \quad Q_{a_2} := \frac{1}{\sqrt{2}}\varepsilon^{a_1b_1}Q_{a_1b_1a_2}\ , \\ S_{A_3a_2} &:= -iS_{a_3b_3a_2}(T_J)^{a_3b_3}\ , \quad S_{a_2} := \frac{1}{\sqrt{2}}\varepsilon^{a_3b_3}S_{a_3b_3a_2}\ . \end{aligned} \tag{2.197}$$

Finally, we introduce the fields

$$\phi_{a_2} = \frac{1}{\sqrt{2}}(Q_{a_2} + iS_{a_2})\ , \qquad \bar{\phi}_{a_2} = \frac{1}{\sqrt{2}}(Q_{a_2} - iS_{a_2})\ . \tag{2.198}$$

The theory has a $U(1)_F$ flavor symmetry that is not completely obvious given the usual structure of flavor symmetries in class $\mathcal{S}$ theories. The fields ϕ and $\bar{\phi}$ have charges $+1$ and -1 respectively under the flavor symmetry, and the remaining fields are neutral.

The BRST cohomology problem for this theory can be set up as in the previous sections. In fact, the analysis may be somewhat simplified by leveraging the $\mathcal{N}=4$ analysis of the previous section. In particular, each loop in the quiver corresponds to a small $\mathcal{N}=4$ superconformal algebra along with a decoupled $SU(2)$ doublet of symplectic bosons. The genus two theory is obtained by gauging the diagonal subgroup of the $SU(2)$ flavor symmetries for each side. Nevertheless, the resulting cohomology problem is substantially more intricate than those of the previous examples, and we will not describe the level-by-level analysis.

Instead, we will take an indirect approach to understand the spectrum of *generators* of this chiral algebra at low levels. In particular, by analyzing various superconformal indices of this theory and comparing with the structure of the HL chiral ring, we will be able to prove that the full chiral algebra must have generators in addition to those related to HL chiral ring generators and the stress tensor. More precisely, by studying the spectrum up to dimension three, we find that there must be additional generators that arise from $\hat{\mathcal{C}}_{1(0,0)}$ multiplets in four dimensions.

The Higgs branch chiral ring for this theory has been analyzed in [77]. It has three generators: a $U(1)_F$ neutral operator of dimension two, which is actually the moment map for $U(1)_F$,

$$M = -\epsilon^{a_2a_2'}\phi_{a_2}\bar{\phi}_{a_2'}\ , \tag{2.199}$$

and two operators of dimension four,

$$\mathcal{O}_1 = 2\ \phi_{a_2}\phi_{a_2'}\ \epsilon^{a_2 b_2}\epsilon^{a_2' b_2'}\ Q_{A_1 b_2}Q_{B_1 b_2'}\ \delta^{A_1 B_1}\ . \tag{2.200}$$

$$\mathcal{O}_2 = 2\ \bar{\phi}_{a_2}\bar{\phi}_{a_2'}\ \epsilon^{a_2 b_2}\epsilon^{a_2' b_2'}\ Q_{A_1 b_2}Q_{B_1 b_2'}\ \delta^{A_1 B_1}\ , \tag{2.201}$$

that have charges $+2$ and -2 under the flavor symmetry. These generators satisfy a flavor neutral relation of dimension eight:

$$\mathcal{O}_1\mathcal{O}_2 = M^4\ . \tag{2.202}$$

It will be helpful for us to write down the Hilbert series [77] for this theory, refined by the $U(1)_F$ flavor symmetry:

$$g(\tau, a) = \frac{1-t^4}{(1-t)(1-a^2t^2)(1-a^{-2}t^2)} = 1 + t + \left(a^2 + a^{-2} + 1\right)t^2 + \left(a^2 + a^{-2} + 1\right)t^3 + \dots\ , \tag{2.203}$$

where a is the $U(1)_F$ fugacity, and t is the fugacity for the dimension of the operator.

The generalized quiver for this theory has closed loops, so there will be additional elements of the HL chiral ring coming from $\mathcal{D}$-type multiplets. The HL index for this theory can be computed by standard methods, and is given by

$$\mathcal{I}_{\mathrm{HL}}(t; a) = 1 + t + (a^2 + a^{-2} - 2a - 2a^{-1} + 1)t^2 + (a^2 + a^{-2} - 2a - 2a^{-1} + 2)t^3 + \dots\ . \tag{2.204}$$

By subtracting off the contributions of the Higgs chiral ring operators (obtained from (2.203)), we can find the contributions of just the $\mathcal{D}$-type multiplets. In turn, we can extract the structure of the $\mathcal{D}$-type generators.[27] All told, at dimension two there are two $\mathcal{D}_{1(0,0)}$ multiplets with $U(1)_F$ charge $+1$ and two with charge -1, and at dimension three there is a single $\mathcal{D}_{\frac{3}{2}(0,\frac{1}{2})}$ multiplet that is $U(1)_F$ neutral. The two-dimensional counterparts of these operators can be defined in an explicit calculation of the BRST cohomology.

Up to dimension three, we have now determined all of the generators of the HL chiral ring. The question is whether these operators (along with the conjugates of the $\mathcal{D}$-type operators), in addition to the two-dimensional stress tensor, are sufficient to explain the full spectrum of the chiral algebra (up to dimension three). The generators are listed in the three blocks of Table 2.6, together with their contribution to the Macdonald index and the quantum numbers of the corresponding Schur operators.

The Macdonald limit of the superconformal index of this theory is obtained from the

[27] We have checked by a computation of the HL cohomology that the HL index captures faithfully the complete spectrum of $\mathcal{D}$-type multiplets up to dimension three.

Multiplet	Index contribution	h	$U(1)_r$	$U(1)_F$
$\hat{\mathcal{B}}_1$	$\frac{t}{1-q}$	1	0	0
$\hat{\mathcal{B}}_2$	$\frac{t^2a^2}{1-q}$	2	0	+2
$\hat{\mathcal{B}}_2$	$\frac{t^2/a^2}{1-q}$	2	0	-2
$2\times\mathcal{D}_{1\,(0,0)}$	$-2\frac{t^2a}{1-q}$	2	$\frac{1}{2}$	+1
$2\times\bar{\mathcal{D}}_{1\,(0,0)}$	$-2\frac{tqa}{1-q}$	2	$-\frac{1}{2}$	+1
$2\times\mathcal{D}_{1\,(0,0)}$	$-2\frac{t^2/a}{1-q}$	2	$\frac{1}{2}$	-1
$2\times\bar{\mathcal{D}}_{1\,(0,0)}$	$-2\frac{tq/a}{1-q}$	2	$-\frac{1}{2}$	-1
$\mathcal{D}_{\frac{3}{2}\,(0,\frac{1}{2})}$	$\frac{t^3}{1-q}$	3	1	0
$\bar{\mathcal{D}}_{\frac{3}{2}\,(\frac{1}{2},0)}$	$\frac{tq^2}{1-q}$	3	-1	0
$\hat{\mathcal{C}}_{0(0,0)}$	$\frac{tq}{1-q}$	2	0	0
$3\times\hat{\mathcal{C}}_{1(0,0)}$	$3\frac{t^2q}{1-q}$	3	0	0

Table 2.6: Chiral algebra generators for the genus two theory with $h\leqslant 3$. The first columns lists the name and multiplicity of the four dimensional multiplets giving rise to the generators. The second column lists the contribution of each multiplet to the Macdonald superconformal index, including the flavor fugacity. The last columns list the two-dimensional quantum numbers of the generators. The first block of the table consists of Higgs chiral ring generators, the second the remaining HL chiral and anti-chiral ring generators, the third the two-dimensional stress tensor, and the last block the extra generators deduced from the superconformal index.

following contour integral,

$$\begin{aligned}\mathcal{I}_{\mathrm{MD}}(q,t;a)=\oint[db_1][db_2][db_3]\,P.E.\Bigg[&\frac{\sqrt{t}}{1-q}\left[\left(\chi^{\mathbf{3}}(b_1)\chi^{\mathbf{2}}(b_3)+\chi^{\mathbf{3}}(b_2)\chi^{\mathbf{2}}(b_3)\right)\right.\\&\left.+(a+a^{-1})\chi^{\mathbf{2}}(b_3)\right]+\left(\frac{-t-q}{1-q}\right)\left(\chi^{\mathbf{3}}(b_1)+\chi^{\mathbf{3}}(b_2)+\chi^{\mathbf{3}}(b_3)\right)\Bigg]\ ,\end{aligned}\tag{2.205}$$

and the expansion including all operators up to dimension three is as follows,

$$\begin{aligned}\mathcal{I}_{\mathrm{MD}}(q,t;a)=&\,1+t+(a^2+a^{-2}-2a-2a^{-1}+1)t^2+(-2a-2a^{-1}+2)qt+\\&+(a^2+a^{-2}-2(a+a^{-1})+2)t^3+(3-2(a+a^{-1}))q^2t\\&+(a^2+a^{-2}-4(a+a^{-1})+5)t^2q+\dots\ .\end{aligned}\tag{2.206}$$

We find that not all of the terms in this expansion can be accounted for by enumerating normal ordered products of generators and their descendants. In particular, from the list of known generators, the only operators that could contribute as t^2q to the index (with no flavor fugacity) are the normal-ordered product of a $\hat{\mathcal{B}}_1$ and a $\hat{\mathcal{C}}_{0(0,0)}$ and the derivative of the normal-ordered product of two $\hat{\mathcal{B}}_1$ operators. This leaves a contribution of $3t^2q$ remains to be explained. We can thus conclude that there are at least three new operators, and they must all must correspond to $\hat{\mathcal{C}}_{1,(0,0)}$ multiplets that are uncharged under the flavor symmetry. We have included these as the last entry in Table 2.6. The argument presented above shows that at least these three multiplets must be present, however it does not take into account possible cancellations in the index, which could hide even more additional generators.

2.6 Beyond Lagrangian theories

Although the discussion of the previous section focused on theories admitting Lagrangian descriptions, the correspondence between $\mathcal{N}=2$ SCFTs and chiral algebras is of course much more general. In particular, the vast landscape of superconformal theories of class $\mathcal{S}$, most of which are non-Lagrangian in nature, will be mapped to an intricate and interesting class of chiral algebras. The purpose of this section is to draw a sketch of the class of chiral algebras defined by this map. Most of the features discussed here follow from the general structure of class $\mathcal{S}$ and the correspondence with chiral algebras. We do however include a few specific claims that will be left unsubstantiated here, but which are explained in the more complete analysis of [75]. To begin, we offer a quick reminder of the salient features of $\mathcal{N}=2$ SCFTs of class $\mathcal{S}$.

2.6.1 A review of class $\mathcal{S}$ in four dimensions

Class $\mathcal{S}$ theories [17, 18] are those that arise from compactification of any of the $\mathcal{N}=(2,0)$ six-dimensional superconformal theories on a Riemann surface $\mathcal{C}$, known as the *UV curve*, possibly with the inclusion of real codimension two defect operators at points of $\mathcal{C}$.[28] We will be interested in the case of *superconformal* theories of class $\mathcal{S}$, which means that the mass parameters associated to defect operators will all be set to zero. The conformal manifold of a theory of class $\mathcal{S}$ is equal to the complex structure moduli space of the UV curve, with boundaries at which the curve degenerates corresponding to physical limits in which a gauge coupling goes to zero and a free vector multiplet decouples from the rest of the spectrum.

For our purposes, the most useful way to think about these theories is in terms of a set of four-dimensional "building block" theories associated to three-punctured spheres, or *trinions* [17]. Such a theory can be denoted $T_{\mathfrak{g}}^{(\rho_1,\rho_2,\rho_3)}$, where $\mathfrak{g}$ is the lie algebra of the underlying

[28]We restrict to the case of *regular* defects in all that follows. These are defects that are specified by an embedding $\rho:\mathfrak{su}(2)\to\mathfrak{g}$, where $\mathfrak{g}$ is the simply laced Lie algebra that labels the six-dimensional theory. Such a defect supports a flavor symmetry equal to the centralizer of the embedded $\mathfrak{su}(2)$ subalgebra of $\mathfrak{g}$.

six-dimensional theory, and the ρ_i label the defects at the three punctures. When all three embeddings are trivial, the theory is sometimes simply denoted $T_{\mathfrak{g}}$ (or T_N for the case that $\mathfrak{g} = A_{N-1}$). These building blocks can be assembled into more complex theories in a manner that is represented by a generalized quiver diagram such as those displayed in §2.5.4. The shape of the generalized quiver is necessarily a tropical limit of the corresponding UV curve, with different tropical limits corresponding to different S-duality frames of the same theory.

A number of known features of the building block theories can be used to predict the structure of the associated chiral algebras. In the interest of simplifying the discussion, we shall henceforth restrict to the case where $\mathfrak{g} = A_{N-1}$. The maximal building block (that is, the one with the largest flavor symmetry group) is then the T_N theory mentioned above. We begin by reviewing its properties.

Generically, T_N has $SU(N)_1 \times SU(N)_2 \times SU(N)_3$ flavor symmetry, and central charges [78, 79]

$$c_{4d} = \frac{N^3}{6} - \frac{N^2}{4} - \frac{N}{12} + \frac{1}{6} , \qquad k_{4d}^{SU(N)} = 2N = 2h^\vee . \tag{2.207}$$

When $N = 2$, this is just the theory of free trifundamental half-hypermultiplets that appeared in the example of §2.5.4, so the associated chiral algebra is already known. In the special case of the T_3 theory, the global symmetry is enhanced to E_6 and this is the classic theory of [12]. In that case, the four-dimensional level for the E_6 symmetry is $k_{4d}^{E_6} = 6$.

The generators of the Higgs branch chiral ring are known for these theories. There are always dimension two moment maps $\mu_{i=1,2,3}$ that transform in the adjoint of $SU(N)_i$ and obey the relation

$$\operatorname{Tr} \mu_1^k = \operatorname{Tr} \mu_2^k = \operatorname{Tr} \mu_3^k , \quad k = 2, \ldots, N . \tag{2.208}$$

These are supplemented by generators $Q_{(k)}$ of dimension $k(N-k)$ for $k = 1, \ldots, N-1$, which transform in the $(\wedge^k, \wedge^k, \wedge^k)$ representation of $SU(N)_1 \times SU(N)_2 \times SU(N)_3$, where $\wedge^k$ denotes the k-fold antisymmetric tensor representation. For $N = 2$ the only operator of this type is $Q_{(1)}$, which is the free hypermultiplet itself. The moment maps are actually composites of this basic operator. For the $N = 3$ case the operators are $Q_{(1)}$ and $Q_{(2)}$, which are the additional moment maps of E_6. For higher values of N, these are genuine new generators of the Higgs branch chiral ring, all with dimension greater than two. Some of the relations amongst these higher generators and the moment maps have been derived in [80], though we do not list them here. In the case of the E_6 theory, the full set of Higgs branch relations are precisely those that define the Joseph ideal for the E_6 one-instanton moduli space.

The trinion theories with reduced punctures (*i.e.*, with nontrivial defining embeddings ρ_i) can be thought of as arising by coupling the theory with a maximal puncture to a certain superconformal tail and then turning on specific Higgs branch vacuum expectation values [81, 82]. Though we do not write down the explicit formulae, the central charges for these theories can be computed for any choice of defining representations [72]. Important special cases are the trinions for which the theory that results from reducing the punctures of the non-Lagrangian T_N theory is described in terms of free fields. A canonical example is the theory where ρ_1 and ρ_2 are trivial, but ρ_3 is the subregular embedding of $\mathfrak{su}(2)$ into $\mathfrak{su}(N)$. In this case puncture three is known as a *minimal* punctures, and the resulting trinion theory is that of N free hypermultiplets.

Finally, we mention that index considerations suggest that there are no $\mathcal{D}$-type multiplets for these theories, in which case the HL chiral ring is just the Higgs chiral ring [13, 80].

2.6.2 An outline of class $\mathcal{S}$ chiral algebras

We now turn to the class of chiral algebras that form the image of the class $\mathcal{S}$ SCFTs under the map χ. In parallel with the full four-dimensional story, there will be a set of basic building block chiral algebras corresponding to the sphere with three maximal punctures. These will be the chiral algebras $\chi[\,T_N\,]$. General aspects of the chiral algebra correspondence allow us to predict a number of properties of these theories. The two-dimensional central charge is fixed by the usual proportionality with the four-dimensional conformal anomaly,

$$c_{2d} = -2N^3 + 3N^2 + N - 2 \ . \tag{2.209}$$

Additionally, these chiral algebras have $\widehat{\mathfrak{su}}(n)^3_k$ affine symmetry with

$$k_{2d} = -h^\vee \ . \tag{2.210}$$

It is interesting to note that this is precisely the level that is relevant for the connection between two-dimensional vertex algebras and the geometric Langlands program (see, *e.g.*, [83]). In addition to the generating currents of the affine flavor symmetry, the chiral algebra will have additional generators $\chi[Q_{(k)}]$ of holomorphic dimension $h = \frac{1}{2}k(N-k)$ transforming in the appropriate representations of the flavor symmetries.

For the case of the T_3 theory, the Higgs chiral ring generators are just the E_6 moment maps. The relations are generated by the E_6 Joseph ideal, and correspondingly the central charges of this theory saturate the appropriate unitarity bounds of §2.4.2. In particular, this means that the stress tensor is not an independent generator, but rather is equivalent to the Sugawara stress tensor of the E_6 current algebra (see §2.4.3). Given our prior experience in §2.5.1, it is natural to make a preliminary conjecture concerning the description of the T_3 chiral algebra:

Conjecture 4. *The chiral algebra for the rank one E_6 theory, also known as T_3, is isomorphic to the E_6 affine Lie algebra at level $k_{2d} = -3$.*

It is difficult to directly address this conjecture, since we do not have the free-field realization of this chiral algebra that was present for Lagrangian theories. Nevertheless, a variety of indirect checks have been performed and are presented in [75].

The chiral algebras associated to more general punctured Riemann surfaces can be realized in a procedure that parallels the gluing construction in four dimensions. In particular, for a given generalized quiver construction we start with a number of copies of $\chi[T_N]$ along with $SU(N)$ ghost small algebras, and then perform the BRST reduction associated to four-dimensional gauging to define the chiral algebra. Because the chiral algebra that is associated to a given four-dimensional theory is independent of the exactly marginal couplings, the chiral algebras associated to a given UV curve will not depend on the complex structure moduli of the curve, and in particular will not depend on the choice of generalized quiver within a given topological class. Thus, there will be a generalized topological quantum field theory that associates a chiral algebra to any appropriately decorated Riemann surface. This is very much in the spirit of [84] and [85], where the superconformal index and the symplectic holomorphic variety of the Higgs branch, respectively, were used to define a generalized TQFT via class $\mathcal{S}$. Associativity of the gluing imposes highly nontrivial requirements on the chiral algebra of the elementary T_N building block. There are three *a priori* inequivalent gauging procedures of two T_N theories that must lead to the unique theory associated to the four-punctured sphere. From the $2d$ perspective, the BRST complexes associated to the different gaugings must give the same cohomology. In the simple case of T_2, this follows at once from Conjecture 1, as the $\widehat{\mathfrak{so}}(8)$ current algebra is manifestly independent of the choice of gluing.

Having focused thus far on the case of maximal punctures, we should also consider chiral algebras $\chi[T_N^{(\rho_1,\rho_2,\rho_3)}]$ associated to the non-maximal theories. The task of reducing the rank of a puncture can be accomplished directly within the two-dimensional chiral algebra setting. We propose that the chiral algebra for the theory $T_N^{(\rho_1,\rho_2,\rho_3)}$ is determined by quantum Drinfeld-Sokolov (DS) reduction of the T_N theory with respect to the three embeddings. In the canonical setting, quantum DS reduction is an operation that is performed on an affine Lie algebra in order to produce a different $\mathcal{W}$-algebra as the cohomology of an appropriate BRST operator. In the present setting, the reduction is performed on a theory with an affine Lie subalgebra, so one may think of this as quantum DS reduction with modules. The generalization is conceptually straightforward, but somewhat involved technically. This proposal passes several checks, most notably that the central charges of the reduced theory precisely reproduce the expected answers. The behavior of the class $\mathcal{S}$ chiral algebras under the reduction of punctures imposes additional powerful constraints on the form of these two-dimensional theories. In particular, complete reduction of a puncture (corresponding to choosing a maximal embedding ρ) must lead to the chiral algebra for the theory with one fewer puncture. Similarly, reducing one maximal puncture in $\chi[T_N]$ to a minimal punctures must lead to the free hypermultiplet chiral algebra. A detailed discussion will be presented in [75].

The connection between reducing the rank of a puncture and quantum DS reduction has made previous appearances in the context of the AGT correspondence [86, 87], and the fact that the same procedure is used here suggests a deeper connection between the chiral algebras defined here and those that appear in the AGT correspondence.

2.7 Open questions

We have outlined the main features of a new surprising correspondence between the four-dimensional $\mathcal{N} = 2$ superconformal field theories and chiral algebras. It should be apparent that there is a great deal more to learn about this rich structure. There are many aspects that should be clarified further, and many natural directions in which the construction could be generalized. We will simply provide a concise list of what we consider to be the most salient open questions, some of which are currently under investigation.

- For the Lagrangian examples considered in §2.5, as well as the class $\mathcal{S}$ examples sketched in §2.6, we have made specific conjectures for the description of the resulting chiral algebras as $\mathcal{W}$-algebras. We hope that some of these conjectures can be proved by more advanced homological-algebraic techniques.

- A detailed analysis of the $\hat{\mathcal{B}}_1$ four-point function that compared $4d$ and $2d$ perspectives led to powerful new unitarity bounds that must be obeyed in any interacting $\mathcal{N} = 2$ SCFT with flavor symmetry. It is likely that applying the same methods to more general correlators will lead to further unitarity constraints.

- A better understanding of the implications of four-dimensional unitarity may help clarify what sort of chiral algebra can be associated to a four-dimensional theory. A sharp characterization of the class of chiral algebras that descend from four-dimensional SCFTs could prove invaluable, both as a source of structural insights and as a possible first step towards a classification program for $\mathcal{N} = 2$ SCFTs.

- We have seen that the four-dimensional operators that play a role in the chiral algebra are closely related to the ones that contribute to the Schur and Macdonald limits of the superconformal index. While the Schur limit has been interpreted in §2.4.4 as an index of the chiral algebra, the additional grading that appears in the Macdonald index is not natural in the framework that we have developed. It would be interesting if the additional refinement of the Macdonald index could be captured by a deformation of the chiral algebra structure, perhaps along the lines of [88].

- It seems inevitable that extended operators will ultimately find a place in our construction. We expect that codimension-two defects orthogonal to the chiral algebra plane will play the role of vertex operators transforming as non-trivial modules of the chiral algebra. One could also apply the tools developed here to study protected operators that live on conformal defects that fill the chiral algebra plane.

- As it was presented here, the definition of a protected chiral algebra appears to use extended superconformal symmetry in an essential way. Nevertheless, one wonders whether some aspects of this structure may survive away from conformality, perhaps after putting the theory on a nontrivial geometry.

- A related question is whether some aspects of our construction for Lagrangian theories may be accessible to the techniques of supersymmetric localization. The chiral algebra itself may emerge after an appropriate localization of the four-dimensional path integral.

- In many examples, the structure of the $4d$ Higgs branch appears to play a dominant role in determining the structure of the associated chiral algebra. It is an interesting question whether there is a sense in which the chiral algebra is an intrinsic property of the Higgs branch, possibly with some additional structure added as decoration.

- The structure that we have utilized in this article does not admit a direct generalization to odd space-time dimensions. However, a philosophically similar approach leads to a correspondence between three-dimensional $\mathcal{N} = 4$ superconformal field theories and one-dimensional topological field theories. The topological field theory captures twisted correlators of three-dimensional BPS operators whose positions are constrained to a line. We hope to return to investigate this structure in the future.

- The cohomological approach to chiral algebras that was successfully pursued in this article can be repeated in two-dimensional theories with at least $\mathcal{N} = (0, 4)$ superconformal symmetry and six-dimensional theories with $\mathcal{N} = (2, 0)$ superconformal symmetry [28]. As it was in the four-dimensional case, correlation functions of the six-dimensional chiral algebra should provide the jumping off point for a numerical bootstrap analysis of the elusive $(2, 0)$ theories.

- Combining the extension of this story to six dimensions with the inclusion of defect operators has the potential to provide a direct explanation for the AGT relation between conformal field theory in two-dimensions and $\mathcal{N} = 2$ supersymmetric field theories in four dimensions.

Chapter 3

Chiral Algebras for Trinion Theories

The contents of this chapter appear in [2]: "Chiral Algebras for Trinion Theories",
M. Lemos and W. Peelaers,
arXiv:1411.3252 [hep-th], JHEP **1502**, 113 (2015)
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3.1 Introduction and conclusions

In chapter 2 and Ref. [28] it was shown that even-dimensional extended superconformal field theories (SCFTs)[1] contain a protected subsector that is isomorphic to a two-dimensional chiral algebra. This subsector is obtained by restricting operators to be coplanar and treating them at the level of cohomology with respect to a particular nilpotent supercharge, obtained as a combination of a supercharge and a superconformal charge of the theory. In showing the existence of the chiral algebra one relies only on the symmetries of the theory and there is no need to have a Lagrangian description — a fact that was used to study the chiral algebras associated with the six-dimensional $(2,0)$-theory in [28] and with those obtained from four-dimensional theories of class $\mathcal{S}$ in [75]. In this chapter we will focus on the chiral algebras associated with the so-called trinion or T_n theories of class $\mathcal{S}$.

Chiral algebras of class $\mathcal{S}$, *i.e.*, the collection of chiral algebras associated with four-dimensional theories of class $\mathcal{S}$ [17, 18], were argued to take the form of a generalized topological quantum field theory (TQFT) in [75]. Within this TQFT, gluing, the operation associated to four-dimensional exactly marginal gauging, is achieved by solving a BRST cohomology problem, and partially closing a puncture is implemented via a quantum Drinfeld-Sokolov reduction. Furthermore, just as the isolated, strongly interacting T_n theories, *i.e.*, the theories whose UV-curve is a sphere with three punctures of maximal type, are the basic building blocks of class $\mathcal{S}$ theories, so are their associated chiral algebras the basic building

[1] More precisely $\mathcal{N}=(2,0)$ in $d=6$, $\mathcal{N}\geq 2$ in $d=4$, and "small" $\mathcal{N}=(0,4)$ and $\mathcal{N}=(4,4)$ in $d=2$.

blocks of said TQFT. Characterizing the T_n chiral algebras is thus a prerequisite for an in principle complete understanding of chiral algebras of class $\mathcal{S}$.

However, while the existence of a chiral algebra inside a generic $\mathcal{N}=2$ SCFT can be argued in general terms, a complete characterization of its generators is currently lacking.[2] As for a partial characterization, it was argued in 2 that one is guaranteed to have at least generators in one-to-one correspondence with the Higgs branch chiral ring generators.[3] In particular, the T_n Higgs branch chiral ring contains as generators three moment map operators, one for each factor in the T_n flavor symmetry algebra $\bigotimes_{i=1}^{3}\mathfrak{su}(n)_i$, and it was shown in 2 that their corresponding chiral algebra generators are three affine currents with affine levels $k_{2d,i}$ determined in terms of the four-dimensional flavor central charges $k_{4d,i}$ as $k_{2d,i}=-\frac{k_{4d,i}}{2}$. These central charges are equal for the three factors, $k_{4d,i}=2n$, and thus the affine current algebras $\widehat{\mathfrak{su}(n)}$ have critical level $k_{2d}\equiv k_{2d,i}=-n$. The remaining generators of the T_n Higgs branch chiral ring give rise to additional generators of the chiral algebra, which must be primaries of the affine Kac-Moody (AKM) algebras.

It was also shown in chapter 2 that the existence of a four-dimensional stress tensor implies that the chiral algebra must contain a meromorphic stress tensor. Therefore the global $\mathfrak{sl}(2)$ conformal algebra enhances to a Virasoro algebra, with the central charge fixed in terms of the four-dimensional c-anomaly coefficient by $c_{2d}=-12c_{4d}$. However, the stress tensor is not necessarily a new generator of the chiral algebra, as it could be a composite operator (*i.e.*, obtained from normal-ordered products of the generators and of their derivatives). Since the AKM current algebras are at the critical level, they do not admit a normalizable Sugawara stress tensor, and therefore the stress tensor can only be a composite if additional dimension two singlet composites can be constructed. This is only possible (and in fact happens) for $n=2$ and 3.

In the first part of this chapter we perform a detailed study of the graded partition function of the T_n chiral algebra, which can be computed thanks to its equality to the so-called Schur limit of the $\mathcal{N}=2$ superconformal index [13, 46], and which shows that the collection of generators listed so far is not complete for $n>4$ (see section 3.2). Motivated by this analysis, we conjecture the complete set of generators to be as follows:

Conjecture 5 (T_n chiral algebra)**.** *The T_n chiral algebra $\chi(T_n)$ is generated by*

- *The set of operators, $\mathcal{H}$, arising from the Higgs branch chiral ring:*
  - $\rightarrow$ *Three $\widehat{\mathfrak{su}(n)}$ affine currents J^1, J^2, J^3, at the critical level $k_{2d}=-n$, one for each factor in the flavor symmetry group of the theory,*

[2]For Lagrangian theories, this problem can (in principle) be circumvented by explicitly constructing the full chiral algebra from the basic known chiral algebras associated with the free hypermultiplet and vector multiplet.

[3]More generally, in the terminology of 2, all generators of the so-called Hall-Littlewood chiral ring give rise to generators of the chiral algebra. For class $\mathcal{S}$ theories with acyclic generalized quivers, such as the T_n theories, the Hall-Littlewood chiral ring equals the Higgs branch chiral ring.

→ *Generators* $W^{(k)}$, $k = 1, \ldots, n-1$ *in the* $(\wedge^k, \wedge^k, \wedge^k)$ *representation of* $\bigotimes_{j=1}^{3} \mathfrak{su}(n)_j$, *where* $\wedge^k$ *denotes the* $k-$*index antisymmetric representation of* $\mathfrak{su}(n)$. *These generators have dimensions* $\frac{k(n-k)}{2}$,

- *Operators* $\mathcal{O}_i$, $i = 1, \ldots n-1$, *of dimension* $h_i = i+1$ *and singlets under* $\bigotimes_{j=1}^{3} \mathfrak{su}(n)_j$, *with the dimension* 2 *operator corresponding to the stress tensor* T *of central charge* $c_{2d} = -2n^3 + 3n^2 + n - 2$,

modulo possible relations which set some of the operators listed above equal to composites of the remaining generators.

In other words, if one were to start with all generators of the above conjecture, one would find that some of them could be involved in null relations with composite operators, thereby being redundant. For example, in the chiral algebra associated with T_2, *i.e.*, the theory of eight free half-hypermultiplets, the affine currents and the stress tensor can be written as composites of the dimension $\frac{1}{2}$ generator $W^{(1)}$. For the case of T_3, which corresponds to the E_6 theory of [12], convincing evidence was provided in [75] that its chiral algebra $\chi(T_3)$ is fully generated by operators originating from the Higgs branch chiral ring. The stress tensor can be written as a composite and also, although not explicitly constructed in [75], the dimension three singlet operator is accounted for as a composite. For $n > 3$, as argued above, the stress tensor cannot be a composite of generators in $\mathcal{H}$, but the remaining dimension $3, \ldots, n$ singlet generators could still be. In the case of the T_4 chiral algebra the dimensions three and four singlet generators are redundant, as will be shown in section 3.3.

Our aim in the second part of this chapter is to verify Conjecture 5 for T_4, in which case the chiral algebra is generated by the operators in $\mathcal{H}$ and the stress tensor, by explicitly constructing an associative algebra with these generators. Our approach to bootstrap this problem is to write down the most general operator product expansions (OPEs) between the generators, and to demand associativity of the operator product algebra by imposing the Jacobi-identities. Since chiral algebras are very rigid, one can hope that these constraints are sufficiently stringent to completely fix the operator algebra, as was famously shown to be the case for the first time for the $\mathcal{W}_3$ algebra in [89] (see for example [49] for a review of other cases). We indeed find that the OPEs are completely and uniquely fixed. The analysis of the Jacobi-identities becomes technically involved in several instances, and as a result we can only claim that the conditions analyzed are necessary for an associative operator product algebra. However we believe that the remaining Jacobi-identities provide redundant constraints. As an interesting by-product of the explicit T_4 chiral algebra, we can compute four-dimensional Higgs branch chiral ring relations, which appear as null relations in the chiral algebra setting. Some of these relations are already known in the literature, (*e.g.*, [80, 90]), and recovering them here provides a further check of the chiral algebra, while others are new.

As mentioned, four-dimensional Higgs branch chiral ring relations can be obtained from null relations in the chiral algebra. The explicit construction of $\chi(T_4)$ we present here thus provides a new, conceptually clear method to obtain all Higgs branch chiral ring relations for the T_4 theory. It seems plausible that once their structure is understood, they can be generalized to arbitrary T_n. In this chapter we obtain all $\chi(T_4)$ null relations of dimension smaller than four, already uncovering new Higgs branch chiral ring relations, but the procedure can be taken further. For example, it would be possible to verify the recently proposed null relation of [91], as well as uncover further unknown ones. Furthermore, as will be elaborated upon in the next sections, our interpretation of the $\chi(T_n)$ chiral algebra partition function also predicts the existence of certain types of null relations, facilitating the task of explicitly computing them in the chiral algebra setting.

Further checks of the $\chi(T_4)$ chiral algebra could be performed by partially closing punctures (via a quantum Drinfeld-Sokolov (qDS) reduction (see [75])) to obtain the free hypermultiplet, the E_7 theory of [67], or more generally the other fixtures of [72]. For example, the chiral algebra associated with the E_7 theory is conjectured to be described by an affine $\widehat{\mathfrak{e}_7}$ current algebra at level $k_{2d} = -4$ and it is easy to convince oneself that the qDS procedure associated with the relevant $\mathfrak{su}(2)$ embedding will indeed result in dimension one currents corresponding to the decomposition of the $\mathfrak{e}_7$ adjoint representation. As shown in [75], to complete the reduction argument, certain null relations need to exist in order to remove redundant generators in the reduced algebra. Such null relations are expected to descend from those of $\chi(T_4)$.

The construction of $\chi(T_4)$ here made use of the constraints arising from associativity of the operator algebra. It would also be interesting to study if the theory space bootstrap, as introduced in [75], which imposes instead associativity of the TQFT structure, might result in a complementary route to construct the chiral algebra. In particular with an eye towards a construction of $\chi(T_n)$, for $n > 4$, an alternative (or a combined) approach might prove useful.

The organization of this chapter is as follows. In section 3.2 we analyze the partition function of $\chi(T_n)$ employing its equality to the superconformal index of T_n theories, and show how it motivates Conjecture 5, as well as some other expectations about the chiral algebra. In section 3.3 we present the explicit construction of the T_4 chiral algebra and give explicit expressions for various null relations. We also show how our expectations deduced from the superconformal index are realized for T_4. The readers interested only in the explicit construction of $\chi(T_4)$ can safely skip section 3.2 as section 3.3 is mostly independent from it. Appendix D contains some technical details on the relation between critical affine characters and the superconformal index, and in appendix E we collect all singular OPEs defining the chiral algebra $\chi(T_4)$.

3.2 T_n indexology

In this section we analyze the partition function of the T_n chiral algebra, which gives insights into its generators and relations. By writing the partition function in a suggestive way we can justify Conjecture 5 and infer some properties of the structure of the chiral algebra, such as its null relations.

As shown in 2, the graded partition function of the chiral algebra $\chi(T_n)$ equals the so-called Schur limit of the superconformal index of the T_n theory [13, 46]. We work under the assumption that all generators are bosonic and thus the grading is immaterial. In appendix D we show that the index can be written in a way suggestive of its interpretation as a two-dimensional partition function as

$$\mathcal{Z}_{\chi(T_n)}(q;\mathbf{x}_i) = \sum_{\mathfrak{R}_\lambda} q^{\langle\lambda,\rho\rangle} C_{\mathfrak{R}_\lambda}(q) \prod_{i=1}^{3} \mathrm{ch}_{\mathfrak{R}_\lambda}(q,\mathbf{x}_i)\,. \tag{3.1}$$

Here $\mathbf{x}_i$ denote flavor fugacities conjugate to the Cartan generators of the $\mathfrak{su}(n)_i$ flavor symmetry associated with each of the three punctures, and the sum runs over all irreducible $\mathfrak{su}(n)$ representations $\mathfrak{R}_\lambda$ of highest weight λ. The summand contains the product of three copies — one for each puncture — of $\mathrm{ch}_{\mathfrak{R}_\lambda}(q,\mathbf{x})$, the character of the critical irreducible highest weight representation of the affine current algebra $\widehat{\mathfrak{su}(n)}_{-n}$ with highest weight $\hat{\lambda}$, whose restriction to $\mathfrak{su}(n)$ is the highest weight λ [92].[4] Furthermore, ρ is the Weyl vector and $\langle\cdot,\cdot\rangle$ denotes the Killing inner product. Finally, the structure constants $C_{\mathfrak{R}_\lambda}(q)$ can be written as

$$C_{\mathfrak{R}_\lambda}(q) = \mathrm{P.E.}\left[2\sum_{j=1}^{n-1}\frac{q^{d_j}}{1-q} + 2\sum_{j=1}^{n-1}(n-j)\,q^j - 2\sum_{j=2}^{n}\sum_{1\le i<j} q^{\ell_i-\ell_j+j-i}\right]\,, \tag{3.2}$$

where $\ell_{i=1,\ldots,n}$ denote the lengths of rows of the Young tableau describing representation $\mathfrak{R}_\lambda$ with $\ell_n = 0$, d_j are the degrees of invariants, *i.e.* $d_j = j+1$ for $\mathfrak{su}(n)$, and finally P.E. denotes the standard plethystic exponential

$$\mathrm{P.E.}\,[f(x)] = \exp\left(\sum_{m=1}^{\infty}\frac{f(x^m)}{m}\right)\,. \tag{3.3}$$

Let us provide some preliminary interpretative comments:

- We have obtained an expression for the partition function (3.1) that is manifestly organized in terms of modules of the direct product of the three critical affine current algebras $\bigotimes_{i=1}^{3}(\widehat{\mathfrak{su}(n)_i})_{-n}$. Indeed, the factor $q^{\langle\lambda,\rho\rangle}\prod_{i=1}^{3}\mathrm{ch}_{\mathfrak{R}_\lambda}(q,\mathbf{x}_i)$ in (3.1) captures threefold AKM primaries of dimension $\langle\lambda,\rho\rangle$, transforming in representations $(\mathfrak{R}_\lambda,\mathfrak{R}_\lambda,\mathfrak{R}_\lambda)$, including for example all the $W^{(k)}$, and all of their AKM descendants.
- The role of the structure constants is to encode additional operators beyond those captured by the threefold AKM modules. In particular, in the term $\mathfrak{R}_{\lambda=0}$ in the sum

[4] Our notation here and in appendix D follows that of [93].

over representations, the structure constant $C_{\mathfrak{R}_{\lambda=0}}(q) = \text{P.E.}\left[2\sum_{j=1}^{n-1}\frac{q^{d_j}}{1-q}\right]$ encodes two sets of additional operators of dimensions $d_j = j+1$, for $j = 1, \ldots, n-1$, (and their $\mathfrak{sl}(2)$ descendants) acting on the vacuum module. These operators can either be new generators, or obtained as singlet composites of the generators captured by the AKM modules, which themselves are not present in the modules. Let us now describe these two sets:

1. The fact that the three AKM current algebras are at the critical level implies that all the Casimir operators $\text{Tr}(J^1)^k,\ \text{Tr}(J^2)^k,\ \text{Tr}(J^3)^k$ with $k = 2, 3, \ldots, n$ are null within their respective AKM algebra, and therefore that their action is not included in the affine modules. However, these operators do not remain null in the full chiral algebra, as it contains a stress tensor as well. In fact, null relations set all Casimirs equal $\text{Tr}(J^1)^k = \text{Tr}(J^2)^k = \text{Tr}(J^3)^k$.[5] These $n-1$ Casimirs correspond to the first set of operators reinstated by the structure constants.
2. The second set of operators motivates our conjecture that there can be extra generators $\mathcal{O}_i$ with precisely dimensions $h_i = d_i = i+1$.

A more detailed discussion of these statements, and the interpretation of the two remaining factors in (3.2) is given in the remainder of this section. Readers not interested in this technical analysis can safely skip the remainder of this section.

The AKM modules

Ignoring for a moment the structure constants, each term in the sum over representations $\mathfrak{R}_\lambda$ of (3.1) captures the states in the direct product of three critical affine modules with primary transforming in representation $(\mathfrak{R}_\lambda, \mathfrak{R}_\lambda, \mathfrak{R}_\lambda)$. The dimension of the threefold AKM primary is implemented by the factor $q^{\langle\lambda,\rho\rangle}$, yielding

$$h_{(\mathfrak{R}_\lambda,\mathfrak{R}_\lambda,\mathfrak{R}_\lambda)} = \langle\lambda,\rho\rangle = \sum_{i=1}^{n-1}\frac{n-(2i-1)}{2}\ell_i\,. \tag{3.4}$$

These pairings of dimension and representations include all the threefold AKM primary generators $W^{(k)},\ k = 1, \ldots, n-1$ in Conjecture 5. (Note that the currents themselves are AKM descendants of the identity operator and appear in the vacuum module.) We expect that the remaining threefold AKM primaries in the sum over $\mathfrak{R}_\lambda$ all arise from combinations of normal-ordered products of generators in $\mathcal{H}$ (the set of generators originating from the Higgs branch chiral ring generators), and do not give rise to additional generators. It is clear that for each representation $(\mathfrak{R}_\lambda, \mathfrak{R}_\lambda, \mathfrak{R}_\lambda)$ one can write down a composite operator of

[5]The existence of these null relations follows directly from the existence of relations on the Higgs branch chiral ring setting the Casimir operators formed out of the moment map operators of the three flavor symmetries equal [80]. The corresponding chiral algebra null relations will be recovered in the next section.

the $W^{(k)}$, transforming in such representation, and with the appropriate dimension. Then, it seems plausible that such operator can always be made into a threefold AKM primary by — if necessary — adding composites of the remaining operators in $\mathcal{H}$. We have checked this statement in a few low-dimensional examples for T_4 (see equation (3.15) for an explicit example). All in all, the AKM modules capture the generators $W^{(k)}$, as well as other threefold AKM primaries obtained as their normal-ordered product, and all of their AKM descendants.

The structure constants

The structure constants (3.2) encode additional operators on top of those captured by the AKM modules already described. Let us start by analyzing the factor

$$\text{P.E.}\left[2\sum_{j=1}^{n-1}\frac{q^{d_j}}{1-q}\right]. \tag{3.5}$$

When inserted in (3.1), it encodes two sets of operators of dimensions d_j and their $\mathfrak{sl}(2)$ descendants (taken into account by the denominator $\frac{1}{1-q}$), normal-ordered with all operators in any given AKM module $(\mathfrak{R}_\lambda, \mathfrak{R}_\lambda, \mathfrak{R}_\lambda)$. As described before, one set adds back the Casimir operators $\text{Tr}(J^1)^k = \text{Tr}(J^2)^k = \text{Tr}(J^3)^k$ of the AKM algebras,[6] and the second set motivates the claim that there can be additional generators $\mathcal{O}_{i=1,\ldots,n-1}$ of dimensions $h_i = d_i = i + 1$.[7] However, one should bear in mind that in some cases one can construct (non-null) non-AKM-descendant singlet operators as composites of the $W^{(k)}$ of dimensions h equal to one of these dimensions. Since the only singlet operator in the sum over AKM modules, which is not an AKM descendant, corresponds to the identity operator, such operators must be accounted for by (3.5). This leaves two possibilities: it is either equal (or set equal by a null relation) to a composite of smaller dimensional $\mathcal{O}_i$ operators and/or of Casimirs, and consequently taken into account by the plethystic exponentiation in (3.5). Or it must take the place of the would-be generator $\mathcal{O}$ of dimension h. In other words, if one were to include $\mathcal{O}$, one would find a null relation between this would-be generator and the composite of $W^{(k)}$. As was mentioned before, the simplest example is the stress tensor $T \equiv \mathcal{O}_1$, which for T_2 and T_3 is a composite, but for $T_{n\geq 4}$ must be a new generator. In the next section we will show that for T_4 the generators of dimension three and four are absent, as the type of composites described above exist. However for $n \geq 5$ it is not possible to write such a composite of dimension three, and $\mathcal{O}_2$ must be a generator.

[6] Both the Casimirs and the Casimirs normal-ordered with threefold AKM primaries are new threefold AKM primaries, since they were null if one were to consider each AKM current algebra in isolation.

[7] For readers familiar with the classification of four-dimensional superconformal multiplets of [52], these generators arise from four-dimensional operators in the $\hat{\mathcal{C}}$ multiplets.

We now turn to the next factor in the structure constants (3.2)

$$\text{P.E.}\left[2\sum_{k=1}^{n-1}(n-k)\,q^k\right]. \tag{3.6}$$

Recalling that at the critical level the stress tensor is not obtained from the Sugawara construction, the critical modules do not contain derivatives[8] of the threefold AKM primaries, although the full chiral algebra must. Similarly, the action of the modes $(\mathcal{O}_i)_{-1}$, $i=2,\ldots,n-1$, $(\mathcal{O}_i)_{-2}$, $i=2,\ldots,n-1$, $(\mathcal{O}_i)_{-3}$, $i=3,\ldots,n-1$, $\ldots$ of the remaining singlet operators of the chiral algebra are not yet included. The number of modes we have to take into account at grade k is precisely given by $n-k$, and these modes are added by one of the factors in (3.6). The other factor adds back similar modes of the Casimir operators of the AKM current algebras, an explicit example of which will be given in the next section (see (3.14)). It is clear that these modes cannot be added for all representations: for example, they cannot be added when considering the vacuum, since it is killed by all of them. Similarly, (and here we restrict to $n>2$) the only grade one modes acting on any of the $W^{(k)}$ that do not kill it must be the ones which correspond to either acting on it with a derivative, or normal-ordering it with a current, since these are the only ways one can write a dimension $\frac{k(n-k)}{2}+1$ composite in representation $(\wedge^k,\wedge^k,\wedge^k)$.[9] These facts are taken into account by the factor

$$\text{P.E.}\left[-2\sum_{j=2}^{n}\sum_{1\leq i<j}q^{\ell_i-\ell_j+j-i}\right], \tag{3.7}$$

which must subtract such relations, as well as other possible relations specific of each representation. Indeed, it is for example easy to verify that (3.6) and (3.7) cancel each other for the vacuum module. For representations $(\wedge^k,\wedge^k,\wedge^k)$ only two q terms survive in the plethystic exponential in the product of (3.6) and (3.7), which means that we are left with two grade one modes. One might have expected four grade one modes: one corresponding to acting with a derivative and three to normal-ordering with the three currents, but, as we will see in the next section, normal-ordering the three currents with $W^{(k)}$ (making an operator in representation $(\wedge^k,\wedge^k,\wedge^k)$) results in equal operators up to nulls (see equations (3.11) and (3.14)).

As a final observation we note that the sum in (3.1) only runs over flavor symmetry representations of the type $(\mathfrak{R}_\lambda,\mathfrak{R}_\lambda,\mathfrak{R}_\lambda)$, and the structure constants (3.2) cannot alter flavor symmetry information. Therefore the partition function predicts that any operator transforming in a representation $(\mathfrak{R}_{\lambda_1},\mathfrak{R}_{\lambda_2},\mathfrak{R}_{\lambda_3})$ with not all equal λ_i cannot be a three-

[8] *i.e.*, the action of the mode L_{-1}. As is common practice we use the mode expansion $\mathcal{O}(z)=\sum_n \frac{\mathcal{O}_n}{z^{n+h}}$ of an operator $\mathcal{O}$ of dimension h, and L_n denotes the modes of the stress tensor T.

[9] Note that the normal-ordered product $(JW^{(k)})$ in representation $(\wedge^k,\wedge^k,\wedge^k)$ is absent in the critical module.

fold AKM primary. More precisely, if we encounter an operator in unequal representations $(\mathfrak{R}_{\lambda_1}, \mathfrak{R}_{\lambda_2}, \mathfrak{R}_{\lambda_3})$ it must either be an AKM descendant, or obtained from one via the operators taken into account by the structure constants (namely by the action of any operators contributing to (3.5) and (3.6)). We will get back to this point in the next section (around example (3.12)).

3.3 The T_4 chiral algebra

For the chiral algebra associated with the T_4 theory, Conjecture 5 states that the collection of generators $\mathcal{G}$ contains three $\widehat{\mathfrak{su}(4)}$ affine currents at the critical level $k_{2d} = -4$, which we denote by $(J^1)^{b_1}_{a_1}$, $(J^2)^{b_2}_{a_2}$, $(J^3)^{b_3}_{a_3}$, two dimension $\frac{3}{2}$ generators, $W^{(1)}$ and $W^{(3)}$, in the tri-fundamental and tri-antifundamental representations of the flavor symmetry group respectively, which we rename $W_{a_1a_2a_3}$ and $\widetilde{W}^{b_1b_2b_3}$, and one dimension two generator, $W^{(2)}$, in the $\mathbf{6}\times\mathbf{6}\times\mathbf{6}$ representation which we denote explicitly as $V_{[a_1b_1][a_2b_2][a_3b_3]}$. Here $a_i, b_i, c_i, \ldots = 1, 2, 3, 4$ are (anti)fundamental indices corresponding to the flavor symmetry factor $\mathfrak{su}(4)_{i=1,2,3}$. Moreover, we must add the stress tensor T as an independent generator, with central charge $c_{2d} = -78$, but we claim that the dimension three and four singlets operators can be obtained as composites. As will be shown later the dimension three operator is argued to be a Virasoro primary involving $W\widetilde{W}|_{\text{sing}}$, where $|_{\text{sing}}$ means we take the singlet combination, while the dimension four generator is a Virasoro primary combination involving $VV|_{\text{sing}}$. We summarize the conjectured generators in Table 3.1.

generator $\mathcal{G}$	$h_\mathcal{G}$	$\mathcal{R}_\mathcal{G}$
$(J^1)^{a_1}_{b_1}$	1	$(\mathbf{15},\mathbf{1},\mathbf{1})$
$(J^2)^{a_2}_{b_2}$	1	$(\mathbf{1},\mathbf{15},\mathbf{1})$
$(J^3)^{a_3}_{b_3}$	1	$(\mathbf{1},\ \mathbf{1},\mathbf{15})$
T	2	$(\mathbf{1},\mathbf{1},\mathbf{1})$
$W_{a_1a_2a_3}$	$\frac{3}{2}$	$(\mathbf{4},\mathbf{4},\mathbf{4})$
$\widetilde{W}^{a_1a_2a_3}$	$\frac{3}{2}$	$(\bar{\mathbf{4}},\bar{\mathbf{4}},\bar{\mathbf{4}})$
$V_{[a_1b_1][a_2b_2][a_3b_3]}$	2	$(\mathbf{6},\mathbf{6},\mathbf{6})$

Table 3.1: T_4 generators $\mathcal{G}$, their dimension $h_\mathcal{G}$ and their $\mathfrak{su}(4)^3$ representation $\mathfrak{R}_\mathcal{G}$.

As mentioned before, our strategy for finding the T_4 chiral algebra is a concrete implementation of the conformal bootstrap program. We start by writing down the most general OPEs for this set of generators consistent with the symmetries of the theory, and in particular we impose that the three different flavor symmetry groups appear on equal footing. This of course implies that the three flavor currents have the same affine level, simply denoted by

k_{2d}. The OPEs of all the generators with the stress tensor are naturally fixed to be those of Virasoro primaries with the respective dimensions. Moreover, all generators listed in Table 3.1, with the exception of the stress tensor[10], are affine Kac-Moody primaries of the three current algebras, transforming in the indicated representation. Therefore their OPEs with the currents are also completely fixed. In the self-OPEs of the AKM currents and the stress tensor, we could fix the affine level and the central charge to the values corresponding to the $\chi(T_4)$ chiral algebra, $k_{2d} = -4$ and $c_{2d} = -78$. Instead we leave them as free parameters and try to fix them the same way as any other OPE coefficient. For the remaining OPEs we write all possible operators allowed by the symmetries of the theory with arbitrary coefficients. Our expectation is that this chiral algebra is unique, and that by imposing associativity one can fix all the OPE coefficients, including k_{2d} and c_{2d}. This indeed turns out to be true. Some of the resulting OPEs are quite long so we collect them all in appendix E.[11]

The next step is to fix all the arbitrary coefficients by imposing Jacobi-identities, implementing in this way the requirement that the operator algebra is associative. Concretely, we impose on any combination of three generators A, B, C the Jacobi-identities (see, *e.g.*, [94])

$$[A(z)\,[B(w)C(u)]] - [B(w)\,[A(z)C(u)]] - [[A(z)B(w))]\,C(u)] = 0\,, \tag{3.8}$$

for $|w-u| < |z-u|$, where $[A(z)B(w)]$ denotes taking the singular part of the OPE of $A(z)$ and $B(w)$, and where we already took into account that our generators are bosonic and no extra signs are needed. It is important to note that the Jacobi-identities do not need to be exactly zero, but they can be proportional to null operators. Since null operators decouple, associativity of the algebra is not spoiled. For analyzing the Jacobi-identities we make use of the `Mathematica` package described in [55]. Even so, the analysis is quite cumbersome due to the large number of fields appearing in the OPEs and the necessity of removing null relations, especially so for the Jacobi-identities involving the generator V. These null relations are not known *a priori*, therefore part of the task consists of obtaining all null operators at each dimension and in a given representation of the flavor symmetry. Due to these technical limitations we have only found necessary conditions for the Jacobi-identities to be satisfied, not sufficient ones. Nevertheless these conditions turn out to fix completely all the OPE coefficients, including the level and the central charge of the theory, meaning the chiral algebra with this particular set of generators is unique. After all coefficients are fixed, the remaining Jacobi-identities analyzed serve as a test on the consistency of our chiral algebra. We have checked a large enough set of Jacobi-identities to be convinced that the remaining ones will not give any additional constraints. If that is the case we have found an associative operator

[10]It is clear that the stress tensor cannot be an AKM primary, as the OPE between a dimension one operator (the current) and the stress tensor must have necessarily a $\frac{1}{z^2}$ pole. It is also not an AKM descendant, since at the critical level it cannot be given by the Sugawara construction.

[11]In there and in what follows we adopt the standard conventions for the normal-ordering of operators such that $\mathcal{O}_1\mathcal{O}_2\ldots\mathcal{O}_{\ell-1}\mathcal{O}_\ell = (\mathcal{O}_1\,(\mathcal{O}_2\ldots(\mathcal{O}_{\ell-1}\mathcal{O}_\ell)\,...))$.

algebra with the same set of generators and the same central charges as conjectured for the T_4 chiral algebra. A further check that the chiral algebra we constructed corresponds indeed to the T_4 chiral algebra can be performed by comparing the partition function of the former to the one of the latter (which is nothing else than the superconformal index of T_4). Whereas in section 3.2 we have exploited the index to motivate our claim about the full set of generators of the T_4 chiral algebra, in what follows we perform a partial check of the equality of the actual partition function of the constructed chiral algebra with the index by comparing the null states of the chiral algebra to the ones predicted by the superconformal index up to dimension $\frac{7}{2}$. Even if there were generators that we have missed in this analysis, the facts that the generators in our chiral algebra must be present, and that the chiral algebra we constructed is closed (assuming we have solved all constraints from the Jacobi-identities), imply that we have found a closed subalgebra of the full T_4 chiral algebra.

h	mult.	$\mathcal{R}$	h	mult.	$\mathcal{R}$
2	2	$(\mathbf{1},\mathbf{1},\mathbf{1})$	$\frac{7}{2}$	10	$(\mathbf{4},\mathbf{4},\mathbf{4})$, $(\bar{\mathbf{4}},\bar{\mathbf{4}},\bar{\mathbf{4}})$
$\frac{5}{2}$	2	$(\mathbf{4},\mathbf{4},\mathbf{4})$, $(\bar{\mathbf{4}},\bar{\mathbf{4}},\bar{\mathbf{4}})$		2	$(\mathbf{36},\mathbf{4},\mathbf{4})$, $(\bar{\mathbf{36}},\bar{\mathbf{4}},\bar{\mathbf{4}})$, and perms.
3	4	$(\mathbf{1},\mathbf{1},\mathbf{1})$		1	$(\mathbf{20},\mathbf{20},\mathbf{4})$, $(\bar{\mathbf{20}},\bar{\mathbf{20}},\bar{\mathbf{4}})$, and perms.
	2	$(\mathbf{6},\mathbf{6},\mathbf{6})$		3	$(\mathbf{20},\mathbf{4},\mathbf{4})$, $(\bar{\mathbf{20}},\bar{\mathbf{4}},\bar{\mathbf{4}})$, and perms.
	1	$(\mathbf{6},\mathbf{6},\mathbf{10})$, and perms.			
	3	$(\mathbf{15},\mathbf{1},\mathbf{1})$, and perms.			
	1	$(\mathbf{15},\mathbf{15},\mathbf{1})$, and perms.			

Table 3.2: Quantum numbers and multiplicities of T_4 null operators up to dimension $\frac{7}{2}$.

For practical purposes, it is useful to rewrite the partition function of the T_n chiral algebra (3.1) alternatively as

$$\mathcal{Z}_{\chi(T_n)}(q;\mathbf{x}_i) = \text{P.E.}\left[\frac{1}{1-q}\sum_{\text{generators }\mathcal{G}} q^{h_{\mathcal{G}}}\chi^{\mathfrak{su}(n)^3}_{\mathcal{R}_{\mathcal{G}}}(\mathbf{x}_i)\right] - \sum_{\text{nulls }\mathcal{N}} q^{h_{\mathcal{N}}}\chi^{\mathfrak{su}(n)^3}_{\mathcal{R}_{\mathcal{N}}}(\mathbf{x}_i)\,, \tag{3.9}$$

in terms of a piece that describes the generators $\mathcal{G}$ of dimensions $h_{\mathcal{G}}$ transforming in representations $\mathcal{R}_{\mathcal{G}}$ of the $\mathfrak{su}(n)^3$ flavor symmetry and their $\mathfrak{sl}(2)$ descendants, and a term that subtracts off explicitly the null operators $\mathcal{N}$, of dimension $h_{\mathcal{N}}$ and in representation $\mathcal{R}_{\mathcal{N}}$. By comparing the expansion in powers of q of (3.1) with that of (3.9) (and under the assumption that the full list of generators is as in Table 3.1) we can predict how many nulls to expect in each representation and at each dimension. In Table 3.2 we summarize the resulting quantum numbers of the low-lying null operators $\mathcal{N}$. We have explicitly constructed the null operators corresponding to the entries in Table 3.2; the full list is given in Tables 3.3 and 3.4, where we have defined S^i to be the quadratic Casimir $S^i = (J^i)^{b_i}_{a_i}(J^i)^{a_i}_{b_i}$.

$h_{\mathcal{N}}$	$\mathcal{R}_{\mathcal{N}}$	Null relations
2	$(\mathbf{1},\mathbf{1},\mathbf{1})$	$(J^1)^{b_1}_{a_1}(J^1)^{a_1}_{b_1} = (J^2)^{b_2}_{a_2}(J^2)^{a_2}_{b_2} = (J^3)^{b_3}_{a_3}(J^3)^{a_3}_{b_3}$
5/2	$(\mathbf{4},\mathbf{4},\mathbf{4})$	$(J^1)^{b_1}_{a_1}W_{b_1a_2a_3} = (J^2)^{b_2}_{a_2}W_{a_1b_2a_3} = (J^3)^{b_3}_{a_3}W_{a_1a_2b_3}$
3	$(\mathbf{1},\mathbf{1},\mathbf{1})$	$(J^1)^{b_1}_{a_1}(J^1)^{c_1}_{b_1}(J^1)^{a_1}_{c_1} = (J^2)^{b_2}_{a_2}(J^2)^{c_2}_{b_2}(J^2)^{a_2}_{c_2} = (J^3)^{b_3}_{a_3}(J^3)^{c_3}_{b_3}(J^3)^{a_3}_{c_3}$ $\partial\left((J^1)^{b_1}_{a_1}(J^1)^{a_1}_{b_1}\right) = \partial\left((J^2)^{b_2}_{a_2}(J^2)^{a_2}_{b_2}\right) = \partial\left((J^3)^{b_3}_{a_3}(J^3)^{a_3}_{b_3}\right)$
3	$(\mathbf{6},\mathbf{6},\mathbf{6})$	$(J^1)^{c_1}_{[a_1}V_{[b_1]c_1][a_2b_2][a_3b_3]} = (J^2)^{c_2}_{[a_2}V_{[a_1b_1][b_2]c_2][a_3b_3]} = (J^3)^{c_3}_{[a_3}V_{[a_1b_1][a_2b_2][b_3]c_3]}$
3	$(\mathbf{10},\mathbf{6},\mathbf{6})$	$W_{(a_1[a_2[a_3}W_{b_1)b_2]b_3]} = -\frac{1}{4}J^{c_1}_{(a_1}V_{[\lvert c_1\rvert b_1)][a_2b_2][a_3b_3]}$
3	$(\mathbf{15},\mathbf{1},\mathbf{1})$	$(J^1)^{b_1}_{a_1}(J^1)^{d_1}_{c_1}(J^1)^{c_1}_{d_1} = (J^1)^{b_1}_{a_1}(J^2)^{b_2}_{a_2}(J^2)^{a_2}_{b_2} = (J^1)^{b_1}_{a_1}(J^3)^{b_3}_{a_3}(J^3)^{a_3}_{b_3}$ $(W_{a_1a_2a_3}\tilde{W}^{b_1a_2a_3} - \text{trace}) = \frac{1}{16}(J^1)^{b_1}_{a_1}(J^3)^{b_3}_{a_3}(J^3)^{a_3}_{b_3} + \frac{1}{16}(J^1)^{b_1}_{a_1}T$ $+\left[((J^1)^{c_1}_{a_1}\partial(J^1)^{b_1}_{c_1} - \text{trace}) + ((J^1)^{b_1}_{c_1}\partial(J^1)^{c_1}_{a_1} - \text{trace})\right]$ $-\frac{1}{4}((J^1)^{c_1}_{a_1}(J^1)^{d_1}_{c_1}(J^1)^{b_1}_{d_1} - \text{trace})$
3	$(\mathbf{15},\mathbf{15},\mathbf{1})$	$(W_{a_1a_2a_3}\tilde{W}^{b_1b_2a_3} - \text{traces}) = \frac{1}{4}\left[(J^1)^{b_1}_{a_1}\partial(J^2)^{b_2}_{a_2} + (J^2)^{b_2}_{a_2}\partial(J^1)^{b_1}_{a_1}\right]$ $-\frac{1}{16}\left[((J^1)^{b_1}_{a_1}(J^2)^{c_2}_{a_2}(J^2)^{b_2}_{c_2} - \text{trace}) + ((J^2)^{b_2}_{a_2}(J^1)^{c_1}_{a_1}(J^1)^{b_1}_{c_1} - \text{trace})\right]$

Table 3.3: Explicit null relations up to dimension three, which can be uplifted to four-dimensional Higgs branch chiral ring relations. Representations which are not real give rise to a similar null in the complex conjugate representation, and representations which are not equal in the three flavor groups give rise to similar null relations with permutations of the flavor group indices. Note that these, together with Table 3.4, are in one-to-one correspondence to the null relations subtracted from the index given in Table 3.2.

Null relations in the two-dimensional chiral algebra can be uplifted to four-dimensional Higgs branch chiral ring relations, a partial list of which is given in [80], by setting to zero all derivatives and generators not coming from the Higgs branch chiral ring (in particular the stress tensor and the other singlet generators, if present as independent generators). The nulls of Tables 3.3 and 3.4 allow one to recover the low-dimensional Higgs branch chiral ring relations in [80], and to find additional ones. Let us give a few illustrative examples.

$h_{\mathcal{N}}$	$\mathcal{R}_{\mathcal{N}}$	Null relations
$7/2$	$(\mathbf{20},\mathbf{4},\mathbf{4})$	$8\widetilde{W}^{f_1b_2b_3}V_{[d_1(e_1][a_2b_2][a_3b_3]}\ \epsilon_{\lvert f_1\rvert a_1)b_1c_1}$ $= 9(J^1)^{f_1}_{[a_1}(J^1)^{g_1}_{b_1}W_{(c_1]a_2a_3}\ \epsilon_{d_1)e_1f_1g_1} - 2(J^1)^{f_1}_{[d_1}(J^3)^{b_3}_{a_3}W_{(e_1]a_2b_3}\ \epsilon_{\lvert f_1\rvert a_1)b_1c_1}$ $+3\partial(J^1)^{f_1}_{[d_1}W_{(e_1]a_2a_3}\ \epsilon_{\lvert f_1\rvert a_1)b_1c_1} + 6\partial\left((J^1)^{f_1}_{[d_1}W_{(e_1]a_2a_3}\right)\ \epsilon_{\lvert f_1\rvert a_1)b_1c_1}$ $(J^1)^{f_1}_{[d_1}(J^2)^{b_2}_{a_2}W_{(e_1]b_2a_3}\ \epsilon_{\lvert f_1\rvert a_1)b_1c_1} = (J^1)^{f_1}_{[d_1}(J^3)^{b_3}_{a_3}W_{(e_1]a_2b_3}\ \epsilon_{\lvert f_1\rvert a_1)b_1c_1}$ $= (J^1)^{f_1}_{[d_1}(J^1)^{g_1}_{(e_1]}W_{\lvert g_1\rvert a_2a_3}\ \epsilon_{\lvert f_1\rvert a_1)b_1c_1}$
$7/2$	$(\mathbf{20},\mathbf{20},\mathbf{4})$	$\widetilde{W}^{f_1f_2b_3}V_{[d_1(e_1][d_2(e_2][a_3b_3]}\ \epsilon_{\lvert f_1\rvert a_1)b_1c_1}\ \epsilon_{\lvert f_2\rvert a_2)b_2c_2}$ $= -\frac{1}{2}(J^1)^{f_1}_{[d_1}(J^2)^{f_2}_{[d_2}W_{(e_1](e_2]a_3}\ \epsilon_{\lvert f_1\rvert a_1)b_1c_1}\ \epsilon_{\lvert f_2\rvert a_2)b_2c_2}$
$7/2$	$(\mathbf{36},\mathbf{4},\mathbf{4})$	$(J^1)^{b_1}_{(a_1}(J^2)^{b_2}_{a_2}W_{c_1b_2a_3}\ \epsilon_{d_1)b_1e_1f_1} = (J^1)^{b_1}_{(a_1}(J^3)^{b_3}_{a_3}W_{c_1a_2b_3}\ \epsilon_{d_1)b_1e_1f_1} =$ $(J^1)^{b_1}_{(a_1}(J^1)^{h_1}_{c_1}W_{\lvert h_1\rvert a_2a_3}\ \epsilon_{d_1)b_1e_1f_1}$
$7/2$	$(\mathbf{4},\mathbf{4},\mathbf{4})$	$S^1W_{a_1a_2a_3} = S^2W_{a_1a_2a_3} = S^3W_{a_1a_2a_3}$ $8\widetilde{W}^{b_1b_2b_3}V_{[b_1a_1][b_2a_2][b_3a_3]} = 2(J^1)^{b_1}_{a_1}(J^3)^{b_3}_{a_3}W_{b_1a_2b_3} + 9TW_{a_1a_2a_3}$ $+15\partial\left((J^3)^{b_3}_{a_3}W_{a_1a_2b_3}\right) - \frac{9}{2}\partial^2W_{a_1a_2a_3} - \frac{3}{2}S^1W_{a_1a_2a_3}$ $-9\left((J^1)^{b_1}_{a_1}\partial W_{b_1a_2a_3} + (J^2)^{b_2}_{a_2}\partial W_{a_1b_2a_3} + (J^3)^{b_3}_{a_3}\partial W_{a_1a_2b_3}\right)$ $(J^1)^{b_1}_{a_1}(J^2)^{b_2}_{a_2}W_{b_1b_2a_3} = (J^1)^{b_1}_{a_1}(J^3)^{b_3}_{a_3}W_{b_1a_2b_3} = (J^2)^{b_2}_{a_2}(J^3)^{b_3}_{a_3}W_{a_1b_2b_3}$ $= (J^1)^{b_1}_{a_1}(J^1)^{c_1}_{b_1}W_{c_1a_2a_3} = (J^2)^{b_2}_{a_2}(J^2)^{c_2}_{b_2}W_{a_1c_2a_3} = (J^3)^{b_3}_{a_3}(J^3)^{c_3}_{b_3}W_{a_1a_2c_3}$ $\partial\left[(J^1)^{b_1}_{a_1}W_{b_1a_2a_3}\right] = \partial\left[(J^2)^{b_2}_{a_2}W_{a_1b_2a_3}\right] = \partial\left[(J^3)^{b_3}_{a_3}W_{a_1a_2b_3}\right]$

Table 3.4: Explicit null relations at dimension 7/2, which can be uplifted to four-dimensional Higgs branch chiral ring relations. Representations which are not real give rise to a similar null in the complex conjugate representation, and representations which are not equal in the three flavor groups give rise to similar null relations with permutations of the flavor group indices. Note that these, together with Table 3.3, are in one-to-one correspondence to the null relations subtracted from the index given in Table 3.2.

A simple calculation shows that the null relations

$$\mathrm{Tr}(J^1)^2 = \mathrm{Tr}(J^2)^2 = \mathrm{Tr}(J^3)^2\,, \tag{3.10}$$

hold true. Each of these operators separately is null within its respective critical current algebra, but thanks to the presence of the stress tensor T in the full chiral algebra, one finds that only their differences are null. Similarly, we have explicitly recovered the analogous relation for the third order Casimir operators. These null relations are just two instances of the general null relations setting equal the Casimir operators of the three current algebras, which are similarly valid for general T_n. The corresponding Higgs branch chiral ring relations on the moment map operators are well-known (see for example [80]).

Another nice set of null relations is obtained by acting with a current on the generators $W^{(k)}$:

$$\begin{aligned}
&(J^1)_{a_1}^{b_1} W_{b_1 a_2 a_3} = (J^2)_{a_2}^{b_2} W_{a_1 b_2 a_3} = (J^3)_{a_3}^{b_3} W_{a_1 a_2 b_3}\,,\\
&(J^1)_{b_1}^{a_1} \widetilde{W}^{b_1 a_2 a_3} = (J^2)_{b_2}^{a_2} \widetilde{W}^{a_1 b_2 a_3} = (J^3)_{b_3}^{a_3} \widetilde{W}^{a_1 a_2 b_3}\,,\\
&(J^1)_{[a_1}^{c_1} V_{[b_1]c_1][a_2 b_2][a_3 b_3]} = (J^2)_{[a_2}^{c_2} V_{[a_1 b_1][b_2]c_2][a_3 b_3]} = (J^3)_{[a_3}^{c_3} V_{[a_1 b_1][a_2 b_2][b_3]c_3]}\,.
\end{aligned} \tag{3.11}$$

Null relations of this type are expected to be valid in general T_n as well, and extend the ones listed in [80] for $W^{(1)}, W^{(n-1)}$. Some of the null relations presented in Tables 3.3 and 3.4 are direct consequences of these nulls, obtained by either acting with derivatives or normal-ordering them with other operators, but others are new. For example, the last two nulls given in Table 3.3 are not obtained from previous nulls, and they give rise to known Higgs branch chiral ring relations (they precisely turn into the relations given in equation (2.7) of [80] after setting all derivatives and the stress tensor to zero, and taking into account the different normalizations of the two- and four-dimensional operators). All null relations involving the generator V in Tables 3.3 and 3.4 give rise to new Higgs branch chiral ring relations.

As mentioned, when computing Jacobi-identities one might find that some of them are not zero on the nose, but end up being proportional to null states. In practice this happens quite often, and we find that consistency of the Jacobi-identities relies precisely on the existence of some of these nulls. For example, when examining the Jacobi-identities involving W, $\widetilde{W}$ and V, one encounters the following null relation:

$$W_{(a_1[a_2[a_3} W_{b_1)b_2]b_3]} = -\frac{1}{4} J_{(a_1}^{c_1} V_{[|c_1|b_1)][a_2 b_2][a_3 b_3]}\,, \tag{3.12}$$

which only exists at $k_{2d} = -4$.

We can now check a prediction made in section 3.2, namely that any operator transforming in a representation $(\mathfrak{R}_{\lambda_1}, \mathfrak{R}_{\lambda_2}, \mathfrak{R}_{\lambda_3})$ for not all equal $\mathfrak{R}_{\lambda_i}$ must be an AKM descendant (or be obtained from an AKM descendant by acting on it with the operators which contribute to the structure constants). The operator $W_{(a_1[a_2[a_3} W_{b_1)b_2]b_3]}$ would seem to contradict this statement, since it transforms in the representation $(\mathbf{10}, \mathbf{6}, \mathbf{6})$, and it clearly cannot be obtained from an AKM descendant. Fortunately, there is no contradiction with the superconformal index as this operator is set equal to an AKM descendant by the null relation (3.12). More generally, we have verified in several cases that threefold AKM primaries either appear in representations of the type $(\mathfrak{R}_\lambda, \mathfrak{R}_\lambda, \mathfrak{R}_\lambda)$, or are null. Moreover we have checked that all operators in representations which are not of the type $(\mathfrak{R}_\lambda, \mathfrak{R}_\lambda, \mathfrak{R}_\lambda)$ are either AKM descendants or obtained from them by acting with the operators which contribute to the structure constants, such as a derivative, or normal-ordering it with the stress tensor. A direct consequence of this interpretation of the partition function is that we can predict the existence of certain types of relations: whenever we can write an operator in a representation not of the type $(\mathfrak{R}_\lambda, \mathfrak{R}_\lambda, \mathfrak{R}_\lambda)$ which is neither a descendant nor obtained from one in the manner described above, there has to be a null relation involving it. Since null operators are threefold AKM primary, obtaining AKM primaries in said representation provides a faster way to write down the null combinations than to diagonalize norm matrices.

Finally we must point out that there exist operators that are not AKM descendants and can never take part in an AKM primary combination. We already encountered such an example, namely the stress tensor: since it is not of Sugawara type it cannot be an AKM descendant, and the requirement that the AKM currents are Virasoro primaries implies that it also is not an AKM primary. Since the only other dimension two singlets are given by the quadratic Casimir operators, which have zero OPEs with the currents, one concludes that it is impossible to make an AKM primary combination involving the stress tensor. Another example of an operator which cannot be involved in any AKM primary combination is $(W\widetilde{W})|_{\text{sing.}}$. We expect that the existence of this operator, as well as $(VV)|_{\text{sing.}}$ is precisely the reason why the T_4 chiral algebra does not require (Virasoro primary) singlet generators of dimension three and four to close. Although these operators are not Virasoro primaries on their own, they take part in Virasoro primary combinations, of dimensions three and four respectively, which are not AKM primaries. Note that by being neither AKM primaries nor descendants, their contribution to the partition function is necessarily encrypted in the structure constant. As explained in section 3.2, their contribution is indeed captured by the P.E. $\left[\frac{q^3+q^4}{1-q}\right]$ factor in the T_4 structure constant (see (3.2)).

Looking at these operators it is natural to ask if the stress tensor and the Virasoro primary singlet operators obtained from $(W\widetilde{W})|_{\text{sing.}}$ and $(VV)|_{\text{sing.}}$ form a closed subalgebra. If such an algebra closes, it must correspond to the $\mathcal{W}_4$ algebra, which is the unique (up to the choice of central charge) closed algebra with such a set of generators [95, 96]. In principle this could be checked using our explicit construction; however, it is computationally challenging and we have not pursued it. More generally, one could wonder whether the set of operators $\mathcal{O}_i$ in Conjecture 5 could form a closed subalgebra, which then should be a $\mathcal{W}_n = \mathcal{W}(2, 3, \ldots, n)$ algebra with central charge $c_{2d} = -2n^3 + 3n^2 + n - 2$. In particular, one should also be able to test this statement for $\chi(T_3)$ using the explicit construction of [75], in which case one would obtain the $\mathcal{W}_3$ algebra of [89].

In section 3.2 we argued that the structure constant factor of (3.6) would add negative modes of the current algebra Casimir operators. Now we can give an explicit example: the operator $(J^1)_{a_1}^{b_1} W_{b_1 a_2 a_3}$, which precisely at the critical level becomes an AKM primary, and thus is not included in the critical module of $W_{a_1a_2a_3}$. Taking the OPE of S^1 with $W_{a_1a_2a_3}$ we find

$$\begin{aligned} S^1(z) W_{a_1a_2a_3}(0) &\sim \frac{15}{4}\frac{W_{a_1a_2a_3}}{z^2} + 2\frac{(J_1)_{a_1}^{b_1} W_{b_1a_2a_3}}{z}\,, \\ \Longleftrightarrow \left[(S^1)_m, (W_{a_1a_2a_3})_n\right] &= \frac{15(m+1)}{4}(W_{a_1a_2a_3})_{m+n} + 2\left((J_1)_{a_1}^{b_1} W_{b_1a_2a_3}\right)_{m+n}\,, \end{aligned} \tag{3.13}$$

where $(\mathcal{O})_n$ denote the modes of operator $\mathcal{O}$, which in the case of $\left((J_1)_{a_1}^{b_1} W_{b_1a_2a_3}\right)_{m+n}$ correspond to the modes of the normal-ordered product. Acting with the $(S_1)_{-1}$ mode of S_1 on the AKM primary yields

$$(S^1)_{-1}|W_{a_1a_2a_3}\rangle = (S^1)_{-1}(W_{a_1a_2a_3})_{-\frac{3}{2}}|0\rangle = 2\left((J_1)_{a_1}^{b_1} W_{b_1a_2a_3}\right)_{-\frac{5}{2}}|0\rangle\,, \tag{3.14}$$

which exactly adds $(J^1)_{a_1}^{b_1} W_{b_1a_2a_3}$.[12]

When analyzing the superconformal index we also argued that threefold AKM primaries in the sum over AKM modules, that do not correspond to generators $W^{(k)}$ must be obtained by normal-ordered products of generators of Higgs branch chiral ring origin. We can now give explicit examples. Let us start by considering representation $(\mathbf{15}, \mathbf{15}, \mathbf{15})$, for which the corresponding primary must have dimension three. As described in the previous section we can always write down a composite operator with the right quantum numbers, in this case it is $W\widetilde{W}|_{(\mathbf{15},\mathbf{15},\mathbf{15})}$. Even though this operator is not a threefold AKM primary, the following combination is:

$$W\widetilde{W}|_{(\mathbf{15},\mathbf{15},\mathbf{15})} + \frac{1}{64}(J^1)(J^2)(J^3)\,, \tag{3.15}$$

and it is precisely this combination that is accounted for by the $\mathfrak{R} = \mathbf{15}$ term in (3.1). Other examples at dimension three correspond to $(\mathbf{10}, \mathbf{10}, \mathbf{10})$ (and its conjugate), in which case the threefold AKM primary is simply $WW|_{(\mathbf{10},\mathbf{10},\mathbf{10})}$ (and $\widetilde{W}\widetilde{W}|_{(\overline{\mathbf{10}},\overline{\mathbf{10}},\overline{\mathbf{10}})}$).

[12] Recalling that the first null relation in (3.11), sets equal $(J^1)_{a_1}^{b_1} W_{b_1a_2a_3} = (J^2)_{a_2}^{b_2} W_{a_1b_2a_3} = (J^3)_{a_3}^{b_3} W_{a_1a_2b_3}$, this term and $L_{-1}|W_{a_1a_2a_3}\rangle$ (which produces $\partial W_{a_1a_2a_3}$) account for all the powers of q in the structure constants, since for the fundamental representation only $2q$ survives in the plethystic exponential after combining (3.6) and (3.7). It can be shown that the OPEs of higher dimensional Casimirs with $W_{a_1a_2a_3}$ do not produce anything new.

Chapter 4

The $\mathcal{N} = 2$ superconformal bootstrap

The contents of this chapter appear in [3]: "The $\mathcal{N} = 2$ superconformal bootstrap"
C. Beem, M. Lemos, P. Liendo, L. Rastelli and B. C. van Rees,
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4.1 Introduction

In this work we initiate the conformal bootstrap program for four-dimensional conformal field theories with $\mathcal{N} = 2$ supersymmetry. These theories are extraordinarily rich, both physically and mathematically, and have been studied intensively from many viewpoints. Nevertheless, we feel that a coherent picture is still missing. We hope that the generality of the conformal bootstrap framework will allow such a picture to be developed. We also feel the time is ripe for such an investigation – the recent explosion of results for $\mathcal{N} = 2$ superconformal field theories (SCFTs) calls out for a more systematic approach, while the methods first introduced in [6] have reinvigorated the conformal bootstrap [31, 32, 97–101] with a powerful and flexible toolkit for studying conformal field theories with a great deal of generality.

The first examples of $\mathcal{N} = 2$ superconformal field theories (SCFTs) were relatively simple gauge theories with matter representations chosen so that the beta functions for all gauge couplings would vanish. Since then, the library of known theories has grown in size, with the new additions including many Lagrangian models [11], but remarkably also many theories that appear to admit no such description. In particular, the class $\mathcal{S}$ construction of [17, 18] gives rise to an enormous landscape of theories, most of which resist description by conventional Lagrangian field theoretic techniques. Despite this abundance, the current catalogue seems fairly structured, and one may reasonably suspect that a complete classification of $\mathcal{N} = 2$ superconformal field theories (SCFTs) will ultimately be possible. The development of the $\mathcal{N} = 2$ superconformal bootstrap seems an indispensable step towards this ambitious goal.

Our first task is to introduce an abstract operator-algebraic language for $\mathcal{N}=2$ SCFTs. In this reformulation, we retain only the vector space of local operators (organized into representations of the superconformal algebra), and the algebraic structure on this vector space defined by the operator product expansion. From this viewpoint, we can see that a theory is free (or contains a free factor) if its operator spectrum includes higher spin currents; we can see that a theory has a Higgs branch of vacua if its operator algebra includes an appropriate chiral ring that is the coordinate ring of an affine algebraic variety; and so on and so forth. Representation theory of the $\mathcal{N}=2$ superconformal algebra proves an invaluable tool, as its shortened representations neatly encode different facets of the physics. This algebraic viewpoint is remarkably rich, and we have have dedicated the next section to its extensive presentation.

Once equipped with the proper language, we can make an informed decision on where and how to employ numerical bootstrap methods. We explain that there are three classes of four-point functions that should be the starting point for any systematic exploration of this type: the stress-tensor four-point function; the moment map four-point function; and the four-point function of $\mathcal{N}=2$ chiral operators. In the present work, we report on numerical investigations into specific examples of the latter two classes. The requisite superconformal block expansion for the first correlator, which is the most universal, is not yet available, so this case is left for future work. The moment map four-point function is related to the flavor symmetry of the theory, and we focus on the cases of $\mathfrak{su}(2)$ and $\mathfrak{e}_6$. The $\mathfrak{su}(2)$ case is clearly the simplest and is a natural starting point, while $\mathfrak{e}_6$ case is interesting because exceptional flavor symmetries cannot appear in any Lagrangian field theory, and $\mathfrak{e}_6$ is (among others) the simplest case after $\mathfrak{su}(2)$. On the other hand, the four point function of $\mathcal{N}=2$ chiral operators gives us access to a very different aspect of the physics, namely the Coulomb branch chiral ring.

There are two broad types of questions that we can hope to address by bootstrap methods. First of all, we can *constrain the space of consistent $\mathcal{N}=2$ SCFTs.* There are a number of universal structures that appear throughout the $\mathcal{N}=2$ catalogue that cannot be satisfactorily explained in the abstract bootstrap language. Are Coulomb branch chiral rings always freely generated? Are central charges bounded from below by those of free theories, or are there exotic theories with even lower central charges? Is every $\mathcal{N}=2$ conformal manifold parametrized by gauge couplings? As we will see, these questions can sometimes be connected with the constraints of crossing symmetry, and then numerical analysis can provide (partial) answers.

Our second motivation is to *learn more about specific $\mathcal{N}=2$ SCFTs.* There are many cases where supersymmetry can tell us a lot about an $\mathcal{N}=2$ SCFT even when we have no Lagrangian description. In many examples we know, *e.g.*, the central charges (including flavor central charges), the spectrum of protected operators, and some OPE coefficients associated with protected operators. This partial knowledge can be used as *input* for a numerical bootstrap analysis. Optimistically, we may hope that this protected data and the constraints of crossing symmetry are enough to determine the theory uniquely. The bootstrap may then allow us to effectively solve the theory along the lines of what has been done for the three dimensional Ising CFT [7, 33, 102]. Because the bootstrap is completely nonperturbative in nature, it is a natural tool for studying intrinsically strongly coupled (non-Lagrangian) theories. In fact, when it comes to studying unprotected operators in a non-Lagrangian theory, the bootstrap is really the only game in town.

The detailed organization of the chapter can be found in the table of contents. In the first part (sections 2-4) we develop the algebraic viewpoint and the details of the superconformal block expansion for the two classes of correlators that we consider, while in the second part (sections 5-8) we present our numerical investigations. Several appendices complement the main text with technical and reference material.

4.2 The $\mathcal{N} = 2$ superconformal bootstrap program

In the bootstrap approach to conformal field theories, one adopts an abstract viewpoint that takes the algebra of local operators as the primary object. On the other hand, the majority of conventional wisdom and communal intuition about $\mathcal{N} = 2$ field theories arises from a Lagrangian – or at least quasi-Lagrangian – perspective. This leads to something of a disconnect. The bootstrap perspective is likely to be unfamiliar to many experts in supersymmetric field theory, while amongst readers with a background in the conformal bootstrap the additional structure that follows from $\mathcal{N} = 2$ supersymmetry may not be well known. In this section we will try to bridge this divide.

4.2.1 The insufficiency of Lagrangians

Let us recall some aspects of Lagrangian $\mathcal{N} = 2$ field theories, which provide a historical foundation of the subject and help to guide our thinking even for the non-Lagrangian theories discussed below. The building blocks of an $\mathcal{N} = 2$ four-dimensional Lagrangian are vector multiplets, transforming in the adjoint representation of a gauge group G, and hypermultiplets (the *matter content*), transforming in some representation R of G.[1] For the theory to be microscopically well-defined, the gauge group should contain no abelian factors,[2] so we can take G to be semi-simple,

$$G = G_1 \times G_2 \times \cdots G_n \; . \tag{4.1}$$

[1]More generally, for appropriate choices of gauge group one can allow for "half-hypermultiplets", *i.e.*, $\mathcal{N} = 1$ chiral multiplets, transforming in pseudo-real representations of G. See, *e.g.*, [11] for a recent discussion.

[2]An exception is when no hypermultiplet is charged under the abelian factors, in which case there are decoupled copies of the free vector multiplet SCFT in the theory.

To each simple factor G_i is associated a complexified gauge coupling $\tau_i \in \mathbb{C}$, Im $\tau_i > 0$, and for each choice of $(G, R, \{\tau_i\})$ there is a unique, classically conformally invariant $\mathcal{N} = 2$ Lagrangian. For the *quantum* theory to be conformally invariant, the matter content must be chosen so that the one loop beta functions for the gauge couplings vanish. Thanks to $\mathcal{N} = 2$ supersymmetry, this is also a sufficient condition at the full quantum level.

The classification of the pairs (G, R) that lead to $\mathcal{N} = 2$ SCFTs can therefore be reduced to a purely combinatorial problem, whose complete solution has been described recently in [11]. The simplest examples are $\mathcal{N} = 2$ superconformal QCD, which has gauge group $G = SU(N_c)$ and $N_f = 2N_c$ hypermultiplets in the fundamental representation, and $\mathcal{N} = 4$ super Yang-Mills theory (which can be regarded as an $\mathcal{N} = 2$ SCFT), for which G is any simple group and the hypermultiplets transform in the adjoint representation.

The *conformal manifold* of a CFT is the space of theories that can be realized by deforming a given CFT by exactly marginal operators. In a slight abuse of terminology we often refer to the conformal manifold of an $\mathcal{N} = 2$ SCFT as the (not necessarily proper) submanifold of the full conformal manifold where in addition the full $\mathcal{N} = 2$ supersymmetry is preserved. For a Lagrangian theory this submanifold coincides with the space of gauge couplings $\{\tau_i\}$, up to the discrete identifications induced by generalized S-dualities.[3] The conformal manifold comes endowed with a metric – the Zamolodchikov metric – which is Kähler and with respect to which the weak coupling points (where some $\tau_i \to \infty$ in some S-duality frame) are at infinite distance as measured from the interior. Thus the conformal manifold of any $\mathcal{N} = 2$ Lagrangian SCFT is non-compact with boundaries where gauge couplings are turned off.

Lagrangian theories also always possess nontrivial moduli spaces of supersymmetric vacua. The simplest parts of the moduli space are the *Coulomb branch* and the *Higgs branch*. The Coulomb branch consists of vacua where the complex scalar fields φ_i in the vector multiplets acquire nonzero vacuum expectation values (vevs), while the complex scalars $(q, \tilde{q})$ in the hypermultiplets are set to zero – this branch is characterized by the fact that $SU(2)_R$ is unbroken, while $U(1)_r$ is broken. Alternatively, on the Higgs branch only the hypermultiplet scalars get nonzero vevs, and this branch is characterized by $SU(2)_R$ breaking with $U(1)_r$ preserved. There can also be *mixed branches* where the entire R-symmetry is broken, though we will not have much to say about mixed branches in this work.

The best way to parametrize these moduli spaces is by the vevs of gauge-invariant combinations of the elementary fields. The Coulomb branch is parametrized by the vevs of operators of the form $\{\mathrm{Tr}\,\varphi^k\}$. These operators form a freely generated ring, called the *Coulomb branch chiral ring*, with generators in one-to-one correspondence with the Casimir invariants of the gauge group. Similarly, the Higgs branch can be parametrized by the vevs of gauge invariant composites of the hypermultiplet scalars. These operators also form a finitely generated ring, the *Higgs branch chiral ring*. The Higgs branch chiral ring is generally *not*

[3] Because the action of S-duality can have fixed points in the space of gauge couplings, the conformal manifold may have orbifold points, so it may not really be a *manifold*.

freely related, but rather has relations so that the Higgs branch acquires a description as an affine complex algebraic variety. Alternatively, the Higgs branch can be expressed as a Hyperkähler quotient [103].

Isolated SCFTs and quasi-Lagrangian theories

Lagrangian SCFTs make up only small subset of all SCFTs. A wealth of strongly coupled $\mathcal{N}=2$ SCFTs with no marginal deformations are known to exist – by virtue of being isolated, they cannot have a conventional Lagrangian description. One particularly elegant way to find such isolated theories is through generalized S-dualities of the kind discussed in [54]. By taking a Lagrangian theory and dialing a marginal coupling all the way to infinite strength, one may recover a weakly gauged *dual* description which involves one or more isolated SCFTs and a set of vector multiplets to accomplish the gauging. In this dual description the gauging procedure is described in what we may call a *quasi-Lagrangian* fashion: the isolated SCFT is treated as a non-Lagrangian black box with a certain flavor symmetry, which is allowed to talk to the vector multiplets through minimal coupling of the conserved flavor current of the isolated SCFT to the gauge field. The one-loop beta function for each simple gauge group factor is given by

$$\beta = -h^{\vee} + 4k \ , \tag{4.2}$$

where $h^{\vee}$ is the dual Coxeter number of the group and k the flavor central charge, defined from the two-point function of the conserved flavor current. This determines a simple condition for when non-Lagrangian theories can be. (Of course, this expression for β applies also to the Lagrangian case, where the flavor current is a composite operator made of the hypermultiplet fields.)

The web of generalized S-dualities for large classes of theories can be elegantly described through the class $\mathcal{S}$ constructions of [17, 18]. These theories arise from twisted compactifications of the six-dimensional $(2,0)$ theories on a punctured Riemann surface, with additional discrete data specified at each puncture. The marginal deformations of the four-dimensional theory correspond to the moduli of the Riemann surface, and weakly gauged theories arise if the Riemann surface degenerates. In this picture the isolated theories correspond to three-punctured spheres which have no continuous moduli. They do, however, depend on the discrete data at the three punctures as well as on a choice of $\mathfrak{g} \in \{A_n, D_n, E_n\}$ for the six-dimensional ancestor theory. In this way several infinite classes of isolated theories can be constructed. A few of these theories turn out to be equal to theories of free hypermultiplets, but most cases do not admit a Lagrangian description.

Another large class of isolated theories are the Argyres-Douglas fixed points [104] which describe the infrared physics at special points on the Coulomb branch of another $\mathcal{N}=2$ theory. At these distinguished points several BPS particles with mutually non-local charges become simultaneously massless, which precludes any Lagrangian description of the infrared theory. Alternatively, many Argyres-Douglas fixed points can be constructed in class $\mathcal{S}$ by allowing for irregular singularities on the UV curve [105]. Argyres-Douglas theories have also recently been used as building blocks in a quasi-Lagrangian set-up [106].

In order to describe the currently known landscape of $\mathcal{N}=2$ SCFTs, then, it is clearly not sufficient to only consider Lagrangians with hypermultiplets and vector multiplets. We can certainly accommodate any theory in a framework which takes as fundamental the spectrum and algebra of local operators. This is the basic starting point for the bootstrap approach that we take in this chapter. The remainder of this section is dedicated to the development of such a framework.

4.2.2 The bootstrap philosophy

In the bootstrap approach, we take a (super)conformal field theory to be characterized by its local operator algebra.[4] The aim is then to understand the constraints imposed upon such algebras by (super)conformal invariance, associativity, and unitarity. This approach dates back to the foundational papers of [31, 32, 97–101]. See, *e.g.*, [6, 107] for modern expositions. We will briefly recall the general logic, while placing particular emphasis on the role played by short representations of the conformal algebra. In the next subsection we describe the special features that arise in the $\mathcal{N}=2$ superconformal case.

The local operators $\{\mathcal{O}_i(x)\}$ of a CFT form a vector space that is endowed with a product that gives it something like an associative algebra structure. The product for local operators is known as the Operator Product Expansion (OPE), and takes the schematic form

$$\mathcal{O}_1(x)\mathcal{O}_2(y) = \sum_k c_{12k}(x-y)\mathcal{O}_k(y) \ . \tag{4.3}$$

Any correlation function of separated local operators in flat spacetime $\mathbb{R}^d$ can be evaluated by successive applications of the OPE, which is an absolutely convergent expansion. The OPE follows as a straightforward consequence of the *state/operator correspondence*.[5] To each local operator is associated a state, obtained by acting on the vacuum with the operator inserted at the origin,

$$\mathcal{O}(x) \to |\mathcal{O}\rangle := \mathcal{O}(0)|0\rangle \ , \tag{4.4}$$

and conversely each state defines a unique local operator,

$$|\psi\rangle \to \mathcal{O}_\psi(x) \ . \tag{4.5}$$

As customary, we will use the language of operators or states interchangeably.

[4] In adopting this perspective, we are therefore willfully ignoring the complications associated with including non-local observables – such as Wilson line operators in conformal gauge theory – and non-trivial spacetime geometries.

[5] See [108] for a recent discussion.

To completely specify a CFT at the level of correlators of local operators, it is therefore sufficient to list the set of local operators (that is, the set of their quantum numbers) and the structure constants appearing in their OPEs. Conformal invariance streamlines the presentation of this information. First, it allows the local operators to be assembled into conformal families, each of which transforms as a highest weight representation of the conformal algebra $\mathfrak{so}(d,2)$. The highest weight state, known as the conformal primary, is annihilated by all raising operators in the conformal algebra, notably the special conformal generators K_μ. Specializing to the four-dimensional case, a representation $\mathcal{R}[\Delta, j_1, j_2]$ of $\mathfrak{so}(4,2) \cong \mathfrak{su}(2,2)$ is labelled by the quantum numbers of the primary, namely its conformal dimension Δ and its Lorentz spins (j_1, j_2). If the theory enjoys an additional global symmetry G_F, then the local operators can be further organized into G_F representations, labelled by some flavor symmetry quantum numbers f, and the full representations are then denoted as $\mathcal{R}[\Delta, j_1, j_2; f]$. Conformal symmetry also restricts the spacetime dependence of the functions $c_{ijk}(x)$ appearing in the OPE (4.3). In particular, the functions $c_{ijk}(x)$ are uniquely determined in terms of the quantum numbers of the representations $\mathcal{R}_i$, $\mathcal{R}_j$, and $\mathcal{R}_k$ and the coefficients λ^s_{ijk} that parametrize their three-point functions.[6] All told, the data that fully specify the local theory amount to a countably infinite list

$$\{a_i, \lambda^s_{ijk}\} \ , \quad a_i := (\Delta, j_1, j_2, f)_i \ . \tag{4.6}$$

These data are constrained by the requirements that the theory be unitary and that the OPE be associative. The hypothesis underlying the conformal bootstrap is that these constraints are so powerful that they can completely determine the local data given some minimal physical input. In practice, one expects that the input will include the global symmetry of the theory and some simple spectral assumptions such as the number of relevant operators.

Unitarity and shortening

We first recall the constraints imposed by unitarity. Non-trivial[7] unitary representations of $\mathfrak{so}(4,2)$ are required to satisfy the following *unitarity bounds*,

$$\begin{aligned} &\Delta \geqslant j_1 + j_2 + 2 \quad &&\text{for} \quad &&j_1 j_2 \neq 0 \ , \\ &\Delta \geqslant j_2 + 1 &&\text{for} &&j_1 = 0 \ , \\ &\Delta \geqslant j_1 + 1 &&\text{for} &&j_2 = 0 \ . \end{aligned} \tag{4.7}$$

[6] In the simplest case of three spacetime scalars (with no additional flavor charges), the three-point function is completely fixed up to a single overall coefficient λ_{ijk}. In general there are multiple parameters λ^s_{ijk}, $s = 1, \ldots \text{mult}(ijk)$, where the (finite) multiplicity $\text{mult}(ijk)$ is given by the number of independent conformally covariant tensor structures that can be built from the three reps $\mathcal{R}_{i,j,k}$.

[7] We use the qualification "non-trivial" to exclude the vacuum representation, which consists of a single state with $\Delta = j_1 = j_2 = 0$.

Generic representations are denoted as $\mathcal{A}_{\Delta,j_1,j_2}$. Non-generic, or *short*, representations occur when the norm of a conformal descendant state in the Verma module built over some conformal primary is rendered null by a conspiracy of quantum numbers. This happens precisely when the unitarity bounds are saturated, leading the following list of short representations:

$$\begin{aligned}
\mathcal{C}_{j_1,j_2} &: \quad \Delta = j_1 + j_2 + 2 \ , \\
\mathcal{B}^L_{j_1} &: \quad \Delta = j_1 + 1 \ , \quad j_2 = 0 \ , \\
\mathcal{B}^R_{j_2} &: \quad \Delta = j_2 + 1 \ , \quad j_1 = 0 \ , \\
\mathcal{B} &: \quad \Delta = 1 \ , \quad j_1 = j_2 = 0 \ .
\end{aligned} \tag{4.8}$$

All of these representations have null states at level one with the exception of $\mathcal{B}$, which has a null state at level two.

The presence of short representations in the spectrum of a CFT is connected to the existence of free fields and symmetries in the theory. In particular, the primaries of $\mathcal{B}$-type representations are decoupled free fields, and as such are not of much interest when studying interacting CFTs. For example, the primary of a $\mathcal{B}$ representation is a free scalar field $\phi(x)$. Modding out by the null state at level two imposes the operator constraint

$$P^\mu P_\mu \phi = \Box\phi(x) = 0 \ , \tag{4.9}$$

which is nothing but the free scalar equation of motion. Similarly, $\mathcal{B}^\star_{\frac{1}{2}}$ multiplets have as their primaries free Weyl fermions; the null state level one imposes the free equation of motion

$$\begin{aligned}
\mathcal{B}^L_{\frac{1}{2}} &: \qquad \partial^{\alpha\dot{\alpha}}\psi_\alpha(x) = 0 \ , \\
\mathcal{B}^R_{\frac{1}{2}} &: \qquad \partial^{\alpha\dot{\alpha}}\tilde{\psi}_{\dot{\alpha}}(x) = 0 \ .
\end{aligned} \tag{4.10}$$

On the other hand, $\mathcal{C}$-type representations have various *conserved currents* as their primaries; their level-one null state is the consequence of a conservation equation,

$$\partial^{\alpha_1\dot{\alpha}_1} J_{\alpha_1\cdots\alpha_{2j_1}\ \dot{\alpha}_1\cdots\dot{\alpha}_{2j_2}}(x) = 0 \ . \tag{4.11}$$

Conserved currents with spin $j_1 + j_2 > 2$ are *higher-spin currents*, which are a hallmark of free CFTs [53, 109]. For the purposes of the bootstrap, we will usually impose by hand that no such multiplets appear. Conserved currents with $(j_1, j_2) = (1, \frac{1}{2})$ and $(j_1, j_2) = (\frac{1}{2}, 1)$ give rise to an enhancement of the conformal algebra to a *super*conformal algebra – when these operators are present one should therefore be taking full advantage of the power of superconformal symmetry.

Thus, amongst the short representations of $\mathfrak{so}(4,2)$, those which may be present in an interacting non-supersymmetric CFT are $\mathcal{C}_{1,1}$ and $\mathcal{C}_{\frac{1}{2},\frac{1}{2}}$. In the former case, the conformal primary is the stress tensor $T_{\mu\nu}$. In the latter case, the conformal primary is a conserved current J_μ, so the presence of such multiplets portend the existence of continuous global symmetries.

Locality in the operator algebra

An important remark is in order. When characterizing CFTs by their local operator algebra, certain ingredients which are usually automatically present in a Lagrangian context are no longer necessarily compulsory. For example, one need not assume that the local algebra includes a stress tensor at all. Indeed, there are interesting local algebras, such as the algebra of local operators supported on conformal defects in a higher-dimensional CFT, in which the stress tensor is *not* present. The presence of a stress tensor is clearly connected with the notion of *locality* in the CFT, and we will take the existence of a unique stress tensor (that is, the existence of a unique conformal representation of type $\mathcal{C}_{1,1}$) as part of the *definition* of a local CFT.

Similarly, in the Lagrangian context a continuous global symmetry implies the existence of a conserved current in the operator spectrum. We will assume the validity of this claim even in the non-Lagrangian context:

Conjecture 6 (CFT Noether "theorem")**.** *In a local CFT, to any continuous global symmetry is associated a conserved current in the operator algebra that generates the symmetry.*

Clarifying the conceptual status of this "theorem" is an important open problem. On one hand, one may take it as part of the definition of what it means for a CFT to be local, in which case this is a tautology. Alternatively, it is possible that the theorem may be derived from general principles in a suitable axiomatic framework.[8] Whatever the case may be, the proof of such a statement is of interest in part due to its reinterpretation via AdS/CFT, which is the statement that there are no continuous global symmetries in AdS quantum gravity.

Canonical data

The data associated to short representations of the conformal algebra carries particular physical significance. The three-point function of the stress tensor depends on three parameters, two of which can be identified with the two coefficients appearing in the conformal anomaly, conventionally denoted by a and c. The a coefficient gives a measure of the degrees of freedom of the theory and serves as a height function in theory space: for two CFTs connected by RG flow, $a_{\rm UV} > a_{\rm IR}$ [47, 48]. However, since a can only be extracted from the stress

[8] It is unclear whether the axioms for the algebra of local operators should be sufficient for this purpose. It is possible that the existence of a conserved current could follow from the assumptions that the operator algebra is invariant under a continuous symmetry *and* that there is a stress tensor. Alternatively, the framework may need to be enlarged, perhaps allowing for correlation functions in non-trivial geometries, subject to suitable locality assumptions.

tensor three-point function, it is rather difficult to access by bootstrap methods – one would generally need to consider correlation functions involving external stress tensors, which are very complicated [110]. By contrast, if one uses the canonical normalization for the stress tensor, its *two-point* function is proportional to c. The c coefficient will then appear in any four-point function containing an intermediate stress tensor, making its presence ubiquitous in the bootstrap literature. Using "conformal collider" observables, it was argued in [111] that in a general unitarity CFT the ratio of conformal anomaly coefficients must obey the bounds[9]

$$\frac{1}{3} \leqslant \frac{a}{c} \leqslant \frac{31}{18} \,. \tag{4.12}$$

The lower bound is saturated by the free scalar CFT, the upper bound by the free vector CFT. There is strong evidence that these free CFTs are the *only* theories saturating the bounds [113].

Similarly, the two-point function of canonically normalized currents depends on a parameter k often called the *flavor central charge* that can be identified with an 't Hooft anomaly for the corresponding global symmetry [114, 115]. This parameter appears in the OPE of conserved currents as follows,

$$J^A_\mu(x) J^B_\nu(0) \sim \frac{3k}{4\pi^4}\delta^{AB}\frac{x^2 g_{\mu\nu} - 2x_\mu x_\nu}{x^8} + \frac{2}{\pi^2}\frac{x_\mu x_\nu f^{AB}{}_C x \cdot J^C(0)}{x^6} + \dots \,. \tag{4.13}$$

Like the c central charge, the flavor central charge makes frequent appearances in the bootstrap because it controls the contribution of the conserved current in a correlation function of charged operators.

In a sense, the data associated to the spectrum of conserved currents and stress tensors and their associated anomaly coefficients is the most basic data associated to a conformal field theory. We designate this data as the *canonical data* for the CFT. It is natural to organize an exploration of the space of conformal field theories in terms of these parameters, and if one wants to study a particular theory in detail this data is an obvious starting point. This has not always been the approach in the existing bootstrap literature thus far, but that is at least in part because the natural observables through which to pursue such a strategy would be the four point functions of conserved currents and stress tensors. At a technical level, these are much more complex observables than the correlators of spacetime scalars.

The numerical bootstrap approach

Intuitively, associativity of the operator algebra is a tremendous constraint. However, aside from the case of two-dimensional CFTs where the global conformal symmetry algebra en-

[9]The argument uses positivity of energy correlators in a unitarity theory, which is a reasonable physical assumption (see also [112]). It would be interested to recover the HM bounds by conformal bootstrap methods. This will likely have to await for the complete conformal block analysis of the stress tensor four-point function, a challenging technical problem.

hances to two copies of the infinite-dimensional Virasoro algebra, it seems very difficult to extract useful information from these conditions. The way forward was shown in [6], where the focus was shifted away from trying to *solve* the associativity problem and towards obtaining *constraints* for, *e.g.*, the spectrum of local operators or their OPE coefficients in a unitary CFT. The prototypical bounds that can be obtained in this way are uppers bound for the dimension of lowest-lying operator of a given spin, or a lower bound on the c central charge of a theory, all given some input about the spectrum of scalar operators.

In order to test associativity it suffices to investigate four-point functions in a given CFT, where the OPE can be taken in three essentially inequivalent ways by fusing different pairs of operators together. For each choice one finds a representation of the four-point function as a sum over conformal blocks [107], with one block for each conformal multiplet that appears in both OPEs. The statement that these three decompositions have to sum to exactly the same result is known as *crossing symmetry*. It was shown in [6] that useful bounds can be extracted already from the requirement of crossing symmetry for a single four-point function involving four identical scalar operators. Such an analysis is conspicuously tractable – as opposed to trying to solve all of the infinitely many crossing symmetry constraints simultaneously, we simply find the conditions that follow from a finite subset of those constraints. The structure of four-point functions and their OPE decompositions are severely constrained by conformal symmetry – see, *e.g.*, [107] for an introductory exposition.

The work of [6] has been extended in numerous directions, and bounds have been obtained in theories with and without supersymmetry and in various spacetime dimensions. Further numerical bootstrap results can be found for example in [7, 33–37, 102, 116–136]. An essential ingredient in the numerical analysis is the (super)conformal block decomposition of a four-point functions. These structure have been investigated in various cases in, *e.g.*, [108, 110, 137–149]. In related work, [150–153] obtained nontrivial constraints for the operator spectrum by considering in particular the OPE in the limit where operators become lightlike separated.

4.2.3 Operator algebras of $\mathcal{N} = 2$ SCFTs

The superconformal case follows largely the same conceptual blueprint as the non-supersymmetric case, where we replace the conformal algebra $\mathfrak{so}(4,2)$ with the superconformal algebra is $\mathfrak{su}(2,2|2)$. The maximal bosonic subalgebra is just the conformal algebra $\mathfrak{so}(4,2) \equiv \mathfrak{su}(2,2)$ times the R-symmetry algebra $SU(2)_R \times U(1)_r$. Additionally there are sixteen fermionic generators – eight Poincaré supercharges and eight conformal supercharges – denoted as $\{\mathcal{Q}^{\mathcal{I}}_{\alpha}, \widetilde{\mathcal{Q}}_{\mathcal{I}\dot{\alpha}}, \mathcal{S}^{\alpha}_{\mathcal{J}}, \widetilde{\mathcal{S}}^{\mathcal{J}\dot{\alpha}}\}$ where $\mathcal{I} = 1, 2$, $\alpha = \pm$, and $\dot{\alpha} = \dot{\pm}$ are $SU(2)_R$, $\mathfrak{su}(2)_1$, and $\mathfrak{su}(2)_2$ indices, respectively.

The spectrum of local operators can be organized in highest weight representations of $\mathfrak{su}(2,2|2)$ whose highest weight states, known as superconformal primaries, are annihilated by all lowering operators of the superconformal algebra – in particular, by all the conformal supercharges $\mathcal{S}$. These representations are labelled by the quantum numbers $[\Delta, j_1, j_2, R, r]$

of the superconformal primary; the additional labels R and r that extend the ordinary conformal case are the eigenvalues of the Cartan generators of $SU(2)_R$ and $U(1)_r$. We will also consider theories that are invariant under additional flavor symmetry $\mathfrak{g}_F$ (a semi-simple Lie algebra commuting with $\mathfrak{su}(2,2|2)$), which introduces additional flavor quantum numbers f. In summary, the local data for an $\mathcal{N}=2$ SCFT are

$$\{a_i, \lambda^s_{ijk}\} \ , \quad a_i := [\Delta, j_1, j_2, R, r; f]_i \ . \tag{4.14}$$

In analogy with the conformal case, the coefficients λ^s_{ijk} encode the information needed to completely reconstruct the *superspace* three-point function[10] $\langle \mathcal{R}_i(x_1,\theta_1)\, \mathcal{R}_j(x_2,\theta_2)\, \mathcal{R}_k(x_3,\theta_3)\rangle$.

Unitarity and shortening

The unitary representation theory of the $\mathcal{N}=2$ superconformal algebra is more elaborate than that of the ordinary conformal algebra. The unitarity bounds are now given by

$$\begin{aligned} &\Delta \geqslant \Delta_i \ , && j_i \neq 0 \ , \\ &\Delta = \Delta_i - 2 \quad \text{or} \quad \Delta \geqslant \Delta_i \ , && j_i = 0 \ , \end{aligned} \tag{4.15}$$

where we have defined

$$\Delta_1 := 2 + 2j_1 + 2R + r \ , \qquad \Delta_2 := 2 + 2j_2 + 2R - r \ . \tag{4.16}$$

The unitary representations of $\mathfrak{su}(2,2|2)$ have been classified in [14, 52, 154]. Short representations occur when one or more of these bounds are saturated, and the different ways in which this can happen correspond to different combinations of Poincaré supercharges that can annihilate the highest weight state of the representation. There are again two types of shortening conditions, the $\mathcal{B}$ type and the $\mathcal{C}$ type. Each type now has four incarnations corresponding to the choice of chirality (left or right-moving) and the choice of $SU(2)_R$ component:

$$\mathcal{B}^{\mathcal{I}} \quad : \qquad \mathcal{Q}^{\mathcal{I}}_{\alpha}|\psi\rangle = 0 \ , \quad \alpha = 1,2 \ , \tag{4.17}$$

$$\bar{\mathcal{B}}_{\mathcal{I}} \quad : \qquad \widetilde{\mathcal{Q}}_{\mathcal{I}\dot{\alpha}}|\psi\rangle = 0 \ , \quad \dot{\alpha} = 1,2 \ , \tag{4.18}$$

$$\mathcal{C}^{\mathcal{I}} \quad : \qquad \begin{cases} \epsilon^{\alpha\beta}\mathcal{Q}^{\mathcal{I}}_{\alpha}|\psi\rangle_{\beta} = 0 \ , & j_1 \neq 0 \ , \\ \epsilon^{\alpha\beta}\mathcal{Q}^{\mathcal{I}}_{\alpha}\mathcal{Q}^{\mathcal{I}}_{\beta}|\psi\rangle = 0 \ , & j_1 = 0 \ , \end{cases} \tag{4.19}$$

[10] In the conformal case, the λ^s_{ijk} can be extracted from the three-point function of the conformal primaries, because descendant operators are simply derivatives of the primaries and their three-point functions contain no extra information. In general this is no longer the case with superconformal symmetry: knowledge of the three-point functions of the superconformal primaries does not always suffice. But at an abstract level there is no difference: what matters are superconformally covariant structures that can be built from the three representations.

$$\bar{\mathcal{C}}_{\mathcal{I}} \quad : \qquad \begin{cases} \epsilon^{\dot\alpha\dot\beta}\widetilde{\mathcal{Q}}_{\mathcal{I}\dot\alpha}|\psi\rangle_{\beta} = 0\ , & j_2 \neq 0\ , \\ \epsilon^{\dot\alpha\dot\beta}\widetilde{\mathcal{Q}}_{\mathcal{I}\dot\alpha}\widetilde{\mathcal{Q}}_{\mathcal{I}\dot\beta}|\psi\rangle = 0\ , & j_2 = 0\ . \end{cases} \tag{4.20}$$

Some authors refer to $\mathcal{B}$-type conditions as shortening conditions, and to $\mathcal{C}$-type conditions as *semi*-shortening conditions, to highlight the fact that a $\mathcal{B}$-type condition is twice as strong. We refer to Appendix B for a tabulation of all allowed combinations of (semi-)shortening conditions and for naming conventions for the resulting representations.

Because of the proliferation of short representations in the $\mathcal{N} = 2$ context, there is potentially much more "canonical data" than in the non-supersymmetric case. Indeed, these many short representations are closely related to various nice features theories with $\mathcal{N} = 2$ supersymmetry. Here we focus primarily on three classes of short representations that have particularly straightforward connections to familiar physical characteristics of $\mathcal{N} = 2$ theories. These representations have the distinction of obeying the maximum number of shortening or semi-shortening conditions that can simultaneously be imposed (two and four, respectively). In the notations of [52], they are:

- $\mathcal{E}_r$: Half-BPS multiplets "of Coulomb type". These obey two $\mathcal{B}$-type shortening conditions of the same chirality: $\mathcal{B}^1 \cap \mathcal{B}^2$. In other terms, they are $\mathcal{N} = 2$ chiral multiplets, annihilated by the action of *all* left-handed supercharges.[11]

- $\hat{\mathcal{B}}_R$: Half-BPS multiplets "of Higgs type". These obey two $\mathcal{B}$-type shortening conditions of opposite chirality: $\mathcal{B}^1 \cap \bar{\mathcal{B}}_2$. These types of operators are sometimes called "Grassmann-analytic".

- $\hat{\mathcal{C}}_{0(j_1,j_2)}$: The stress tensor multiplet (the special case $j_1 = j_2 = 0$) and its higher spin generalizations. These obey the maximal set of semi-shortening conditions: $\mathcal{C}^1 \cap \mathcal{C}^2 \cap \bar{\mathcal{C}}_1 \cap \bar{\mathcal{C}}_2$.

The CFT data associated to these representations encodes some of the most basic physical information about an $\mathcal{N} = 2$ SCFT. We now look at each in more detail, starting from the third and most universal class, which contains the stress tensor multiplet.

Stress tensor data

The maximally semi-short multiplets $\hat{\mathcal{C}}_{0(j_1,j_2)}$ contain conserved tensors of spin $2 + j_1 + j_2$. For $j_1 + j_2 > 0$, such multiplets are not allowed in an interacting CFT, and we will always impose their absence from the double OPE of the four-point functions under consideration.

The $\hat{\mathcal{C}}_{0(0,0)}$ representation includes a conserved tensor of spin two, which we identify as the stress tensor of the theory. By definition, a *local* $\mathcal{N} = 2$ SCFT will contain exactly one

[11] We are focusing on the *scalar* $\mathcal{E}_r$ multiplets – $\mathcal{E}_r := \mathcal{E}_{r(0,0)}$ in the notations of Table B.1. Representation theory allows for $\mathcal{N} = 2$ chiral multiplets $\mathcal{E}_{r(0,j_2)}$ with $j_2 \neq 0$, but such exotic multiplets do not occur in any known $\mathcal{N} = 2$ SCFT. See [155] for a recent discussion.

$\hat{\mathcal{C}}_{0(0,0)}$ multiplet.[12] We will usually assume that the theories that we study are local, but we'll also briefly explore non-local theories, which have no stress tensor and thus no $\hat{\mathcal{C}}_{0(0,0)}$ multiplet.

The superconformal primary of $\hat{\mathcal{C}}_{0(0,0)}$ is a scalar operator of dimension two that is invariant under all R-symmetry transformations. The other bosonic primaries in the multiplet are the conserved currents for $SU(2)_R \times U(1)_r$ and the stress tensor itself. An analysis in $\mathcal{N} = 2$ superspace [156] reveals that three-point function of $\hat{\mathcal{C}}_{0(0,0)}$ multiplets involves two independent structures, whose coefficients can be parametrized in terms of the a and c anomalies. The $\mathcal{N} = 2$ version of the Hofman-Maldacena bounds reads

$$\frac{1}{2} \leqslant \frac{a}{c} \leqslant \frac{5}{4} . \tag{4.21}$$

The lower bound is saturated by the free hypermultiplet theory, and the upper bound by the free vector multiplet theory. By a generalization of the analysis of [113], one should be able to argue that these are the only $\mathcal{N} = 2$ SCFTs saturating the bounds.

In this work we will not study the four-point function of the stress tensor multiplet, because the requisite superconformal block expansion has not yet been worked out. We will, however, have indirect access to the c anomaly coefficient. As in the non-supersymmetric case, if one chooses the canonical normalization for the stress tensor then two-point function of $\hat{\mathcal{C}}_{0(0,0)}$ multiplets will depend on c only. The c coefficient will make an appearance in all four point functions that we study, since $\hat{\mathcal{C}}_{0(0,0)}$ appears in their double OPE.

Coulomb and Higgs branches

As indicated by our choice of terminology, the two types of half-BPS multiplets – $\mathcal{E}_r$ and $\hat{\mathcal{B}}_R$ – are closely related to the Coulomb and Higgs branches of the moduli space of vacua, respectively. In Lagrangian theories, the superconformal primaries in the $\mathcal{E}_r$ multiplets are the gauge-invariant composites of vector multiplet scalars that parameterize the Coulomb branch, and the superconformal primaries in the $\hat{\mathcal{B}}_R$ multiplets are the gauge-invariant composites of hypermultiplet scalars that parameterize the Higgs branch.

We should call attention to the fact that a satisfactory understanding of the phenomenon of spontaneous conformal symmetry breaking has not yet been developed in the language of CFT operator algebras. In principle, the local data should contain all necessary information to describe the phases of the theory where conformal symmetry is spontaneously broken. A method to extract this information is, however, presently not known. Even the basic question of *whether* a given CFT possesses nontrivial vacua remains out of reach. Since all known examples of vacuum manifolds in CFTs occur in supersymmetric theories, one might

[12] A caveat to this definition of locality is that in the tensor product of two local theories there will be two stress tensor multiplets. For the purposes of the conceptual discussion here we restrict our attention to theories that are not factorizable in this manner – we might call such theories *simple*.

speculate that supersymmetry is a necessary condition for spontaneous conformal symmetry breaking.

We are now ready to look in more detail at the CFT data encoded in the two classes of BPS multiplets.

Coulomb branch data

We will refer to the data associated to $\mathcal{E}_r$ multiplets as *Coulomb branch data*. By passing to the cohomology of the left-handed Poincaré supercharges, one finds a commutative ring of operators known as the *Coulomb branch chiral ring*, the elements of which can be identified with the superconformal primaries of $\mathcal{E}_r$ multiplets. In all known examples, this ring is exceedingly simple, and it is natural to formulate a conjecture that the ring is always as simple as it is in the examples:[13]

Conjecture 7 (Free generation of the Coulomb chiral ring)**.** *In any $\mathcal{N}=2$ SCFT, the Coulomb branch chiral ring is freely generated.*

This conjecture can in principle be translated into a statement about the OPE coefficients of the $\mathcal{E}_r$ multiplets. For instance, a simple consequence is that no $\mathcal{E}_r$ superconformal primary can square to zero in the chiral ring, so an $\mathcal{E}_{2r}$ operator must appear with nonzero coefficient in the OPE of the $\mathcal{E}_r$ with itself. Precisely this kind of statement can be tested by numerical bootstrap methods, as we will describe in Section 4.7.

The number of generators of the Coulomb branch chiral ring is usually referred to as the *rank* of the theory. The set $\{r_1,\ldots r_{\rm rank}\}$ of $U(1)_r$ charges of these chiral ring generators is one of the most basic invariants of an $\mathcal{N}=2$ SCFT. Unitarity implies $r\geqslant 1$, with $r=1$ only in the case of the free vector multiplet, so we will always assume $r>1$. In Lagrangian SCFTs, the r_i are all integers, but there are several non-Lagrangian models that possess $\mathcal{E}_r$ multiplets with interesting fractional values of r. We are not aware of any examples where $U(1)_r$ charges take irrational values.

It is widely believed that the Coulomb branch of the moduli space of any $\mathcal{N}=2$ SCFT is parameterized by assigning independent vevs to each of the Coulomb branch chiral ring generators. We will generally operate under the assumption that this statement is true, which amounts to assuming the validity of the following conjecture.

Conjecture 8 (Geometrization of the Coulomb chiral ring)**.** *The Coulomb chiral ring is isomorphic to the holomorphic coordinate ring on the Coulomb branch.*

We note that the union of Conjecture 7 and Conjecture 8 implies that the Coulomb branch of any $\mathcal{N}=2$ SCFT just $\mathbb{C}^r$, with r the rank of the theory.

[13] To the best of our knowledge, this conjecture was first explicitly stated in the literature by Yuji Tachikawa in [10].

At present we are not sure how one might establish Conjecture 8 using bootstrap methods due to the obstacle of spontaneous conformal symmetry breaking discussed above. However, once one has found their way onto the Coulomb branch, the powerful technology of Seiberg-Witten (SW) theory becomes applicable. The effective action for the low-energy $U(1)^{\text{rank}}$ gauge theory on the Coulomb branch is characterized by geometric data (in the simplest cases, this is the SW curve, more generally it is some abelian variety). There are well-developed techniques to determine the SW geometry, which apply to most Lagrangian examples and to several non-Lagrangian cases as well. In turn, the SW geometry determines a wealth of physical information, such as the spectrum of massive BPS states. Unfortunately, how to translate this information into CFT data remains an unsolved problem.[14]

In [158], Shapere and Tachikawa (ST) proved a remarkable formula that relates the a and c central charges to the generating r-charges $\{r_1, \ldots r_{\text{rank}}\}$,

$$2a - c = \frac{1}{4}\sum_{i=1}^{\text{rank}}(2r_i - 1) \ . \tag{4.22}$$

The ST sum rule holds in all known examples, and it is tempting to conjecture that it is a general property of all $\mathcal{N} = 2$ SCFTs. The derivation of [158] requires that the SCFTs in question be realized at a point on the moduli space a Lagrangian theory. The result can then be extended to all SCFTs connected to that class of theories by generalized S-dualities. In particular, this includes a large subset of theories of class $\mathcal{S}$.

According to the ST sum rule, a theory with zero rank necessarily has $a/c = 1/2$, which is the value saturating the lower HM bound. As remarked above, there are strong reasons to believe that the only SCFT saturating this bound is the free hypermultiplet theory. However, since the whole logic of [158] relies on the existence of a Coulomb branch, this reasoning is circular. An interacting SCFT of zero rank would be rather exotic, but we do not know how to rule it out with present methods.

The special case of the $\mathcal{E}_2$ multiplet is particularly significant. The top component of the multiplet, obtained by acting with four right-moving supercharges on the superconformal primary,[15] $\mathcal{O}_4 \sim \tilde{Q}^4\mathcal{E}_2$ is a scalar operator of dimension four. This operator provides an exactly marginal deformation of the SCFT that preserves the full $\mathcal{N} = 2$ supersymmetry. (By CPT symmetry, there is also a complex conjugate operator $\overline{\mathcal{O}}_4 \sim Q^4\bar{\mathcal{E}}_{-2}$). The converse is also true: any $\mathcal{N} = 2$ supersymmetric exactly marginal operator $\mathcal{O}_4$ must be the top component of an $\mathcal{E}_2$ multiplet. It follows that the number of $\mathcal{E}_2$ multiplets is equal to the (complex) dimension of the conformal manifold of the theory. In a Lagrangian theory, there is an $\mathcal{E}_2$ multiplet for each simple factor of the gauge group, and the exactly marginal operator

[14] See however [157] for a relation between the spectrum of BPS states on the Coulomb branch and a certain partition function (evaluated at the conformal point), which appears to be closely related to the superconformal index.

[15] In an abuse of notation, we are denoting the superconformal primary with the same symbol $\mathcal{E}_2$ that represents the whole multiplet.

$\mathcal{O}_4 \sim \mathrm{Tr}(F^2 + i\tilde{F}^2)$ (where F is the Yang-Mills field strength) is dual to the complexified gauge coupling.

Another true feature of all Lagrangian SCFTs (and many non-Lagrangian ones in class $\mathcal{S}$) is that they can be constructed by taking isolated building blocks with no marginal deformations (such as hypermultiplets in the Lagrangian case, or T_N theories in the class $\mathcal{S}$ case) and gauging global symmetry groups for which the beta function will vanish. A natural conjecture is that this feature is indeed universal:

Conjecture 9 (Decomposability)**.** *Any $\mathcal{N} = 2$ SCFT with an n-dimensional conformal manifold can be constructed by gauging n simple factors in the global symmetry group of a collection of isolated $\mathcal{N} = 2$ SCFTs.*

Of course such a decomposition need not be unique – the existence of inequivalent decompositions of the same theory is what is often called "generalized S-duality". Note that the validity of this conjecture would imply the absence of compact conformal manifolds for $\mathcal{N} = 2$ SCFTs.[16]

Higgs branch data

In a similar vein, the $\hat{\mathcal{B}}_R$ multiplets are expected to encode the information about the Higgs branch of the theory. The $\hat{\mathcal{B}}_R$ superconformal primaries, which are also $SU(2)_R$ highest weights, form the *Higgs branch chiral ring.* In all known examples this ring describe by a finite set of generators obeying polynomial relations. The algebraic variety defined by this ring is then expected to coincide with the Higgs branch of vacua. This expectation can be formalized as follows:

Conjecture 10 (Geometrization of the Higgs chiral ring)**.** *In any $\mathcal{N} = 2$ SCFT, the Higgs branch chiral ring is isomorphic to the holomorphic coordinate ring on the Higgs branch of vacua.*

The Higgs branch of vacua is hyperkähler, so there are actually a $\mathbb{CP}^1$ worth of holomorphic coordinate rings on it depending on the choice of complex structure. The choice of complex structure corresponds to a choice of Cartan element in $SU(2)_R$, so we have implicitly made the choice already.

In this chapter we will focus on the simplest non-trivial[17] case of these multiplets, the $\hat{\mathcal{B}}_1$ multiplet. This multiplet plays a distinguished role, because it encodes the information about the continuous global symmetries of the theory. Indeed, the multiplet contains a conserved current,

$$J_{\alpha\dot{\alpha}} = \epsilon^{\mathcal{J}\mathcal{K}} \mathcal{Q}^{\mathcal{I}}_{\alpha} \widetilde{\mathcal{Q}}_{\mathcal{J}\dot{\alpha}} \phi_{\mathcal{I}\mathcal{K}} \ , \tag{4.23}$$

[16] In the $\mathcal{N} = 1$ case the existence of compact conformal manifolds has recently been established in [159]. The methods used there cannot easily be generalized to the $\mathcal{N} = 2$ case.

[17] $\hat{\mathcal{B}}_{\frac{1}{2}}$ describes a free hypermultiplet.

where $\phi_{\mathcal{I}\mathcal{J}}$ is the operator of lowest dimension in the $\hat{\mathcal{B}}_1$ multiplet. It is an $SU(2)_R$ triplet and is often referred to as the *moment map* operator. (The superconformal primary is the highest $SU(2)_R$ weight ϕ_{11}.) The current $J_{\alpha\dot{\alpha}}$ generates a continuous symmetry global symmetry, and is thus necessarily in the adjoint representation of some Lie group G_F. Vice versa, if the theory enjoys a continuous global symmetry, it follows from Conjecture 6 that the CFT contains an associated conserved current $J_{\alpha\dot{\alpha}}$, and one can show that in an interacting $\mathcal{N}=2$ SCFT such a current must necessarily belong to a $\hat{\mathcal{B}}_1$ multiplet. Indeed, one can survey the list of superconformal representations and identify all the ones that contain conserved spin one currents that are also $SU(2)_R \times U(1)_r$ singlets. The list is very short: $\hat{\mathcal{B}}_1$ and $\hat{\mathcal{C}}_{0(\frac{1}{2},\frac{1}{2})}$. The latter multiplet has a conserved current as its superconformal primary, but also contains conserved spin three conserved current among its descendants, so by our usual criterion it is not allowed in an interacting SCFT. What's more, $\hat{\mathcal{B}}_1$ representations cannot combine with other short representations to form long representations, so the $\hat{\mathcal{B}}_1$ content of a theory is an invariant on the conformal manifold, except for possible enhancements in singular limits where some free subsector decouples (such as the zero coupling limit of a gauge theory) and $\hat{\mathcal{C}}_{0(\frac{1}{2},\frac{1}{2})}$ multiplets may split off from long multiplets hitting the unitarity bound.

As we have already mentioned in the context of exactly marginal gauging of SCFTs, to each simple non-abelian factor of the global symmetry group is associated a *flavor central charge* k, defined from the OPE coefficient of the conserved current with itself (4.13). Thus the most basic data associated to the $\hat{\mathcal{B}}_1$ representations in an SCFT are the global symmetry group $G_F = G_1 \times \ldots G_k$ and the corresponding flavor central charges.

Chiral algebra data

It was recognized in chapter 2 (see also [75] and chapter 3) that the local operator algebra of any $\mathcal{N}=2$ SCFT admits a closed subsector isomorphic to a two-dimensional chiral algebra. The operators that play a role in the chiral algebra are the so-called *Schur operators*, which (by definition) obey the conditions[18]

$$\Delta - (j_1 + j_2) - 2R = 0 \ , \qquad j_2 - j_1 - r = 0 \ . \tag{4.24}$$

Schur operators are found in the following short representations,

$$\hat{\mathcal{B}}_R \ , \quad \mathcal{D}_{R(0,j_2)} \ , \quad \bar{\mathcal{D}}_{R(j_1,0)} \ , \quad \hat{\mathcal{C}}_{R(j_1,j_2)} \ . \tag{4.25}$$

One should in particular note the absence of the $\mathcal{E}_r$ multiplets from this structure. Each supermultiplet in this list contains precisely one Schur operator: for the $\hat{\mathcal{B}}_R$ multiplets, the Schur operator is the superconformal primary itself, while for the other multiplets in (4.25) it

[18] In fact one can show that the first condition implies the second in a unitary theory.

is a superconformal descendant.[19] When inserted on a fixed plane $\mathbb{R}^2 \subset \mathbb{R}^4$, parametrized by the complex coordinate z and its conjugate $\bar{z}$, and appropriately *twisted* (the twist identifies the right-moving global conformal algebra $\overline{\mathfrak{sl}(2)}$ acting on $\bar{z}$ with the complexification of $\mathfrak{su}(2)_R$ algebra), Schur operators have *meromorphic* correlation functions. The rationale behind this construction is that twisted Schur operators are closed under the action of a certain nilpotent supercharge, $\mathbb{Q} := \mathcal{Q}^1_- + \widetilde{\mathcal{S}}^{\dot{-}}_1$, and they have well-defined meromorphic OPEs at the level of $\mathbb{Q}$ cohomology. This is precisely the structure that defines a two-dimensional chiral algebra.

We refer the reader to 2 for a comprehensive explanation of this construction. Here we mainly wish to emphasize that the *chiral algebra data* (*i.e.*, the Schur operators and their three-point functions) are a very natural generalization of the Higgs data. Since they are subject to associativity conditions expressed by meromorphic equations, the chiral algebra data can be often determined exactly given some minimum physical input.

The simplest example, and the one that will play a role in this chapter, is the case of moment maps. Moment maps transform in the adjoint representation of the flavor symmetry group, and in the associated chiral algebra they correspond to affine Kac-Moody currents, where the level k_{2d} of the affine current algebra is related to the four-dimensional flavor central charge k by the universal relation

$$k_{2d} = -\frac{k}{2} \ . \tag{4.26}$$

The four-point function of affine currents completely determined by meromorphy and crossing symmetry. In the present context, it admits a reinterpretation as a certain meromorphic piece of the full moment map four-point function. Crucially, this meromorphic piece contains the complete information about the contribution of short representations to the double OPE of the four-point function.[20] All in all, combining the constraints of four-dimensional unitarity with the ability to solve exactly for the contributions of short representations leads to novel unitarity bounds for the level k and the trace anomaly coefficient c that are valid in any interacting $\mathcal{N} = 2$ SCFT. These bounds will play a significant role in the analysis of Section 4.6.

4.2.4 A first look at the landscape: theories of low rank

The ultimate triumph of the $\mathcal{N} = 2$ bootstrap program would be the classification of $\mathcal{N} = 2$ SCFTs. If the decomposability conjecture of Section 4.2.3 holds true, then this problem is

[19] For example, the Schur operator in a $\hat{\mathcal{C}}_{0(0,0)}$ multiplet is a single component of the $SU(2)_R$ conserved current.

[20] To be able to uniquely reconstruct the contribution of the short representations from the meromorphic function, one must make the now-familiar assumption that the theory does not contain higher-spin conserved currents.

reduced to the enumeration of elementary building block theories with no conformal manifold. Still, this is completely out of reach at present, and any attempt at a direct attack on the classification problem would be premature. We are still very much in an exploratory phase.

To organize our explorations we may characterize theories by their *rank* – *i.e.*, the dimension of their Coulomb branch or the number of generators in the Coulomb branch chiral ring. Theories with low rank by and large have smaller values for their central charges than their higher-rank counterparts, so this may be a reasonable measure of the complexity of a theory. From the bootstrap point of view, theories with small central charges are attractive as targets for numerical study.

The rank zero case is probably trivial. The simplest conjecture is that the only $\mathcal{N}=2$ SCFT with no Coulomb branch is the free hypermultiplet theory. This would be compatible with the universal validity of the Shapere-Tachikawa bound.

For rank one, we can start by reviewing the list of established theories. This survey will prove useful in our efforts to interpret the numerical bootstrap results reported in later sections. The classic rank one theories are the SCFTs that arise on a single D3 brane probing an F-theory singularity with constant dilaton [12, 64–68]. There are seven such singularities, denoted by H_0, H_1, H_2, D_4, E_6, E_7, E_8. With the exception of the theory associated to the $D4$ singularity, which is an $SU(2)$ gauge theory with $N_f=4$ fundamental flavors, these theories are all isolated non-Lagrangian SCFTs. They have an alternative realization is in class $\mathcal{S}$, where they are associated to punctured spheres with certain special punctures – see, *e.g.*, [17, 79, 105, 160, 161].

Basic properties of these rank one SCFTs are summarized in Table 4.1. Their flavor symmetry algebra $\mathfrak{h}$ is given by the Lie algebra of the same name (with $H_i \to A_i$; the H_0 theory has no flavor symmetry). From the F-theory realization it is manifest that the Higgs branch of each theory is the one-instanton moduli space for the corresponding flavor symmetry group. As algebraic varieties, these Higgs branches are generated by the $\mathfrak{h}$ moment maps subject to a set of quadratic relations known as the Joseph relations. Relatedly, the flavor central charge k and the c anomaly saturate the unitarity bounds derived in 2. It was argued in Section 4 of chapter 2 that this is strong evidence that the protected chiral algebra is the affine Lie algebra $\hat{\mathfrak{h}}_{k_{2d}}$ at level $k_{2d}=-\frac{k}{2}$.[21]

Another well-known rank one $\mathcal{N}=2$ SCFT is $\mathcal{N}=4$ super Yang-Mills theory with gauge group $SU(2)$. Regarded as an $\mathcal{N}=2$ theory, it has flavor symmetry $\mathfrak{h}=\mathfrak{su}(2)$, the commutant of $SU(2)_R\times U(1)_r$ in the full $SU(4)$ R-symmetry. There are three more recent additions to the list of rank one theories. They were initially discovered in [71] by considering the strong coupling limit of Lagrangian theories and then given a class $\mathcal{S}$ re-interpretation in [162, 163]. In these theories the Coulomb branch is generated by an $\mathcal{E}_r$ multiplet with $r=3\,,4\,,6$. These are the same values as in the E_6, E_7 and E_8 theories in Table 4.1, but the

[21] We mention in passing, as this will play a role later, that each of these theories admits a rank N generalization, physically realized on the worldvolume of N parallel D3 branes probing the same F-theory singularity. The Higgs branches of the higher rank theories are the moduli spaces of rank-N $\mathfrak{h}$-instantons, with global symmetry $\mathfrak{h}\otimes\mathfrak{su}(2)$ for $N\geqslant 2$.

flavor symmetries for these new theories are smaller. Given the serendipitous discovery of these "new" rank one theories, one may rightly view with suspicion the claim that the list of known rank one theories is complete. How could we find out?

A systematic study of rank one $\mathcal{N} = 2$ SCFTs has been undertaken by Argyres and collaborators [164, 165] using Seiberg-Witten technology.[22] Let us give a quick informal summary of this approach. The Coulomb branch chiral ring of a rank one theory is by definition generated by a single operator $\mathcal{E}_{r_0}$. Assuming the validity of Conjecture 7, this operator should not be nilpotent, and further assuming Conjecture 8, its vacuum expectation value

$$u := \langle \mathcal{E}_{r_0} \rangle \ , \tag{4.27}$$

parametrizes the Coulomb branch of vacua. For $u \neq 0$, the theory admits a low-energy description in terms of an effective $U(1)$ gauge theory, whose data are encoded in a family of elliptic curves [168, 169],

$$y^2 = x^3 + f(u, m_i)x + g(u, m_i) \ , \tag{4.28}$$

and in a meromorphic one form $\lambda_{\rm SW}(u, m_i)$, subject to certain consistency conditions. The complex parameters $\{m_i\}$ are mass parameters, dual to the Cartan generators of the flavor symmetry algebra $\mathfrak{h}$ of the theory. For zero masses, the curve must take a scale invariant form, *i.e.*, it must transform homogeneously if one rescales x, y and u with the appropriate weights. The scaling weight of u is nothing but the conformal dimension $\Delta = r_0$ of $\mathcal{E}_{r_0}$. The possible scale-invariant curves are then given by a subset of Kodaira's classification of stable degenerations of elliptic curves depending holomorphically on a single complex parameter. There turn out to be seven cases, and they are the same as the F-theory singularities with constant dilaton. Starting from the scale-invariant curve, one can construct its mass deformations (which must be compatible with the existence of the meromorphic one-form $\lambda_{\rm SW}$), and infer the flavor symmetry algebra $\mathfrak{h}$. It turns out that for a given scale invariant curve there can be numerous inequivalent mass deformations [164, 165]. The "canonical" rank one theories of Table 4.1 correspond to the *maximal* mass deformation, but submaximal deformations with smaller flavor symmetry are also possible. An example of this phenomenon that we have already implicitly encountered is the submaximal deformation of the $D4$ singularity, with $\mathfrak{h} = \mathfrak{su}(2) \subset \mathfrak{so}(8)$, which corresponds to $\mathcal{N} = 4$ SYM with gauge group $SU(2)$. The "new" rank one theories of [71, 162] are recognized as submaximal deformations of the E_6, E_7 and E_8 Kodaira singularities, but several other possibilities also appear to be consistent[23] [165]. In the absence of an independent physical construction (in class $\mathcal{S}$ or otherwise), it is *a priori* unclear whether the mere existence of a Seiberg-Witten geometry guarantees the existence of a full fledged SCFT. The bootstrap approach should be able to shed light on this question, at the very least by providing some consistency checks of the candidate models.

[22] The rank two case is considerably more involved [166, 167].

[23] We are grateful to P. Argyres for sharing some of the results of [165] with us prior to publication.

G	H_0	H_1	H_2	D_4	E_6	E_7	E_8
$\mathfrak{h}$	–	$\mathfrak{su}(2)$	$\mathfrak{su}(3)$	$\mathfrak{so}(8)$	$\mathfrak{e}_6$	$\mathfrak{e}_7$	$\mathfrak{e}_8$
$h^\vee$	–	2	3	6	12	18	30
k	$\frac{12}{5}$	$\frac{8}{3}$	3	4	6	8	12
c	$\frac{11}{30}$	$\frac{1}{2}$	$\frac{2}{3}$	$\frac{7}{6}$	$\frac{13}{6}$	$\frac{19}{6}$	$\frac{31}{6}$
a	$\frac{43}{120}$	$\frac{11}{24}$	$\frac{7}{12}$	$\frac{23}{24}$	$\frac{41}{24}$	$\frac{59}{24}$	$\frac{95}{24}$
r_0	$\frac{6}{5}$	$\frac{4}{3}$	$\frac{3}{2}$	2	3	4	6

Table 4.1: Properties of rank one SCFTs associated to maximal mass deformations of the Kodaira singularities [54, 70, 170]. We list the name of the singularity, the flavor symmetry algebra $\mathfrak{h}$ and its dual Coxeter number $h^\vee$, the flavor central charge k, the c and a anomaly coefficients, and the $U(1)_r$ charge r_0 of the Coulomb branch chiral ring generator.

In summary, even for rank one the situation is not completely settled. There are several established theories and a growing list of possible additional models. A complete elucidation of the rank one case should be a benchmark for our understanding of the $\mathcal{N}=2$ landscape.

4.3 The moment map four-point function

As our first observable of interest we take the four-point function of moment map operators. As explained in the previous section, these are the superconformal primaries for representations containing conserved currents for global symmetries (the $\hat{\mathcal{B}}_1$ multiplets). This is in some sense the paradigmatic observable by means of which we can investigate SCFTs with flavor symmetries. The moment map operators are spacetime scalars of conformal dimension two, and they transform in the adjoint representation of $SU(2)_R$ while being neutral with respect to $U(1)_r$. Like the conserved currents in the same multiplet, they transform in the adjoint representation of the flavor symmetry group G_F. We denote these operators $\phi^A_{(\mathcal{I}\mathcal{J})}(x)$, where $\mathcal{I},\mathcal{J}=1,2$ are fundamental indices for $SU(2)_R$ and $A=1,\ldots,\dim G_F$ is an adjoint index for G_F.

The purpose of the present section is to describe the structure of this correlation function and to formulate its (super)conformal block decomposition. Let us briefly outline the general trajectory of this analysis. The four-point function of moment map operators can initially be organized to reflect the constraints of conformal symmetry, $SU(2)_R$ symmetry, and G_F flavor symmetry. In practice this means decomposing the general correlator into a number of functions of conformal cross ratios that encode the contributions of operators with fixed transformation properties under $SU(2)_R$ and G_F in the conformal block expansion. These functions are further constrained by superconformal Ward identities [41] (see also [42, 43]).

The ultimate result of these Ward identities is that the functions corresponding to different $SU(2)_R$ channels are not independent, but rather the full four-point function is algebraically determined in terms of a set of meromorphic functions $f_i(z)$ and unconstrained functions $\mathcal{G}_i(z,\bar{z})$, where the index i runs over the irreps that appear in the tensor product of two copies of the adjoint representation of G_F,

$$\mathrm{Adj}(G_F) \otimes \mathrm{Adj}(G_F) =: \bigoplus_{i=1}^{n} \mathfrak{R}_i(G_F) \ . \tag{4.29}$$

The meromorphic functions are identical to the four-point functions of affine currents in two dimensions 2, and are completely determined by the flavor central charge. The unconstrained functions $\mathcal{G}_i(z,\bar{z})$ are best considered in a superconformal partial wave expansion. They can be split into two parts which we call $\mathcal{G}_i^{\mathrm{short}}(z,\bar{z})$ and $\mathcal{G}_i^{\mathrm{long}}(z,\bar{z})$. The former functions encode the contributions of protected operators appearing in the OPE of two moment maps, and under mild assumptions[24] they can be completely determined in terms of the central charges k and c by reading off the relevant CFT data from the (now fixed) meromorphic functions. The latter functions encode the spectrum and OPE coefficients of unprotected operators, about which we generally have scant knowledge. The point of the numerical analysis of Section 4.6 will be to constrain the CFT data encoded in the functions $\mathcal{G}_i^{\mathrm{long}}(z,\bar{z})$ using crossing symmetry.

4.3.1 Structure of the four-point function

The appearance of the four-point function in question can be cleaned up a bit by introducing some auxiliary structure. Following [43], we eliminate the explicit $SU(2)_R$ indices on $\phi^A_{(\mathcal{I}\mathcal{J})}(x_i)$ in favor of complex polarization vectors $t^{\mathcal{I}}$ in terms of which we define

$$\varphi^A(t,x) := \phi^A_{(\mathcal{I}\mathcal{J})}(x) t^{\mathcal{I}} t^{\mathcal{J}} \ . \tag{4.30}$$

With these conventions, conformal symmetry and R-symmetry demand that the four-point function of moment map operators be of the form

$$\langle \varphi^A(t_1,x_1)\varphi^B(t_2,x_2)\varphi^C(t_3,x_3)\varphi^D(t_4,x_4)\rangle = \frac{(t_1\cdot t_2)^2(t_3\cdot t_4)^2}{x_{12}^4 x_{34}^4} G^{ABCD}(u,v;w) \ , \tag{4.31}$$

where u and v are (standard) conformally invariant cross-ratios,

$$u := \frac{x_{12}^2 x_{34}^2}{x_{24}^2 x_{13}^2} =: z\bar{z} \ , \qquad v := \frac{x_{14}^2 x_{23}^2}{x_{24}^2 x_{13}^2} =: (1-z)(1-\bar{z}) \ , \tag{4.32}$$

[24]The assumption in question is that there are no higher spin conserved currents appearing in the conformal block decomposition. This is expected to hold true for any interacting theory.

w is the unique $SU(2)_R$−invariant "cross-ratio" of the auxiliary variables,

$$w := \frac{(t_1 \cdot t_2)\,(t_3 \cdot t_4)}{(t_1 \cdot t_3)\,(t_2 \cdot t_4)} \ , \tag{4.33}$$

and we have defined the contraction $t_i \cdot t_j := \epsilon_{\mathcal{I}\mathcal{J}} t_i^{\mathcal{I}} t_j^{\mathcal{J}}$.

The flavor symmetry of the correlator can be captured by introducing a complete basis P_i^{ABCD} of invariant tensors. We can always choose this basis such that the label i runs over the various irreducible representations $\mathfrak{R}_i$ of G_F that appear in the tensor product of two copies of the adjoint representation of G_F, with the P_i^{ABCD} projectors onto this representation. We may then write

$$G^{ABCD}(u,v;w) = \sum_{i \in \mathrm{Adj}\otimes\mathrm{Adj}} G_i(u,v;w) P_i^{ABCD} \ , \tag{4.34}$$

and the projectors themselves satisfy

$$P_i^{ABCD} P_j^{DCEF} = \delta_{ij} P_i^{ABEF} \ , \qquad P_i^{ABBA} = \dim(R_i) \ . \tag{4.35}$$

For each representation $\mathfrak{R}_i$ one can decompose the corresponding $G_i(u,v;w)$ into three terms corresponding to the three $SU(2)_R$ channels. In terms of the auxiliary variable w we find

$$G_i(u,v;w) = \sum_{R=0}^{2} a_{i,R}(u,v) P_R(y) \ , \tag{4.36}$$

where we have defined $y = \frac{2}{w} - 1$, and the $P_R(y)$ are Legendre polynomials

$$P_0(y) = 1 \ , \qquad P_1(y) = y \ , \qquad P_2(y) = \frac{1}{2}\left(3y^2 - 1\right) \ . \tag{4.37}$$

Each of the $a_{i,R}(u,v)$ has a conventional conformal block decomposition that encodes the exchanged conformal families in the appropriate flavor and R-symmetry representations. These conformal blocks are actually grouped together in *super*conformal blocks, as we will explain further below.

The consequences of superconformal covariance for this four-point function have been analyzed in detail in [41–43]. Because supersymmetry transformations commute with flavor symmetries, the superconformal Ward identities apply to each $G_i(u,v;w)$ independently. The end result of the analysis in those papers is neatly encapsulated in the following specialization condition,

$$G_i(u,v;w)\big|_{w=\bar{z}} = f_i(z) \ , \qquad G_i(u,v;w)\big|_{w=z} = f_i(\bar{z}) \ , \tag{4.38}$$

where it is the same meromorphic function f_i appearing in both expressions. We note here that this condition can also be seen to follow from the existence of the superconformal twist introduced in 2. In terms of these meromorphic functions, one then finds that the the $G_i(u, v; w)$ take the following form [43],

$$G_i(u,v;w) = \frac{z(w-\bar{z})f_i(\bar{z}) - \bar{z}(w-z)f_i(z)}{w(z-\bar{z})} + \left(1-\frac{z}{w}\right)\left(1-\frac{\bar{z}}{w}\right)\mathcal{G}_i(u,v) \ . \tag{4.39}$$

Upon decomposing this expression in the basis of Legendre polynomials of y, one recovers expressions for the various R-symmetry channels in terms of f_i and $\mathcal{G}_i$,

$$\begin{aligned} a_{i,2}(u,v) &= \frac{u\,\mathcal{G}_i(u,v)}{6} \ , \\ a_{i,1}(u,v) &= \frac{u(f_i(z)-f_i(\bar{z}))}{2(z-\bar{z})} - \frac{(1-v)\mathcal{G}_i(u,v)}{2} \ , \\ a_{i,0}(u,v) &= \mathcal{G}_i(u,v)\left(\frac{v+1}{2}-\frac{u}{6}\right) - \frac{u}{2(z-\bar{z})}\left(\frac{(2-z)f_i(z)}{z} - \frac{(2-\bar{z})f_i(\bar{z})}{\bar{z}}\right) \ . \end{aligned} \tag{4.40}$$

We see that (for a given flavor symmetry channel) the functions $a_{i,R}(u, v)$ are not independent; instead they are all determined in terms of the meromorphic function $f_i(z)$ and a single unconstrained function $\mathcal{G}_i(u, v)$.

Constraints of crossing symmetry

As a consequence of Bose symmetry, the four-point function must be invariant under arbitrary permutations of the four inserted operators. For the functions $G_i(u, v; y)$, these permutations lead to the following relations,

$$(x_1,t_1) \longleftrightarrow (x_2,t_2) \qquad \Longrightarrow \qquad G_i\left(\frac{u}{v},\frac{1}{v};-y\right) = (-1)^{\mathrm{symm}(i)} G_i(u,v;y) \ , \tag{4.41}$$

$$(x_1,t_1) \longleftrightarrow (x_3,t_3) \qquad \Longrightarrow \qquad \frac{1-2y+y^2}{4} G_i\left(v,u;\frac{y+3}{y-1}\right) = \frac{v^2}{u^2} G_j(u,v;y) F_i^{\ j} \ . \tag{4.42}$$

The first of these is called the *braiding relation*, while we refer to the second as the *crossing symmetry equation*. We have introduced the notation $\mathrm{symm}(i)$ which is equal to zero or one if representation i appears in the symmetric or antisymmetric tensor product of two copies of the adjoint, respectively. The matrix $F_i^{\ j}$ relates the projectors in one channel with the projectors in the crossed channel:

$$P_i^{ABCD} = P_j^{CBAD} F_i^{\ j} \ , \tag{4.43}$$

and is related to "Wigner's 6j-coefficients" (see, *e.g.*, [63]). In the cases considered in the present work this matrix satisfies $F_i^{\ j} F_j^{\ k} = \delta_i^{\ k}$.

The corresponding constraints for the functions $f_i(z)$ and $\mathcal{G}_i(u, v)$ are obtained from (4.41) and (4.42) by using the solution for $G_i(u,v;y)$ from (4.39) and reading off the constraints term by term in a y-expansion. This exercise was already worked out without the flavor symmetry structure in [43]. Upon including flavor symmetry indices we find two sets of relations involving only the meromorphic functions,

$$f_i(z) = (-1)^{\mathrm{symm}(i)} f_i\left(\frac{z}{z-1}\right) \ , \qquad z^2 f_i(1-z) = (1-z)^2 f_j(z) F_i^{\ j} \ , \tag{4.44}$$

one braiding equation involving only the two-variable functions,

$$\mathcal{G}_i(u,v) = (-1)^{\mathrm{symm}(i)} \frac{1}{v} \mathcal{G}_i\left(\frac{u}{v}, \frac{1}{v}\right) \ . \tag{4.45}$$

There is one additional non-trivial crossing symmetry relation for the unconstrained function,

$$\begin{aligned}&(z-\bar{z})(1-z)^2(1-\bar{z})^2 F_j^{\ i} \mathcal{G}_j(u,v) + z^2\bar{z}^2(\bar{z}-z)\mathcal{G}_i(v,u)\\ &+z\bar{z}\Big(z(\bar{z}-1)f_i(1-z) - \bar{z}(z-1)f_i(1-\bar{z})\Big) = 0 \ .\end{aligned} \tag{4.46}$$

This is the equation that we will investigate numerically. Before doing so we have to first compute its superconformal block decomposition and solve the other crossing symmetry equations, in particular the last equation in (4.44). We will discuss these two topics in the next two subsections.

Fixing the meromorphic functions

By meromorphicity, the single-variable functions $f_i(z)$ are fixed completely by the structure of their singularities. The only physically allowable singularities occur when two of the inserted operators collide, *i.e.*, at $z=0$, $z=1$, and $z\to\infty$. The equations in (4.44) relate the singularities at these three points, so it suffices to specify the singular behavior of $f_i(z)$ near, say, $z=0$. This simple crossing symmetry problem is reminiscent of what arises in the study of two-dimensional meromorphic conformal field theories. Indeed, a compelling physical picture that explains the relationship between this crossing symmetry problem and the two dimensional case has been presented in 2. There it was shown that the functions $f_i(z)$ are precisely equal to the four-point functions of an extended chiral algebra that can be isolated by working at the level of cohomology relative to a particular nilpotent supercharge. Indeed, the equations (4.44) are exactly the crossing equations one encounters in studying chiral algebra four-point functions.

In 2 it was found that the moment maps $\phi^A_{(\mathcal{I}\mathcal{J})}(x)$ are related to dimension one affine currents in the corresponding chiral algebra. These affine currents generate an affine Kac-Moody (AKM) algebra $\widehat{G_F}$. The level k_{2d} of this AKM algebra is related to the four-dimensional flavor central charge k as

$$k_{2d} = -\frac{k}{2} \ . \tag{4.47}$$

Many more details about these chiral algebras can be found in 2 (see also [28, 75]). For our purposes here we need only know that the chiral algebra completely determines the one-variable part of the four-point function $f^{ABCD}(z)$ to be the four-point function of affine currents, which for any group G_F takes the form:

$$f^{ABCD}(z) = \delta^{AB}\delta^{CD} + z^2\delta^{AC}\delta^{BD} + \frac{z^2\delta^{AD}\delta^{CB}}{(1-z)^2} + \frac{2z}{k}f^{ACE}f^{BDE} + \frac{2z}{k(z-1)}f^{ADE}f^{BCD} \ . \tag{4.48}$$

Note that the normalization here is such that the current has a canonical two-point function, so the level k appears in the denominator in this expression.

4.3.2 Superconformal partial wave expansion

So far we have understood the functional form of the four-point function as follows from $\mathfrak{su}(2,2|2)$ symmetry and an analysis of the associated chiral algebra. The next step is to consider the superconformal partial wave expansion of the correlator.

The supersymmetric OPE of a $\hat{\mathcal{B}}_1$ representation with itself has been studied in [171]. The approach taken in that paper was to analyze all possible three-point functions of two $\hat{\mathcal{B}}_1$ representations with a third *a priori* generic representation in harmonic superspace. The result can be summarized in the following "fusion rule",

$$\hat{\mathcal{B}}_1 \times \hat{\mathcal{B}}_1 \sim \mathbf{1} + \hat{\mathcal{B}}_1 + \hat{\mathcal{B}}_2 + \hat{\mathcal{C}}_{0(j,j)} + \hat{\mathcal{C}}_{1(j,j)} + \mathcal{A}^{\Delta}_{0,0(j,j)} \ . \tag{4.49}$$

This fusion rule can be further refined by taking into account flavor symmetry representations, which lead to some additional constraints. For example, long multiplets can appear in all possible flavor symmetry representations but the stress tensor multiplet $\hat{\mathcal{C}}_{0(0,0)}$ can only appear as a flavor singlet. The precise selection rules are summarized in Table F.2.

Each superconformal multiplet $\mathcal{X}$ in flavor representation i that appears on the right-hand side of (4.49) must contribute a finite number of conventional conformal blocks to each of the three functions $a_{i,R}(u,v)$ with $0 \leqslant R \leqslant 2$. We denote these contributions as $a^{\mathcal{X}}_{i\,R}(u,v)$. For this particular four-point function the coefficients of the conventional conformal blocks are all related by the superconformal Ward identities [41], and we end up with just a single undetermined OPE coefficient (squared) for each superconformal block. This leads to the decomposition:

$$G^{ABCD}(u,v;y) = \sum_i P_i^{ABCD} \sum_{\mathcal{X} \text{ in rep } i} (-1)^{\text{symm}(i)} \lambda^2_{i\,\mathcal{X}} \left(\sum_{R=0}^{2} P_R(y) a^{\mathcal{X}}_{i\,R}(u,v) \right) \tag{4.50}$$

where the term in parentheses is the superconformal block. The factor of $(-1)^{\text{symm}(i)}$ follows from reflection positivity. In a unitary theory the $\lambda_{i\,\mathcal{X}}$ are real and their square is therefore always positive.

The complete set of superconformal blocks for this four-point function was obtained in [41]. It is most naturally given in terms of the functions $\mathcal{G}_i(u,v)$ and $f_i(z)$ introduced above, which is presented in Table 4.2, where $\ell = 2j_1 = 2j_2 = 2j$ since all the multiplets appearing in this OPE have $j_1 = j_2$. In the table $G_\Delta^{(\ell)}(u,v)$ denotes the four-dimensional conformal

Multiplet	Contribution to $\mathcal{G}_i^{\mathcal{X}}(u,v)$	Contribution to $f_i^{\mathcal{X}}(z)$	Restrictions
Id.	0	1	
$\mathcal{A}^{\Delta}_{0,0j,j)}$	$6u^{\frac{\Delta-\ell}{2}}G^{(\ell)}_{\Delta+2}(u,v)$	0	$\Delta \geqslant \ell+2$
$\hat{\mathcal{C}}_{0(j,j)}$	0	$2g_{2j+2}(z)$	$j \geqslant 0$
$\hat{\mathcal{B}}_1$	0	$2g_1(z)$	
$\hat{\mathcal{C}}_{1(j,j)}$	$6uG^{(\ell+1)}_{\ell+5}(u,v)$	$-12g_{2j+3}(z)$	$j \geqslant 0$
$\hat{\mathcal{B}}_2$	$6uG^{(0)}_4(u,v)$	$-12g_2(z)$	

Table 4.2: Superconformal blocks for the $\mathcal{E}_{r_0}$ four point function in the $\hat{2}$ channel.

block which is given by (F.1) in our conventions, and

$$g_\ell = \left(-\frac{z}{2}\right)^{\ell-1} z\,{}_2F_1(\ell,\ell;2\ell;z) \tag{4.51}$$

is a two-dimensional conformal block in a chiral algebra, as we discuss in more detail below. Through (4.40), the contribution of each superconformal multiplet to the $a_{i,R}(u,v)$ is obtained from the contribution of said multiplet to $\mathcal{G}_i(u,v)$ and $f_i(z)$. This is worked out in detail in Appendix F.1.

From inspection of Table 4.2 it follows that the decomposition into superconformal blocks of a given four-point function can be ambiguous. For example, a long multiplet at the unitarity bound $\Delta = \ell + 2$ contributes in exactly the same manner as a combination of two short multiplets. These ambiguities can be understood from the fact that these two multiplets can recombine to form a long multiplet according to[25]

$$\mathcal{A}^{\Delta=2j+2}_{0,0(j,j)} \simeq \hat{\mathcal{C}}_{0(j,j)} \oplus \hat{\mathcal{C}}_{\frac{1}{2}(j-\frac{1}{2},j)} \oplus \hat{\mathcal{C}}_{\frac{1}{2}(j,j-\frac{1}{2})} \oplus \hat{\mathcal{C}}_{1(j-\frac{1}{2},j-\frac{1}{2})}\ . \tag{4.52}$$

[25]Appendix B provides an overview of all the recombination rules for the unitary irreps of $\mathfrak{su}(2,2|2)$.

Only the first and last multiplet are allowed in the OPE of two scalars. For the case $j = 0$ we can use $\hat{\mathcal{C}}_{1(j-\frac{1}{2},j-\frac{1}{2})} = \hat{\mathcal{B}}_2$ and we get $\simeq \hat{\mathcal{C}}_{0(0,0)} + \hat{\mathcal{B}}_2$ on the right-hand side. These ambiguities will be fixed below.

The braiding relations (4.45) together with Table 4.2 correlate the allowed spins of multiplet $\mathcal{X}_i$ to symm(i): only even/odd spins appear in $\mathcal{G}_i(u,v)$ for a representation appearing in the symmetric/antisymmetric tensor product, respectively. This follows from the braiding relations from the individual conformal blocks, $G_\Delta^{(\ell)}(u,v) = (-1)^\ell v^{-\frac{\Delta-\ell}{2}} G_\Delta^{(\ell)}(\frac{u}{v},\frac{1}{v})$. As an example, for flavor singlets the spin of these operators is always even and for the flavor adjoint multiplets these spins are always odd.

While the meromorphic functions $f_i(z)$ receive contributions only from short multiplets, the two-variable functions $\mathcal{G}_i(u,v)$ include contributions from both long and short multiplets. It is then useful to split the two-variable functions into the long and short contributions appearing in a given channel,

$$\mathcal{G}_i(u,v) = \mathcal{G}_i^{\text{short}}(u,v) + \mathcal{G}_i^{\text{long}}(u,v) \ , \tag{4.53}$$

where we have

$$\begin{aligned} \mathcal{G}_i^{\text{short}}(u,v) =& \ 6\lambda^2_{i\,\hat{\mathcal{B}}_2} u G_4^{(0)}(u,v) + \sum_{\ell=0,1,\ldots} 6\lambda^2_{i\,\hat{\mathcal{C}}_{1(j,j)}} (-1)^\ell u\, G_{\ell+5}^{(\ell+1)}(u,v) \ , \\ \mathcal{G}_i^{\text{long}}(u,v) =& \sum_{\Delta\geqslant \ell+2,\ell} 6\lambda^2_{i\,\mathcal{A}^\Delta_\ell} (-1)^\ell u^{\frac{\Delta-\ell}{2}} G_{\Delta+2}^{(\ell)}(u,v) \ . \end{aligned} \tag{4.54}$$

In the next subsection we will show that the coefficients of the short superconformal blocks – and therefore the complete functional form of $\mathcal{G}_i^{\text{short}}(u,v)$ – are completely fixed from the chiral algebra correlator (4.48). All the undetermined information in the four-point function is then contained in $\mathcal{G}_i^{\text{long}}(u,v)$. These are the functions that will be analyzed numerically in Section 4.6.

Fixing the short multiplets

Because the meromorphic functions $f_i(z)$ are completely determined by crossing symmetry (or alternatively by analyzing the associated chiral algebra), their decomposition in chiral blocks of the form (4.51) is determined. We can thus write

$$f_i(z) = \sum_{\ell\geqslant -2} b_{i,\,\ell} (-1)^\ell g_{\ell+2}(z) \ , \tag{4.55}$$

with known coefficients $b_{i,\,\ell}$. Upon examining the contributions of general supermultiplets to $f_i(z)$ in Table 4.2, we see that the chiral OPE coefficients are related to four-dimensional

OPE coefficients of the short multiplets as follows,

$$\begin{aligned}
b_{\mathbf{1},-2} &= \lambda^2_{\mathbf{1},\,\mathrm{Id}}\ , \\
b_{i,-1} &= 2\lambda^2_{i,\,\hat{\mathcal{B}}_1}\ , \\
b_{i,\,0} &= 2\lambda^2_{i,\,\hat{\mathcal{C}}_{0(0,0)}} - 12\lambda^2_{i,\,\hat{\mathcal{B}}_2}\ , \\
b_{i,\,\ell} &= 2\lambda^2_{i,\,\hat{\mathcal{C}}_{0(j,j)}} - 12\lambda^2_{i,\,\hat{\mathcal{C}}_{1(j-\frac{1}{2},j-\frac{1}{2})}}\ , \qquad \ell = 2j \geqslant 1\ .
\end{aligned} \tag{4.56}$$

Note that in the first line, the identity operator can only appear in the flavor singlet channel $i = \mathbf{1}$. If we now further assume that the theory has (a) no higher spin currents, and (b) a unique stress tensor, then one can actually fix the OPE coefficients of all short multiplets. Indeed, the first assumption implies the absence of any $\hat{\mathcal{C}}_{0(j,j)}$ multiplets with $j \geqslant 1$, so in particular

$$\lambda^2_{i,\,\hat{\mathcal{C}}_{0(j,j)}} = 0 \quad \text{for} \quad \ell = 2j \geqslant 1\ . \tag{4.57}$$

Our second assumption implies that there is a unique multiplet of type $\hat{\mathcal{C}}_{0(0,0)}$, which is a flavor singlet, and whose OPE coefficient is fixed in terms of the c central charge according to

$$\lambda^2_{i,\,\hat{\mathcal{C}}_{0(0,0)}} = \frac{\dim G_F}{6c}\delta_{i,\mathbf{1}}\ , \tag{4.58}$$

where $\dim G_F$ is the dimension of G_F. This numerical value follows from conformal Ward identities upon imposing appropriate normalization conventions which we spell out below.

With these additional conditions, we see that (4.56) completely determines the OPE coefficients $\lambda^2_{i,\,\hat{\mathcal{B}}_1}$, $\lambda^2_{i,\,\hat{\mathcal{B}}_2}$, and $\lambda^2_{i,\,\hat{\mathcal{C}}_{1(j,j)}}$, in addition to the coefficient of the identity $\lambda^2_{\mathbf{1},\,\mathrm{Id}}$ which is merely an overall normalization. The remaining undetermined variables in the four-point function are the spectrum of long multiplets $\mathcal{A}^{\Delta}_{\ell}$ and the corresponding OPE coefficients $\lambda^2_{i,\,\mathcal{A}^{\Delta}_{\ell}}$. This demonstrates how the chiral algebra leads to a clear distinction between the contributions of the short multiplets, which we can solve analytically, and the contribution of the long multiplets, which we can determine only numerically.

The precise values of the coefficient $b_{i,\,\ell}$ can be read off from (4.48) after decomposing it in the different flavor symmetry channels, using the projectors P_i^{ABCD}. The form of these projectors generally depends on G_F, see for example [63] for many examples. For the singlet and adjoint representation the projectors always have the same universal form, so we can discuss the corresponding decomposition in full generality.

The projector onto the singlet channel is always given by $P_{\mathbf{1}}^{ABCD} = \frac{1}{\dim G_F}\delta^{AB}\delta^{CD}$, where the normalization is chosen such that the trace of the projector corresponds to the dimension of the representation. The projection of (4.48) in the singlet channel then yields:

$$\begin{aligned} f_{\mathbf{1}}(z) &= \dim G_F + z^2 \left(1 + \frac{1}{(1-z)^2}\right) + \frac{2\psi^2 z^2 h^\vee}{k(z-1)} \\ &= \dim G_F - \sum_{\ell=0,2,\cdots} \frac{2^\ell(\ell+1)(\ell!)^2 \left(2(\ell+1)(\ell+2)k - 4\psi^2 h^\vee\right)}{k(2\ell+1)!} g_{\ell+2}(z) \ , \end{aligned} \tag{4.59}$$

where $h^\vee$ is the dual Coxeter number of G_F, and ψ^2 the length squared of the longest root. In a similar vein, the adjoint projector is fixed to be $P^{ABCD}_{\text{Adj.}} = \frac{1}{\psi^2 h^\vee} f^{ABE} f^{EDC}$, which traces to $\dim G_F$, and so we find that for any flavor group:

$$\begin{aligned} f_{\text{Adj.}}(z) &= -\frac{(z-2)z\left(\frac{h^\vee\psi^2 z}{k} - \frac{h^\vee\psi^2}{k} + z^2\right)}{(z-1)^2} \\ &= -\frac{2\psi^2 h^\vee}{k} g_1(z) + \sum_{\ell=1,3,\cdots} \frac{2^\ell(\ell+1)(\ell!)^2 \left(2(\ell+1)(\ell+2) - \frac{2h^\vee\psi^2}{k}\right)}{(2\ell+1)!} g_{\ell+2}(z) \ . \end{aligned} \tag{4.60}$$

Equations (4.59) and (4.60) determine an infinite number of coefficients $b_{i\,\ell}$. It is worthwhile to analyze the coefficients of the first few low-lying operators in more detail.

Let us begin with the identity operator, which only appears in the singlet channel. From equations (4.56) and (4.59) we find

$$\lambda^2_{\mathbf{1},\,\text{Id}} = \dim G_F \ . \tag{4.61}$$

The explicit factor $\dim G_F$ cancels against the same factor in the projector and therefore the operator is unit normalized, so

$$\langle \phi^A(t_1, x_1)\phi^B(t_2, x_2)\rangle = \frac{(t_1\cdot t_2)\delta^{AB}}{|x_1 - x_2|^4} \tag{4.62}$$

in our conventions.

Next we consider the $\ell = -1$ term in (4.55). This block corresponds to the $\hat{\mathcal{B}}_1$ multiplet and therefore appears only in the adjoint flavor channel. From (4.56) and (4.60) we obtain that

$$\lambda^2_{\hat{\mathcal{B}}_1} = \frac{\psi^2 h^\vee}{k} \ . \tag{4.63}$$

In Table F.1 in Appendix F we expanded the superconformal block into a sum of conventional conformal blocks, and with the given coefficient we find the correct contribution of the flavor current conformal block for a four-point function of adjoint fields, see, *e.g.*, [34, 35].

At the next order in (4.55) we find the coefficients $b_{i,0}$ which according to (4.56) get contributions from $\hat{\mathcal{C}}_{0(0,0)}$ and $\hat{\mathcal{B}}_2$ multiplets. As we mentioned above, the former multiplet contains the stress tensor and we can fix its coefficient in terms of the central charge. In a general CFT, the contribution of the stress tensor conformal block to the four-point function of a scalar of dimension 2 is, *e.g.*, [34, 35]

$$x_{12}^4 x_{34}^4 \langle \phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\rangle \sim \frac{4}{90c} u G_4^{(2)} \,. \tag{4.64}$$

According to the third entry in Table F.1 this conformal block appears in the superconformal block with a factor of $\frac{4}{15}$. After adding an additional factor $\dim G_F$ in order to cancel the corresponding factor in the singlet projector we recover (4.58). Using this equation in conjunction with (4.56) and the expression of $b_{\mathbf{1},\,0}$ that can be read off from (4.59), we find that for any flavor group

$$\lambda^2_{\mathbf{1},\,\hat{\mathcal{B}}_2} = \frac{1}{12}\left(\frac{\dim G_F}{3c} - \frac{4\psi^2 h^\vee}{k} + 4\right) \,. \tag{4.65}$$

This coefficient must be positive for unitarity theories, and so we obtain a constraint on the allowed values of k and c for a given flavor group G_F:

$$\frac{\dim G_F}{c} \geqslant \frac{12\psi^2 h^\vee}{k} - 12 \,. \tag{4.66}$$

This is one of the unitary bounds obtained in 2. Its saturation corresponds to the absence of the $\hat{\mathcal{B}}_2$ multiplet in the singlet representation, which implies a relation in the Higgs branch chiral ring of these theories.

Finally, from the last line of (4.56) we see that for $j > 1$ the two multiplets contributing to the meromorphic function are $\hat{\mathcal{C}}_{0(j,j)}$ and $\hat{\mathcal{C}}_{1(j-\frac{1}{2},j-\frac{1}{2})}$. As we already mentioned before, the $\hat{\mathcal{C}}_{0(j,j)}$ multiplets contain conserved currents of spin higher than two and are not expected to be present in an interacting theory and we can set the corresponding OPE coefficients to zero. In the singlet channel this for example directly leads to

$$\lambda^2_{\mathbf{1},\,\hat{\mathcal{C}}_{1(0,0)}} = \frac{2}{5}\left(2 - \frac{\psi^2 h^\vee}{3k}\right) \,, \tag{4.67}$$

whose positivity implies another bound of 2,

$$k \geqslant \frac{\psi^2 h^\vee}{6} \,. \tag{4.68}$$

This bound can also be found by using similar arguments in the adjoint channel.

In what follows we will fix the normalization of the longest root to be $\psi^2 = 2$.

4.3.3 $\mathfrak{su}(2)$ global symmetry

The first special case of the above structure that we will investigate is the case of global symmetry algebra $\mathfrak{su}(2)$. This is quantitatively the simplest case because it is the unique simple algebra for which only three irreducible representations appear in the tensor product of two copies of the adjoint representation – in particular, we have

$$\mathbf{3} \otimes \mathbf{3} = \mathbf{1} \oplus \mathbf{3} \oplus \mathbf{5} \ . \tag{4.69}$$

Interesting examples of $\mathcal{N}=2$ superconformal theories with $\mathfrak{su}(2)$ flavor symmetry are the theory of a single $D3$ brane probing an H_1 singularity in F-theory as well as the theories of any number $n>1$ of $D3$ branes probing any of the F-theory singularities listed above. (Recall also that the theory of a single free hypermultiplet is invariant under an $\mathfrak{su}(2)_F$ flavor symmetry.)

The projectors onto each of the representations in (4.69) are easy to compute, see, *e.g.*, [63],

$$\begin{aligned}
P_{\mathbf{1}}^{ABCD} &= \frac{1}{3}\delta^{AB}\delta^{CD} \ , \\
P_{\mathbf{3}}^{ABCD} &= \frac{1}{2}\left(\delta^{AD}\delta^{CB} - \delta^{AC}\delta^{BD}\right) \ , \\
P_{\mathbf{5}}^{ABCD} &= \frac{1}{2}\left(\delta^{AD}\delta^{CB} + \delta^{AC}\delta^{BD}\right) - P_{\mathbf{1}}^{ABCD} \ ,
\end{aligned}$$

where $A = 1\ldots3$ is an adjoint index. From [63] the F matrix can be computed as

$$F_i^{\ j} = \frac{1}{\dim(j)} P_i^{BDCA} P_j^{ABDC} \ , \tag{4.70}$$

where $\dim(j) = P_j^{ABBA}$. We will arrange the rows and columns of F such that $i,j = 1,2,3$ correspond to the $\mathbf{1}$, $\mathbf{3}$, $\mathbf{5}$ channels respectively. The F matrix for $\mathfrak{su}(2)$ is then,

$$F = \begin{pmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 1 & \frac{1}{2} & -\frac{1}{2} \\ \frac{5}{3} & -\frac{5}{6} & \frac{1}{6} \end{pmatrix} \ . \tag{4.71}$$

We can now use equation (4.48) and compute the $f_i(z)$ functions,

$$\begin{aligned} f_{\mathbf{1}}(z) &= \frac{3 - 6z + (5 - \frac{8}{k})z^2 - (2 - \frac{8}{k})z^3 + z^4}{(1-z)^2} \,, \\ f_{\mathbf{3}}(z) &= \frac{-\frac{8}{k}z + \frac{12}{k}z^2 + (2 - \frac{4}{k})z^3 - z^4}{(1-z)^2} \,, \\ f_{\mathbf{5}}(z) &= \frac{(2 + \frac{4}{k})(z^2 - z^3) + z^4}{(1-z)^2} \,. \end{aligned} \tag{4.72}$$

We have chosen conventions in which the flavor central charge of the free hypermultiplet is $k = 1$.

As described in the previous subsection, we can use this expression to solve for the $b_{i,\ell}$ coefficients in the expansion (4.55). By demanding that the stress tensor is unique and that the theory does not contain higher spin currents we find the OPE coefficients of all the semishort multiplets. After performing the infinite sums in the $\mathcal{G}_i^{\text{short}}(u,v)$ we will be left with a crossing equation involving only long operators. The final expressions for $\mathcal{G}_i^{\text{short}}(u,v)$ are given in (F.9). The singlet and quintuplet channels are symmetric and so they involve only even spins in the expansion of $f_i(z)$ and $\mathcal{G}_i(u,v)$, while the triplet channel is antisymmetric and contains only odd spins.

4.3.4 $\mathfrak{e}_6$ global symmetry

As a second case we consider theories with global symmetry $\mathfrak{e}_6$. This flavor symmetry group also arises in the F-theory singularities described above. From the point of view of the crossing symmetry relations, this is actually the second simplest case because five irreducible representations appear in the square of the adjoint representation,

$$\mathbf{78} \otimes \mathbf{78} = \mathbf{1} \oplus \mathbf{650} \oplus \mathbf{2430} \oplus \mathbf{78} \oplus \mathbf{2925} \,, \tag{4.73}$$

whereas for all other simple groups (aside from $\mathfrak{su}(2)$) there are five or more representations. The projection tensors for $\mathfrak{e}_6$ can be found in [63], in terms of which the F matrix can be computed using (4.70):

$$F = \begin{pmatrix} \frac{1}{78} & \frac{1}{78} & \frac{1}{78} & \frac{1}{78} & \frac{1}{78} \\ \frac{25}{3} & -\frac{7}{24} & \frac{5}{24} & \frac{25}{12} & -\frac{1}{6} \\ \frac{405}{13} & \frac{81}{104} & \frac{29}{104} & -\frac{135}{52} & -\frac{9}{26} \\ 1 & \frac{1}{4} & -\frac{1}{12} & \frac{1}{2} & 0 \\ \frac{75}{2} & -\frac{3}{4} & -\frac{5}{12} & 0 & \frac{1}{2} \end{pmatrix} . \tag{4.74}$$

The indices of the above matrix $F_i^{\ j}$ run over the ordered set of irreps $i, j \in \{\mathbf{1}, \mathbf{650}, \mathbf{2430}, \mathbf{78}, \mathbf{2925}\}$.

Positivity of the coefficient of the $\hat{\mathcal{B}}_2$ multiplet in the **650** representation requires

$$k \geqslant 6 \ , \tag{4.75}$$

with saturation of the bound occurring when the coefficient of $\hat{\mathcal{B}}_2$ goes to zero. The absence of this multiplet corresponds to a relation in the Higgs branch chiral ring 2. The only known theory with $k = 6$ is the rank one E_6 theory.

As before we now compute the $f_i(z)$ functions, which are given in (F.10). Once again we can use these expressions to solve for the coefficients of the short multiplets and to perform the infinite sums in $\mathcal{G}_i^{\text{short}}(u, v)$, the final results are given in (F.11). We note that the channels **1**, **650** and **2430** appear in the symmetric tensor product, while channels **78** and **2925** appear in the antisymmetric tensor product. As such the former will only include even spins and the latter only odd spins.

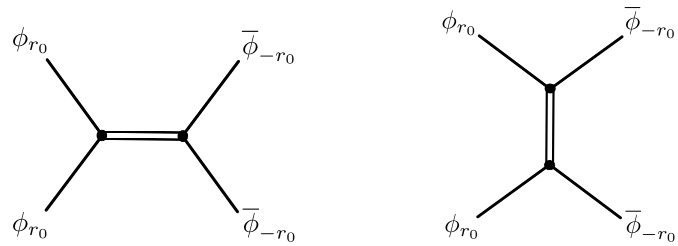
4.4 The $\mathcal{E}_r$ four-point function

Our second observable of interest is the four-point function of $\mathcal{N} = 2$ chiral operators, *i.e.*, the superconformal primaries of $\mathcal{E}_{r_0}$ multiplets. These multiplets were introduced in Section 4.2 as being connected to the *Coulomb data* of a theory. We recall that these superconformal primaries are spacetime scalars with non-zero $U(1)_r$ charge r_0 that are neutral with respect to $SU(2)_R$ and that have conformal dimension $\Delta = r_0$. We will denote the operator of interest as ϕ_{r_0}, with the conjugate anti-chiral operator being $\bar{\phi}_{-r_0}$. Unitarity requires $\Delta \geqslant 1$. In principle, one would like to focus on *generators* of the Coulomb branch chiral ring. Our methods are such that it is not easy to distinguish between generators and composites. However, if we take $r_0 \leqslant 2$, then unitarity dictates that ϕ_{r_0} must be a chiral ring generator.

We will be investigating the four-point function of a single chiral operator and its conjugate,

$$\left\langle \phi_{r_0}(x_1)\bar{\phi}_{-r_0}(x_2)\phi_{r_0}(x_3)\bar{\phi}_{-r_0}(x_4)\right\rangle \ . \tag{4.76}$$

The general procedure is now analogous to that of the previous section. We should determine what operators can be exchanged in each channel and find the corresponding superconformal blocks. In contrast to the previous section, here we are dealing with operators that are invariant under any flavor symmetries in the theory but that are nontrivially charged under $U(1)_r$. Although this is an R-symmetry, the role it plays in this correlator will be largely that of a $SO(2)$ flavor symmetry, with some minor differences that we discuss below. After obtaining the superconformal blocks in all channels we have to work out the constraints imposed by crossing symmetry. The $\mathcal{E}_r$ multiplets are not involved with the chiral algebra data of a theory. This means that unlike the previous section, we are not able to fix the coefficients of all the short and semi-short multiplets being exchanged. Of all the short and



(a) Chiral OPE channel. (b) Nonchiral OPE channel.

Figure 4.1: The two inequivalent OPE channels for the $\mathcal{E}_r$ four-point function.

semi-short multiplets appearing in the partial wave expansion, the only coefficient we are able to fix is that of the stress tensor, which must appear in the $\phi_{r_0} \times \bar{\phi}_{-r_0}$ OPE. This gives us a handle on the central charge c of the theory, which together with the dimension of the external operators $\Delta = r_0$ are the two parameters we can tune. We can therefore in principle derive bounds or other constraints as a function of the pair (r_0, c), but we will sometimes leave the central charge arbitrary in order to obtain bounds that are universally valid for all central charges, or alternatively in order to bound c itself.

Another short operator of special interest is the superconformal primary ϕ_{2r_0} of the $\mathcal{E}_{2r_0}$ multiplet, which appears in the $\phi_{r_0} \times \phi_{r_0}$ OPE as part of the Coulomb branch chiral ring. The corresponding conformal block appears with a nontrivial coefficient that is not protected by supersymmetry. As we will see this multiplet is isolated and thus we will be able to bound this coefficient both from below and from above. In this way we will also be able to probe relations on the Coulomb branch chiral ring of the type $\phi_{r_0}\phi_{r_0} \sim 0$.

Finally, let us note that exactly marginal deformations of an $\mathcal{N} = 2$ SCFT that preserve the full $\mathcal{N} = 2$ superconformal invariance lie in $\mathcal{E}_2$ multiplets. More specifically, the deforming operators are obtained by acting with all four anti-chiral supercharges $\widetilde{Q}_{\mathcal{I}\dot{\alpha}}$ on the superconformal primary of those multiplets. Theories with $\mathcal{E}_2$ multiplets in their spectrum therefore necessarily lie on a conformal manifold of positive dimension. As we will review below, the coefficient of the $\mathcal{E}_4$ multiplet in the OPE of two $\mathcal{E}_2$ multiplets is related to the curvature of the Zamolodchikov metric on this conformal manifold. This curvature is thus a natural target for numerical investigation.

4.4.1 Structure of the four-point function

In contrast to the case of moment maps, there are now two qualitatively different OPE channels to consider depending on whether we take the non-chiral OPE $\phi_{r_0}(x_1) \times \bar{\phi}_{-r_0}(x_2)$ or the chiral OPE $\phi_{r_0}(x_1) \times \phi_{r_0}(x_2)$ (see Fig. 4.1). We now describe the various selection rules for superconformal representations appearing in these two channels, as well as the corresponding superconformal blocks.

The $\phi_{r_0}(x_1) \times \bar{\phi}_{-r_0}(x_2)$ channel

We begin with the selection rules for the non-chiral OPE. The problem simplifies due to the fact that an operator $\mathcal{O}(x_3)$ can participate in a non-zero three-point function $\langle\phi_{r_0}(x_1)$ $\bar{\phi}_{-r_0}(x_2)\mathcal{O}(x_3)\rangle$ only if the superconformal primary of the multiplet to which it belongs also participates in such a non-vanishing three-point function. A sketch of the derivation of this result can be found in Appendix F.2.1.

Selection rules for the $U(1)_r$ and $SU(2)_R$ require that any such operator $\mathcal{O}(x_3)$ be an $SU(2)_R$ singlet and have $r_{\mathcal{O}} = 0$. To appear in the OPE of two scalars they must also have $j_1 = j_2 = j$. Taken together, these conditions imply the following selection rule:

$$\mathcal{E}_{r_0} \times \bar{\mathcal{E}}_{-r_0} \sim \mathbf{1} + \hat{\mathcal{C}}_{0,(j,j)} + \mathcal{A}^{\Delta}_{0,0(j,j)} \ . \tag{4.77}$$

Note that the structure of the OPE we present here is *only* for the superconformal primaries of the $\mathcal{E}_{r_0}$ and $\bar{\mathcal{E}}_{-r_0}$ multiplets, despite our abuse of notation in using the name of the full multiplet on the left-hand side of the above equation. The superconformal blocks for these multiplets have been computed in [146]. They are given by the general formula

$$\begin{aligned}\mathcal{G}^{\mathrm{sc}}_{\Delta,\ell}(z,\bar{z}) := \frac{(z\bar{z})^{\frac{1}{2}(\Delta-\ell)}}{z-\bar{z}} \quad & \left(\left(-\frac{z}{2}\right)^{\ell} z\, {}_2F_1\left(\frac{1}{2}(\Delta+\ell)\,,\frac{1}{2}(\Delta+\ell+4)\,;\Delta+\ell+2;z\right)\right. \\ & \left. {}_2F_1\left(\frac{1}{2}(\Delta-\ell-2)\,,\frac{1}{2}(\Delta-\ell+2)\,;\Delta-\ell;\bar{z}\right)\right) - z \leftrightarrow \bar{z}\Big) \ ,\end{aligned} \tag{4.78}$$

with Δ and $\ell = 2j$ denoting the dimension and spin of the superconformal primary of each multiplet. The spin can be either even or odd. Note that the superconformal blocks for the $\hat{\mathcal{C}}_{0(j,j)}$ representations are simply the specialization of (4.78) to the case $\Delta = \ell + 2$, while the block for the identity operator is just a constant as usual. These superconformal blocks can of course be written as a finite sum of conventional conformal blocks – we provide such a decomposition in Appendix F.2.3.

The $\phi_{r_0}(x_1) \times \phi_{r_0}(x_2)$ OPE

We now turn to the chiral OPE. In this case only $SU(2)_R$ singlets with $r_{\mathcal{O}} = 2r$ and $j_1 = j_2 = j$ are allowed, and the spin $\ell = 2j$ is required to be even because we are considering the OPE of two identical scalars. The complete selection rules for this channel are worked out in Appendix F.2.2, where it is shown that one conformal family per superconformal multiplet can contribute, implying the superconformal blocks are then equal to the standard conformal blocks corresponding to that family. The complete list of superconformal multiplets that can appear in this OPE is derived in the aforementioned appendix. All told we find the following selection rules, where for simplicity we momentarily assume that $r_0 > 1$,

$$\mathcal{E}_{r_0} \times \mathcal{E}_{r_0} \sim \mathcal{A}_{0,2r_0-2(j,j)} + \mathcal{E}_{2r_0} + \mathcal{C}_{0,2r_0-1(j,j+1)} + \mathcal{B}_{1,2r_0-1(0,0)} + \mathcal{C}_{\frac{1}{2},2r_0-\frac{3}{2}(j,j+\frac{1}{2})} \ . \tag{4.79}$$

Once again we note that these selection rules only necessarily hold true for the superconformal primaries of the $\mathcal{E}_{r_0}$ multiplets. The corresponding superconformal blocks for these multiplets are given in Table 4.3. The blocks for certain additional short multiplets that are allowed when $r_0 = 1$, are presented in the second part of the table. Note that the $\mathcal{C}_{\frac{1}{2},2r_0-\frac{3}{2}(j,j+\frac{1}{2})}$

Multiplet	Contribution to $\mathcal{G}_{\hat{i}=\hat{2}}(u,v)$	Restrictions
$\mathcal{A}_{0,2r_0-2(j,j)}$	$u^{\frac{\Delta-\ell}{2}}\, G_{\Delta}^{(\ell=2j)}(u,v)$	$\Delta \geqslant 2+2r_0+\ell$
$\mathcal{E}_{2r_0}$	$u^{r_0}\, G_{2r_0}^{(0)}(u,v)$	
$\mathcal{C}_{0,2r_0-1(j,j+1)}$	$u^{r_0}\, G_{2r_0+\ell}^{(\ell=2j+2)}(u,v)$	$\ell \geqslant 2$
$\mathcal{B}_{1,2r_0-1(0,0)}$	$u^{r_0+1}\, G_{2r_0+2}^{(0)}(u,v)$	
$\mathcal{C}_{\frac{1}{2},2r_0-\frac{3}{2}(j,j+\frac{1}{2})}$	$u^{r_0+1}\, G_{2r_0+\ell+2}^{(\ell=2j+1)}(u,v)$	$\ell \geqslant 2$
$\hat{\mathcal{C}}_{\frac{1}{2}(j,j+\frac{1}{2})}$	$u^2\, G_{\Delta=\ell+4}^{(\ell)}$	$\ell \geqslant 2;\ r_0 = 1$
$\hat{\mathcal{C}}_{0(j,j+1)}$	$u\, G_{\Delta=\ell+2}^{(\ell)}$	$\ell \geqslant 2;\ r_0 = 1$
$\mathcal{D}_{1(0,0)}$	$u^2\, G_{\Delta=4}^{(\ell=0)}$	$r_0 = 1$

Table 4.3: Superconformal blocks for the $\mathcal{E}_{r_0}$ four point function in the chiral channel.

and $\mathcal{B}_{1,2r_0-1(0,0)}$ classes of short representations lie at the unitarity bound for long multiplets, and their superconformal blocks are simply the specializations of the long multiplet block to appropriate values of Δ and ℓ. The $\mathcal{E}_{2r_0}$ and $\mathcal{C}_{0,2r_0-1(j,j+1)}$ representations, on the other hand, are separated from the continuous spectrum of long multiplets by a gap. The two short multiplets that are available only when $r_0 = 1$ contribute with the same blocks as some of the other blocks appearing in Table 4.3.

4.4.2 Crossing symmetry

To formulate the crossing symmetry condition for this correlator we will treat $U(1)_r$ as an $SO(2)$ global symmetry – this is similar to the approach used in [35] to study the four-point function of chiral operators in $\mathcal{N}=1$ SCFTs. In this approach, the fields ϕ_{r_0} and $\bar{\phi}_{-r_0}$ are combined in the fundamental representation of $SO(2)$ with charge $|r_0|$, which we denote as $\mathbf{2}_{|r_0|}$. This representation has the following tensor product with itself,

$$\mathbf{2}_{|r_0|} \otimes \mathbf{2}_{|r_0|} = \left(\mathbf{1} \oplus \mathbf{2}_{|2r_0|}\right)_{\text{symm.}} \oplus \mathbf{1}_{\text{antisymm.}}\ , \tag{4.80}$$

where the subscripts denote which representations appear in the symmetrized tensor product and which appear in the antisymmetrized tensor product. The crossing symmetry discussion of Section 4.3.1 is now directly applicable, with the crossing matrix $F_i{}^j$ given by

$$\begin{pmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 1 & 0 & -1 \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{pmatrix} , \tag{4.81}$$

where the ordering of the rows and columns is the same as in (4.80). The crossing equation then takes the form

$$(z-\bar{z})((1-z)(1-\bar{z}))^{r_0} F_j{}^i \mathcal{G}_j(z,\bar{z}) + (\bar{z}-z)(z\bar{z})^{r_0}\mathcal{G}_i(1-z,1-\bar{z}) = 0 , \tag{4.82}$$

where each $\mathcal{G}_i(z,\bar{z})$ can be expanded in the blocks relevant for the $SO(2)$ channel i. As usual, the braiding relation requires that only operators of even spin appear in the symmetric channels while only operators of odd spin appear in the antisymmetric channel.

In more conventional terms, the symmetric traceless channel $\mathbf{2}_{|2r_0|}$ encodes the operators appearing in the $\phi_{r_0} \times \phi_{r_0}$ OPE. This channel can therefore be expanded entirely in terms of the conformal blocks $G_\Delta^{(\ell)}$ given in Table 4.3 with even spins. The singlet channels, on the other hand, describe the $\phi_{r_0} \times \bar{\phi}_{-r_0}$ OPE, with the symmetric channel getting all the even spin conformal block contributions and the antisymmetric channel getting the odd spin ones. The blocks in these latter two channels are related by supersymmetry because the $U(1)_r$ symmetry is part of the superconformal algebra, and conformal families from the same superconformal multiplet appear in both channels. To wit, in the symmetric singlet channel we have contributions from the superconformal primaries appearing in (4.77) with even spin, together with their even spin superconformal descendants, and the even spin superconformal descendants of odd spin superconformal primaries. For the antisymmetric channel the opposite takes place. We have therefore broken the superconformal blocks (4.78) apart, splitting them by the parity of the spin, with each channel enjoying a "partial" superconformal block.

This splitting of superconformal blocks can be ameliorated by a change of basis. Let us define

$$\mathcal{G}_{\hat{1},\hat{3}} := \mathcal{G}_1 \pm \mathcal{G}_3 , \qquad \mathcal{G}_{\hat{2}} := \mathcal{G}_2 . \tag{4.83}$$

All conformal blocks arising from the same superconformal multiplet are now grouped together, with each superconformal multiplet from the singlet channels appearing twice: once each in $\mathcal{G}_{\hat{1}}$ and $\mathcal{G}_{\hat{3}}$. The channels $\hat{1}$ and $\hat{3}$ are almost identical – the only difference is an extra minus sign for all the odd spin conformal blocks in $\mathcal{G}_{\hat{3}}$ due to the extra minus sign in (4.83). There are two ways to insert this minus sign. The first option is to decompose the superconformal block (4.78) into ordinary conformal blocks and insert extra factors of $(-1)^\ell$

in front of every block. However the second option is more efficient. We recall that ordinary conformal blocks satisfy the following braiding relation:

$$\left(\frac{z}{z-1}\frac{\bar{z}}{\bar{z}-1}\right)^{\frac{\Delta-\ell}{2}} G_{\Delta}^{(\ell)}\left(\frac{z}{z-1},\frac{\bar{z}}{\bar{z}-1}\right) = (-1)^{\ell}(z\bar{z})^{\frac{\Delta-\ell}{2}} G_{\Delta}^{(\ell)}(z,\bar{z})\ . \tag{4.84}$$

We can thus insert the necessary factors of $(-1)^\ell$ by substituting $z \to \frac{z}{z-1}$ and $\bar{z} \to \frac{\bar{z}}{\bar{z}-1}$ in the superconformal block (4.78). We thus find that a supermultiplet in the singlet channel contributes to the four-point function as follows,

$$\mathcal{G}_{\hat{i}=1}(z,\bar{z}) \sim \mathcal{G}^{sc}_{\Delta,\ell}(z,\bar{z})\,, \qquad \mathcal{G}_{\hat{i}=3}(z,\bar{z}) \sim \mathcal{G}^{sc}_{\Delta,\ell}\left(\frac{z}{z-1},\frac{\bar{z}}{\bar{z}-1}\right)\ , \tag{4.85}$$

with the same OPE coefficient appearing in both channels. The operators contributing to the doublet channel still contribute to $\mathcal{G}_{\hat{2}}(z,\bar{z})$ as before.

The relevant crossing equation is now the same as in (4.82) but with $\hat{i}$ and $\hat{j}$ replacing i and j, and with the flavor matrix $F_i{}^j$ replaced by

$$F_{\hat{i}}{}^{\hat{j}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & \frac{1}{2} & 0 \end{pmatrix}\ . \tag{4.86}$$

This is the same as the crossing equation that was previously derived for chiral operators in $\mathcal{N}=1$ SCFTs [35]. This matrix squares to one, which is relevant to the numerical implementation described in the next section.

When $\Delta=\ell=0$ in the $\hat{i}=\hat{1},\hat{3}$ channels we get the contribution of the identity operator, which we set equal to two. This fixes the normalization of the external operators to be one as is conventional (the factor of two arises from the projector onto the singlet, similarly to the discussion in the previous section). The contribution of the stress tensor is contained in the superconformal block (4.78) with $\Delta=2$ and $\ell=0$. When expanded in ordinary conformal blocks, the contribution is given by

$$\mathcal{G}^{\hat{\mathcal{C}}_{0,(0,0)}}_{\hat{i}=1}(z,\bar{z}) = uG_2^{(0)}(u,v) - \frac{u}{2}G_3^{(1)}(u,v) + \frac{2u}{30}G_4^{(2)}(u,v)\ . \tag{4.87}$$

The coefficient of this block can be fixed in terms of the central charge c as was done in the previous section. Namely, fixing the coefficient of $uG_4^{(2)}(u,v)$ requires that this superconformal block should appear with coefficient $\frac{r_0^2}{3c}$ (again, a factor of two comes from the projector onto the singlet). The "braided" superconformal block appears with the same coefficient.

Free theory expansion

A simple illustration of the superconformal block decomposition procedure is the explicit analysis of free field theory. Namely we consider the theory of a free vector multiplet, and we study the four-point function of the chiral scalar which has $r_0 = 1$. Decomposing the free field correlator in terms of the superconformal blocks described above we find the following channel expansions,

$$\begin{aligned}
\mathcal{G}_{i=\hat{1}}(z,\bar{z}) &= \sum_{\ell=0}^{\infty} \frac{(\ell+2)(\ell!)^2(-2)^\ell}{(2\ell+1)(2\ell)!} \mathcal{G}^{\text{sc}}_{\ell+2,\ell}(z,\bar{z})\,, \\
\mathcal{G}_{i=\hat{2}}(z,\bar{z}) &= \sum_{\ell=0}^{\infty} \frac{\left((-1)^\ell+1\right)(\ell!)^2(-2)^\ell}{(2\ell)!} u G^{(\ell)}_{\ell+2}(u,v)\,, \\
\mathcal{G}_{i=\hat{3}}(z,\bar{z}) &= \sum_{\ell=0}^{\infty} \frac{(\ell+2)(\ell!)^2(-2)^\ell}{(2\ell+1)(2\ell)!} \mathcal{G}^{\text{sc}}_{\ell+2,\ell}\left(\frac{z}{z-1},\frac{\bar{z}}{\bar{z}-1}\right)\,.
\end{aligned} \tag{4.88}$$

We can immediately verify that the only difference between channel $\hat{1}$ and $\hat{3}$ is the replacement $z \to \frac{z}{z-1}$ and $\bar{z} \to \frac{\bar{z}}{\bar{z}-1}$. We can also ferret out the stress tensor block $\mathcal{G}^{\text{sc}}_{2,0}(z,\bar{z})$ and see that it appears with coefficient two. This suggests a central charge of $\frac{1}{6}$, which is correct for the theory of a free vector multiplet. For future reference, we also note that the $\mathcal{E}_2$ block, which is the $uG^{(0)}_2$ term in the $\hat{2}$ channel, comes with coefficient two.

4.5 Operator bounds from crossing symmetry

The output from the previous two sections was a collection of crossing symmetry equations and their (super)conformal block decompositions. In this section we describe the numerical methods by which these equations can be used to extract useful information about $\mathcal{N}=2$ SCFTs. We follow the approach of [35], where the original numerical analysis of [6] was recast as a semidefinite programming problem.

Each of the nontrivial crossing symmetry equations can be put into the general form

$$\mathcal{H}_i(z,\bar{z}) + F_i^{\ j}\mathcal{H}_j(1-z,1-\bar{z}) = 0\ . \tag{4.89}$$

Here and below, summation over repeated indices is always implied. The functions $\mathcal{H}_i(z,\bar{z})$ can always be written as

$$\mathcal{H}_i(z,\bar{z}) = \mathcal{G}_i(z,\bar{z}) - a_i(z,\bar{z})\ , \tag{4.90}$$

where the $a_i(z,\bar{z})$ are some known functions that have been fixed analytically, and the $\mathcal{G}_i(z,\bar{z})$

have a decomposition of the form

$$\mathcal{G}_i(z,\bar{z}) = \sum_{k_i} \lambda_{k_i}^2 \widetilde{G}_{\Delta_{k_i}}^{(\ell_{k_i})}(z,\bar{z}) \ . \tag{4.91}$$

The coefficients $\lambda_{k_i}^2$ are real, positive numbers, and the $\widetilde{G}_{\Delta}^{(\ell)}(z,\bar{z})$ are roughly the superconformal blocks, the precise form of which depends on the crossing symmetry equation in question.[26] The sum is over all operators that appear in the i'th channel, and the matrix $F_i{}^j$ is related to Wigner's $6j$ symbol for the relevant global symmetry group and in particular is involutory: $F_i{}^j F_j{}^k = \delta_i{}^k$, for the cases considered here.

As in [6], we will analyze these equations by considering the action of certain linear functionals upon them. We may introduce one linear functional ϕ^i for each channel i. The functionals that we consider are defined by taking linear combinations of various numbers of derivatives of the function and evaluating at the symmetric point $z = \bar{z} = 1/2$, *i.e.*,

$$\phi^i[f_i(z,\bar{z})] = \sum_{m,n} \alpha_{mn}^i \partial_z^m \partial_{\bar{z}}^n f_i(z,\bar{z})\big|_{z=\bar{z}=\frac{1}{2}} \ . \tag{4.92}$$

The matrices $\frac{1}{2}(\delta_i{}^j \pm F_i{}^j)$ are projectors onto the positive and negative eigenspaces of $F_i{}^j$, so we can split the coefficients into even and odd parts, $\alpha_{mn}^i = \alpha_{mn,+}^i + \alpha_{mn,-}^i$, satisfying

$$\frac{1}{2}\alpha_{\pm}^j(\delta_j{}^i \pm F_j{}^i) = \alpha_{\pm}^i \,, \qquad \frac{1}{2}\alpha_{\pm}^j(\delta_j{}^i \mp F_j{}^i) = 0 \ . \tag{4.93}$$

Upon acting with our functionals on both sides of (4.89), we find the following equation,

$$\sum_{m,n}(\alpha_{mn,+}^i + \alpha_{mn,-}^i)\left(\delta_i{}^j + (-1)^{m+n} F_i{}^j\right)\partial_z^m \partial_{\bar{z}}^n \mathcal{H}_j(z,\bar{z})\big|_{z=\bar{z}=\frac{1}{2}} = 0 \ . \tag{4.94}$$

Only those terms with $m+n$ even in ϕ_+^i and those with $m+n$ odd in ϕ_-^i have a nontrivial action on the crossing symmetry equation (4.89). Without loss of generality, we can therefore set the other terms to zero. With this choice now implicit, the action of the functional on the crossing symmetry equation can be succinctly written as

$$\begin{aligned} 0 &= \phi^i\left[\mathcal{H}_i(z,\bar{z}) + F_i{}^j \mathcal{H}_j(1-z,1-\bar{z})\right] \\ &= 2\sum_{m,n}\left(\alpha_{mn,+}^i + \alpha_{mn,-}^i\right)\partial_z^m \partial_{\bar{z}}^n \mathcal{H}_i(z,\bar{z})\big|_{z=\bar{z}=\frac{1}{2}} \\ &= 2\phi^i\left[\mathcal{H}_i(z,\bar{z})\right] \ . \end{aligned} \tag{4.95}$$

[26]These $\widetilde{G}$ functions are not exactly the superconformal blocks of the previous sections, but rather they include simple prefactors that have been absorbed in their definition. This is not particularly important for the discussion here.

The nontrivial relations between the different global symmetry channels have been completely accounted for by the eigenvector constraints (4.93), which are simple algebraic constraints that are easily solved in any given case.

By construction, all the functions appearing in this relation are symmetric under the exchange of z and $\bar{z}$, so we lose nothing by restricting the coefficients of the functionals to obey $m \leqslant n$. We obtain a finite-dimensional functional space by introducing a cutoff $\Lambda \in \mathbb{N}$, and demanding that $m+n \leqslant \Lambda$. For each i, we then find $(1+\lfloor\frac{\Lambda}{2}\rfloor)(2+\lfloor\frac{\Lambda-1}{2}\rfloor)$ independent derivative combinations. The total number of independent coefficients $\alpha^i_{mn,\pm}$ is then determined by multiplying the number of derivative combinations with $m+n$ even by the number of positive eigenvalues of $F_i{}^j$ and the number of derivative combinations with $m+n$ odd by the number of negative eigenvalues and then taking the sum. Quantitatively, if the total number of channels is c and $F_i{}^j$ has b positive eigenvalues, then the dimension of the space of functionals is given by

$$\dim_{\Lambda,c,b} = \frac{c}{2}\left(1+\left\lfloor\frac{\Lambda-1}{2}\right\rfloor\right)\left(2+\left\lfloor\frac{\Lambda-1}{2}\right\rfloor\right)+\frac{b}{2}\left(1+(-1)^{\Lambda}\right)\left(1+\left\lfloor\frac{\Lambda}{2}\right\rfloor\right)~. \tag{4.96}$$

This is the dimension of the space in which we will be performing a numerical search, and is therefore an important measure of the complexity of the numerical problem. For large Λ the dimension behaves approximately like $c/2$ times the total number of derivative combinations.

The numerical results presented in subsequent sections are the results of two different strategies. The aim of the first strategy is to provide an upper bound for the lowest dimension operator in a given channel and with a given spin that may appear in a solution of crossing symmetry. For instance, we may want to find an upper bound $\Delta_0^\star$ for the dimension of the first scalar operator in channel $\hat{i}$. Such a bound follows immediately from the existence of a functional possessing the following properties:

$$\begin{aligned}
&\phi^j\left[\widetilde{G}^{(\ell)}_{\Delta}(z,\bar{z})\right] \geqslant 0~, && \forall\,(\Delta,\ell)~\text{in channel}~j\neq\hat{i}~,\\
&\phi^{\hat{i}}\left[\widetilde{G}^{(\ell)}_{\Delta}(z,\bar{z})\right] \geqslant 0~, && \forall\,(\Delta,\ell)~\text{in channel}~\hat{i}~\text{with}~\ell>0~,\\
&\phi^{\hat{i}}\left[\widetilde{G}^{(0)}_{\Delta}(z,\bar{z})\right] \geqslant 0~, && \forall\,\Delta\geqslant\Delta_0^\star~\text{in channel}~\hat{i}~,\\
&\sum_i \phi^i\left[a_i(z,\bar{z})\right] \leqslant 0~.
\end{aligned} \tag{4.97}$$

It is implicit in this description that the functional need not be positive for scaling dimensions below the unitarity bound, since such operators cannot be present in the type of solution of crossing symmetry that we are aiming to constrain. The optimal bound that can be obtained by this method at a given cutoff will then be the minimal value of $\Delta_0^\star$ for which such a functional exists.

The aim of the second strategy is to provide an upper bound for value of a particular

OPE coefficient, say $\lambda^2_{k_{\hat{i}}}$ which multiplies the block corresponding to an operator with dimension $\Delta_{k_{\hat{i}}}$ and spin $\ell_{k_{\hat{i}}}$ in channel $\hat{i}$. Such a bound follows from performing the following optimization over the space of functionals:

$$\begin{aligned} \phi^j\left[\widetilde{G}^{(\ell)}_{\Delta}(z,\bar{z})\right] &\geqslant 0, \qquad\qquad \forall(\Delta,\ell) \text{ in all channels } j\ , \\ \phi^{\hat{i}}\left[\widetilde{G}^{(\ell_{k_{\hat{i}}})}_{\Delta_{k_{\hat{i}}}}(z,\bar{z})\right] &= 1\ , \\ \text{minimize } \sum_i \phi^i\left[a_i(z,\bar{z})\right]&\ . \end{aligned} \tag{4.98}$$

If the minimum is positive and equal to, say, M, then we obtain an upper bound

$$\lambda^2_{k_{\hat{i}}} \leqslant M\ . \tag{4.99}$$

If, on the other hand, the minimum turns out to be negative then we are effectively back to the previous case and there can be no solution to crossing symmetry. In our analysis, we often apply this minimization procedure to the block corresponding to the stress tensor multiplet. The OPE coefficient for that block is inversely proportional to the c central charge, so an upper bound on the OPE coefficient translates to a lower bound for c.

Finally, there are some cases where the quantum numbers $(\Delta_{k_{\hat{i}}}, \ell_{k_{\hat{i}}})$ of an operator of interest are isolated, in the sense that the corresponding conformal block is not continuously connected to the set of blocks for which the functional is required to be positive in (4.98). This is common in supersymmetric CFTs because of the various distinguished short multiplets whose scaling dimensions lie strictly below the unitarity bound for generic representations with the same Lorentz and R-symmetry quantum numbers. In such cases we can flip the sign on the second line of (4.98) and instead require

$$\phi^{\hat{i}}[\widetilde{G}^{(\ell_{k_{\hat{i}}})}_{\Delta_{k_{\hat{i}}}}(z,\bar{z})] = -1\ . \tag{4.100}$$

In such a case, a negative value for M provides a *lower bound* on the corresponding OPE coefficient,

$$\lambda^2_{k_{\hat{i}}} \geqslant -M\ . \tag{4.101}$$

Here the result of the optimization is only meaningful if $M \leqslant 0$. because unitarity constrains the coefficient is nonnegative.

In each of the cases just described, the search for functionals of appropriate type can be reduced to a semidefinite programming problem [35]. We review this story in Appendices G and H, where we also offer additional details about our particular numerical implementation.

4.6 Results for the moment map four-point function

The four-point function of moment map operators depends on a choice of global symmetry G_F, the associated flavor central charge k, and the trace anomaly coefficient c. Under mild assumptions, the contributions of all short multiplets that appear in the conformal block decomposition are completely determined by those parameters. For each such triple (G_F, k, c) there is then a corresponding crossing symmetry relation for the CFT data associated to long multiplets that can be subjected to numerical analysis.

We have restricted our attention to flavor symmetries $\mathfrak{su}(2)$ and $\mathfrak{e}_6$. From the perspective of the bootstrap equations, these are the least complicated of all simple algebras because the number of irreps appearing the tensor product of two copies of the adjoint is the lowest (three for $\mathfrak{su}(2)$ and five for $\mathfrak{e}_6$). Moreover, since every non-abelian semi-simple Lie algebra has $\mathfrak{su}(2)$ as a subalgebra, the $\mathfrak{su}(2)$ bounds are in a sense universal and must hold for any $\mathcal{N} = 2$ superconformal field theory with a non-abelian flavor symmetry.

Below we will first discuss how in certain regions of the (c, k) plane the crossing symmetry equations can never be satisfied by a unitary theory, irrespective of the precise spectrum of long multiplets. Recall that certain combinations of c and k are already excluded by the unitarity bounds that follow from the chiral algebra 2. We will show that the numerical analysis carves out an even smaller region. Within the allowed region in the (c, k) plane we then obtain bounds on operators in various Lorentz and flavor symmetry representations. We finally focus on values of c and k that correspond to known theories and compute more detailed bounds for the scaling dimensions of unprotected operators.

4.6.1 $\mathfrak{su}(2)$ global symmetry

Before presenting the results of the numerical analysis, it is useful to review our expectations based on our present knowledge of $\mathcal{N} = 2$ theories with $\mathfrak{su}(2)$ flavor symmetry. Let us consider the projection of the landscape of such SCFTs to the plane spanned by the two central charges c and k. Every point on this plane then falls into one of three categories. First, there are points where a solution to crossing symmetry cannot exist because it would violate a known unitarity bound. Second, there are points where a solution to crossing symmetry is guaranteed to exist because it can in principle be constructed from known theories. All the other points then fall in the third category where we do not *a priori* know if a solution exists. These three regions are charted in Fig. 4.2 and we discuss each of them below.

Besides positivity of c and k, there are additional unitarity bounds that originate from the chiral algebra construction of 2. For $G_F = \mathfrak{su}(2)$ these bounds are given by

$$k \geqslant \frac{2}{3}\,, \qquad k \geqslant \frac{16c}{1+4c}\,. \tag{4.102}$$

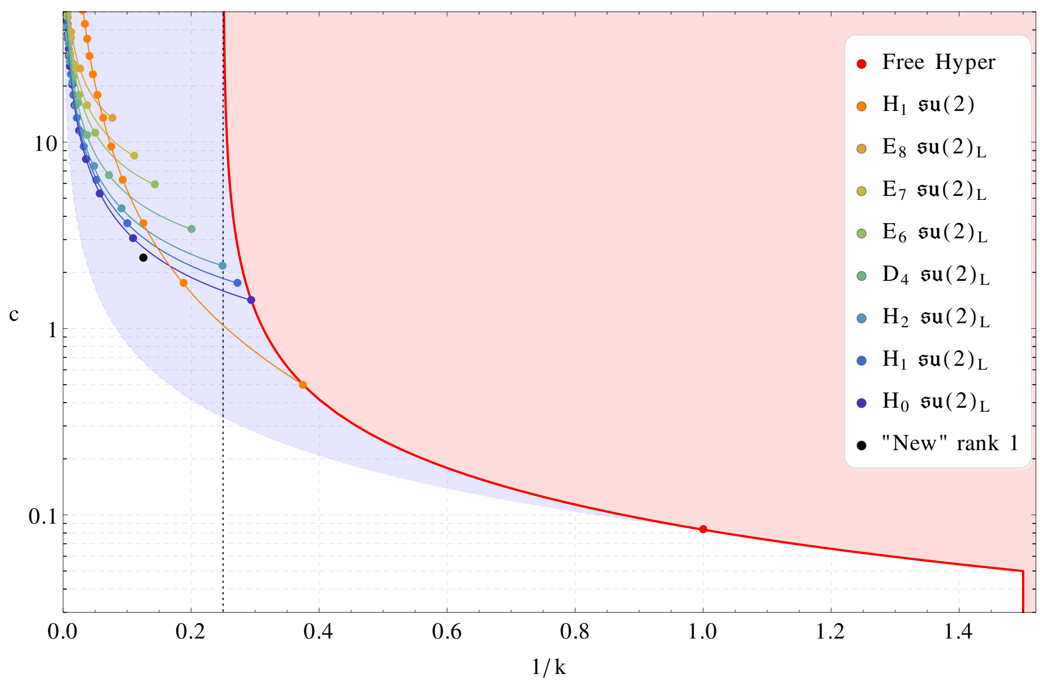


Figure 4.2: The (c, k) plane for theories with an $\mathfrak{su}(2)$ flavor symmetry. The red region on the right is excluded by analytic unitarity bounds, whereas we are guaranteed to have valid solutions to the crossing symmetry constraints in the blue region. The curves connect points corresponding to theories related to F-theory singularities of different rank, which increases with c. We show the (c, k) values corresponding the $\mathfrak{su}(2)_L$ flavor symmetry of all F-theory singularities with rank $N \geqslant 2$, and also to $\mathfrak{su}(2)$ flavor symmetry of the rank $N \geqslant 1$ H_1 theory. The "new" rank one theory is one of the theories obtained in [71]. It has a product flavor symmetry with one factor being $\mathfrak{su}(2)$, which is the one whose value of k is shown in the plot. The vertical dotted line corresponds to the value of k for the codimension two defect of the six-dimensional $(2, 0)$ theory of type A_1, which effectively has $c \to \infty$.

We refer to these bounds as the *analytic bounds*, and the regions that they exclude in the (c, k) plane are shaded in red in our plots.

Theories that saturate the analytic bounds have some special properties. For example, if the second of the analytic bounds is saturated then there can be no $\hat{\mathcal{B}}_2$ multiplet contributing to the moment map four point function in the singlet channel, which implies a relation in the Higgs branch chiral ring. Examples of theories with this feature are the theory of a free hypermultiplet with $(c, k) = (\frac{1}{12}, 1)$ and the rank one Argyres-Douglas theory with $(c, k) = (\frac{1}{2}, \frac{8}{3})$, which is the rightmost point of type H_1 in Fig. 4.2. Notice that the two bounds in (4.102) intersect at $(c, k) = (\frac{1}{20}, \frac{2}{3})$. The equivalent intersection point for $\mathfrak{e}_6$ flavor symmetry corresponds precisely to the Minahan-Nemeschansky theory [67]. It is natural to ask if a theory might exist at the intersection point for $\mathfrak{su}(2)$ flavor symmetry.

The second region contains all pairs (c, k) that correspond to a known $\mathcal{N} = 2$ SCFT. The region is however not limited to just those points, because we can take *linear combinations* of known solutions as well: the sum of two solutions to crossing symmetry, with relative weights that sum to one, is again a good solution to crossing symmetry (at the level of a single four-point function). Since the central charges appear in four-point functions only through OPE coefficients that are proportional to $1/c$ or $1/k$, a solution constructed in this way has effective central charges

$$\frac{1}{c_{\text{eff}}} = \frac{\alpha}{c_1} + \frac{1-\alpha}{c_2}\,, \qquad\qquad \frac{1}{k_{\text{eff}}} = \frac{\alpha}{k_1} + \frac{1-\alpha}{k_2}\,, \tag{4.103}$$

in terms of central charges (c_i, k_i) of the two original solutions and a weight factor $0 \leqslant \alpha \leqslant 1$. In the $(\frac{1}{c}, \frac{1}{k})$ plane, the values of c and k that can be realized as linear combinations in this way span the convex hull of all points corresponding to known theories. This region is shaded in blue in Fig. 4.2. It is effectively spanned by three points: the free hypermultiplet at $(c, k) = (\frac{1}{12}, 1)$, the generalized free field theory solution where c and k are both infinity, and the four-point function on a codimension two defect in the six-dimensional $(2, 0)$ theory of type A_1 where c is infinite and $k = 4$. We will discuss these three points in more detail below. We have computed the values of c and k for many other known theories but were not able to find any instance corresponding to a point outside the blue region in Fig. 4.2.

We should emphasize that taking linear combinations of solutions to crossing symmetry is not the same thing as taking correlation functions of operators in the tensor product of two theories. In particular, there is no guarantee that a linear combination of solutions can be realized in a complete $\mathcal{N} = 2$ SCFT. We can however be sure that our kind of numerical analysis will not rule out any points corresponding to linear combinations of solutions. A more sophisticated bootstrap analysis might exclude them, but we leave this direction for future work.

Plotting the entire set of known $\mathcal{N} = 2$ superconformal theories with at least $\mathfrak{su}(2)$ flavor symmetry is a daunting task, so we have opted to show only a subset. In Fig. 4.2 we show in particular the location of the theories that describe the low-energy behavior of N D3 branes probing F-theory singularities. As we discussed in Section 4.2.4, there are seven types of these singularities and they are denoted by the corresponding global symmetry group of the SCFT: H_0, H_1, H_2, D_4, E_6, E_7, E_8 (with $H_i \to A_i$). The theories with $N > 1$ have an additional $\mathfrak{su}(2)_L$ flavor symmetry.

The full set of central charges of these theories was calculated in [70] as a function of N using holography. Let us denote by k the flavor central charge of the global symmetry group indicated by the name of the theory, and by k_L the level of the additional $\mathfrak{su}(2)_L$ for the theories with rank greater than one. Then the relevant central charges are given by

$$\begin{aligned}
c\ &= \frac{1}{2}N^2 r_0 + \frac{3}{4}N(r_0-1) - \frac{1}{12}\ , \\
k\ &= 2Nr_0\ , \\
k_L &= N^2 r_0 - N(r_0-1) - 1\ , \qquad\qquad N \geqslant 2\ ,
\end{aligned} \tag{4.104}$$

where the vale of r_0 for each of the seven types is given in Table 4.1. The resulting values of c and k for these theories are plotted in Fig. 4.2.

Our plan for the remainder of this subsection can now be formulated as follows. We will first focus on the unshaded, third region in Fig. 4.2. Could there be theories hidden in this region, or some of these points be excluded? We will see that the latter is true, and we can numerically obtain a universal lower bound on c for each value of k. For the remaining allowed region, which includes the entire blue region in Fig. 4.2, we find upper bounds for the dimension of several unprotected operators as a function of c and k. The numerical analysis necessary to generate these bounds was computationally rather demanding because of the two-dimensional parameter space, which limited the value of Λ for which the computation was feasible to a maximum of $\Lambda = 18$. For restricted values of c and k that are of particular interest, we generated superior bounds by going as high as $\Lambda = 22$. In particular, we chose to study the H_0 and H_1 curves shown in Fig. 4.2. We also studied the point at $k = 4$ and $c = \infty$, which corresponds to an interesting defect SCFT. Bounds for the $\mathfrak{e}_6$ curve are postponed until the next subsection for the purposes of comparison to bounds extracted from the $\mathfrak{e}_6$ moment map four-point function.

Constraints on c and k

To constrain the (c, k) plane we employed the second numerical method described in the previous section. For a given value of k we normalize the functional by demanding that it evaluate to one on the contribution of protected operators to (4.46) that are proportional to the inverse central charge $1/c$. We then minimize its value when acting on the remaining protected contribution to crossing. The upper bound that we obtain in this way for $1/c$ then corresponds to a lower bound on the central charge.[27]

[27] Bounds obtained in this way for the central charge, and more generally for OPE coefficients, have been studied in the literature starting with [34, 118, 119].

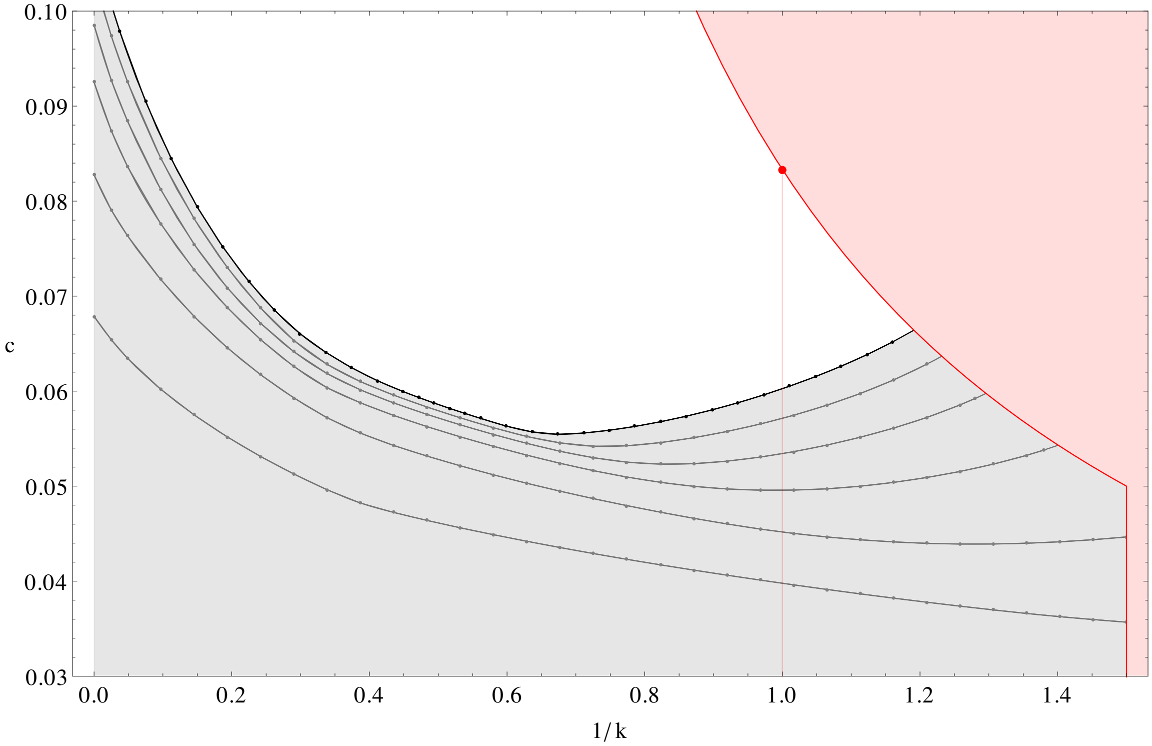


Figure 4.3: Bounds for the central charge c of a theory with $\mathfrak{su}(2)$ flavor symmetry as a function of the flavor central charge k. These bounds are a consequence of crossing symmetry for the $\hat{\mathcal{B}}_1$ four-point function. The red regions on the right are excluded by the analytic bounds (4.102), and the gray region at the bottom is the numerically excluded region. The gray and black lines correspond to the numerical bounds, shown for $\Lambda = 10, 14, \ldots, 30$, with the strongest bound (black line) corresponding to $\Lambda = 30$. The curves are interpolations through the data points shown in the figure. The red dot denotes the free hypermultiplet theory.

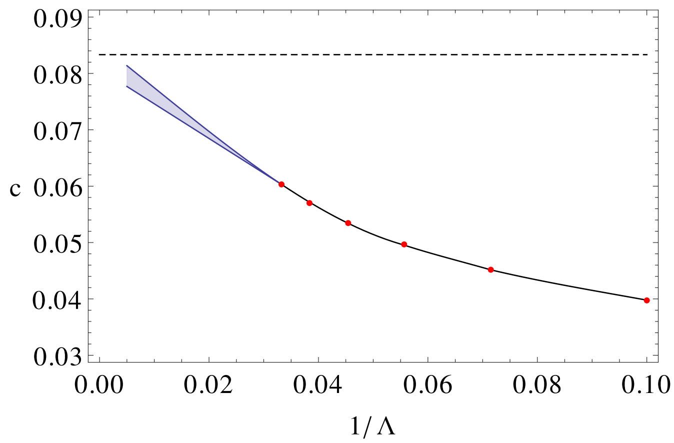


Figure 4.4: Minimum allowed value of c for a theory with $\mathfrak{su}(2)$ flavor symmetry and $k = 1$ as a function of the (inverse of) the maximum number of derivatives. The red dots are our data points, and the blue curves are possible extrapolations to infinite Λ intended to guide the eye. The dashed line corresponds to the central charge of the free hypermultiplet $c = \frac{1}{12}$.

The results of this program are shown in Fig. 4.3. The numerically excluded region is shaded in gray. This result was obtained with $\Lambda = 30$, *i.e.*, by taking at most 30 derivatives in the z or $\bar{z}$ directions. Bounds for smaller Λ are indicated with gray curves. One interesting and very general lesson to be drawn from Fig. 4.3 is that the analytic and the numerical bounds complement each other, and the most stringent constraints can only be obtained by using both methods. For example, the numerical analysis eliminates the possibility of a unitary SCFT existing at the intersection point $(c, k) = (\frac{1}{20}, \frac{2}{3})$ of the two analytic bounds given in (4.102). We also find that for all values of k, there exists a universal lower bound on the central charge,

$$c \geqslant 0.055 \ , \tag{4.105}$$

for any $\mathcal{N} = 2$ SCFT with a non-abelian flavor symmetry. For comparison we may note that for a free hypermultiplet $c = 1/12 = 0.0833\ldots$. From Fig. 4.3 it seems that there may in fact be a solution to crossing symmetry with roughly this minimum value of the central charge, because the global minimum of the exclusion curve at $1/k \simeq 0.68$ seems rather stable against increasing Λ. We are however not aware of any $\mathcal{N} = 2$ SCFT (with or without non-abelian flavor symmetry) with such a low central charge.

A feature of these our numerical bounds that we will be repeated both here and in the next section is that they are non-optimal, meaning that they display substantial dependence on Λ for the values of the cutoff considered. This is in contrast with, *e.g.*, the three-dimensional investigations in [33]. In that paper the bounds converge much faster and on the scales that we consider here they are essentially constant at $\Lambda = 22$.[28] Notice that with $\Lambda = 30$ and three flavor symmetry channels we have a functional with 392 components, which surpasses even the 231 components used in the precision work on the three-dimensional Ising model [102]. Apparently this crossing symmetry problem is numerically more expensive. We cannot currently offer a good explanation as to why this is the case.

A natural way to deal with the relatively poor convergence is to extrapolate our results from finite to infinite Λ.[29] In this way we can generate a rough guess of where the best possible bound may lie. Fig. 4.4 shows an example of such an extrapolation. The minimum allowed value of c for $k = 1$ is plotted as a function of the cutoff Λ, and a possible extrapolation to infinite cutoff is sketched. The dashed line in the figure corresponds to the central charge which saturates the analytic bound at $k = 1$ (corresponding to the free hypermultiplet with $c = \frac{1}{12}$). It seems plausible that in the $\Lambda \to \infty$ limit the numerical bounds will intersect the analytic bound at this point.

Dimension bounds for $\mathfrak{su}(2)$

We now focus on the allowed region in the (c, k) plane and generate bounds for the dimension of the first unprotected operator appearing in the $\hat{\mathcal{B}}_1 \times \hat{\mathcal{B}}_1$ OPE. In the tensor product of two copies of the adjoint representation of $\mathfrak{su}(2)$ one finds three irreps: the singlet, triplet, and quintuplet. We will report on bounds for the dimension of the first unprotected operator of lowest spin in each of these channels.

Singlet channel

In Fig. 4.5 we present the upper bound for the scaling dimension Δ of the first unprotected scalar operator in the singlet channel, for all allowed values of the central charges. The values shown are an interpolation through a total of 572 data points, distributed on a square grid with finer resolution near the edges. The cutoff for this analysis is $\Lambda = 18$. The surface so obtained appears smooth and monotonic, with the bound getting stronger when approaching the wall that represents the analytic bound or and at large central charge.

The bounds shown in Fig. 4.5 are completely universal – any four-dimensional $\mathcal{N} = 2$ SCFT with at least $\mathfrak{su}(2)$ flavor symmetry corresponds to a point somewhere inside the allowed region. We will discuss several examples of such theories below, but as a zeroth-order check we confirm that our bounds are consistent with some elementary solutions to crossing symmetry.

At the infinite point $(1/c, 1/k) = (0, 0)$ the stress tensor and the flavor current decouple, their OPE coefficients being $\lambda_T \sim \frac{1}{c}$ and $\lambda_J \sim \frac{1}{k}$ respectively. A well-known solution to crossing symmetry for which these operators are absent is generalized free field theory, for

[28] In [33] the cutoff is defined differently – $\Lambda = 22$ here corresponds to $n_{\max} = 11$ there.

[29] We do not currently have theoretical control of the dependence of the numerical bounds on Λ, but we hope the apparent smoothness of the numerical results is enough to justify such extrapolations.

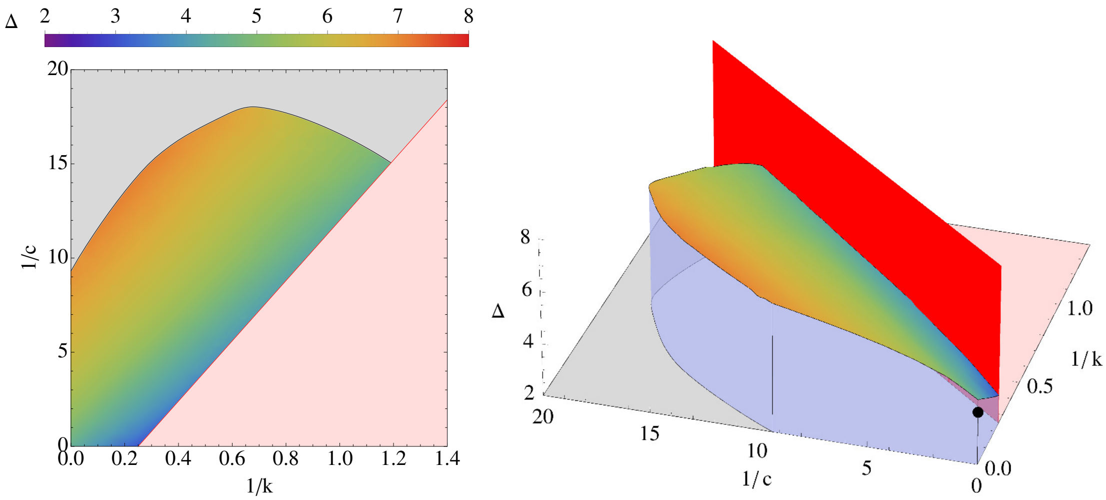


Figure 4.5: Upper bounds for the dimension of the first unprotected singlet scalar operator in theories with $\mathfrak{su}(2)$ flavor symmetry, as a function of $1/k$ and $1/c$. The cutoff used for this plot was $\Lambda = 18$. The two- and a three-dimensional plots are generated with the same data set. The gray and light red surfaces in the figure are the excluded regions from Fig. 4.3, and the vertical red wall is added help visualize the constraints imposed by the analytic bounds. The black dot is the generalized field theory solution to crossing.

which the four-point function is a sum of disconnected pieces,

$$\langle \phi_1(x_1) \ldots \phi_4(x_4) \rangle = \langle \phi_1(x_1)\phi_2(x_2) \rangle \langle \phi_3(x_3)\phi_4(x_4) \rangle + \text{two permutations} . \tag{4.106}$$

Specializing this solution to the four-point function of moment map operators, we find that the first operator in the conformal block decomposition has dimension four. As is indicated in Fig. 4.5, the generalized free field solution is consistent with the numerical upper bound which gives $\Delta \leqslant 4.47$ at this point. The numerical bound is similarly consistent with the theory of a free hypermultiplet with $(c, k) = (\frac{1}{12}, 1)$, since the first unprotected singlet scalar in the corresponding four-point function again has dimension four and numerically we have $\Delta \leqslant 4.38$. Finally, we can take a linear combination of the two solution with positive weights that sum to one. This results in a valid solution to crossing symmetry along the straight line in Fig. 4.5 that runs from the origin to the free-field point, with a first unprotected singlet scalar operator that always has dimension four. Again, this is consistent with the numerical bound which is greater than four everywhere above this line. Much like the bound on c sketched in Fig. 4.4, we expect these bounds to decrease substantially as Λ is increased, and to converge to four along this line as $\Lambda \to \infty$. An extrapolation in Λ for $(1/c, 1/k) = (0, 0)$ (not shown) bolsters this intuition. Similar extrapolation experiments suggest that the bound should end up below 4 for all values in the (c, k) plane between the analytic bound and the

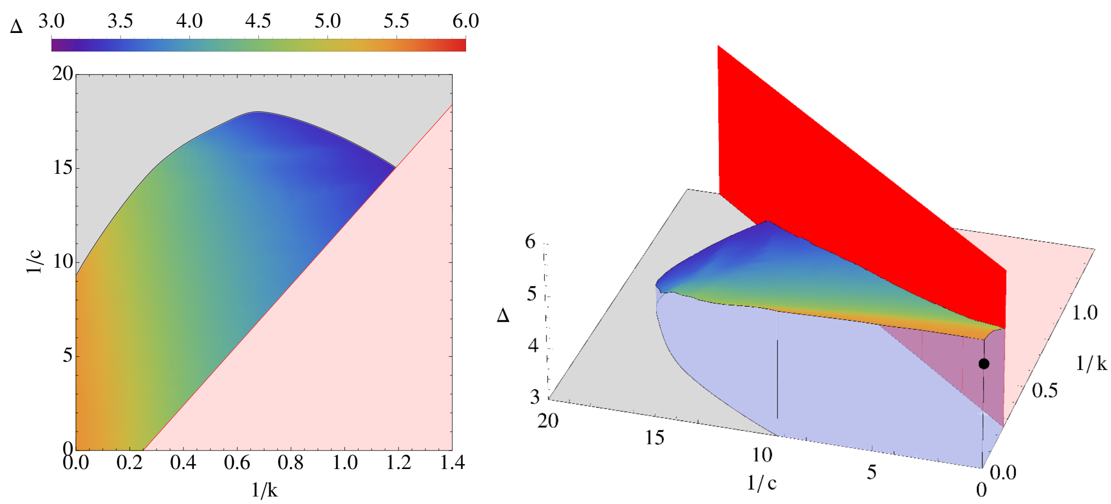


Figure 4.6: Upper bounds for the dimension of the first unprotected spin one multiplet in the triplet channel of a theory with $\mathfrak{su}(2)$ flavor symmetry, for all allowed values of c and k, presented both as a three-dimensional plot and as a density plot. The gray and light red surfaces in the figure are the excluded regions from Fig. 4.3. These bounds were obtained with $\Lambda = 18$ and 547 data points in the (c, k) plane.

interpolating solution of the previous paragraph.

Although we have presented the two results in Figs. 4.3 and 4.5 as independent results, they are in fact related. Indeed, the bound on the first scalar operator drops sharply to the unitarity bound when we venture inside the numerically excluded region of Fig. 4.3. Such a drop indicates that there does not exist any spectrum that is simultaneously consistent with unitarity and crossing, and delineating the region where this happens is another way to obtain the numerically excluded region in Fig. 4.3. The c-minimization approach used to generate Fig. 4.3 is much more efficient, and could consequently be performed at higher values of Λ.

Triplet and quintuplet channels

We now present numerical results for the triplet and quintuplet channels. The triplet appears in the antisymmetric combination of two adjoints, so only odd spins can be exchanged. In this case we bound the dimension of the first unprotected spin one operator appearing in the $\hat{\mathcal{B}}_1 \times \hat{\mathcal{B}}_1$ OPE. This bound is shown in Fig. 4.6 for the allowed range of c and k. Note that the unitarity bound for a spin one multiplet is $\Delta \geqslant 3$. The numerical upper bound is again represented by a smooth surface, with weaker bounds appearing at larger values of k. In the limit where both c and k go to infinity the bound is close to 5, which is the value for generalized free field theory.

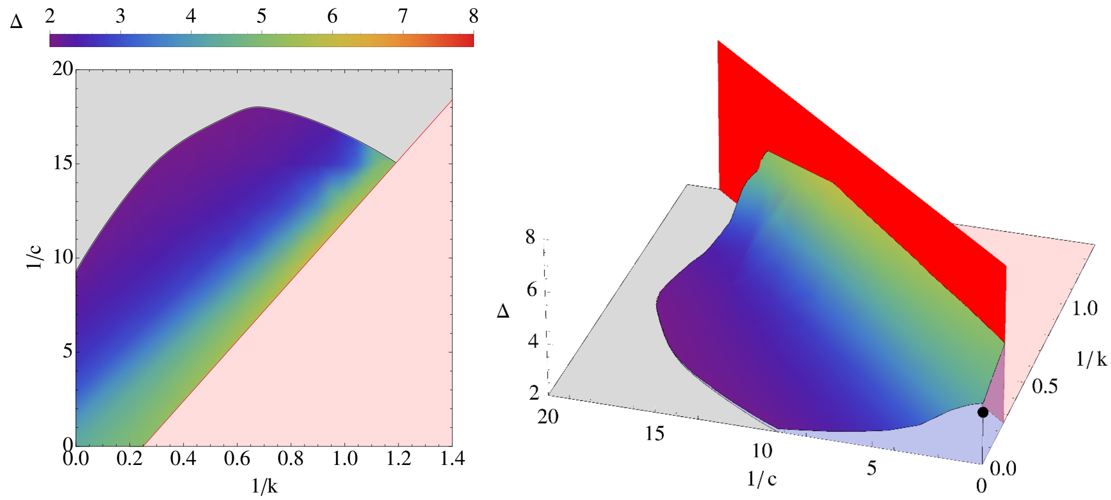


Figure 4.7: Upper bounds for the dimension of the first unprotected scalar in the quintuplet channel of a theory with $\mathfrak{su}(2)$ flavor symmetry, for all allowed values of c and k, presented both as a three-dimensional plot and as a density plot. The gray and light red surfaces in the figure are the excluded regions from Fig. 4.3. This plot was obtained with $\Lambda = 18$ and 398 data points in the (c, k) plane.

The quintuplet channel is again symmetric, so the exchanged operators will have even spin as they did in the singlet channel. We have generated upper bounds for the dimension of the first scalar operator. These are shown in Fig. 4.7 as a three-dimensional plot and a density plot. The behavior of the bounds when approaching the minimum allowed values of c and k is different from the other two channels – in this case the bound drops smoothly to the unitarity bound at $\Delta = 2$. As either c or k are increased the bound gets weaker, and when they both go to infinity the bound is near $\Delta = 4$, which is the correct value in generalized free field theory.

We note that the triplet and quintuplet bounds approach the unitarity bound near the minimum of the exclusion curve of Fig. 4.3 at $1/k \simeq 0.68$. This is a strong indication that the solution to crossing symmetry at that point has higher spin currents, which we would generally associate to a free theory. Because the central charge is not that of a free hypermultiplet, one may suspect that this point is not related to a physical theory.

Bounds for theories of interest

In the previous subsections we discussed bounds on operator dimensions for the entire (c, k) plane that were obtained with a cutoff $\Lambda = 18$. We will now turn to a discussion of stronger bounds, obtained with $\Lambda = 22$, which we computed only for specific values of c and k that correspond to theories of interest. In this subsection we present operator dimension bounds along the curves in the (c, k) plane that correspond to the H_0 and H_1 theories shown in Fig. 4.2. In the next subsection we will discuss the defect theory at infinite c and $k = 4$ that corresponds to the dotted line in Fig. 4.2.

For the H_0 theories with $N \geqslant 2$ the only flavor symmetry is $\mathfrak{su}(2)_L$. We can trace the results shown in Fig. 4.5 along the H_0 curve in Fig. 4.2 to recover upper bounds for the dimension of the first unprotected scalar singlet in these theories. This slice is displayed in Fig. 4.8. The H_1 theories with $N \geqslant 2$ have two independent $\mathfrak{su}(2)$ symmetries with different

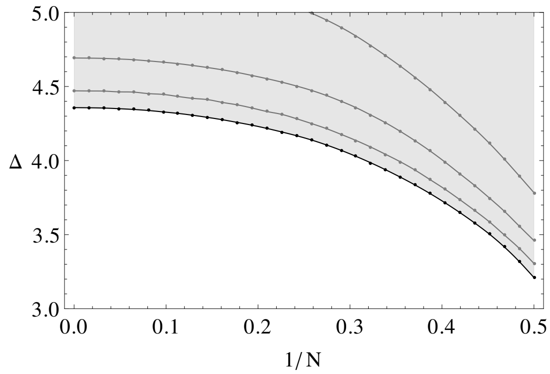


Figure 4.8: Upper bounds for the first unprotected scalar in the theories of type H_0 as a function of the inverse rank. The bounds are extracted from the four-point function of the $\mathfrak{su}(2)_L$ flavor symmetry moment map and are valid only for $N \geqslant 2$. The different lines correspond to cutoff values $\Lambda = 10, 14, \ldots, 22$, with the strongest bound shown as the black line.

flavor central charges. We derived bounds for the two different cases by following the two different curved labelled H_1 in Fig. 4.2. Both of the singlet scalar bounds so obtained are shown in Fig. 4.9.

In all of our plots corresponding to lines of interesting theories, we have shown the progression of the bounds as a function of the cutoff. This gives a feeling for how close to the optimal bound we have gotten – information that is absent from the plots of the previous section where all the results came from analyses with $\Lambda = 18$ was shown. In general, there seems to be some distance yet to go before the bounds will have effectively converged. In particular, in the infinite-rank limit $N \to \infty$ the stress tensor and flavor current decouple from the OPE expansion and the bounds should reach the generalized free field theory value $\Delta = 4$. The difference between the $\Lambda = 22$ bounds at large N and the generalized free field theory value offer a simple proxy for how far we have yet to go.

Despite slow convergence, we may naively extrapolate our bounds to generate estimates for their optimal values. In particular, for the rank one H_1 theory there is a single $\mathfrak{su}(2)$ flavor factor and we might expect that the bound generated by studying the corresponding four-point function of moment maps to be saturated by this theory. Extrapolation for this value of c and k leads to a conjectural optimal bound in the range of $3.2 - 3.4$. Moving on to the rank two case, there are now two independent bounds extracted from the two $\mathfrak{su}(2)$ flavor symmetries. These two bounds could conceivably apply to the same operator. In other words, the same unprotected scalar singlet has no particular reason not to appear in both moment map four-point functions. However, the two bounds appear to be unrelated. The $\mathfrak{su}(2)_L$ bound dominates at low ranks, while the ordinary $\mathfrak{su}(2)$ bound dominates for higher ranks.

Similar bounds to those derived here can be obtained for the triplet and the quintuplet channels by the same methods, though we have not done so here.

Bounds for defect SCFTs

An interesting aspect of the analytic bound (4.102) is that as $c \to \infty$ the bound on k stays finite and we have $k \geqslant 4 = 2h^\vee$. The limit where $c \to \infty$ and k remains finite should correspond to a theory without a stress tensor but with conserved global symmetry current. This kind of physics can be found on certain defects or interfaces in higher-dimensional theories where the global symmetry is confined to the defect but energy can leak into the bulk. There is in fact a natural set of defects that preserves $\mathcal{N} = 2$ superconformal invariance in four dimensions, namely the codimension two defects in the six-dimensional $(2, 0)$ SCFTs (see, *e.g.*, [172] and references therein). For a $(2, 0)$ theory of type $\mathfrak{g} \in \{A_n, D_n, E_n\}$, the possible defects are labelled by an embedding $\rho : \mathfrak{sl}(2) \to \mathfrak{g}$. The degrees of freedom localized on the defect carry a flavor symmetry $\mathfrak{h}$ which is the commutant of the image of ρ. When ρ is trivial, the flavor symmetry is just $\mathfrak{g}$ and the corresponding flavor central charge is then

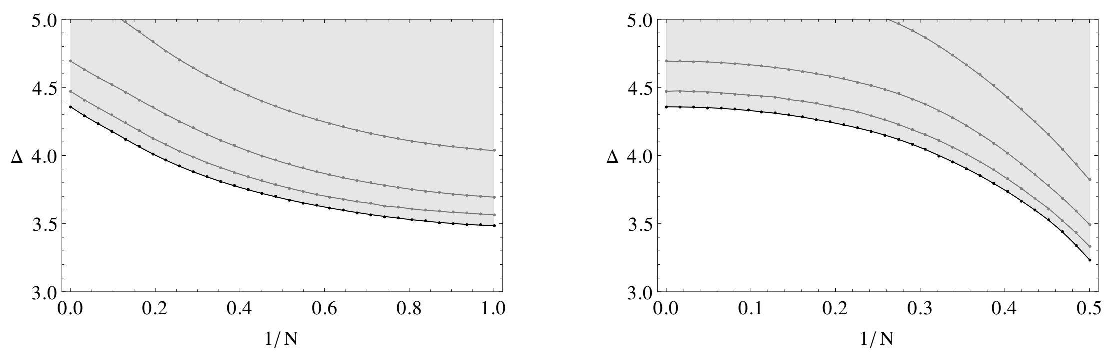


Figure 4.9: Bounds for the dimension of the first unprotected spin zero multiplet in the singlet channel for the H_1 theories, as a function of the inverse of the rank of the theories. The left plot comes from studying the four-point function of the ordinary $\mathfrak{su}(2)$ flavor symmetry moment map and is valid for all $N \geqslant 1$. The right plot comes from the four-point function of the $\mathfrak{su}(2)_L$ flavor symmetry moment map and valid only for $N \geqslant 2$. The different lines correspond to $\Lambda = 10, 14, \ldots, 22$, with the strongest bound shown as the black line.

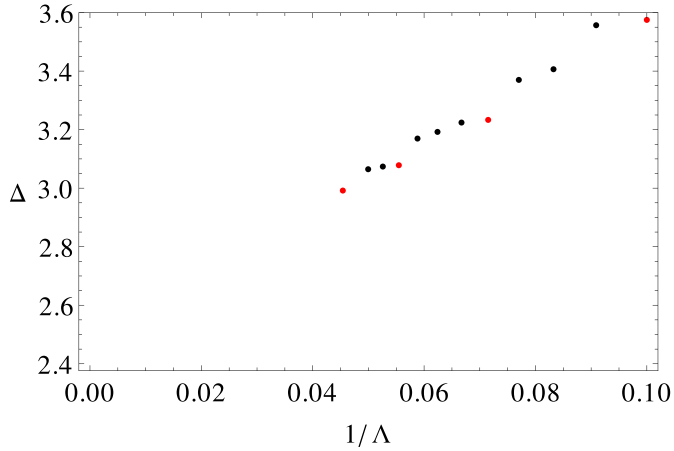


Figure 4.10: Bound on the first unprotected scalar in the singlet channel for a theory with $k = 4$ and infinite central charge, as a function of the cutoff. The red dots correspond to derivatives 10 to 22 in steps of four, and the black dots to the remaining values of Λ ranging from 10 to 20.

given by $k = 2h^\vee$. The bounds that we obtain at the point $k = 2h^\vee$ with $c = \infty$ therefore constrain the spectrum of unprotected operators living on such a surface operator. Since we consider $\mathfrak{su}(2)$ flavor symmetry, this bound is valid for the defects of the $(2, 0)$ theory of type A_1.

In Fig. 4.10 we show the upper bound for scalar singlets in the defect theory as a function of the inverse cutoff. The best bound is given by 2.99, and naive extrapolation suggests a relatively low value for the optimal bound somewhere between 2.5 and 2.9. It is natural to suspect that this is indeed the value of the first unprotected singlet scalar on the defect.

We notice that the bound in Fig. 4.10 displays a step-like behavior whenever $\Lambda - 2$ is a multiple of four, corresponding to the red dots in the figure. Given our lack of theoretical control over the behavior of the bound as a function of the cutoff, we cannot currently offer any theoretical explanation for this quasi-periodicity. It however suggests an extrapolation scheme based on a restricted data set where Λ increases in steps of four. This is what was done in generating Fig. 4.4.

4.6.2 $\mathfrak{e}_6$ global symmetry

Our second investigation focuses on theories with $\mathfrak{e}_6$ global symmetry. Let us again begin by making a rough sketch of the landscape of such theories as seen by the moment map four-point function. We show such a sketch in Fig. 4.11. There are analytic bounds for the central charges of theories with $\mathfrak{e}_6$ global symmetry arising from the chiral algebra of 2.

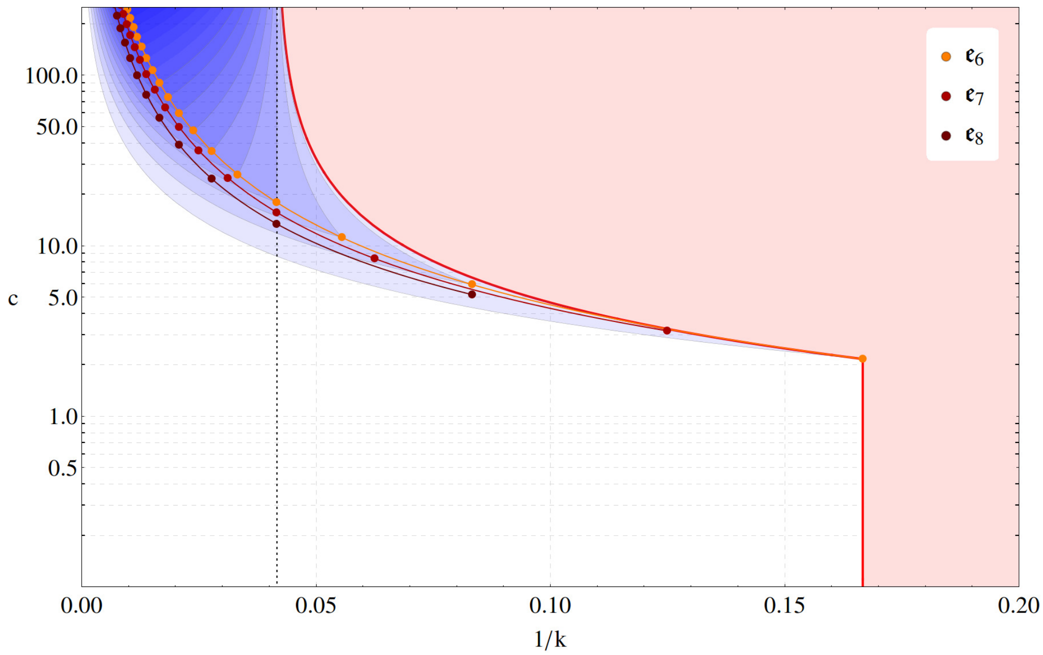


Figure 4.11: An overview of the of known theories with $\mathfrak{e}_6$ flavor symmetry, shown here as points in the plane spanned by their c and k central charges. The red region is excluded by analytic central charge bounds. The vertical dotted line designates the value $k = 6$ with is the value of the central charge for the maximal defect SCFT in the six-dimensional $(2,0)$ theory of type $\mathfrak{e}$. The blue wedges with vertices at each of the $\mathfrak{e}_6$ theories are the region of the plane for which solutions of crossing symmetry can be realized as linear combinations of the four-point function for the theory at the vertex and those of generalized free field theory and the defect theory.

These are given by

$$k \geqslant 6 \ , \qquad k \geqslant \frac{48c}{13+2c} \ . \tag{4.107}$$

The region excluded by these bounds is shown in red in Fig. 4.11. We have also plotted several known families of theories whose flavor symmetry contains an $\mathfrak{e}_6$ factor, namely the theories originating from F-theory singularities of type $\mathfrak{e}_n$ for $n = 6, 7, 8$ and for all ranks. The existence of these theories gives a collection of solutions to crossing symmetry with various values of c and k. By taking linear combinations of these solutions, one can find solutions of crossing symmetry with (c, k) values anywhere inside the blue region in Fig. 4.11. In particular, for each irreducible solution there is a wedge corresponding to linear combinations of that solution with the generalized free field theory solution and the defect solution at $k = 6$ and $c \to \infty$. These wedges are shown for the $\mathfrak{e}_6$ theories.

For the purposes of numerical analysis, the fact that there are now five irreps in the

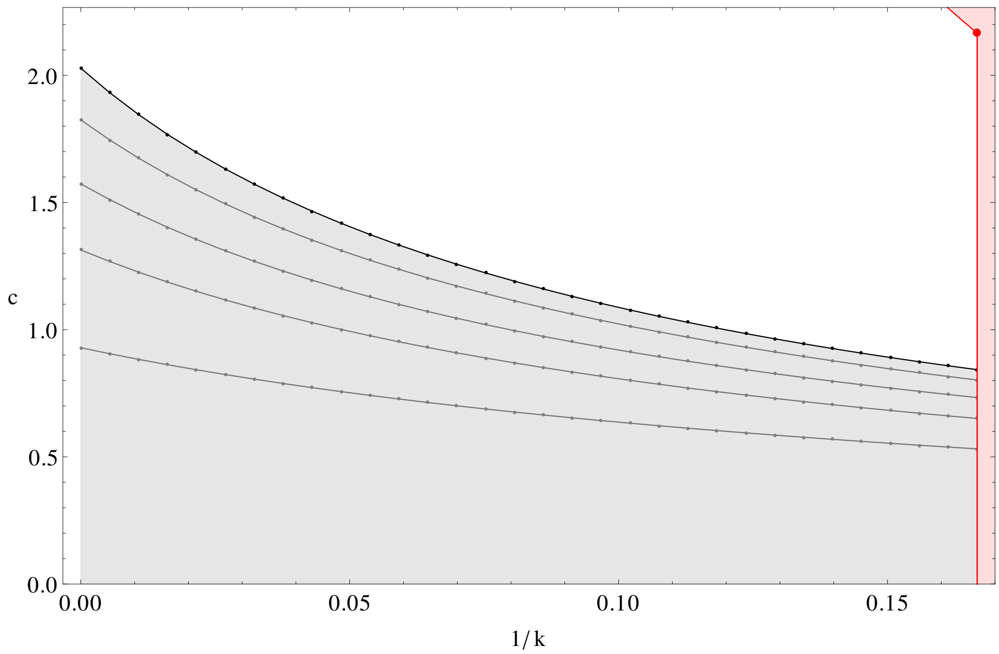


Figure 4.12: Bound on the central charge c of a theory with $\mathfrak{e}_6$ flavor symmetry as a function of the flavor central charge k, obtained from the $\hat{\mathcal{B}}_1$ four-point function. The shaded red regions on the right are the analytic bounds given in (4.107), and the shaded gray region at the bottom is the numerically excluded region. The gray and black lines correspond to the numerical bounds, shown for 10 to 26 derivatives in steps of four, with the strongest bound (black line) corresponding to 26 derivatives. The red dot at the intersection of the two analytic bound corresponds to the rank one $\mathfrak{e}_6$ theory [12].

tensor product of two copies of the adjoint representations makes the search space larger than the $\mathfrak{su}(2)$ case for a given value of Λ. As such, the maximum value of Λ that we were able to reach is lower and the $\mathfrak{su}(2)$ case.

Constraints on c and k

We have obtained numerical lower bounds on c as a function of k following the same approach as in the $\mathfrak{su}(2)$ case. Here we considered a maximum cutoff of $\Lambda = 26$. The lower bound is displayed in Fig. 4.12 as a function of the (inverse) flavor current central charge. The regions shaded in red are again the ones ruled out by the analytic bounds (4.107). We see that independent of k, any $\mathcal{N} = 2$ SCFT with at least $\mathfrak{e}_6$ flavor symmetry has

$$c \geqslant 0.83 \; . \tag{4.108}$$

In contrast to the case of $\mathfrak{su}(2)$ global symmetry, for $\mathfrak{e}_6$ there is a theory living at the intersection of the two analytic bounds. This is the rank one $\mathfrak{e}_6$ theory of Minahan and Nemeschansky. One may wonder whether there is another theory with $k = 6$ but with a lower value of c. In Fig. 4.13 we show the lower bound on c for $k = 6$ derived from the moment map four-point function as a function of Λ. Though the bounds still seem to be improving, it appears unlikely that the optimal bound will reach the value $c = \frac{13}{6}$ (the value of the rank one $\mathfrak{e}_6$ theory). Instead, our best estimate for the optimal value of the central charge bound is somewhere between 1.1 and 1.2. We are not aware of a theory with such a low central charge and (at least) $\mathfrak{e}_6$ flavor symmetry. It would be interesting to determine whether the solution to crossing symmetry being approximated here contains higher spin currents.

Dimension bounds in the singlet channel

Bounds for the first unprotected scalar in the singlet channel as a function of the (inverse) central charges are shown in Fig. 4.14. The range of central charges allowed by unitarity is limited by (4.107). Our plot therefore starts at $k = 6$, and the vertical red wall delimits the region allowed by the second bound in (4.107). The gray region in Fig. 4.14 for low values of the central charge represents central charges excluded by the numerical bounds of the previous section. As both central charges go to infinity we expect the bound to go to the generalized free field theory value of $\Delta = 4$, which we denoted with a black dot in the figure. This point is consistent with the numerical bounds, and naive extrapolation of the numerical results (not shown) suggests convergence towards $\Delta = 4$.

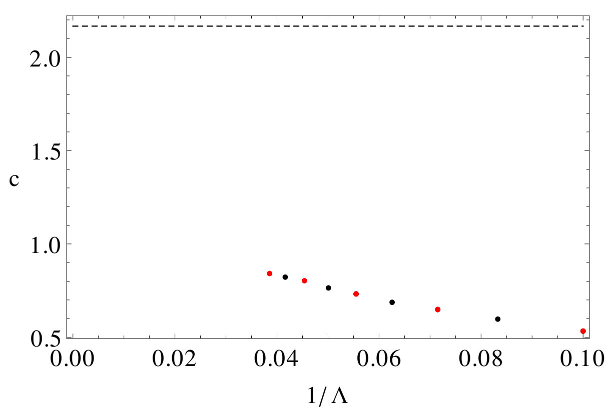


Figure 4.13: Lower bounds on c for a theory with $\mathfrak{e}_6$ flavor symmetry and $k = 6$ as a function the inverse cutoff Λ. The red dots correspond to derivatives $10, 14, \ldots, 26$, while the black dots show the remaining even values of Λ ranging from 10 to 24. The dashed line at $c = \frac{13}{6}$ marks the central charge of the rank one $\mathfrak{e}_6$ theory.

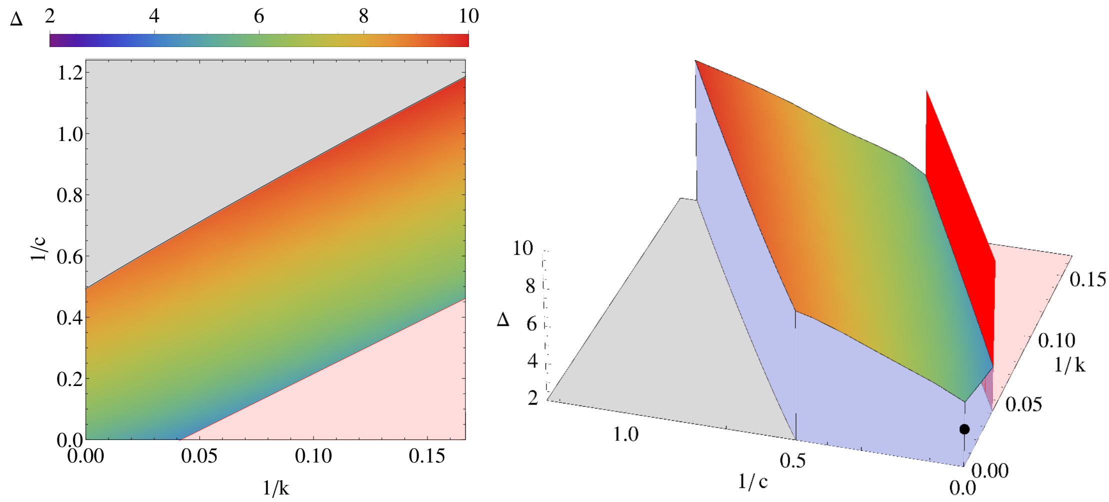


Figure 4.14: Upper bounds for the dimension of the first unprotected scalar in the singlet channel of a theory with $\mathfrak{e}_6$ flavor symmetry as a function of the inverse of the central charges. These bounds were generated with $\Lambda = 16$. The vertical red wall corresponds to the second analytic unitarity bounds in (4.107), with the excluded region being the top right corner. The plot starts at $\frac{1}{k} = \frac{1}{6}$, where the first analytic unitarity bound is saturated.

Bounds for theories of interest

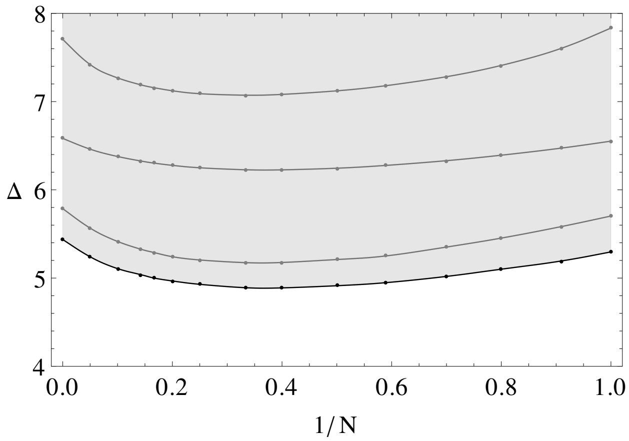
We now specialize to the values in the (c, k) plane that correspond to the theories of $D3$ branes probing F-theory singularities with $\mathfrak{e}_6$ flavor symmetry. The central charges for these theories, shown in orange in Fig. 4.11, are given by [70]

$$\begin{aligned} c &= \frac{3}{4}N^2 + \frac{3}{2}N - \frac{1}{12} \ , \\ k &= 6N \ , \end{aligned} \tag{4.109}$$

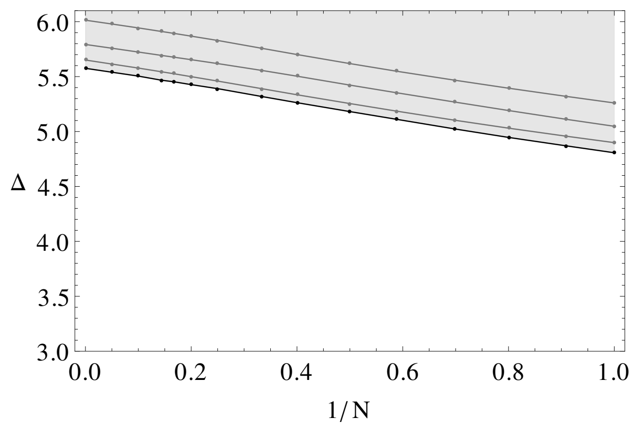
where N is the rank of the theory. All theories with rank $N \geqslant 2$ have an extra $\mathfrak{su}(2)_L$ flavor symmetry, with $k_L = 3N^2 - 2N - 1$.

We derived upper bounds for the dimensions of the first unprotected operators of lowest spin in each of the flavor symmetry channels appearing in the tensor product of two adjoints. For the case of symmetric representations (singlet, **650**, **2430**) we therefore obtain a bound for spin zero operators, and for antisymmetric representations (**78**, **2925**) we bound spin one operators. These bounds are displayed in Fig. 4.15. They are still far from optimal, but serve to give us a feeling for the general shape of things. It would be interesting to improve our numerical search power to the point where these bounds would converge.

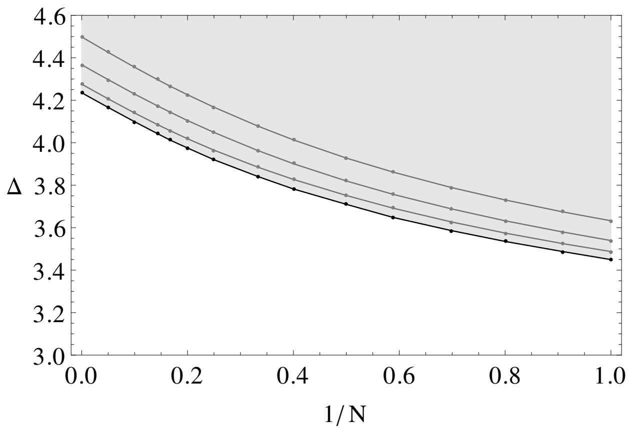
We should compare the singlet bounds in Fig. 4.15a for rank $N \geqslant 2$ to the bounds obta-



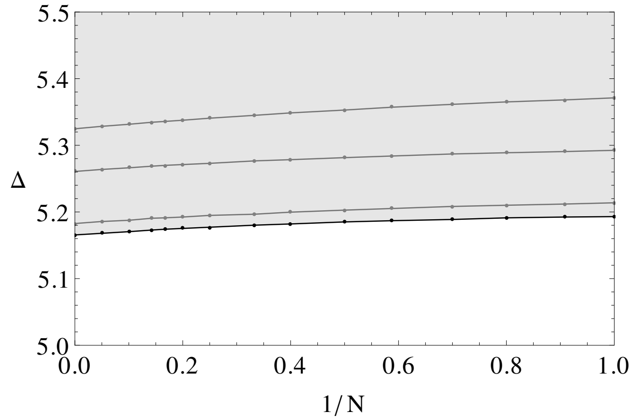
(a) Singlet, scalar



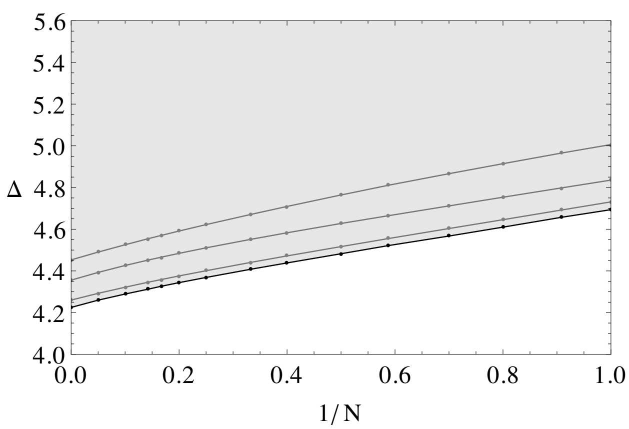
(b) **78**, spin 1



(c) **650**, scalar



(d) **2925**, spin 1



(e) **2430**, scalar

Figure 4.15: Bound for the dimension of the first unprotected spin zero multiplet in the singlet, **78**, **650**, **2925** and **2430** channels for the theories with flavor symmetry $\mathfrak{e}_6$ arising from F-theory singularities, as a function of the inverse of the rank of the theories. The different cutoffs are $\Lambda = 10, 12, \ldots, 16$, with the strongest bound given by the black line.

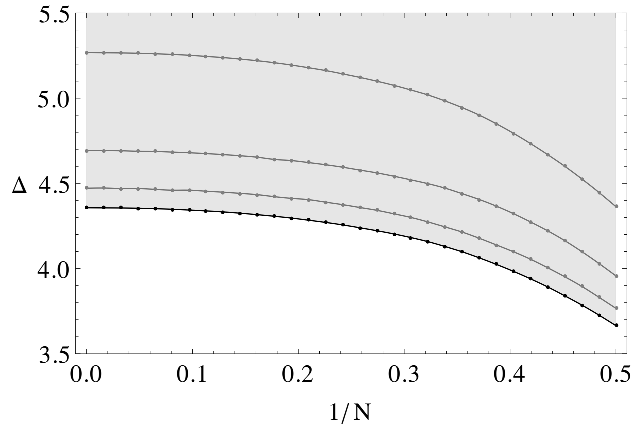


Figure 4.16: Upper bounds for the dimension of the first unprotected scalar in the singlet channel of the $\mathfrak{su}(2)_L$ moment map correlator for the $\mathfrak{e}_6$ theories, as a function of the inverse of the rank of the theories. The cutoff is increased from 10 to 22 in steps of four, with the strongest bound given by the black line.

ined from the $\mathfrak{su}(2)_L$ flavor symmetry of those theories. In principle the same unprotected operators may contribute to the four-point functions of both sets of moment maps, so if the bounds recovered from both correlators are related to these rank N theories then they should agree to some extent. These bounds are shown in Fig. 4.16. The two sets of bounds appear to have nothing in common. It is hard to say whether this is a feature of the space of solutions to crossing symmetry or a consequence of inadequate numerical power.

The rank one theory

We performed a higher precision analysis at the point $k = 6$ and $c = \frac{13}{6}$, which are the central charges of the $\mathfrak{e}_6$ Minahan-Nemeschansky theory. It is plausible that this theory is the unique theory with these central charges and $\mathfrak{e}_6$ flavor symmetry. What's more, because of the location of these central charges in a corner of the allowed region of the (c, k) plane, there can be no pollution at this point by solutions of crossing symmetry that are linear combinations of other irreducible solutions. This gives us some room to be optimistic that the numerical bounds at this point will converge to physical values that correspond to the scaling dimensions of operators in this theory.

As a first example we may consider again the bound on the first unprotected singlet scalar. We have plotted this bound as a function of the cutoff in Fig. 4.17. Naive extrapolation suggests an optimal value in the neighborhood of $\Delta \simeq 4.4$ for the first scalar singlet.

We can also explore *simultaneous bounds* for various channels by searching for functionals with $\Delta^{\star}_{\mathfrak{R}_i}$ greater than the unitarity bound for several choices of flavor symmetry channel $\mathfrak{R}_i$. We performed such an analysis for these central charges to derive simultaneous bounds

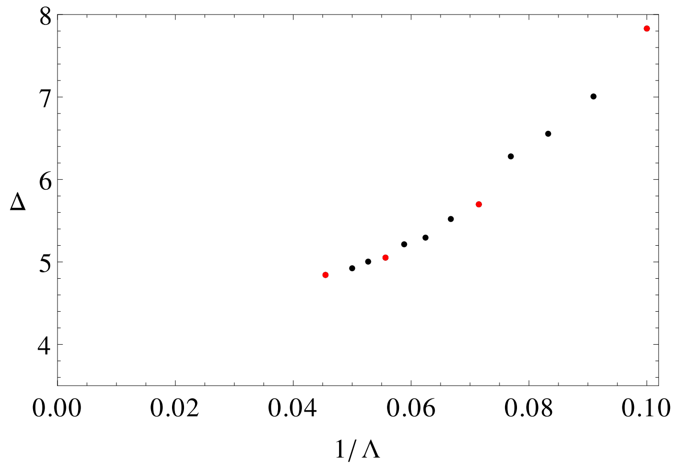


Figure 4.17: Upper bounds for the first unprotected scalar singlet in the rank one $\mathfrak{e}_6$ theory as a function of the inverse cutoff. The points where $\Lambda - 2$ is a multiple of four are shown in red. For the best bound shown, the dimension of the search space in the associated semidefinite program was 366.

for the first scalars in the $\mathbf{1}$, $\mathbf{650}$, and $\mathbf{2430}$ channels. The numerics were performed with $\Lambda = 12$, and the results are shown in Fig. 4.18 in the from of an exclusion plot in the three-dimensional space spanned by the scaling dimensions $(\Delta_{\mathbf{1}}, \Delta_{\mathbf{650}}, \Delta_{\mathbf{2430}})$ of the first operator in those channels. The usual superconformal unitarity bounds constrain us to be in the octant where all these three dimensions are greater than two, but within this octant we have numerically carved out a further excluded region where one or more of the three operator dimensions is too high to satisfy the crossing symmetry equations.

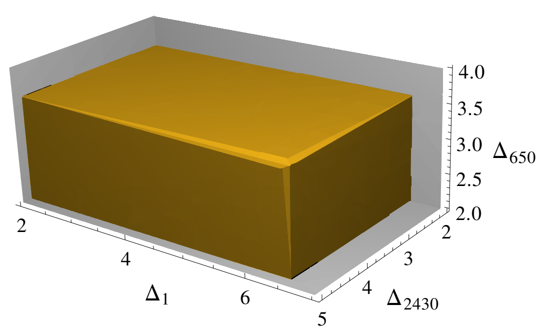


Figure 4.18: Three-dimensional exclusion plot in the octant spanned by the scaling dimensions of the first unprotected scalar in the $\mathfrak{R} = \mathbf{1}, \mathbf{650}, \mathbf{2430}$ representations of $\mathfrak{e}_6$ with $k = 6$ and $c = \frac{13}{6}$. The cutoff used while generating these bounds was $\Lambda = 12$.

Bounds for defect SCFTs

We can again consider the limit where we send $c \to \infty$ with k at the analytic bound, which gives $k = 24$ in this case. In this limit we expect to recover information about the theory living on the codimension two defect corresponding to the trivial embedding in the six-dimensional $(2,0)$ theory of type $\mathfrak{e}_6$. A nontrivial bound for the first unprotected scalar in the singlet channel is given in Fig. 4.19 as a function of the cutoff. Once again we observe some quasi-periodic behavior where the bounds have a sharper jump at every fourth step in the cutoff. By naive extrapolation of the bound we arrive at a rough estimate that the optimal upper bound should be between $\Delta = 3$ and $\Delta = 3.2$.

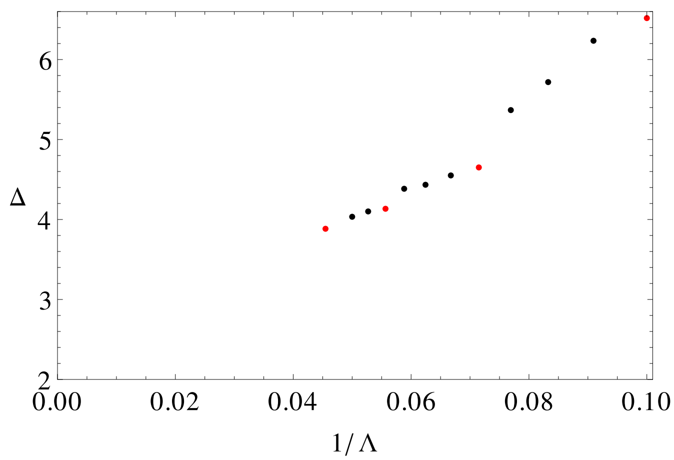


Figure 4.19: Bound on the first unprotected scalar in the singlet channel for a theory with $k = 24$ and infinite central charge as a function of the cutoff. Red dots correspond to cutoff values $\Lambda = 10, 14, \ldots, 22$, while black dots show the remaining cutoff values ranging from $\Lambda = 10$ to $\Lambda = 20$.

4.7 Results for the $\mathcal{E}_r$ four-point function

We now turn to the numerical results obtained for the four-point function of the Coulomb branch operators $\mathcal{E}_{r_0}$. Unlike in the previous section we can vary the dimension of these operators, which we recall is given in terms of their $U(1)_r$ charge r_0 by $\Delta_{\text{ext}} = r_0$. Unitarity requires $r_0 \geqslant 1$. We will consider four-point functions where all operators have equal dimension. A second parameter is again the c central charge which appears in front of the conformal block of the stress tensor multiplet. We will therefore be able to obtain bounds as a function of r_0 and c.

4.7.1 Central charge bounds

Our first bound is again a lower bound for the c central charge, now as a function of the dimension r_0 of the Coulomb branch operators. Assuming the moduli space/chiral ring correspondence for the Coulomb branch, the Shapere-Tachikawa relation provides an analytic lower bound for c. More precisely this bound is obtained combining the ST sum rule (4.22) and the Hofman-Maldacena upper bound (4.21) on $\frac{a}{c}$. If the Coulomb branch, which is assumed to be freely generated, has dimension N with generators of dimension $\{r_1, \ldots, r_N\}$, then we the following bound holds,

$$c \geqslant \frac{1}{6}\sum_{i=1}^{N}(2r_i - 1) \ . \tag{4.110}$$

The fact that only the dimensions of *generators* of the Coulomb branch chiral ring appear in this expression is important. For example, Coulomb branch operators of dimension $r_0 \geqslant 3c + \frac{1}{2}$ are certainly allowed by this bound, they just cannot be generators. On the other hand, a theory that has any Coulomb branch at all must have $c \geqslant \frac{1}{6}$, since the dimension of a Coulomb branch generator cannot be smaller than one. Moreover if $c = \frac{1}{6}$ then the Coulomb branch must have a single generator of dimension $r_0 = 1$, so will necessarily be the theory of a single free vector multiplet.

In setting up the bootstrap for this correlator, there is no straightforward way to insist that the Coulomb branch operators be generators (or that they not be generators, for that matter). Of course, any such operator with $r_0 < 2$ will necessarily a generator because it cannot be a product of two operators with dimensions above the unitarity bound. For $r_0 \geqslant 2$ if we *assume* that we are dealing with a generator, in which case the following analytic bound will be obeyed:

$$c \geqslant \frac{1}{6}(2r_0 - 1) \ . \tag{4.111}$$

Notice that if we drop the generator assumption and consider four-point functions of operators that are not generators, then only the weaker bound $c \geqslant \frac{1}{6}$ applies for $r_0 \geqslant 2$. This bound is in fact saturated at any $r_0 \in \mathbb{N}$ by the operators of the Coulomb branch chiral ring of the free vector multiplet.

In Fig. 4.20 we show the results of a numerical c-minimization procedure as a function of r_0. The analytic bound (4.111) is superimposed in red. For large values of r_0 the analytic bound always dominates over the numerical one, but for $r_0 \lesssim 1.4$ the numerical bound is dominant. Nevertheless, we would like to stress that the analytic bound is contingent upon the validity of the Coulomb branch version of the moduli space/chiral ring correspondence. If there are exceptions to this rule, then the analytic bounds will not hold, whereas the numerical bounds will still necessarily hold true.

As r_0 approaches one the bound drops sharply towards $c = \frac{1}{6}$, the central charge of the free vector. Though it is not clear from the figure, $c = \frac{1}{6}$ is not ruled out for $r_0 = 1$, where

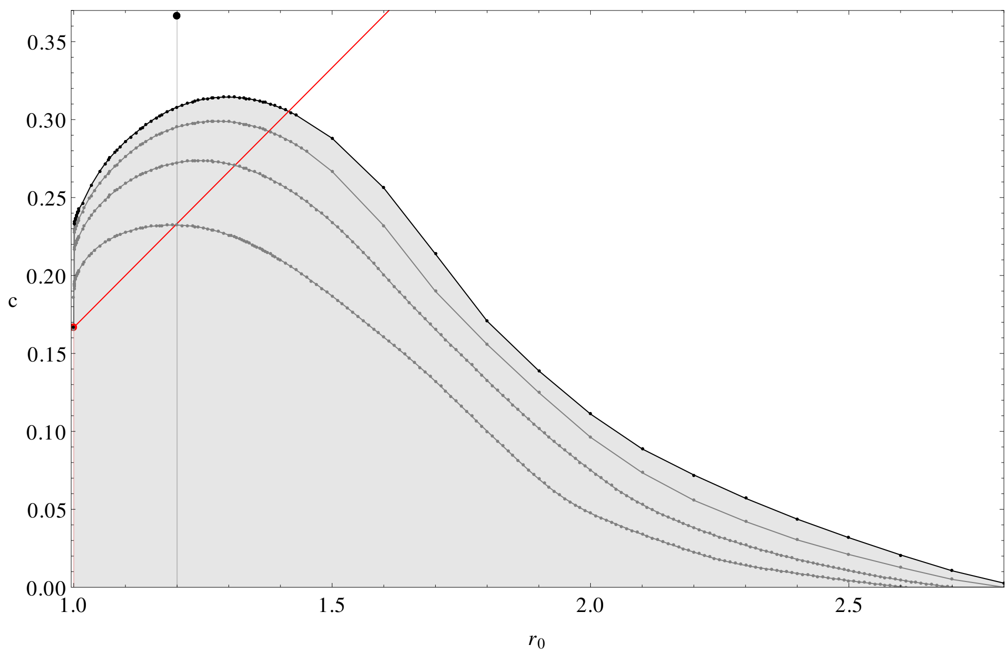


Figure 4.20: Lower bounds for the central charge c of a theory with Coulomb branch operator $\mathcal{E}_{r_0}$ as a function of its dimension r_0. The straight red line is the analytic bound for the case when $\mathcal{E}_{r_0}$ is a Coulomb branch generator, given in (4.111), with the excluded region lying to the right of the line. The shaded gray region is the numerically excluded region, and the gray and black lines correspond to bounds obtained with $\Lambda = 10, 14, 18, 22$, with larger Λ giving the stronger bounds. The red dot denotes the free vector multiplet, and the black dot the rank one H_0 theory.

convergence with Λ is very fast.[30] Away from $r_0 = 1$, convergence as a function of Λ is slower, and the bounds presented here are clearly still quite suboptimal. One interesting question is whether the bound at $r_0 = \frac{6}{5}$ might converge to the $c \geqslant \frac{11}{30}$, with the rank one H_0 theory lying at the boundary. Using our methods, this would require a substantial increase in Λ. Similarly, at $r_0 = 2$ it seems possible that the bound may converge to the free vector value $c = \frac{1}{6}$.

4.7.2 Dimension bounds for non-chiral channel

In the allowed region of the (r_0, c) plane we can bound the dimension of the first unprotected, R-symmetry singlet, scalar operator appearing in the $\mathcal{E}_{r_0}\bar{\mathcal{E}}_{-r_0}$ OPE. Unitarity requires that such an operator have $\Delta \geqslant 2$. When $\Delta = 2$ the long multiplet sits at the unitarity bound

[30] A similar phenomenon was observed in the context of central charge minimization in $\mathcal{N} = 1$ SCFTs [35].

and decomposes into the stress tensor multiplet along with other short multiplets whose OPE coefficients vanish. In order to study local theories we should therefore add the superconformal block with $\Delta = 2$ to the problem by hand and then impose a gap so that the subsequent scalar operator has dimension strictly greater than two.

This situation presents two natural options. First, we can leave the coefficient of the stress tensor block unfixed and simply require that the functional be positive when acting upon it. This approach leads to upper bounds for $\Delta_0^\star$ that are valid for any value of c. Alternatively, we can fix the coefficient of the stress tensor block by hand and in doing so fix the value of the central charge. We can then extract bounds on $\Delta_0^\star$ as a function of c.

Let us make a brief comment about the free vector theory. When $r_0 = 1$ we know that there exists solution with $c = 1/6$, and in this solution there is no other scalar singlet block after the stress tensor multiplet at $\Delta = 2$. Thus at this point in the (r_0, c) plane our numerical procedure will never produce a nontrivial bound for the next operator, since any such bound arise from a functional that would rule out the free field solution.[31] To avoid this singular point in our searches we have studied regions of the (r_0, c) plane with $r_0 \geqslant 1.001$.

Arbitrary central charge

The results of the first strategy are displayed in Fig. 4.21. We find an upper bound on the dimension of the first scalar singlet as a function of $r_0 \geqslant 1.001$, with the bound at a given r_0 being valid for arbitrary values of c. Note that because there is no restriction on the value of c in this approach, there may be approximate solutions to crossing symmetry that influence this plot for which the value of c has been ruled out by (4.111). Indeed, we will find below that excluded central charge values are responsible for the local maximum at r_0 slightly less than two. For higher values of r_0, it seems plausible that the bounds will converge to the generalized free field theory solution indicated in the figure with a dashed line. The results for fixed values of the central charge will shed light on the features of this bound, so we postpone further discussion its shape to the next subsection.

The analogous chiral/anti-chiral OPE for $\mathcal{N} = 1$ SCFTs was considered in [35]. They exclusion curve obtained in that work for the dimension of the first unprotected scalar operator exhibited an interesting "kink". However, in that case a gap larger than two was being imposed, so any theory associated to the kink could not come from an $\mathcal{N} = 2$ theory with a stress tensor multiplet.

Fixed central charge

We turn next to operator dimension bounds for fixed central charge. The results for fixed Λ take the form of a function $\Delta_0^*(r_0, c)$ that is well defined for all points in the (r, c_0) plane that were not excluded by the numerical bounds of Section 4.7.1. This is displayed as a

[31] If for $r_0 = 1$ we do not to include the stress tensor block by hand, then the resulting bound on the first operator dimension would come be very close to two.

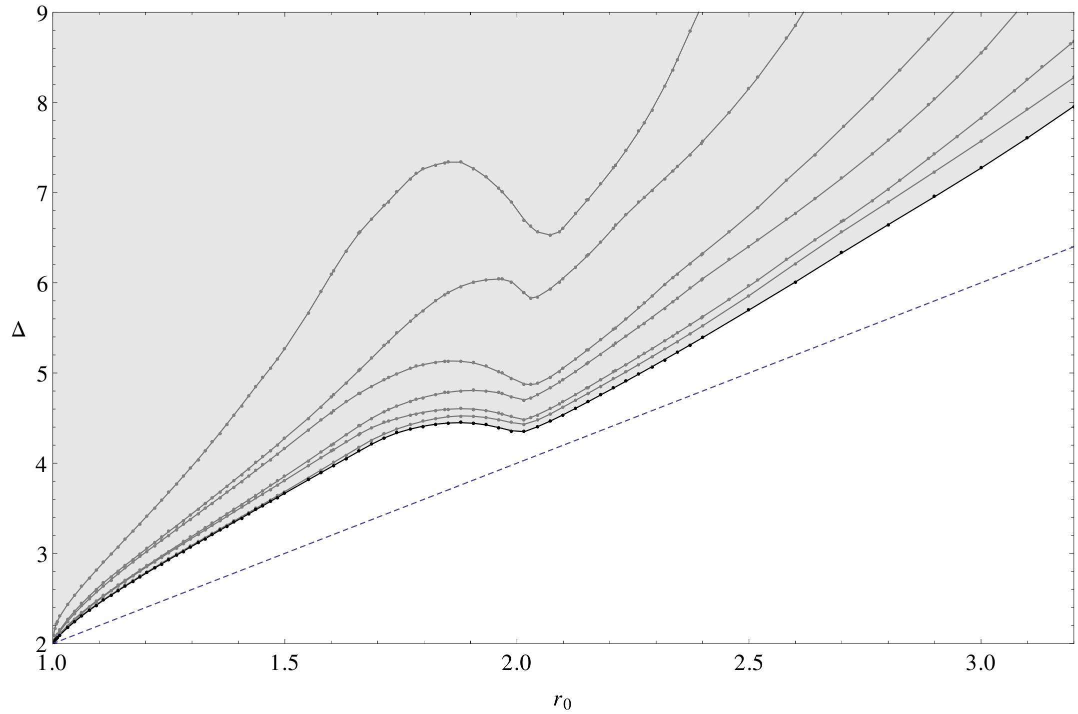


Figure 4.21: Bound on the first scalar in the $\mathcal{E}_{r_0} \times \bar{\mathcal{E}}_{-r_0}$ OPE as a function of r_0 for arbitrary central charge. The lines correspond to $\Lambda = 8, 10, \ldots, 20$, with the strongest bound being the black line. The excluded region is shaded. The dashed line corresponds to generalized free field theory solution, for which $\Delta = 2r_0$.

three-dimensional exclusion plot in Fig. 4.22, which corresponds to $\Lambda = 18$. The red line in Fig. 4.22 corresponds to the analytic bound (4.111), but since it may not hold in all circumstances we have extracted bounds even for points in the plane that violate it.

This exclusion surface was determined in a slightly unconventional manner. Rather than fixing c and r_0 and performing boolean searches to obtain a dimension bound, we fixed r_0 and imposed a gap in the scalar singlet channel and searched for upper and lower bounds on c consistent with that gap.[32] In this way we were able to find bounds for the whole of the plane with only a single numerical search required for each data point.

By taking constant central charge slices of this surface, a feature which is not apparent in Fig. 4.22 comes into view. Several such slices are superimposed in Fig. 4.23, where we the dimension bound is shown as a function of r_0 for various values of the central charge (including infinity). The results that are shown correspond to $\Lambda = 20$. Together with these

[32]Obtaining a lower bound for an OPE coefficient is possible as long as there is a gap between the superconformal block under consideration and the next operator, so this method can be used precisely for bounding the first scalar operator.

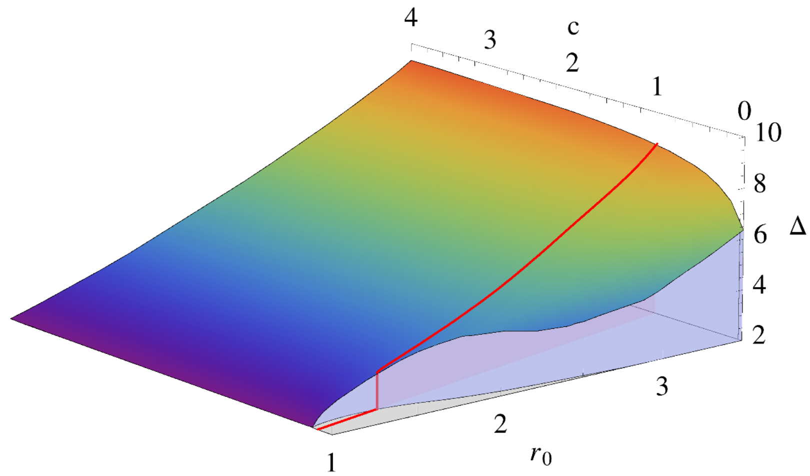


Figure 4.22: Bound on the first scalar in the $\mathcal{E}_{r_0} \times \bar{\mathcal{E}}_{-r_0}$ OPE as a function of the central charge c and dimension of the external operators r_0. These bounds are for $\Lambda = 18$, and are obtained by imposing a gap and minimizing/maximizing the central charge value after imposing a gap in the spectrum. The gray area in the figure is a copy of the excluded region from Fig. 4.20. The red line corresponds to the bound (4.111), and the excluded region, if $\mathcal{E}_{r_0}$ is to be a generator, is the one with smaller central charge.

bounds there is a blue dashed straight line at $\Delta = 2r_0$ corresponding to the generalized free field theory solution, and a thick dashed black line showing to the $\Lambda = 20$ dimension bound for arbitrary central charge. Since the latter line is the best possible bound without fixing the central charge, it envelopes all the lines for fixed c.

Although Fig. 4.23 could have been obtained by interpolating between the data points that define the three-dimensional plane in Fig. 4.22, we chose to revert to performing separate boolean searches for each point as this yields more precise results. As in the results reported in the previous sections, these boolean searches were performed by fixing the stress tensor coefficient in terms of c, imposing a gap in the spectrum and finding whether a functional exists as described in Section 4.5.

These bounds clearly have two qualitatively different regimes, depending on whether r_0 is greater or less than two. For $r_0 > 2$ the bound gets weaker (increases) as the central charge is increased. The weakest bound is just the $c = \infty$ line, and it coincides with the bound with unspecified central charge. In this region convergence is relatively slow, and so it is hard to guess where the bound will end up as the cutoff is lifted. Of course we cannot exclude known solutions, so for $c = \infty$ the bound will not be able to cross the generalized free field theory line. More trivially, for $c = \frac{1}{6}$ the bound will have to allow the points $r_0 = 2n$, $\Delta = 4$

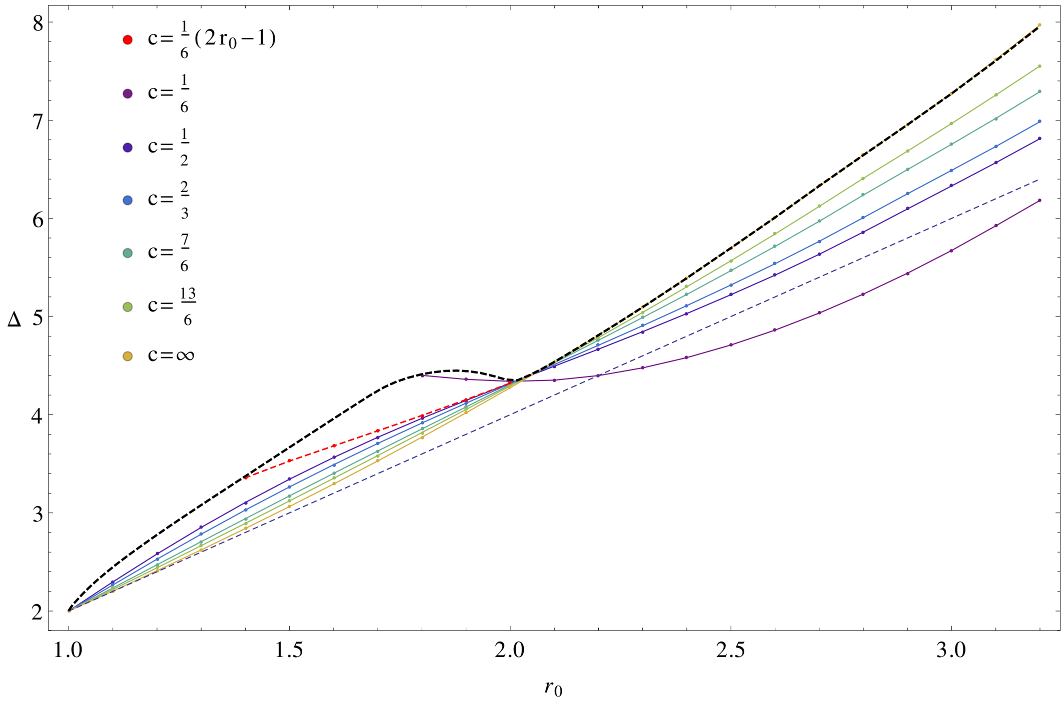


Figure 4.23: Bound on the first unprotected scalar in the $\mathcal{E}_{r_0}\bar{\mathcal{E}}_{-r_0}$ OPE as a function of r_0 for several different values of the central charge, obtained with $\Lambda = 20$. The dashed blue line corresponds to the generalized free field theory solution $\Delta = 2r_0$, the thick dashed black line is the same as in Fig. 4.21, and the red line segment is the bound obtained for the central charge which saturates (4.111). If the central charge of a theory is known then it must correspond to a point below the curve corresponding to that central charge. If the central charge is not known and the theory has a freely generated Coulomb branch, then equation (4.111) together with our numerics dictate that the theory must lie below both the black line and below the red line segment. If we do not know either c or whether the Coulomb branch is freely generated then the theory must still lie below the black curve.

for $n \in \mathbb{N}$.

The point $r_0 = 2$ is particularly interesting. Here the lines for all central charges converge at a value that is close to, and seems to be approaching, $\Delta = 4$. The absence of a stronger bound can be explained by the existence of a one-parameter family of four-point functions – constructed by taking a linear combination of the free field solution and the generalized free field solution – which can realize any $c \geqslant 1/6$ and for which the first scalar operator always has dimension four. However, recall that theories with a chiral operator with $r_0 = 2$ necessarily have a conformal manifold. If these bounds converge to $\Delta = 4$, then it would follow that at any point on any conformal manifold there must be a relevant, unprotected operator with nonzero coefficient in the chiral/anti-chiral OPE. It would be interesting to check this at low order(s) in perturbation theory.

For $r_0 < 2$ the picture is reversed. The bound for infinite central charge still appears to be approaching the generalized free field theory value, but the bounds now grow stronger (decrease) as a function of the central charges. The solution to crossing symmetry along the black line corresponds to the lowest allowed value of the central charge consistent with crossing, which is precisely the bound shown in Fig. 4.20. For example, the $c = 1/6$ line ends on the black curve at the same value of r_0 where 4.20 begins to exclude the value $c = 1/6$, and for even smaller r_0 and fixed c there is no unitary solution anymore.

If $\mathcal{E}_{r_0}$ is a Coulomb branch generator the central charge cannot be arbitrarily small, and in particular must satisfy (4.111). This renders part of the black curve with $r_0 < 2$ unphysical, since it corresponds to solutions with a central charge that violates (4.111). We can correct this by assuming the central charge to be at least $\frac{1}{6}(2r_0 - 1)$. We then obtain a correction to the black curve that is shown in Fig. 4.23 as a red dashed line. Any unitary $\mathcal{N} = 2$ SCFT with a freely generated Coulomb branch must now lie below both the black curve and, because of (4.111), also below the red line segment. This improvement removes the local maximum from Fig. 4.21.

4.7.3 $\mathcal{E}_{2r}$ OPE coefficient bounds

In the chiral OPE channel it is natural to look for constraints on the (squared) OPE coefficient $\lambda^2_{\mathcal{E}_{2r_0}}$ of the $\mathcal{E}_{2r_0}$ multiplet. The conformal block associated to the exchange of this multiplet is given by $G^{(0)}_{2r_0}(z,\bar{z})$, while the next multiplet appearing in the chiral channel has $G^{(0)}_{2r_0+2}(z,\bar{z})$ as its conformal block. Thus the $\mathcal{E}_{2r_0}$ contribution is isolated, and we can look for both upper and lower bounds on its coefficient. These bounds are displayed in Fig. 4.24 for $\Lambda = 22$. Physical theories must lie between the two blue/red sheets. The vertical "wall" corresponds to $c = \frac{1}{6}(2r_0 - 1)$.

As a sanity check, we can compare these numerical bounds to some known theories. The free vector multiplet gives a solution to crossing symmetry with $r_0 = 1$ and $c = \frac{1}{6}$, and from the decomposition (4.88) we can see that $\lambda^2_{\mathcal{E}_{2r_0}} = 2$. This ends up being consistent with the numerical bounds, since at this point both the lower and upper bound are very close to two. Similarly, for infinite c we find the generalized free field solution with an OPE coefficient that is also equal to two - again consistent with the numerical bounds.

It is interesting to observe that the lower bound on this OPE coefficient is strictly positive in a large region of the (r_0, c) plane. In this region, these bounds rigorously exclude the possibility of Coulomb branch chiral ring relations of the form $\mathcal{E}_{r_0}\mathcal{E}_{r_0} \sim 0$. The region of the plane where the lower bound is positive is displayed in Fig. 4.25. It is clear that the bound will improve substantially more at larger Λ.

In interpreting Figs. 4.24, 4.25 there is an important subtlety. In obtaining these bounds we have fixed c to a given value, which corresponds to inserting the superconformal stress

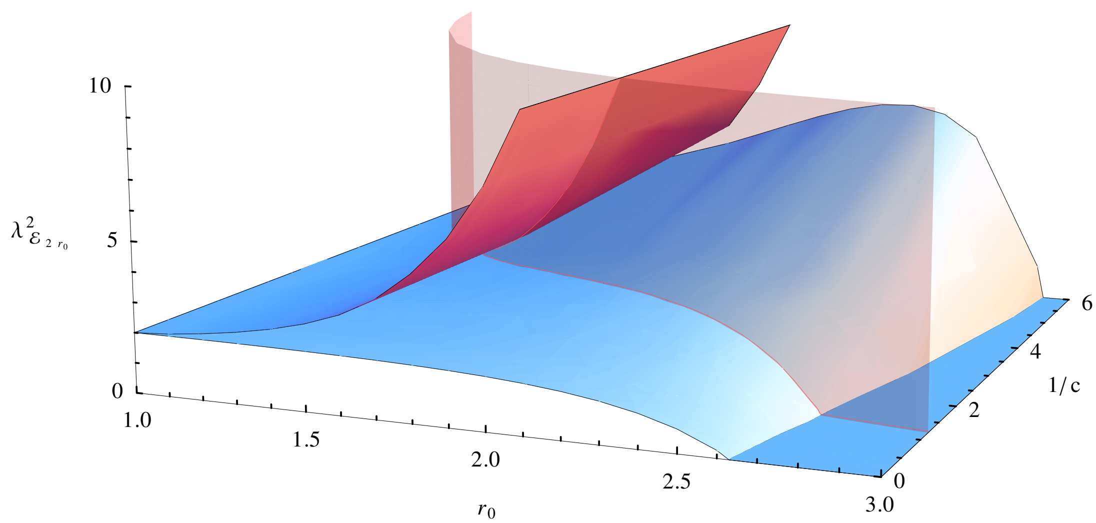


Figure 4.24: Upper and lower bounds on the OPE coefficient squared of $\mathcal{E}_{2r_0}$ as a function of r_0 and $\frac{1}{c}$, corresponding to a cutoff $\Lambda = 22$. The vertical red "wall" corresponds to the bound (4.111), and the excluded region if $\mathcal{E}_{r_0}$ is a Coulomb branch generator is the one with smaller central charge.

tensor block with a fixed coefficient in the appropriate channel. However we have also allowed for arbitrary superconformal blocks for long multiplets in the same channel, both at and above the unitarity bound. A long block at the unitarity bound however reduces exactly to the stress tensor block and can therefore mimic the effect of the stress tensor. Since the coefficient of the stress tensor block is proportional to $1/c$, the bounds obtained for a given value of c are also valid *for all smaller central charges.* In other words, when increasing c the bounds can never improve - instead they either worsen or stay constant. In future searches this issue could be circumvented by imposing a gap in the scalar channel.

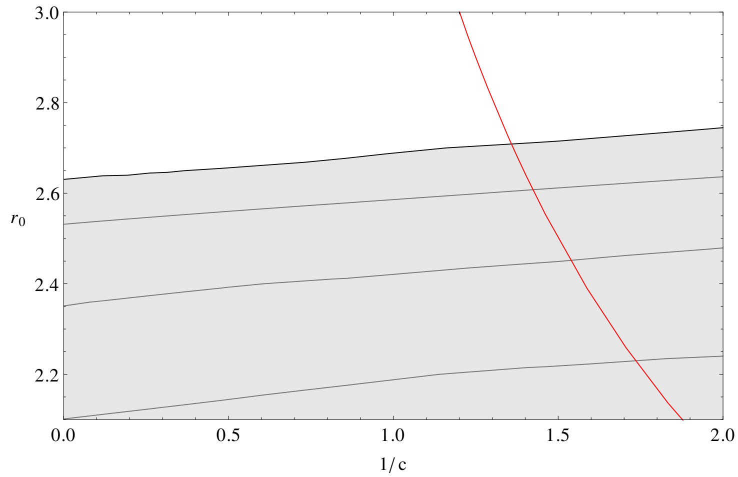


Figure 4.25: Region where the lower bound on the OPE coefficient squared of $\mathcal{E}_{2r_0}$ is strictly positive as a function of r_0 and $\frac{1}{c}$, for Λ varying from 10 to 22 in steps of four. The shading indicates the OPE coefficient squared is positive in that region. The red line corresponds to the unitarity bound (4.111), and the excluded region (if $\mathcal{E}_{r_0}$ is a Coulomb branch generator) is to the right of the line. Note that these results are approximate, as this plot is obtained by an interpolation procedure from results like those shown in Fig. 4.24. The slight wiggles in the lines are likely due to small errors introduced by this procedure.

OPE coefficient bounds and the Zamolodchikov metric

The slice $r_0 = 2$ of Fig. 4.24 is of special interest because of its relation to the curvature of the Zamolodchikov metric on the conformal manifold [173, 174]. Namely, consider an $\mathcal{N} = 2$ SCFT with a moduli space $\mathcal{M}$ of exactly marginal deformations. The different marginal deformations at a given point on $\mathcal{M}$ are the top components of $\mathcal{E}_2$ multiplets (and their complex conjugates) whose superconformal primary we will denote as ϕ_a, $a = 1, \ldots, \dim_{\mathbb{C}}(\mathcal{M})$. The Zamolodchikov metric $g_{a\bar{b}}$ on $\mathcal{M}$ is determined by the two-point functions of these primaries,[33]

$$\langle \phi_a(x) \bar{\phi}_{\bar{b}}(0) \rangle = \frac{g_{a\bar{b}}}{x^4} \ . \tag{4.112}$$

Unit normalizing these operators corresponds to choosing local holomorphic coordinates on $\mathcal{M}$ such that $g_{a\bar{b}} = \delta_{a\bar{b}}$ at the point of interest. In these coordinates, the only non-vanishing four-point function involving the ϕ_a and their complex conjugates is given by

$$\langle \phi_a(x_1) \phi_b(x_2) \bar{\phi}_{\bar{c}}(x_3) \bar{\phi}_{\bar{d}}(x_4) \rangle \ . \tag{4.113}$$

[33] In [174] this is the "metric" written as $g_{a\bar{b}}$, which differs from the actual metric $G_{a\bar{b}}$ studied in that by a factor 192.

The OPE of $\phi_a(x_1)$ and $\phi_b(x_2)$ is regular and correspondingly the first conformal block in the chiral channel for this four-point function is a dimension four scalar that is the superconformal primary of an $\mathcal{E}_4$ multiplet. According to Eqn. (3.13) of [173], the coefficient for this superconformal block is given by

$$\mu_{\mathcal{E}_4\, ab\bar{c}\bar{d}} = -R_{a\bar{c}b\bar{d}} + \delta_{a\bar{c}}\delta_{b\bar{d}} + \delta_{b\bar{c}}\delta_{a\bar{d}} \ . \tag{4.114}$$

where $R_{a\bar{c}b\bar{d}}$ is the Riemann curvature tensor (in the aforementioned distinguished coordinates) of the Zamolodchikov metric on $\mathcal{M}$.[34]

We have obtained upper and lower bounds for the OPE coefficient in the particular four-point function with identical operators, $a = b = c = d$. In that case we have

$$\lambda^2_{\mathcal{E}_4} = \mu_{\mathcal{E}_3\, aa\bar{a}\bar{a}} = 2 - R_{a\bar{a}a\bar{a}} \ . \tag{4.115}$$

When $\dim_{\mathbb{C}}(\mathcal{M}) = 1$, this expression simplifies to

$$\lambda^2_{\mathcal{E}_4} = 2 - \frac{1}{2}R \ , \tag{4.116}$$

with R the Ricci scalar of $g_{a\bar{a}}$. The bounds for $\lambda^2_{\mathcal{E}_4}$ can therefore be interpreted as bounds for the scalar curvature of one-dimensional conformal manifolds.

The $r_0 = 2$ slice of Fig. 4.24 is shown in Fig. 4.26, with the excluded regions shaded in gray. The bounds for lower values of Λ are also shown to indicate the cutoff-dependence. Inside the allowed region we highlighted several points and loci that correspond to known theories. The computation of $\lambda^2_{\mathcal{E}_4}$ for these theories is reviewed in Appendix I.

Even at infinite Λ, the the upper and lower bounds will not be able to penetrate beyond the dashed blue lines. The reason for this is as follows. In the theory of n free vector multiplets one finds a chiral four-point function with $r_0 = 2$ for which

$$\lambda^2_{\mathcal{E}_4} = 2 + \frac{2}{3c} \ , \qquad\qquad \frac{1}{c} < 6 \ . \tag{4.117}$$

This is the lower dashed line in Fig. 4.26. The upper horizontal dashed line, on the other hand, is simply given by $\lambda^2_{\mathcal{E}_4} = 6$, which is the value for the solution corresponding to a single free vector multiplet at $c = 1/6$. The numerical upper bound cannot pass this line because of the aforementioned fact that by design the bound is a non-increasing function of $1/c$.

From the dependence of the bounds on Λ it seems natural to expect that they will eventually converge to the dashed blue lines. If this were to happen, then the purely diagonal

[34] Recall that the Zamolodchikov metric is Kähler and therefore $R_{a\bar{c}b\bar{d}}$ is symmetric under exchange of a and b (as well as exchange of $\bar{c}$ and $\bar{d}$). This is required by the braiding relation of the four-point function.

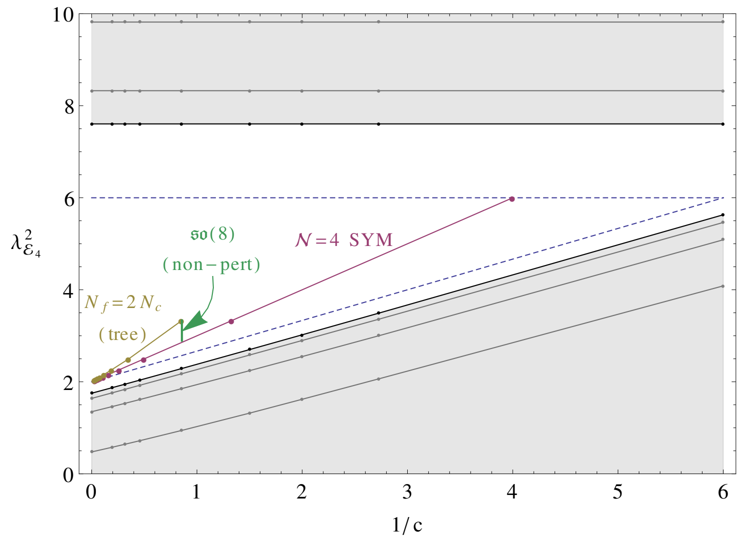


Figure 4.26: Upper and lower bounds on $\lambda^2_{\mathcal{E}_4}$ as a function of the central charge c. Shaded regions are excluded by our numerics, with Λ ranging from 10 to 22 in steps of four (the upper bound for $\Lambda = 10$ is outside the plotted region). The dotted lines are the best possible value of the bounds as dictated by the free vector multiplet solution, and it seems likely that our bounds will converge to these values. We highlighted the known values of the coefficients for $\mathcal{N} = 4$ SYM theories (which are protected), the $\mathcal{N} = 2$ SCQCD theories with gauge group $SU(N_c)$ and $N_f = 2N_c$ fundamental flavors (tree level values only), and finally the special case of $SU(2)$ SCQCD which we call the $\mathfrak{so}(8)$ theory. The line in the latter case shows the range of values that $\lambda^2_{\mathcal{E}_4}$ takes as a function of the exactly marginal coupling, cf. the computation in Appendix I. The individual dots in the colored lines correspond to gauge groups SU(N) (with $N \geqslant 2$), plus the $U(1)$ theory at $c = 1/4$ for $\mathcal{N} = 4$ SYM.

components of the Riemann tensor would have to obey the following bound:

$$-4 \leqslant R_{a\bar{a}a\bar{a}} \leqslant -\frac{2}{3c} \ . \tag{4.118}$$

In particular, for theories with one-dimensional conformal manifolds crossing symmetry appears to dictate that their scalar curvature is always negative. To see if this is also true for higher-dimensional moduli spaces a bound for $R_{a\bar{a}b\bar{b}}$ will be necessary. We plan to investigate the corresponding four-point function in the near future.

4.8 Conclusions

The abstract operator viewpoint offers a unified language for the description of both Lagrangian and non-Lagrangian CFTs. It has also become the entry point for powerful numerical studies in the style of [6]. In this chapter we have advocated for the utility of this viewpoint in studying $\mathcal{N} = 2$ superconformal field theories. We have highlighted the interplay between superconformal representation theory and interesting physics in these theories, and we identified three types of distinguished representations of particular physical interest. Our numerical investigations focused on the four-point functions of two types of multiplets, $\hat{\mathcal{B}}_1$ and $\mathcal{E}_r$. The result was a plethora of numerical unitarity bounds for $\mathcal{N} = 2$ SCFTs involving central charges, operator dimensions, and OPE coefficients.

Our results reveal a number of interesting details about $\mathcal{N} = 2$ superconformal field theories, some of which are new numerical bounds for its observables and some of which make contact with other known facts. For example, we have rigorously established that Coulomb branch chiral operators $\mathcal{E}_r$ with sufficiently low values of r cannot satisfy a certain type of chiral ring relations, and that theories with $\mathfrak{su}(2)$ flavor symmetry must have at least one flavor singlet multiplet of type $\hat{\mathcal{C}}_{1,\ell=1}$ and one flavor triplet multiplet of type $\hat{\mathcal{C}}_{1,\ell=0}$. The latter follows from our numerical exclusion of theories with $k = 2/3$ for which these multiples decouple from the $\hat{\mathcal{B}}_1 \times \hat{\mathcal{B}}_1$ OPE. Similarly, if our extrapolations are on track then we should be able to rule out one complex dimensional conformal manifolds like a smooth two-sphere, for which the Euler characteristic is not compatible with the sign of the curvature of the Zamolodchikov metric. In the future it would be very interesting to look for analytic arguments for some of these statements and to understand if the connection with associativity of the operator algebra can be made analytically tractable. More generally, the combination of both analytic and numerical methods appears to be the most promising way to constrain and explore the landscape of $\mathcal{N} = 2$ superconformal field theories.

Throughout this work, we have observed a strong dependence on the cutoff Λ that determines the size of the numerical problem being investigated. In other words, the bounds derived here – though valid – do not appear to be close to their optimal value. This is not for lack of trying: the strength of our numerical methods is completely on par with (and in some cases exceeds) the state of the art in almost all of the present literature. The strong cutoff dependence therefore appears to be an intrinsic property of bounds extracted from our specific four-point functions. In the near future we are hopeful that better numerical software tailored to the problem at hand will allow for searches with much greater reach and higher precision. Even then, however, it is not clear that the bounds presented here should be expected converge to some limiting value. For example, if the extrapolation shown in Fig. 4.4 is more or less correct, then a cutoff Λ of order $O(100)$ will be necessary to reach a value of the lower bound that is within a few percent of the asymptotic value. The corresponding search space dimension would have to be a factor ten higher than the ones used in this work. Until such methods become computationally feasible, we are stuck with the sorts of extrapolation presented in this work if we want a rough guess for the limiting value of a bound.

With additional work and the development of improved numerical methods, we see a number attractive directions for future work.

Additional correlation functions

An obvious and interesting avenue is to analyze a more diverse collection of four-point functions. The four-point functions of $\hat{\mathcal{B}}_1$ multiplets with a flavor symmetry algebra other than $\mathfrak{su}(2)$ or $\mathfrak{e}(6)$ is a natural choice that would involve very little groundwork on top of what we have reported here. Perhaps the most important extension will be to study the four-point functions of operators in the stress-tensor multiplet. In this case there are several natural candidates.

Recall that the superconformal primary of this multiplet is a dimension two scalar, and among its descendants we find the R-symmetry currents and the stress tensor. The first step towards bootstrapping any operators in this multiplet is to determine the corresponding selection rules and superconformal blocks, and this prerequisite has not yet been fulfilled. This analysis seems quite complicated for the four-point function of the stress tensor multiplet in superspace, but should be tractable for just the four-point function of the superconformal primary. An interesting case of intermediate complexity is the four-point function of $SU(2)_R$ currents, for which the chiral algebra data fixes a large number of OPE coefficients. From either of these four-point functions one may obtain bounds on the a anomaly coefficient, which is a piece of data that was conspicuously absent from the four-point functions considered in this chapter.

Multiple correlation functions

The bounds reported here are valid and must be obeyed by physical theories, but they were derived as a consequence of crossing symmetry for individual correlators. In an honest CFT, crossing symmetry must hold in *all* possible correlators. The simultaneous investigation of multiple correlators in a single numerical program is then a natural next step. The pioneering work of this type was [7], where three-dimensional non-supersymmetric CFTs were studied. With minimal additional assumptions, the mixed correlator approach has the potential to rule out spurious solutions to single-correlator crossing symmetry that have no place in a consistent SCFT. In an optimistic scenario, this would also rule out (presumably spurious) linear combinations of solutions that may saturate the single-correlator bounds for large Λ. In the $\mathcal{N} = 2$ setting one should consider all mixed four-point functions containing a given subset of Coulomb or Higgs branch chiral ring generators. For the Higgs branch chiral ring, the structure of many relevant four-point functions and superconformal blocks have already been worked out in [42, 43].

Theory-specific analysis

In this exploratory work we have taken as general an approach as possible to the $\mathcal{N}=2$ superconformal bootstrap program. In particular, we have avoided making assumptions that might not be shared by all theories. A complementary strategy is to try to specify a particular theory of interest and "zoom in" on that theory in the space of SCFTs. By including as much information as possible about a theory of interest, one hopes to effectively isolate the corresponding solution to crossing symmetry at a boundary of the numerically allowed region. On can then begin to solve that theory at the level of the spectrum of local operators and OPE coefficients.

The numerical results obtained here do not offer much guidance in choosing between known $\mathcal{N}=2$ theories, mostly because of the absence of "kinks" in the bounds. Some natural candidates still present themselves upon further thought. A particularly elegant theory that we think deserves further study is $SU(2)$ SCQCD with $N_f=4$. For this theory the exact OPE coefficients derived in [173] make it possible to use the exactly marginal coupling constant τ as an *input* variable, at least for the four-point function of $\mathcal{E}_r$ multiplets. This opens the way towards exploring the contours of a nontrivial conformal manifold by deriving coupling constant-dependent bounds. This was not possible in the work of [36] on $\mathcal{N}=4$ SYM because in that case the known OPE coefficients are constant on the conformal manifold. The $SU(2)$ SCQCD also enjoys an $\mathfrak{so}(8)$ flavor symmetry, and it would be interesting to compare bounds for the corresponding $\hat{\mathcal{B}}_1$ multiplet with those of the $\mathcal{E}_r$ multiplets. More precisely, in [36] it was conjectured that for certain $\mathcal{N}=4$ SYM theories the coupling-independent bounds were saturated at self-dual values of the coupling. If one can achieve reasonable convergence, it may be possible to check the equivalent conjecture for this theory.

Perhaps the most obvious candidate for targeted bootstrap analysis is the $\mathfrak{e}_6$ theory of Minahan and Nemeschansky [12], which lies at the intersection of two lines where analytic bounds derived from the two-dimensional chiral algebra are saturated. The current numerical analysis does not appear to be extremely constraining, but we expect the more refined strategies that we have mentioned to yield stronger results.

Chapter 5

The $(2,0)$ superconformal bootstrap

The contents of this chapter will appear in [4]: "The $(2,0)$ superconformal bootstrap" C. Beem, M. Lemos, L. Rastelli and B. C. van Rees.

5.1 Introduction and conclusions

In this chapter we present preliminary results on the superconformal bootstrap program for the maximally supersymmetric theory in six dimensions. The analysis follows very similarly to the one of the previous chapter.

In the spirit of section 4.2 we want to organize the operators in representations of the six-dimensional superconformal algebra $\mathfrak{osp}(8^\star|4)$, whose maximal bosonic subalgebra is the conformal algebra times the R-symmetry algebra ($\mathfrak{so}(5)_R$). The classification of these representations has been done in [21, 154, 175, 176] and reviewed and summarized in [28]. Here we simply summarize the the multiplets relevant for our purposes, following their notation for the series of multiplet names. We also denote between square brackets the Dynkin labels, $[d_1, d_2]$, of the $\mathfrak{so}(5)_R$ representation[1] of the superconformal primary of the multiplet. Moreover since we are considering the OPE of scalar operators, only operators in the symmetric-traceless representations can appear (with Dynkin labels $[0,\ell,0]$ of $\mathfrak{su}^\star(4)$) and we refer to the spin ℓ of the operators. Short multiplets obey one of the following shortening conditions, which fixes their dimensions in terms of the remaining quantum numbers:

$$
\begin{aligned}
&\mathcal{A}: && \Delta = 6+\ell+2(d_1+d_2)\,, && \ell \geqslant 0\,,\\
&\mathcal{B}: && \Delta = 4+\ell+2(d_1+d_2)\,, && \ell \geqslant 0\,,\\
&\mathcal{C}: && \Delta = 2+2(d_1+d_2)\,, && \ell = 0\,,\\
&\mathcal{D}: && \Delta = 2(d_1+d_2)\,, && \ell = 0\,.
\end{aligned}
\tag{5.1}
$$

[1] We use $\mathfrak{so}(5)_R$ conventions for the Dynkin labels, so the $\mathbf{5}$ has Dynkin labels $[1,0]$. We note that the order of our Dynkin labels is reversed with respect to [42].

The multiplets of the $\mathcal{D}$ series are $\frac{1}{4}$ BPS, or $\frac{1}{2}$ BPS if the second Dynkin labels of the superconformal primary is zero. All in all we specify the superconformal multiplets by their names which encode which shortening condition the multiplets obey [28], together with the spin and $\mathfrak{so}(5)_R$ representation of the superconformal primary. Long multiplets, which obey no shortening condition, are denoted by $\mathcal{L}$, with the dimension, spin and $\mathfrak{so}(5)_R$ representation of the superconformal primary being indicated. Unitarity requires their dimension to obey

$$\mathcal{L} \ : \qquad \Delta > 6 + \ell + 2(d_1 + d_2)\,, \quad \ell \geqslant 0\,. \tag{5.2}$$

The free theory in six dimensions and with $\mathcal{N} = (2,0)$ consists of the free tensor multiplet. This is a short multiplet, $\mathcal{D}[1,0]$, which has as the superconformal primary a dimension two scalar ϕ^I, where I is a $\mathbf{5}$ index of $\mathfrak{so}(5)_R$. Its superconformal descendants consist of two Weyl fermions, and a two-form with self-dual field strength. The stress tensor of the theory belongs to a multiplet $\mathcal{D}[2,0]$ whose superconformal primary is a dimension four scalar in the $\mathbf{14}$ of $\mathfrak{so}(5)_R$. In terms of free fields it is given by $\Phi^{IJ} = \phi^{(I}\phi^{J)} - \frac{1}{5}\delta^{IJ}\phi^K\phi^K$. This expression is for the theory of a free abelian tensor multiplet, but if one wants an interacting theory there is no such description. However that should not prevent us from applying the bootstrap program.

We take the SCFT to be defined by its local operator algebra and consider an operator which is guaranteed to be present in the theory: the stress tensor, or rather the superconformal primary of the multiplet it belongs to, Φ^{IJ}.

We start by analysing in section 5.2 the four-point function of the superconformal primary of the stress tensor multiplet, and its superconformal partial wave expansion in section 5.3. As before superconformal Ward identities strongly constrain the form of this four-point function, with all five channels corresponding to the various $\mathfrak{so}(5)_R$ representations appearing in the OPE of these operators being determined simply by two functions $h(z)$ and $a(z,\bar{z})$. The crossing equations then split into two sets, one only involving the former function, which can be solved exactly, while the other involves both functions and must be approached numerically. The crossing equation for the meromorphic function $h(z)$ can be solved exactly in terms of the central charge c.[2] Just as before, to obtain this solution we must assume there are no higher spin currents (that is conserved currents of spin larger than two, which in six dimensions have twist four). The situation is different here in respect to the previous chapter because the multiplets containing higher spin currents ($\mathcal{B}[0,0]_\ell$) are separated from the rest of the multiplets by a gap. Therefore they cannot be mimicked by a long multiplet at the unitarity bound, and the assumptions of the absence of higher spin currents implies we are excluding by hand the free theory. Another major difference with respect to the previous chapter is that this time not all OPE coefficients of protected operators can be fixed from this meromorphic $h(z)$ function. As such when setting up the numerical bootstrap we will be looking not only for bounds on the dimensions of unprotected operators, but also on the

[2] In six dimensions there are four different anomaly coefficients. What we call the c central charge corresponds to the one appearing in the stress tensor two point function.

OPE coefficients of protected operators, whose dimensions are fixed, but OPE coefficients we do not know. All the numerical analysis is done in section 5.4, where we also obtain lower bounds on the central charge c of interacting theories. We normalized the central charges such that it is one for the free theory, and as such the theory of type $\mathfrak{g}$ will have central charges [28, 177, 178]

$$c_{\mathfrak{g}} = 4d_{\mathfrak{g}}h^{\vee}_{\mathfrak{g}} + r_{\mathfrak{g}}\,, \tag{5.3}$$

where $d_{\mathfrak{g}}$, $h^{\vee}_{\mathfrak{g}}$ and $r_{\mathfrak{g}}$ are respectively the dimension, dual Coxeter number and rank of $\mathfrak{g}$.

In the large c limit the four-point we are studying has been computed, using supergravity on $AdS_7 \times S^4$ [26, 27], and our bounds appear to be saturated by this solution.

As announced in the introduction we find evidence that the A_1 theory is the unique theory with central charge $c = 25$, and that this is the lowest possible central charge for an interacting theory. Taking this conjecture as a working assumption, we take a first step towards the bootstrapping of the A_1 theory, obtaining bounds on its operator dimensions and OPE coefficients that we expect to be saturated by the actual physical theory. A feature that keeps plaguing our results is the slow convergence of the bounds obtained as the truncation of the crossing equations is increased. However we will see that if the right assumptions are made regarding the A_1 theory our bounds converge faster, providing hope that this theory is within reach of the numerical bootstrap.

Throughout the numerical analysis we will make the case that our bounds are saturated by the physical solutions both as $c \to \infty$ and at $c = 25$. It is then plausible that for the central charges corresponding to the remaining theories our bounds are also saturated by physical theories, and not polluted by some spurious solution to this crossing equation.

If that is not the case then one might need to consider more than one correlator to approach the remaining theories, and attempt to remove spurious solutions in this way. A very natural set of correlators to take would be the generators of the $\frac{1}{2}$-BPS chiral ring, to which the stress tensor multiplet belongs to. For the theory of type $\mathfrak{g}$ these are in correspondence with the Casimirs of $\mathfrak{g}$: they are $\mathcal{D}[n_i, 0]$, with n_i the degree of the i^{th} Casimir of $\mathfrak{g}$ (see, *e.g.*, [28, 179–181]). It might even be the case that by taking the appropriate set of generators of the $\frac{1}{2}$ BPS chiral ring one can find similar uniqueness statements for the remaining theories.

5.2 The four-point function of stress tensor multiplets

As explained in the previous section, we are interested in studying the four-point function of the stress tensor multiplet superconformal primary. These are $\frac{1}{2}$-BPS scalar operators of dimension four, transforming in the **14** of the $\mathfrak{so}(5)_R$. We denote them as $\Phi^{IJ}(x) = \Phi^{\{IJ\}}(x)$ with I, J fundamental $\mathfrak{so}(5)_R$ indices, and the brackets meaning it is a symmetric traceless representation. A convenient way to deal with the $\mathfrak{so}(5)_R$ indices is to contract them with auxiliary vectors Y^I and define

$$\Phi(x,Y) := \Phi^{IJ}(x) Y_I Y_J \,. \tag{5.4}$$

The symmetric tracelessness constraint then translates into

$$Y^I Y_I = 0 \,. \tag{5.5}$$

For example, in this language the two-point function of $\Phi(x,Y)$ takes the form

$$\langle \Phi(x_1,Y_1)\Phi(x_2,Y_2)\rangle = \frac{4(Y_1\cdot Y_2)^2}{x_{12}^8} \,, \tag{5.6}$$

where the normalization is useful later. Following [42] one can take the five-dimensional null vector and solve the null constraint using

$$Y^I = (y^i, \frac{1}{2}(1-y^i y_i), \frac{i}{2}(1+y^i y_i)) \,, \tag{5.7}$$

with y^i a three-dimensional vector. We may then write

$$\langle \Phi(x_1,y_1)\Phi(x_2,y_2)\rangle = \frac{y_{12}^4}{x_{12}^8} \,, \tag{5.8}$$

where, as common for the purposes of the conformal bootstrap we normalize the two-point function to one, which was the reason we added the factor of four in (5.6).

5.2.1 Structure of the four-point function

Let us now examine the four-point function of $\Phi(x,y)$ in more detail. Using the conformal Ward identities one finds that this correlator can be written as [42]

$$\langle \Phi(x_1,y_1)\Phi(x_2,y_2)\Phi(x_3,y_3)\Phi(x_4,y_4)\rangle = \frac{y_{12}^4 y_{34}^4}{x_{12}^8 x_{34}^8} G(z,\bar{z};\alpha,\bar{\alpha}) \,, \tag{5.9}$$

where z and $\bar{z}$ are related to the standard cross-ratios

$$u := \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2} =: z\bar{z} \,, \qquad v := \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2} =: (1-z)(1-\bar{z}) \,, \tag{5.10}$$

and in a similar spirit α and $\bar{\alpha}$ are related to "cross-ratios" in the auxiliary variables Y^I through

$$\frac{1}{\alpha\bar{\alpha}} := \frac{y_{12}^2 y_{34}^2}{y_{13}^2 y_{24}^2} \,, \qquad \frac{(\alpha-1)(\bar{\alpha}-1)}{\alpha\bar{\alpha}} := \frac{y_{14}^2 y_{23}^2}{y_{13}^2 y_{24}^2} \,. \tag{5.11}$$

Although it is not manifest in this notation, the dependence of the full correlator on the Y^I is of course polynomial by construction. This polynomial dependence encodes, in an index free way, the various $\mathfrak{so}(5)_R$ representations appearing in the tensor product of two **14**. The dependence of the function $G(z, \bar{z}; \alpha, \bar{\alpha})$ on α $\bar{\alpha}$ is therefore very constrained, as it must admit a decomposition in functions of u and v that capture the contributions of operators in a fixed $\mathfrak{so}(5)_R$ representation. Furthermore, each of these channels is not completely unconstrained, as the R-symmetry is part of the superconformal algebra, and therefore constrained by superconformal invariance.

The constraints of *superconformal* invariance have been analyzed thoroughly in [42]. The end result is conveniently summarized as

$$G(z, \bar{z}; \alpha, \bar{\alpha}) = u^4 \Delta_2 (z\alpha - 1)(z\bar{\alpha} - 1)(\bar{z}\alpha - 1)(\bar{z}\bar{\alpha} - 1) a(z, \bar{z}) + z^2 \bar{z}^2 \mathcal{H}_1^{(2)}(z, \bar{z}; \alpha, \bar{\alpha}) \,. \quad (5.12)$$

Let us explain the notation. First of all the operator Δ_2 is given by

$$\Delta_2 f(z, \bar{z}) := D_2 u f(z, \bar{z}) := \left(\frac{\partial^2}{\partial z \partial \bar{z}} - \frac{2}{z - \bar{z}} \left(\frac{\partial}{\partial z} - \frac{\partial}{\partial \bar{z}} \right) \right) z\bar{z} f(z, \bar{z}) \,, \quad (5.13)$$

where we also defined an operator D_2. There are no further constraints on the function $a(z, \bar{z})$ from the superconformal Ward identities whereas the function $\mathcal{H}_1^{(2)}(z, \bar{z}; \alpha, \bar{\alpha})$ takes the form [42]

$$\mathcal{H}_1^{(2)}(z, \bar{z}; \alpha, \bar{\alpha}) = D_2 \frac{(z\alpha - 1)(z\bar{\alpha} - 1)h(z) - (\bar{z}\alpha - 1)(\bar{z}\bar{\alpha} - 1)h(\bar{z})}{z - \bar{z}} \,, \quad (5.14)$$

for some *single-variable* function $h(\cdot)$ whose form is not directly constrained by the superconformal Ward identities.

We note that in this language the twist that leads to the chiral algebra obtained in [28] corresponds to (up to a proportionality constant) setting $y_{12} = \bar{z}_{12}$ where $(z, \bar{z})$ are the complex coordinates of the two-plane to which the operators are constrained. (Equivalently in terms of the cross-ratios we set $\alpha = \bar{\alpha} = 1/\bar{z}$). We then obtain

$$\langle \hat{\Phi}(z_1) \hat{\Phi}(z_2) \rangle = \frac{1}{z_{12}^4} \,, \quad (5.15)$$

so the twisted operator $\hat{\Phi}(z)$ has dimension two. Similarly the four-point function becomes

$$\langle \hat{\Phi}(z_1) \hat{\Phi}(z_2) \hat{\Phi}(z_3) \hat{\Phi}(z_4) \rangle = \frac{-z^2 h'(z)}{z_{12}^4 z_{34}^4} \,. \quad (5.16)$$

As expected, the dependence on the two-variable function $a(z, \bar{z})$ completely drops out and we see that the chiral correlator is precisely defined in terms of the derivative $h'(z)$ of the single-variable function introduced above.

5.2.2 Constraints from crossing symmetry

The full correlation function is invariant under permutations of the four operators. By interchanging the first and the second operator in (5.9) we find the constraint

$$G(z,\bar{z};\alpha,\bar{\alpha}) = G\left(\frac{z}{z-1},\frac{\bar{z}}{\bar{z}-1};1-\alpha,1-\bar{\alpha}\right), \tag{5.17}$$

whereas interchanging the first and the third operator gives

$$G(z,\bar{z};\alpha,\bar{\alpha}) = \frac{z^4\bar{z}^4(\alpha-1)^2(\bar{\alpha}-1)^2}{(1-z)^4(1-\bar{z})^4}G\left(1-z,1-\bar{z};\frac{\alpha}{\alpha-1},\frac{\bar{\alpha}}{\bar{\alpha}-1}\right). \tag{5.18}$$

Using equations (5.12) and (5.14) we want to re-write the crossing symmetry constraints above in terms of $a(z,\bar{z})$ and $h(z)$. We do so by analyzing the equation term by term in α and $\bar{\alpha}$. Similarly to [3, 36] we find that each of the above equations gives rise to an independent crossing symmetry constraints for the chiral part of the correlator, namely

$$h'(z) = \frac{1}{(z-1)^2}h'\left(\frac{z}{z-1}\right) = \frac{z^2}{(z-1)^2}h'(1-z)\,. \tag{5.19}$$

With $G(z) = -z^2h'(z)$ this can be rewritten as

$$G(z) = G\left(\frac{z}{z-1}\right) = \left(\frac{z}{z-1}\right)^4 G(1-z)\,. \tag{5.20}$$

This is nothing more than the familiar two-dimensional crossing symmetry equations for a four-point function of a chiral operator of dimension two, and the decoupling of these equations is a direct consequence of the chiral algebra obtained in [28]. We will in addition assume that $G(z)$ is meromorphic in z and admits a regular Taylor series expansion around $z=0$ with integer powers. If we assume that, which we will see in the next section that follows from the superconformal partial wave decomposition,

$$G(z) = \beta_1 + \beta_2 z + \beta_3 z^2 + \beta_4 z^3 + \dots\,, \tag{5.21}$$

then the crossing symmetry constraints determine that $\beta_2 = 0$, $\beta_4 = \beta_3$, and that

$$G(z) = \beta_1\left(1 + z^4 + \frac{z^4}{(1-z)^4}\right) + \beta_3\left(z^2 + z^3 + \frac{z^4}{(1-z)^2} + \frac{z^4}{1-z}\right), \tag{5.22}$$

which implies that

$$h(z) = -\beta_1 \left(\frac{z^3}{3} - \frac{1}{z-1} - \frac{1}{(z-1)^2} - \frac{1}{3(z-1)^3} - \frac{1}{z}\right) - \beta_3 \left(z - \frac{1}{z-1} + \log(1-z)\right) + \beta_5 \,. \tag{5.23}$$

Here β_5 is an integration constant which does not affect $\mathcal{H}_1^{(2)}(z,\bar{z};\alpha,\bar{\alpha})$, and thus we can choose it to any convenient value.

The remaining crossing symmetry equations obtained from (5.17) give the following two equations for the unprotected function:

$$\begin{aligned}
&a(z,\bar{z}) - \frac{1}{(z-1)^5(\bar{z}-1)^5} a\left(\frac{z}{z-1}, \frac{\bar{z}}{\bar{z}-1}\right) = 0\,,\\
&z\bar{z}a\left(z,\bar{z}\right) - \left(z-1\right)\left(\bar{z}-1\right) a\left(1-z, 1-\bar{z}\right) = \frac{1}{(z-\bar{z})^3}\left(\frac{h\left(1-\bar{z}\right) - h(1-z)}{\left(z-1\right)\left(\bar{z}-1\right)} + \frac{h\left(\bar{z}\right) - h(z)}{z\bar{z}}\right) \,.
\end{aligned} \tag{5.24}$$

As we will see the constraint from the first equation is easily solved, while the latter is the final crossing symmetry equation that we will analyze numerically in this chapter.

5.3 Superconformal block decomposition

To investigate the crossing equation (5.24) we must understand which operators appear in the self-OPE of $\Phi(x,Y)$, and how each of them contributes to $G(z,\bar{z};\alpha,\bar{\alpha})$ (and through (5.12) how each of them contributes to the unknown function $a(z,\bar{z})$ and to $h(z)$). Furthermore, since we have solved exactly for $h(z)$, knowing how each operator contributes to it will allow us to recover (under some assumptions) the OPE coefficients of an infinite series of multiplets.

5.3.1 Superconformal partial wave expansion

In the OPE of $\Phi(x,Y)$ with itself we may encounter the following irreducible R-symmetry representations:

$$\begin{aligned}
([2,0]\otimes[2,0])_s &= [0,0]\oplus[2,0]\oplus[4,0]\oplus[0,4]\,,\\
([2,0]\otimes[2,0])_a &= [0,2]\oplus[2,2]\,,
\end{aligned} \tag{5.25}$$

where the first/second line contain the representations appearing in the symmetric/antisymmetric tensor product.

The conformal block decomposition of $G(z,\bar{z};\alpha,\bar{\alpha})$ then takes the following form

$$G(z,\bar{z};\alpha,\bar{\alpha}) = \sum_{r\in R}\left(Y^r(\alpha,\bar{\alpha})\sum_{k_r}\lambda_{k_r}^2\mathcal{G}_{\Delta_{k_r}}^{(\ell_{k_r})}(z,\bar{z})\right), \tag{5.26}$$

with $r \in R = \{[0,0],[2,0],[4,0],[0,4],[0,2],[2,2]\}$ the set of $\mathfrak{so}(5)_R$ representations, k_r labeling the different operators with representation r appearing in the OPE, and $(\lambda_{k_r},\Delta_{k_r},\ell_{k_r})$ denote the OPE coefficient, scaling dimension and spin of the operator, respectively. Finally $Y^r(\alpha,\alpha')$ are harmonic functions that encode the corresponding $\mathfrak{so}(5)_R$ tensor structure, that is, they are obtained from projectors onto representation r, after contracting with (5.7). The exact form of these functions is given in (J.2). The functions $\mathcal{G}_\Delta^{(\ell)}(z,\bar{z})$ are the usual conformal blocks in six dimensions for a correlation function of identical scalars given by $\mathcal{G}_\Delta^{(\ell)}(\Delta_{12},\Delta_{34};z,\bar{z})$ in (J.1) with $\Delta_{12}=\Delta_{34}=0$.

Because of $(2,0)$ supersymmetry the blocks in (5.26) corresponding to operators in the same supermultiplet are grouped together into what we may call superblocks. The superconformal block expansion then becomes

$$G(z,\bar{z};\alpha,\bar{\alpha}) = \sum_{\mathcal{X}}\lambda_{\mathcal{X}}^2\left(\sum_{r\in R}Y^r(\alpha,\bar{\alpha})A_r^{\mathcal{X}}(z,\bar{z})\right), \tag{5.27}$$

where the sum runs over superconformal multiplets $\mathcal{X}$, with only one unknown OPE coefficient squared $\lambda_{\mathcal{X}}^2$ per superconformal multiplet. The constraint on $G(z,\bar{z};\alpha,\bar{\alpha})$ from the superconformal Ward identities, whose solution is given in Eq. (5.12), translates into a constraint on the $A_r^{\mathcal{X}}(z,\bar{z})$. As such each $A_r^{\mathcal{X}}(z,\bar{z})$ is fixed in terms of the contributions of multiplet $\mathcal{X}$ to $a(z,\bar{z})$ and $h(z)$. The explicit form of $A_r(z,\bar{z})$ in terms of $a(z,\bar{z})$ and $h(z)$ is given in Eq. (J.3), and can be obtained by decomposing (5.12) in the projectors $Y^r(\alpha,\bar{\alpha})$ given in (J.2). Furthermore, each $A_r^{\mathcal{X}}(z,\bar{z})$ admits an expansion in six-dimensional conformal blocks $\mathcal{G}_{\Delta_{k_r}}^{(\ell_{k_r})}$ corresponding to the different operators k_r, in $\mathfrak{so}(5)_R$ representation r, that make up the $\mathcal{X}$ multiplet. The $A_r^{\mathcal{X}}(z,\bar{z})$ are what we call superconformal blocks. To fully specify the superconformal blocks one then has to specify how each superconformal multiplet contributes to the $a(z,\bar{z})$ and $h(z)$.

5.3.2 Superconformal blocks

Through analyzing the constraints of superconformal symmetry on three point functions it was determined in [26, 182–184] that seven different types of superconformal multiplets, plus the identity, can make an appearance in the OPE of stress tensor multiplets:

$$\mathcal{D}[2,0]\times\mathcal{D}[2,0] \sim \mathbf{1}+\mathcal{D}[2,0]+\mathcal{D}[4,0]+\mathcal{D}[0,4]+\mathcal{B}[2,0]_\ell+\mathcal{B}[0,2]_\ell+\mathcal{B}[0,0]_\ell+\mathcal{L}[0,0]_{\Delta,\ell}\,. \tag{5.28}$$

In here we have already imposed Bose symmetry, as we are considering the OPE of two identical operators, thereby removing some of the multiplets listed in [26, 182–184] which did not have the right spin parity to appear in this OPE. The superblocks corresponding to all these multiplets can however all be obtained from two basic elements, which we call the *atomic* contribution to $a(z,\bar{z})$ or to $h(z)$. These two functions take the form

$$\begin{aligned} a^{\text{at}}_{\Delta,\ell}(z,\bar{z}) &= \frac{4}{z^6\bar{z}^6(\Delta-\ell-2)(\Delta+\ell+2)}\mathcal{G}^{(\ell)}_{\Delta+4}(0,-2;z,\bar{z})\,, \\ h^{\text{at}}_{\beta}(z) &= \frac{z^{\beta-1}}{1-\beta}F[\beta-1,\beta;2\beta,z]\,, \end{aligned} \tag{5.29}$$

where $\mathcal{G}^{(\ell)}_{\Delta}(\Delta_1-\Delta_2,\Delta_3-\Delta_4;z,\bar{z})$ is the conformal block in a correlation function of four operators with unequal scaling dimensions Δ_i given in (J.1). The functions $A_r(z,\bar{z})$ (obtained from (J.3)) corresponding to each of these building blocks admits a decomposition in a finite number of conformal blocks and with positive coefficients. For $a^{\text{at}}_{\Delta,\ell}(z,\bar{z})$ this leads to the following primary operator content[3]

[0 0]	[0 2]	[2 0]	[0 4]	[2 2]	[4 0]
$(\Delta)_\ell$	$(\Delta+1)_{\ell-1}$	$(\Delta+2)_{\ell-2}$	$(\Delta+2)_\ell$	$(\Delta+3)_{\ell-1}$	$(\Delta+4)_\ell$
$(\Delta+2)_{\ell-2}$	$(\Delta+1)_{\ell+1}$	$(\Delta+2)_\ell$	$(\Delta+4)_{\ell-2}$	$(\Delta+3)_{\ell+1}$	
$(\Delta+2)_\ell$	$(\Delta+3)_{\ell-3}$	$(\Delta+2)_{\ell+2}$	$(\Delta+4)_\ell$	$(\Delta+5)_{\ell-1}$	
$(\Delta+2)_{\ell+2}$	$(\Delta+3)_{\ell-1}$	$(\Delta+4)_{\ell-2}$	$(\Delta+4)_{\ell+2}$	$(\Delta+5)_{\ell+1}$	
$(\Delta+4)_{\ell-4}$	$(\Delta+3)_{\ell+1}$	$(\Delta+4)_\ell$	$(\Delta+6)_\ell$		
$(\Delta+4)_{\ell-2}$	$(\Delta+3)_{\ell+3}$	$(\Delta+4)_{\ell+2}$			
$(\Delta+4)_\ell$	$(\Delta+5)_{\ell-3}$	$(\Delta+6)_{\ell-2}$			
$(\Delta+4)_{\ell+2}$	$(\Delta+5)_{\ell-1}$	$(\Delta+6)_\ell$			
$(\Delta+4)_{\ell+4}$	$(\Delta+5)_{\ell+1}$	$(\Delta+6)_{\ell+2}$			
$(\Delta+6)_{\ell-2}$	$(\Delta+5)_{\ell+3}$				
$(\Delta+6)_\ell$	$(\Delta+7)_{\ell-1}$				
$(\Delta+6)_{\ell+2}$	$(\Delta+7)_{\ell+1}$				
$(\Delta+8)_\ell$					

(5.30)

[3]This is the operator content for $\ell \geq 4$ and $\Delta > 6+\ell$. For smaller values of ℓ, or $\Delta = 6+\ell$ several entries disappear from the list.

whilst for $h^{\text{at}}_\beta(z)$ the same analysis yields

$$\begin{array}{lll} [0\;0] & [0\;2] & [2\;0] \\ \hline (\ell+4)_\ell & (\ell+5)_{\ell+1} & (\ell+6)_{\ell+2} \\ (\ell+6)_{\ell+2} & (\ell+7)_{\ell+3} & \\ (\ell+8)_{\ell+4} & & \end{array} \tag{5.31}$$

where $\beta = \ell + 4$. These tables justify our claim that these are indeed the smallest possible ('atomic') contributions, since for each of the atomic cases there exists an R symmetry channel with only one conformal block. The function $h^{\text{at}}_\beta(z)$ also corresponds to a single block in the chiral algebra correlator $G(z)$, since

$$G^{\text{at}}_\beta(z) = -z^2\partial_z h^{\text{at}}_\beta(z) = z^\beta F[\beta,\beta;2\beta;z]\,, \tag{5.32}$$

which is precisely the familiar form of a chiral $\mathfrak{sl}(2)$ conformal block in two dimensions.

We now proceed to determine the contribution of all supermultiplets in (5.28) to $a(z,\bar z)$ and $h(z)$, which are all linear combinations and/or specializations of $a^{\text{at}}(z,\bar z)$ and $h^{\text{at}}(z)$. The results are listed in table 5.1. We also see that the assumption made when obtaining (5.22), namely that it admits a regular Taylor series expansion around $z = 0$ with integer powers, simply follows from the behavior of $G^{\text{at}}_\beta(z)$ and the contributions of the various multiplets given in 5.1.

atomic	type	Δ	ℓ	comments
$a^{\text{at}}_{\Delta,\ell}(z,\bar z)$	$\mathcal{L}[0,0]$	Δ	ℓ	generic long multiplet, $\Delta > \ell+6$
$a^{\text{at}}_{\ell+6,\ell}(z,\bar z)$	$\mathcal{B}[0,2]$	$\ell+7$	$\ell-1$	$\ell > 0$
$a^{\text{at}}_{6,0}(z,\bar z)$	$\mathcal{D}[0,4]$	8	0	
$h^{\text{at}}_{\ell+4}(z)$	$\mathcal{B}[0,0]$	$\ell+4$	ℓ	higher spin currents, $\ell \geq 0$
$h^{\text{at}}_{2}(z)$	$\mathcal{D}[2,0]$	4	0	stress tensor multiplet
$h^{\text{at}}_{0}(z) = 1$	$\mathbf{1}$	0	0	identity
$a^{\text{at}}_{\ell+4,\ell}(z,\bar z) + 2^{-\ell}h^{\text{at}}_{\ell+4}(z)$	$\mathcal{B}[2,0]$	$\ell+6$	$\ell-2$	$\ell > 0$
$a^{\text{at}}_{4,0}(z,\bar z) + h^{\text{at}}_{4}(z)$	$\mathcal{D}[4,0]$	8	0	

Table 5.1: Superconformal blocks contribution from all superconformal multiplets appearing in the OPE of two stress tensor multiplets. The contributions are determined from the atomic building blocks. Bose symmetry requires that ℓ is an even integer.

We start by noticing that the list of conformal primaries given in (5.30) is nothing but the set of conformal primaries that can appear in the OPE of two $\Phi(x,Y)$ belonging to a superconformal multiplet $\mathcal{L}[0,0]$ whose superconformal primary has dimension $\Delta > \ell+6$ and spin $\ell \geqslant 0$. When $\Delta = \ell+6$ the list given in (5.30) is decreased, and in particular the dimension Δ spin ℓ operator in channel $[0,0]$ which corresponds to the superconformal primary of $\mathcal{L}[0,0]$ disappears.[4] In this case for $\ell > 0$ the list corresponds instead the multiplet $\mathcal{B}[0,2]_{\ell-1}$ with the superconformal primary appearing in channel $[0,2]$ and with dimension $\ell+7$ and spin $\ell-1$. For $\ell = 0$ there are even fewer entries in the table, and it corresponds to the shorter multiplet $\mathcal{D}[0,4]$ whose superconformal primary appears in $[0,4]$ and has dimension eight and spin zero. Similarly the list of operators given in (5.31) corresponds precisely to the conformal primaries belonging to the $\mathcal{B}[0,0]_{\ell\geqslant 0}$ (or $\mathcal{D}[2,0]$ if $\ell = -2$). Naturally the identity must contribute only in the R-symmetry singlet channel and its block is 1. Finally demanding that the operator content of the $A_r(z,\bar{z})$ decomposition in conformal corresponds to the two remaining multiplets of $\mathcal{B}[2,0]_\ell$ and $\mathcal{D}[4,0]$ we find that they must contribute to $a(z,\bar{z})$ and $h(z)$ in the manner listed in table 5.1.

It now follows from properties of the conformal block (J.1) and (5.29) that the first crossing equation in (5.24) is trivially satisfied if one already imposed Bose symmetry in the OPE, *i.e.*, that only $a^{\mathrm{at}}_{\Delta,\ell}(z,\bar{z})$ with ℓ an even integer make an appearance.

Finally we note that the multiplets $\mathcal{B}[0,0]$ contain twist four operators of spin larger than two, which are conserved in six dimensions. These *higher-spin conserved currents* are expected to be absent in interacting CFTs [53, 109]. From the last two rows in the table we see that they can indeed be avoided by pairing up each $h^{\mathrm{at}}_{\ell+4}(z)$ with an $a^{\mathrm{at}}_{\ell+4,\ell}(z,\bar{z})$ for $\ell \geq 0$. The latter term corresponds to a long multiplet below the unitarity bound and consequently the coefficients of the conformal blocks are not all positive. For the given value of the relative coefficient these non-unitary contributions in $a^{\mathrm{at}}_{\ell+4,\ell}(z,\bar{z})$ are however precisely cancelled by the higher spin currents in $h^{\mathrm{at}}_{\ell+4}(z)$ and we recover the given unitary multiplets with positive coefficients.[5]

5.3.3 Solving for the short multiplet contributions

The function $h(z)$ is explicitly known (Eq. (5.23)), and therefore so is its decomposition into an infinite sum of blocks $h^{\mathrm{at}}_\beta(z)$.

Let us start by fixing the two integration constants β_1 and β_3 in (5.23). One of them corresponds to a normalization, which we will fix by demanding that the identity contributes to the correlator as 1, and the other can be fixed in terms of the OPE coefficient of the $\mathcal{D}[2,0]$ multiplet, which is related to the central charge of the theory. The three-point function of

[4] Unitarity requires $\Delta > \ell+6$ for a long multiplet. A long multiplet hits the unitary bound $\Delta = \ell+6$ it decomposes into two short multiplets (see, *e.g.*, [176] for the decomposition rules). However, the superconformal block for only one of these two appears, which is the one we recover here.

[5] This is reminiscent of $\mathcal{N}=4$, see [36].

two identical scalars $\mathcal{O}$ of dimension Δ and the stress tensor is given by [185]

$$\begin{aligned} \langle T_{\mu\nu}(x_1)\mathcal{O}(x_1)\mathcal{O}(x_2)\rangle &= \frac{d\Delta}{S_d(d-1)}\frac{1}{x_{12}^d x_{23}^{2\Delta-d)} x_{31}^d}\left(\frac{x_{23}^2}{x_{12}^2 x_{13}^2}Z_\mu Z_\nu - \frac{1}{d}\eta_{\mu\nu}\right), \quad (5.33)\\ Z_\mu &\equiv \frac{x_{13\,\mu}}{x_{13}^2} - \frac{x_{12\,\mu}}{x_{12}^2}, \quad (5.34) \end{aligned}$$

where $d = 6$ is the number of dimensions and C_T is defined as the two point function of the stress tensor by

$$\begin{aligned} \langle T_{\mu\nu}(x)T_{\alpha\beta}(0)\rangle &= \frac{C_T}{x^{2d}}\mathcal{I}_{\mu\nu\rho\sigma}(x), \quad (5.35)\\ \mathcal{I}_{\mu\nu\rho\sigma}(x) &= \frac{1}{2}\left(I_{\mu\rho}(x)I_{\nu\sigma}(x) + I_{\mu\sigma}(x)I_{\nu\rho}(x) - \frac{1}{d}\delta_{\mu\nu}\delta_{\rho\sigma}\right), \qquad I_{\mu\nu}(x) = \delta_{\mu\nu} - 2\frac{x_\mu x_\nu}{x^2}, \end{aligned}$$

in conventions where $C_T = \frac{84}{\pi^6}$ for the free tensor multiplet [186].[6] We define a central charge c such that a free tensor has $c = 1$ by $C_T =: \frac{84}{\pi^6}c$. This choice for c is the same as in [28] where it was found the chiral algebra central charge is $c_{2d} = c$. In these conventions the A_{N-1} theories have $c = 4N^3 - 3N - 1$.

Combining (5.33) and (5.35) we can fix the coefficient of the stress tensor conformal block (in the conventions of (J.1)), and consequently that of the $\mathcal{D}[2,0]$ multiplet in terms of c, by making use of the decomposition of the $A_r(z,\bar{z})$ in conformal blocks. All in all the coefficients in (5.23) are fixed to be $\beta_1 = 1$ and $\beta_3 = \frac{8}{c}$.[7]

We can now expand $h(z)$ in its atomic contributions (5.29) finding

$$\begin{aligned} h(z) &= h_0^{at}(z) + \sum_{\ell=-2\,,\ell\text{even}}^{\infty} b_\ell\, h_{\ell+4}^{at}(z), \quad (5.36)\\ b_\ell &= \frac{(\ell+1)(\ell+3)(\ell+2)^2\frac{\ell}{2}!\left(\frac{\ell}{2}+2\right)!!\left(\frac{\ell}{2}+3\right)!!(\ell+5)!!}{18(\ell+2)!!(2\ell+5)!!}\\ &\quad + \frac{8}{c}\frac{\left(2^{-\frac{\ell}{2}-1}(\ell(\ell+7)+11)(\ell+3)!!\Gamma\left(\frac{\ell}{2}+2\right)\right)}{(2\ell+5)!!}, \end{aligned}$$

where b_{-2} can be obtained by taking the limit of the above expression which gives $b_{-2} = \frac{8}{c}$. For this decomposition we have to set the integration constant in (5.23) to be $\beta_5 = \beta_1/3 + \beta_3$,

[6] In six dimensions there are three c-type anomaly coefficients, appearing as the coefficients of the three Weyl invariants in six dimensions, and one a-type one appearing as the coefficient of the Euler density. The C_T central charge appearing in our correlation function is related to the c_3 anomaly coefficient of [187].

[7] Note that plugging this solution in (5.22) we find exactly the two-dimensional stress tensor four-point function with $c_{2d} = c$, after multiplying $G(z)$ by an overall prefactor which corresponds to normalizing the twisted operator $\hat{\Phi}(z)$ to have the usual stress tensor normalization.

but as discussed this does not affect the four-point function.

From table 5.1 we see that, if we demand the absence of higher-spin conserved currents (the $\mathcal{B}[0,0]$ multiplets), this decomposition fixes uniquely the OPE coefficients of the $\mathcal{B}[2,0]$ and $\mathcal{D}[4,0]$ multiplets (the OPE coefficient of $\mathcal{D}[2,0]$ was used to fix $h(z)$), in terms of the central charge c. The coefficients of the $\mathcal{B}[0,2]$ and $\mathcal{D}[0,4]$ multiplets are however not fixed in this way, as they do not contribute to $h(z)$.[8]

From Tab. 5.1 we can work out the contribution of the $\mathcal{B}[2,0]$ and $\mathcal{D}[4,0]$ multiplets to $a(z,\bar{z})$ to be

$$a^{\text{short}}(z,zb) = \sum_{\ell=-2\ ,\ell\text{even}}^{\infty} 2^{\ell+2} b_{\ell+2}\, a^{\text{at}}_{\ell+6,\ell+2}(z,\bar{z})\,. \tag{5.37}$$

This means that in the crossing equation (5.24) we can split

$$a(z,\bar{z}) = a^{\text{short}}(z,zb) + a^{\text{unfixed}}(z,\bar{z})\,, \tag{5.38}$$

where we fix explicitly the contributions coming form the $\mathcal{B}[2,0]$ and $\mathcal{D}[4,0]$ multiplets. The function $a^{\text{unfixed}}(z,\bar{z})$ will then have a block decomposition

$$a^{\text{unfixed}}(z,\bar{z}) = \sum_{\substack{\Delta\geqslant\ell+6,\\ \ell\geqslant 0\ ,\ell\text{even}}} \lambda^2_{\Delta,\ell}\, a^{\text{at}}_{\Delta,\ell}(z,\bar{z})\,, \tag{5.39}$$

which includes both the long multiplets and the $\mathcal{B}[0,2]$ and $\mathcal{D}[0,4]$ short multiplets, whose dimensions are protected but OPE coefficients unknown. The spectrum of long multiplets and the OPE coefficients in the above equation will be the subject of our numerical investigations in the following sections.

5.4 Numerics

The numerical investigation of the crossing equation (5.24) is performed as described in section 4.5 of the previous chapter, and as such we do not repeat the procedure here, but simply point out the two differences.

We take the same basis for functionals as before, given in Eq. (4.92), except this time to obtain a finite dimensional subspace we truncate this space as

$$\phi[f(z,\bar{z})] = \sum_{m,n}^{m,n\leqslant\Lambda} \alpha_{mn}\partial_z^m\partial_{\bar{z}}^n f(z,\bar{z})\big|_{z=\bar{z}=\frac{1}{2}}\ , \tag{5.40}$$

instead of taking $m+n\leqslant\Lambda$ as before. Once again, from the symmetry of the problem we

[8]In other words, these multiplets do not contain any $\mathbb{Q}$-chiral operators, and are not captured by the chiral algebra of [28].

only need to take $m \leqslant n$. For practical reasons we multiply the crossing equation (5.24) by $(z-\bar{z})^3$, and we see that only derivative combinations with $m+n$ even give a non-zero result.

Since this time we have a single crossing equation we do not resort to semidefinite programming as in the previous chapter, and instead use simply linear programming. This is the approach taken in the original paper [6] and in many papers that followed. A main difference with respect to semi-definite programming is that the search of a functional cannot be done with a continuous parameter (the dimensions Δ), and as such we discretize it. We have checked that the specific discretization chosen here does not significantly affect the bounds. As before we must truncate the number of spins considered, in the plots presented here we have taken $\ell_{\max} = \Lambda + 20$, and checked that this truncation does not affect significantly the results. To find the functionals described in detail in section 4.5 we used the `IBM ILOG CPLEX` optimizer, interfaced with `Mathematica`.

5.5 Results

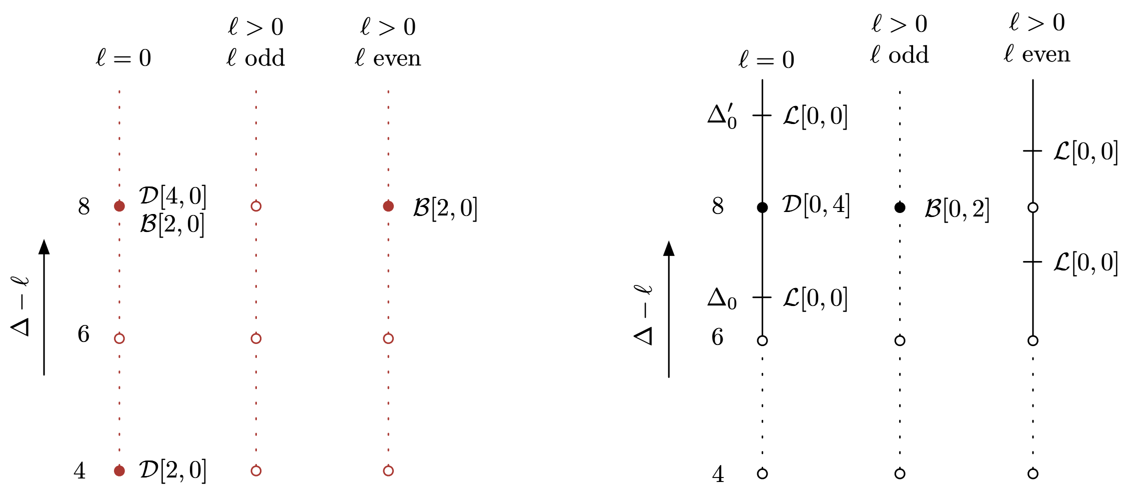


Figure 5.1: Schematic representation of the spectrum of operators appearing in the stress tensor four-point function in a $(2,0)$ theory without higher spin currents. The diagram on the left displays the protected multiplets whose OPE coefficients are predetermined by the chiral algebra. Displayed on the right are multiplets whose OPE coefficients are not predetermined. These multiplets can be either short ($\mathcal{D}[0,4]$ and $\mathcal{B}[0,2]$) or long ($\mathcal{L}[0,0]$), and in the latter case their scaling dimensions are also not predetermined.

In Figure 5.1 we present a reminder of the superconformal multiplets that appear in the stress tensor four-point function. The diagram does not include higher spin current multiplets which we will assume to be absent throughout this section. For the multiplets appearing in the left diagram everything is fixed by the superconformal Ward identities and the chiral algebra in terms of the c central charge of the theory. For the multiplets in the right diagram the following parameters are undetermined:

- the OPE coefficients $\lambda^2_{\mathcal{D}[0,4]}$ and $\lambda^2_{\mathcal{B}[0,2]}$ of the short multiplets;
- the scaling dimensions Δ_ℓ, Δ'_ℓ, ... and OPE coefficients λ_ℓ, λ'_ℓ, ... of all the long ($\mathcal{L}[0,0]$) multiplets.

In subsections 5.5.2-5.5.4 we present numerical results which constrain a subset of these parameters. These constraints will in all cases depend on the c central charge of the theory, which (as we explained previously) enters the crossing symmetry equation via the coefficients of the predetermined multiplets. We will however begin in subsection 5.5.1 with the investigation of a more elementary question:

- are all values of c consistent with crossing symmetry, unitarity, and the absence of higher spin currents?

As we explain below, the negative answer to this question has profound implications for the A_1 theory.

5.5.1 Central charge bounds

Let us start by asking which values of the c central charge are allowed in unitary theories. The numerical methods explained in the preceding section allowed us to obtain a lower bound for c. Our best numerical result can be summarized as the following

Fact. *Every consistent, unitary and local six-dimensional* $(2,0)$ *superconformal theory without higher spin currents has* $c > 15.37$.

This bound was obtained with $\Lambda = 22$ in the conventions of section 5.4. It is most interesting to study the behavior of the bound as a function of Λ. This is shown in Fig. 5.2, which contains in addition to the value quoted above also several data points with smaller values of Λ. From the figure it is clear that the lower bound is most likely to improve upon further increasing Λ, which would however require more numerical resources. We also observe that the points obtained so far approximately follow a straight line, and an ordinary least-squares fit would predict that the bound converges to $c \simeq 25$ as $\Lambda \to \infty$. This is our first indication that the numerical analysis of crossing symmetry is more than a mathematical exercise: the value $c = 25$ is precisely the value corresponding to the A_1 theory, *i.e.* the theory describing the physics of two M5 branes.

Our extrapolation is admittedly rather optimistic. There is currently no precise theory that parametrizes how any of the bounds presented in this chapter depend on Λ, neither in

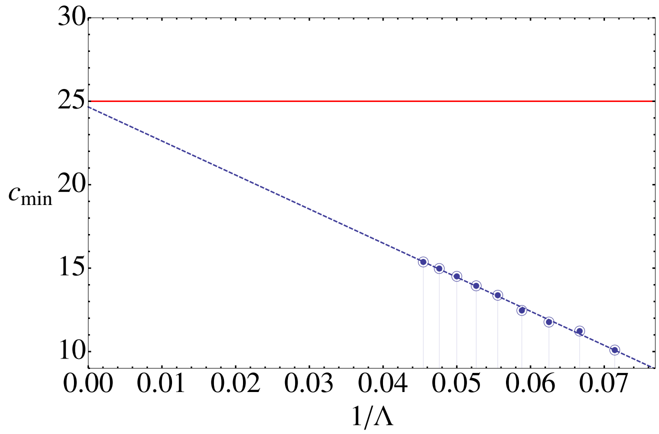


Figure 5.2: Bound on the central charge c as a function of the inverse of Λ, which is a good proxy for the numerical cost of the result. Central charges below the data points are excluded. The dotted line shows a linear extrapolation, which indicates that with infinite numerical power the lower bound might converge to $c \simeq 25$. This is precisely the value for the A_1 theory as indicated by the horizontal line.

the asymptotic regime nor for finite Λ. It is thus far from guaranteed that the behavior will remain (approximately) linear for very large Λ as we assumed in the extrapolation shown in Fig. 5.2. Nevertheless, we would like to venture the following

Speculation. *The lower bound on c converges exactly to 25 as $\Lambda \to \infty$.*

Let us now explain that the validity of this speculation would have some important physical consequences. The numerical problem of finding a lower bound on c has a *dual* formulation where one finds a solution to the truncated crossing symmetry equations rather than a functional. Solving this dual problem is equivalent to proving that a functional does not exist and vice versa. As mentioned in [34, 123] and used extensively in [102], at the lowest possible value of c this dual solution is *unique*, and in all known cases it appears to converge to a complete crossing symmetric four-point function. In our case, this uniqueness implies the following corollary of our speculation.

Corollary. *There is a* unique *crossing symmetric four-point function of the stress tensor multiplet in a unitary six-dimensional $(2,0)$ superconformal theory with $c = 25$ and without higher spin currents. Therefore, at the level of this correlation function, the A_1 theory can be completely bootstrapped.*

Notice that the determination of a single correlation function is no small feat: it contains information about infinitely many scaling dimension of unprotected operators (in this case the R symmetry singlet operators of even spin) and OPE coefficients of all the multiplets appearing in the right diagram in Fig. 5.1. There would then be exceedingly little room for the other crossing symmetry equations to exhibit any freedom whatsoever, and in this scenario the full A_1 theory is likely to be nothing more than the unique solution of the crossing symmetry equations at $c = 25$. In subsection 5.5.4 we will investigate the possibilities for bootstrapping the A_1 theory in more detail, and discuss what we can learn about this theory with finite numerical precision.

5.5.2 Bounds on OPE coefficients

For the allowed values of the central charge we would like to further constrain the dynamical information appearing in the four-point function. In this section we present bounds on the OPE coefficients of the short multiplets ($\mathcal{D}[0,4]$ and $\mathcal{B}[0,2]$, with $\ell = 1, 3, \ldots$) that were not predetermined by the chiral algebra. We would in particular like to see whether these multiplets are necessarily present in physical theories. The conjectured form of the $\frac{1}{4}$-BPS partition function of the A_n theories [181] implies that the $\mathcal{D}[0,4]$ multiplet is present for the theories with $n > 1$ but absent in the A_1 case. In the following we will see how these expectations are consistent with the numerical bounds. In addition, it turns out that the investigation of these OPE coefficients also answers the question of why the central charge bounds of the previous subsection were possible, and we can pinpoint what goes wrong with the crossing symmetry equations as we go below the minimum allowed value of c.

$\mathcal{D}[0,4]$ OPE coefficient bounds

In Fig. 5.3 we constrain the value of the (squared) OPE coefficient of the $\mathcal{D}[0,4]$ multiplet as a function of the central charge. Unitarity trivially requires $\lambda^2_{\mathcal{D}[0,4]} \geqslant 0$, and the upper bound shown in the figure was obtained numerically. Altogether the OPE coefficient is thus required to live in the unshaded region of Fig. 5.3, and we have added a few vertical lines to indicate the allowed range in specific theories. The red and purple lines correspond to the lowest central charge theories of type A_n and D_n respectively, where the central charges are given in Eq. (5.3). The black dots are our strongest bounds, and the grey dots represent bounds obtained with lower values of Λ.

Before proceeding with the interpretation of these results, we would like to point out that the numerical bounds shown here and in the following subsections are not entirely smooth for $\Lambda > 18$. Instead, we find several outlier points which we ascribe to the fact that machine precision is barely sufficient to obtain bounds with these values of Λ. However these “failed searches” occur rather infrequently and the tendency of the bounds as a function of c is still clearly distinguishable. We therefore decided to include all the values up to and including $\Lambda = 22$ in our plots.

The most interesting regions in Fig. 5.3 are the two extremes, corresponding respectively to very small and very large central charges. Let us start by analyzing the former. The leftmost plot of Fig. 5.3 demonstrates that the upper bound crosses zero for small values of

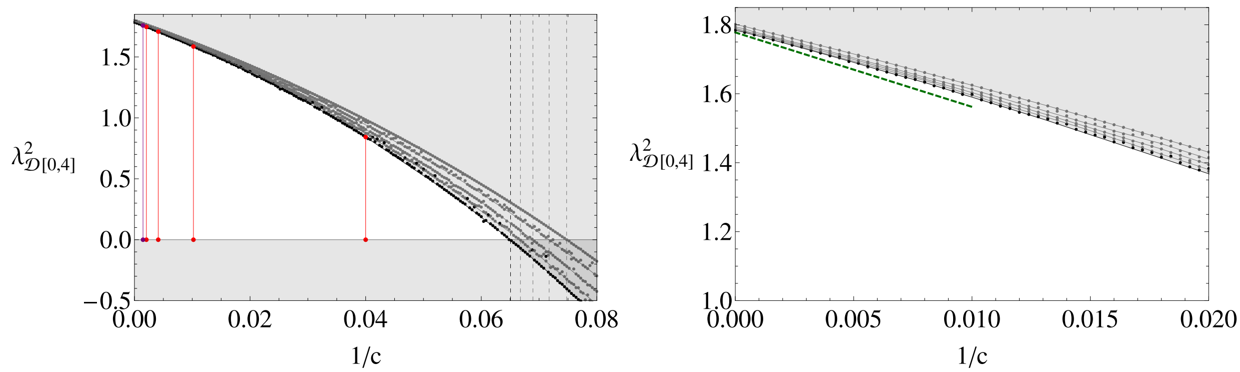


Figure 5.3: Upper bound on the OPE coefficient squared of the $\mathcal{D}[0,4]$ multiplet as a function of the inverse central charge c for increasing number of derivatives $\Lambda = 18, \ldots, 22$, with the strongest bound shown in black. The shaded region is excluded by the numerics and unitarity ($\lambda^2_{\mathcal{D}[0,4]} \geqslant 0$). The red and purple lines correspond to the lowest central charge theories of type A_n and D_n respectively. The vertical dashed lines denote the minimum allowed central charge c_{min} obtained in Fig. 5.2 for the corresponding value of Λ. The smooth curves are interpolations through the data points shown in the figure. The right plot shows a zoom for large central charges, with the dashed green line corresponding to the prediction from supergravity.

the central charge, and we added dashed vertical lines at the crossing points for different values of Λ. To the right of these crossing points the upper bound is negative, effectively forbidding a consistent solution to the crossing symmetry equations. These crossing points therefore translate into a lower bounds for c, which by general consistency arguments are necessarily equal to the bounds shown previously in Fig. 5.2. We therefore rediscover the c_{min} results from the previous subsection, but the vanishing of $\lambda^2_{\mathcal{D}[0,4]}$ at the bound is an additional insight into why the values of c_{min} are special.

As discussed above, the conjectured structure of the $\frac{1}{4}$-BPS partition function implies that the $\mathcal{D}[0,4]$ multiplet is completely absent in the A_1 theory. It therefore most certainly cannot appear in the four-point function, which corroborates nicely with both the vanishing OPE coefficient at c_{min} as well as with the speculation that $c_{min} \to 25$ as $\Lambda \to \infty$. The extrapolation in Fig. 5.2, the vanishing coefficient in Fig. 5.3, and the conjectured $\frac{1}{4}$-BPS partition function are therefore all consistent with the same physical picture. Notice that without the extrapolation to $\Lambda \to \infty$ we have a rigorous bound $0 \leqslant \lambda^2_{\mathcal{D}[0,4]} \leqslant 0.843$ for the A_1 theory corresponding to the rightmost red interval in Fig. 5.3.

We now turn to the large central charge limit, focusing on the the right plot of Fig. 5.3 which shows a zoom for large central charges. In this plot we added a dashed green line showing the OPE coefficient expected from supergravity at $c = \infty$ and the first $1/c$ correction. These OPE coefficients were computed in [184] using the using the four-point function obtained from supergravity on $\mathrm{AdS}_7 \times S^4$ [26, 184]. For the $\mathcal{D}[0,4]$ short multiplet, it should be, in our conventions, $\frac{16}{9} - \frac{340}{63N^3}$, with $c \sim 4N^3$ for the A_N series. From Fig. 5.3 we observe that the supergravity solution lies below our upper bound, which is an important consistency test of our numerics. It is in fact rather striking that the two results are so close, and for infinite Λ the numerical result may well coincide with the supergravity result at very large c. In our opinion this provides another strong indication that the numerical analysis can indeed "mine" the crossing symmetry constraints and recover the physics of the $(2,0)$ theories.[9]

For intermediate values of c one expects the $\mathcal{D}[0,4]$ multiplets to be present for all $c > 25$. This is consistent with our bounds, which we expect to remain strictly positive in this region. More generally one may expect that the $\Lambda \to \infty$ extrapolation of our bounds will again be saturated by the known $(2,0)$ theories, simply because these are the only known solutions to crossing symmetry that would prevent the bounds from decreasing even further. Notice that this would in particular imply that the deviations of our bound from the straight-line behavior at large central charges, *i.e.* the order $1/c^2$ corrections to the bound, should be matched to M-theoretic corrections of eleven-dimensional supergravity.

$\mathcal{B}[0,2]$ OPE coefficient bounds

In this subsection we present upper bounds on the OPE coefficients of several $\mathcal{B}[0,2]$ multiplets. As indicated in the diagram in Fig. 5.1 these appear for all odd spins ℓ, and we will investigate the lowest lying multiplets with $\ell = 1$ and $\ell = 3$. The results are shown in Figs. 5.4 and 5.5. As before, we will discuss in succession the bounds at small and at large central charge.

Let us begin with the bounds near c_{min} which are shown in the left two plots in Figs. 5.4 and 5.5 . Overall consistency of the bounds implies that they have to be zero when $c < c_{min}$, but the way they approach zero is rather different: the $\ell = 1$ bound tends to zero sharply but relatively smoothly, whereas the $\ell = 3$ bound displays genuine step function behavior. This is indicative of the following subtlety concerning the so-called "extremal" solution to crossing symmetry at c_{min}. As we discussed above, this solution is unique but at the same time only approximate given our finite numerical precision. For the case $\ell = 3$, the step function behavior indicates that the corresponding multiplet is actually present in the extremal solution, with a coefficient that is given by its value at the kink. The current value is therefore approximately 19.25, but it is expected to decrease somewhat as Λ increases. The $\ell = 1$ bound, on the other hand, is strictly speaking equal to zero at c_{min} and therefore the corresponding multiplet does not appear in the extremal solution. The bound however increases rather sharply as we move away from c_{min} until a value of around 10, and judging from the left plot of Fig. 5.4 it may well develop the same step function behavior as observed for $\ell = 3$ upon further increasing Λ. In that case the absence of the $\ell = 1$ multiplet is purely a numerical artefact and it will appear, perhaps in an almost discontinuous fashion, with a

[9] A similar match between supergravity results and numerical bounds, including $1/c$ corrections, was observed in [36] for the $\mathcal{N} = 4$ SYM theories.

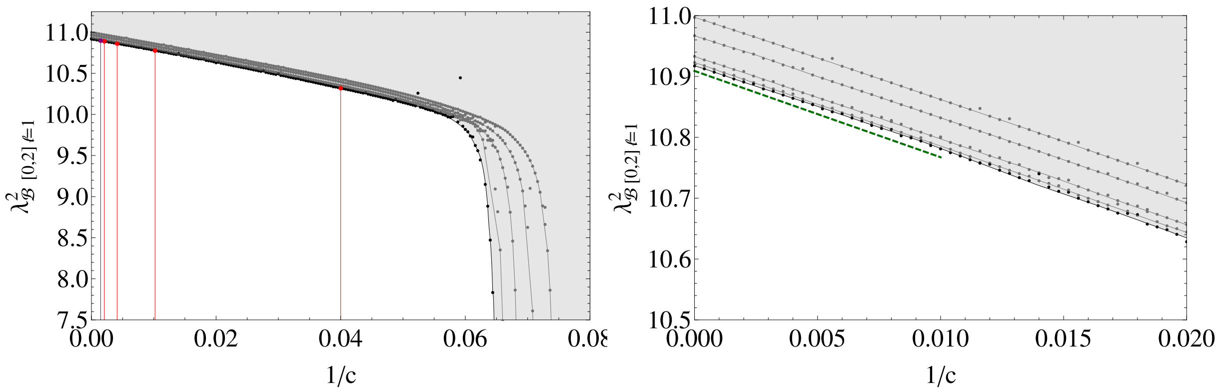


Figure 5.4: Upper bound on the OPE coefficient squared of the $\mathcal{B}[0, 2]$ multiplet with $\ell = 1$ as a function of the inverse central charge c. The different curves correspond to different values of $\Lambda = 18, \ldots, 22$, with the black curve representing the strongest bound. The red and purple lines correspond to the lowest central charge theories of type A_n and D_n respectively. On the right we provide a zoomed in version for very large central charges. The dashed green line corresponds to the supergravity limit.

coefficient of approximately 10 in the extremal solution as $\Lambda \to \infty$. The large central charge behavior is shown in the plots on the right of Figs. 5.4 and 5.5. The dashed lines indicate the supergravity results [184], which in our conventions are given by $\frac{120}{11} - \frac{39}{11N^3}$ and $\frac{256}{13} - \frac{8832}{5005N^3}$ for $\ell = 1$ and $\ell = 3$, respectively. The convergence of the numerical bounds towards the supergravity results is excellent for these multiplets, confirming once more that at least for very large central charges the bounds are sensitive to the physics of the actual $(2, 0)$ theories.

Summary

We end this subsection with a brief summary of the main conclusions related to the OPE coefficient bounds. First of all, the vanishing of the upper bound on the $\mathcal{D}[0, 4]$ multiplet at c_{min} provides additional support for the speculation of subsection 5.5.1. The upper bounds for the $\mathcal{B}[0, 2]$ multiplets with low spins are also consistent with this conjecture, although we would have preferred to see an even sharper transition in the left of Fig. 5.4. This behavior is however likely to improve as Λ increases. For large central charges we find excellent agreement with supergravity, including the $1/c$ corrections. For intermediate central charges it is natural to conjecture that the bounds are saturated by the actual $(2, 0)$ theories, so the upper red dots in Figs. 5.3, 5.4 and 5.5 ought to be close to the physical values in the corresponding theories. It would be very interesting if these results could be verified through other means.

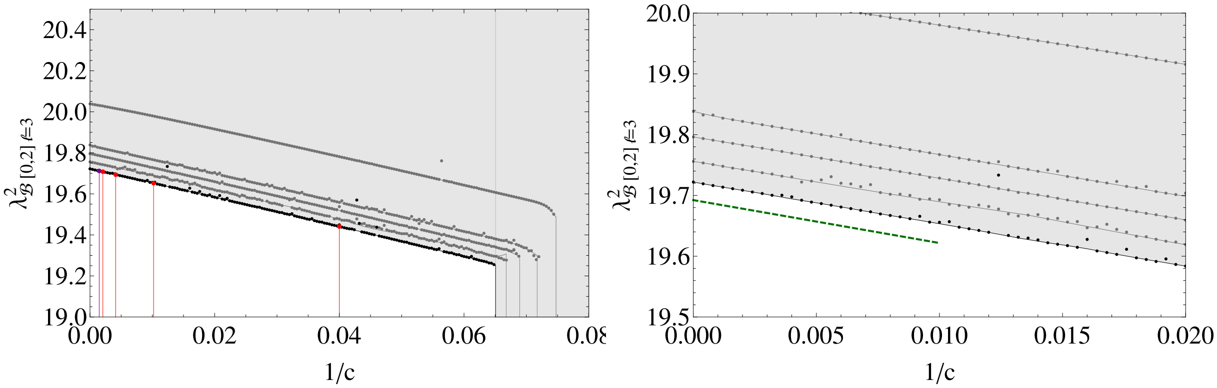


Figure 5.5: Upper bound on the OPE coefficient squared of the $\mathcal{B}[0,2]$ multiplet with $\ell = 3$ (bottom) as a function of the inverse central charge c. The different curves correspond to different values of $\Lambda = 18, \ldots, 22$, with the black curve representing the strongest bound. The red and purple lines correspond to the lowest central charge theories of type A_n and D_n respectively. On the right we provide a zoomed in version for very large central charges. The dashed green line corresponds to the supergravity limit.

5.5.3 Bounds on scaling dimensions

In the previous subsection we have discussed upper bounds on the OPE coefficients of the short multiplets. In this subsection we turn our attention to the long multiplets. As indicated on the right diagram of Fig. 5.1, for the four-point function under consideration these multiplets are necessarily of type $\mathcal{L}[0,0]$ and their superconformal primary has even spin. We will be solely concerned with the scaling dimensions Δ of these multiplets, and leave an investigation of their OPE coefficients to a future study.

Scalar operators

In Fig. 5.6 we present upper bounds on the dimension Δ_0 of the first unprotected scalar operator. We recall that unitarity of the corresponding representation of the superconformal algebra requires that $\Delta_0 \geq 6$. As in the previous subsection, the plots on the left provide an overview for a large range of central charges, and the plots on the right give a zoomed in version of the same bound for very large central charge. The black dots again correspond to the best possible bound, obtained with $\Lambda = 22$, and the shaded area is excluded. The grey dots represent weaker bounds obtained with lower values of Λ. We inserted red, purple and blue vertical lines at those values of c corresponding to known $(2,0)$ theories (the ones with lowest central charge of type A_n, D_n and E_n respectively), and the dashed green line in the plot on the right corresponds to the supergravity solution discussed in [184]. Notice that below the value c_{min} discussed above there is no solution to crossing symmetry whatsoever.

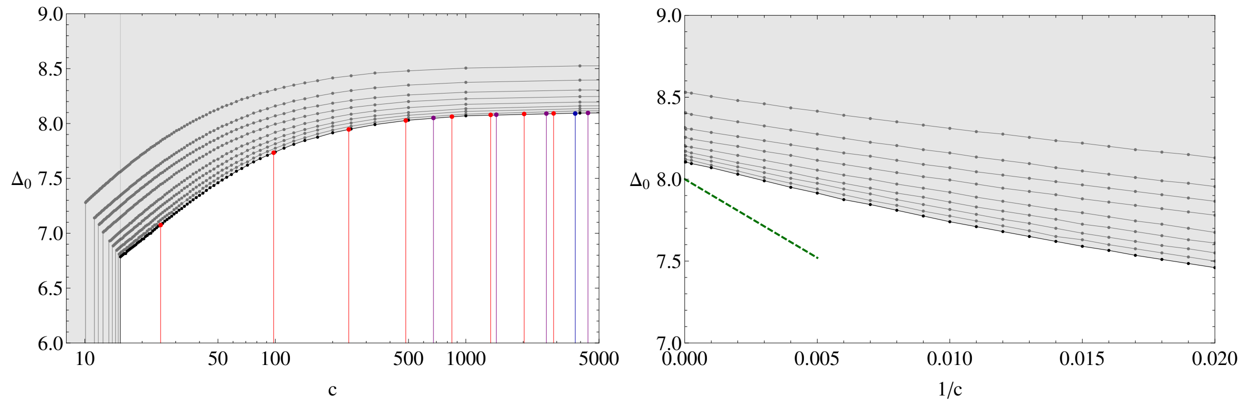


Figure 5.6: Upper bound for the dimension of the first unprotected scalar operator. The different curves correspond to different values of $\Lambda = 18, \ldots, 22$, with the black curve representing the strongest bound, and the shaded region is excluded by the numerics. The vertical red, purple and blue lines correspond to the lowest central charge theories of type A_n, D_n and E_n respectively. The plot on the right displays is a zoomed in result for very large c, with the green dashed line corresponding to the known supergravity solution.

It is then no longer meaningful to ask about upper bounds, and for this reason we find a sharp cutoff on the left of Fig. 5.6.

With our current precision we see that the upper bound is approximately 7.08 for the A_1 theory at $c = 25$ and then increases monotonically until approximately 8.11 at infinite central charge.[10] The latter value is very close to the mean field solution at $\Delta_0 = 8$, to which it presumably would converge with higher Λ. The leading $1/c$ behavior obtained from supergravity does not appear to very closely follow the bound. Although the supergravity result is not excluded by our bound, and therefore at a technical level everything is consistent, the mismatch is nevertheless a little surprising. Indeed, the large c behavior shown in Figs. 5.3, 5.4 and 5.5, and also in Figs. 5.7 and 5.8 below, appear to imply that the bounds will in fact be saturated by the supergravity result. This is also the most natural option from a physical perspective because we do not expect any other theories to exist at very large central charge. It would be interesting to see if the agreement improves upon increasing Λ.

As we emphasized before, for intermediate values of c we have upper bounds for Δ_0 that are valid for all the physical $(2,0)$ theories. It is again natural to assume that these bounds will be saturated by the actual theories and in this way the bounds actually offer a (very rough) estimate of the actual scaling dimensions. In this way we may for example say that the (A_2, A_3, A_4) theories have unprotected $\mathcal{L}[0,0]$ scalar multiplets with primaries of dimensions $\Delta_0 \lesssim (7.7, 7.9, 8.0)$, respectively. (For the A_1 theory we provide a more refined

[10]As discussed in 4 this monotonicity is a generic property of the kind of bounds studied here.

estimate below.) We emphasize that these are the first estimates of unprotected operator dimensions in the $(2,0)$ theories, and it would be very interesting if they could be verified through other means.

Spinning operators

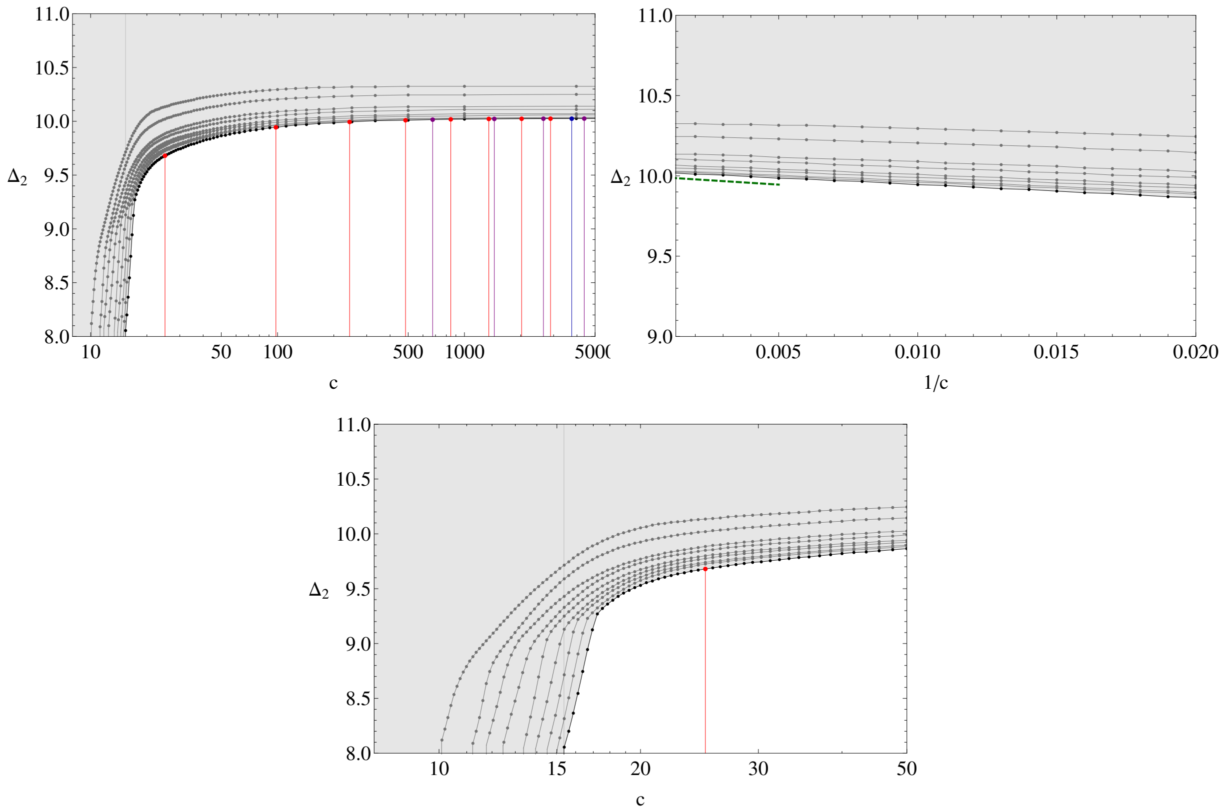


Figure 5.7: Upper bound for the dimension of the first unprotected spin two operator. The different curves correspond to different values of $\Lambda = 18, \ldots, 22$, with the black curve representing the strongest bound, and the shaded region is excluded by the numerics. The vertical red, purple and blue lines correspond to the lowest central charge theories of type A_n, D_n and E_n respectively. The plot on the right is a zoomed in result for very large c, with the green dashed line corresponding to the known supergravity solution. The third plot is a zoom for the small central charge region, where the red line marks the central charge corresponding to the A_1 theory.

Figs. 5.7 and 5.8 present upper bound on the first unprotected spin 2 and spin 4 operators of type $\mathcal{L}[0,0]$ in the $(2,0)$ theories. The structure of these plots is the same as before, and we again would expect these bounds to be saturated by physical theories. This is exemplified at very large c where the bound agrees very well with mean field theory and the $1/c$ correction obtained from supergravity.

In Fig. 5.7 we also provide a zoomed in version of the spin 2 bound for relatively small central charges. In constrast to the scalar and the spin 4 bounds we do not observe step function behavior at c_{min}, but rather a more gradual decrease of the bound towards the unitarity bound. We recall that the $\mathcal{B}[0,2]$ block of spin 1 masquerades itself as an $\mathcal{L}[0,0]$ block of spin 2 at the unitarity bound $\Delta_2 = 8$, so the non-step function behavior in Fig. 5.7 is presumably related to that in the left of Fig. 5.4.

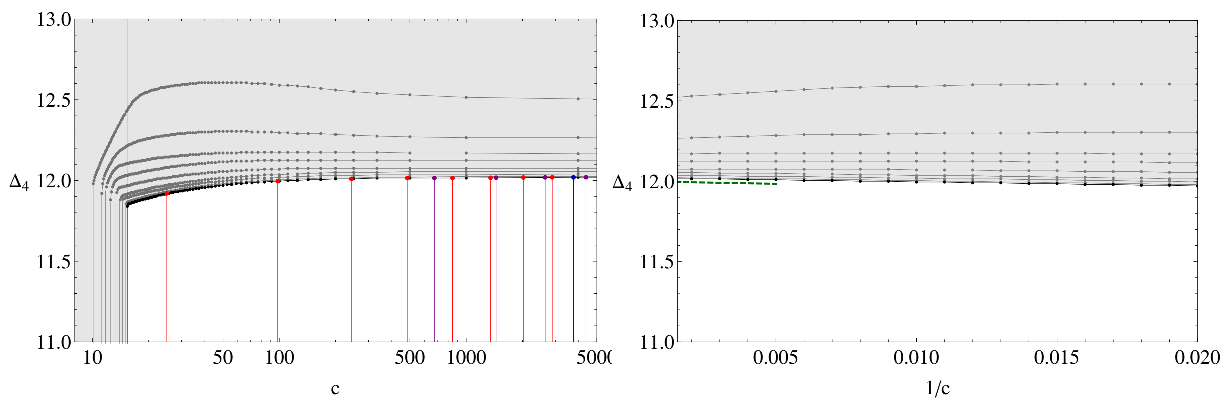


Figure 5.8: Upper bound for the dimension of the first unprotected spin four operator. The different curves correspond to different values of $\Lambda = 18, \ldots, 22$, with the black curve representing the strongest bound, and the shaded region is excluded by the numerics. The vertical red, purple and blue lines correspond to the lowest central charge theories of type A_n, D_n and E_n respectively. The plot on the right is a zoomed in result for very large c, with the green dashed line corresponding to the known supergravity solution.

Although we have not performed a more detailed investigation, the following provides a likely explanation of the behavior in the spin 2 channel.[11] Suppose that the approximate solution to crossing symmetry obtained at c_{min} with finite Λ has a small bias: instead of a $\mathcal{B}[0,2]$ block it has an $\mathcal{L}[0,0]$ block which sits just above the unitarity bound. As in the scalar and spin 4 channel, the presence of such a block would technically imply step function behavior of the bound at c_{min}, but since the block appears only slightly above the unitarity bound the step can be quite small and we would not observe it in Fig. 5.7. This long block is very similar to the $\mathcal{B}[0,2]$ short block of spin 1, and therefore effectively replaces it in the approximate solution to crossing symmetry. In this way the upper bound on the $\mathcal{B}[0,2]$ OPE

[11] This paragraph is rather technical. The noninitiated reader may wish to skip to its last sentence.

coefficient at c_{min} can consistently be zero, which is precisely what is observed in the left figure of Fig. 5.4. Of course we expect the bias to disappear in the limit where $\Lambda \to \infty$. In the current scenario this happens through a decrease of the dimension of the $\mathcal{L}[0,0]$ block towards the unitarity bound, where it degenerates into a $\mathcal{B}[0,2]$ block. At this point we would find a step function at c_{min} in 5.7, and indeed the transition already appears to become sharper for higher Λ. Similarly, the bound in Fig. 5.4 at c_{min} will have to transition towards the dimension of the first unprotected operator, and therefore also become infinitely sharp in the limit of large Λ. In summary, then, the relative smoothness of these particular transition at c_{min} is likely a numerical artefact and we expect to recover genuine step function behavior as $\Lambda \to \infty$.

Combining spins

So far we have restricted to bounding the dimension of a single long multiplet of a given spin at a time, and we have motivated that each of these bound individually is saturated by a physical theory. In particular for $c \to \infty$ we have seen that the spin two and four dimensions bounds agree very well with the expected dimensions from supergravity, while the spin zero had a small mismatch. However if the bounds are in fact saturated by physical theories then we should be able to impose all these gaps simultaneously in more two or more spins. We begin by imposing bounds simultaneously on two spins as it makes the features more apparent. This is shown in Figs. 5.9 for theories with $c = \infty$, and with central charges corresponding to the A_1 and A_2 theories and with $\Lambda = 22$. Unitarity of the superconformal algebra representations requires $(\Delta_0, \Delta_2, \Delta_4) \geqslant (4, 6, 8)$, and the bounds obtained from imposing a single gap at a time restrict the allowed dimensions to be inside the squares delimited by the dashed lines (which simply correspond to the bounds obtained in the previous section). As we impose gaps simultaneously in two channels we numerically carve out part of this square, and the dimensions must now be below the dots shown in Fig. 5.9 for the various central charges.

For $c = \infty$, the bounds on Δ_2 and Δ_4 show a very small dependence on the gap imposed in the other channels, which were precisely the ones that agreed very well with the expect result from generalized free field theory, shown by the green lines in the plots. On the other hand the spin zero bound shows a slight dependence on the gap imposed Δ_2 bringing it close to the generalized free field theory value. It would be interesting to see if this is a feature of working at finite Λ, and if with $\Lambda \to \infty$ the allowed region would take the shape of a rectangle. If that is the case, then even if we are limited to finite Λ we can get a better estimate of the dimensions by imposing simultaneous bounds. Similarly we would expect that it is the dimensions corresponding to the corner of the allowed regions shown in Fig. 5.9 that are saturated by the other known physical theories. We note that the bounds presented here are obtained in the presence of the $\mathcal{D}[0,4]$ short multiplet, which we expect to be present in all theories with the exception of the A_1 theory. The A_1 results reveal a much stronger strong dependence of Δ_0 on Δ_2, and are the focus of our discussion in the next subsection.

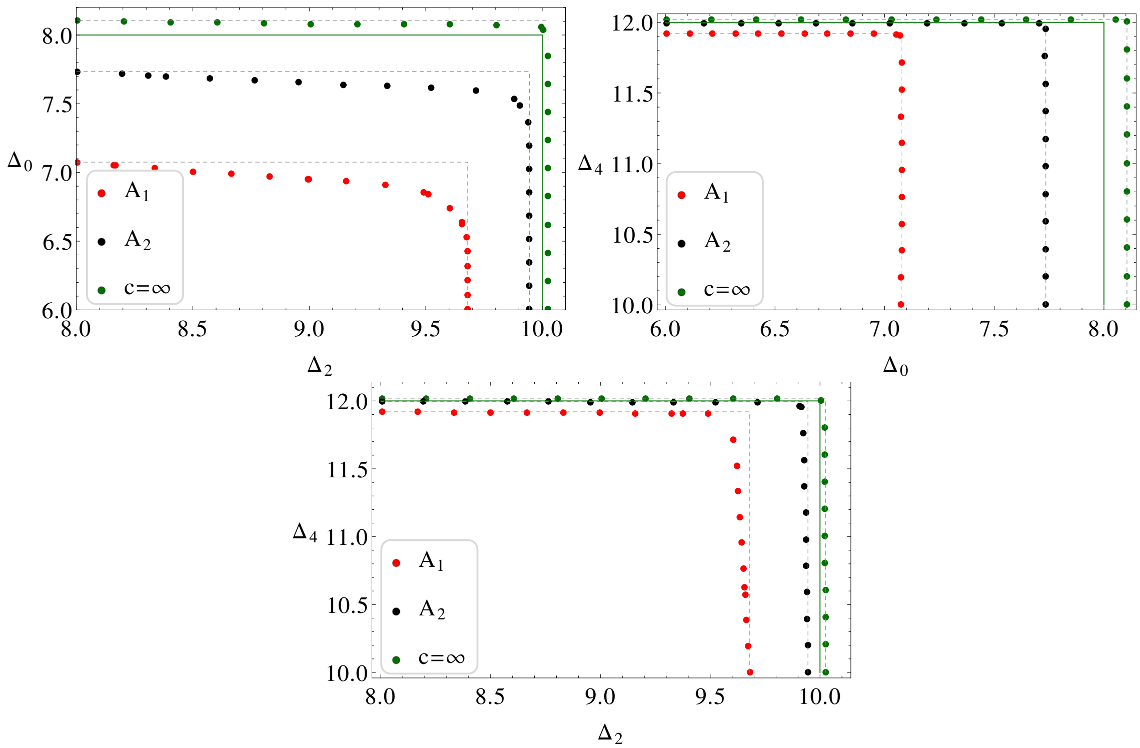


Figure 5.9: Bounds on the spin $0, 2, 4$ superconformal primary dimensions when a gap is imposed in one of the other channels for a cutoff of $\Lambda = 22$. These bounds are for the central charges corresponding to the A_1 and A_2 theories and to the generalized free field theory limit $c = \infty$, and are obtained with the addition of the short multiplet $\mathcal{D}[0, 4]$. The dashed lines show the bounds on $(\Delta_0, \Delta_2, \Delta_4)$ from imposing a single gap at a time, and the full green lines the dimensions expected from the known generalized free field theory. The allowed region corresponds to the inside of the "rectangles" delimited by the dots.

5.5.4 Bootstrapping the A_1 theory

We now turn to the A_1 theory. Picking up on the previous subsection we start by examining what happens when imposing multiple gaps. In Fig. 5.9 we imposed simultaneous gaps in two out of the three lowest spin operators, now we impose on all three of them. As we have motivated we expect that the minimum allowed central charge with $\Lambda = \infty$ is $c = 25$, implying that the theory corresponding to this central charge is unique. As such, the maximum gap we can impose in all three channels simultaneously will correspond to the $(\Delta_0, \Delta_2, \Delta_4)$ values of the physical theory, and we expect the allowed region in the space of these three dimensions should form a cube. Since we are working with a maximum $\Lambda = 22$ and the bounds have not converged yet, the shape of the allowed region is cuboid, but not a cube with completely straight faces, as can be anticipated from Figs. 5.9. The

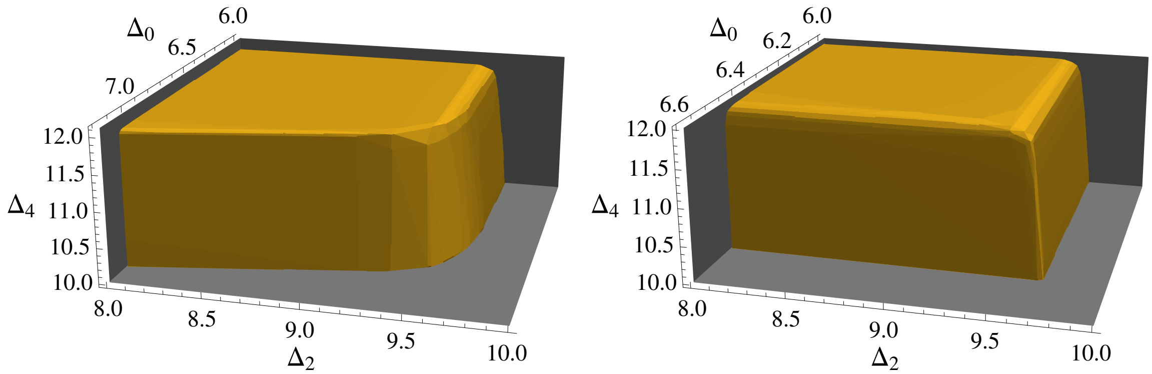


Figure 5.10: Simultaneous bounds on the spin $0, 2, 4$ superconformal primary dimensions for a cutoff of $\Lambda = 22$. These bounds are for the central charge corresponding to the A_1 theory ($c = 25$), and are obtained with (left) and without (right) the addition of the short multiplet $\mathcal{D}[0, 4]$. The allowed region is inside of the region delimited by the yellow surface.

allowed region obtained by imposing simultaneous gaps in $(\Delta_0, \Delta_2, \Delta_4)$ (in the presence of the $\mathcal{D}[0, 4]$ multiplet) is shown in the left side of Fig. 5.10. Indeed the shape of the allowed region resembles a cube but with faces tilted and round edges.

However, as we have discussed in detail in subsection 5.5.2, although we expect the $\mathcal{D}[0, 4]$ short multiplet to be absent for the A_1 theory, the numerical bootstrap allows for its presence at finite Λ, as shown in Fig. 5.3. As such we should obtain bounds by imposing by hand the absence of the $\mathcal{D}[0, 4]$ multiplet. The resulting bounds are shown in the right side of Fig. 5.10. Naturally since we expect that with $\Lambda \to \infty$ the numerical bootstrap will show that this multiplet is absent at $c = 25$, in the same limit the "cubes" on the left and right side of 5.10 should be converging to the same final bound. Requiring the absence of the $\mathcal{D}[0, 4]$ multiplet is then just a trick to overcome the slow convergence of our numerical results, trying to obtain bounds closer to their $\Lambda \to \infty$ values. As expected the bounds get stronger once the $\mathcal{D}[0, 4]$ multiplet is removed, and the allowed region looks much more like a cube indicating that indeed we might be obtaining results closer to the $\Lambda \to \infty$ limit.

Slices of this cube, corresponding to imposing gaps on two channels at a time are shown in Fig. 5.11 where we have also added bounds for increasing smaller values of the cutoff Λ to provide an idea of how far our bounds are from convergence. The bound on the spin $\ell = 0$ dimension is stronger and it no longer shows any dependence on the gap imposed for $\ell = 2$ contrasting Fig. 5.9. We note that since no gap is imposed in Δ_0, the final plot in 5.11 is just the same as the final plot of Fig. 5.9. This is a consequence of allowing for scalar operators of all dimensions starting at the unitary bound, because a scalar long multiplet approaching the unitarity bound mimics precisely the $\mathcal{D}[0, 4]$ short operator. This bound on Δ_2 does show some dependence on Δ_4, and from the discussion above we are led to believe

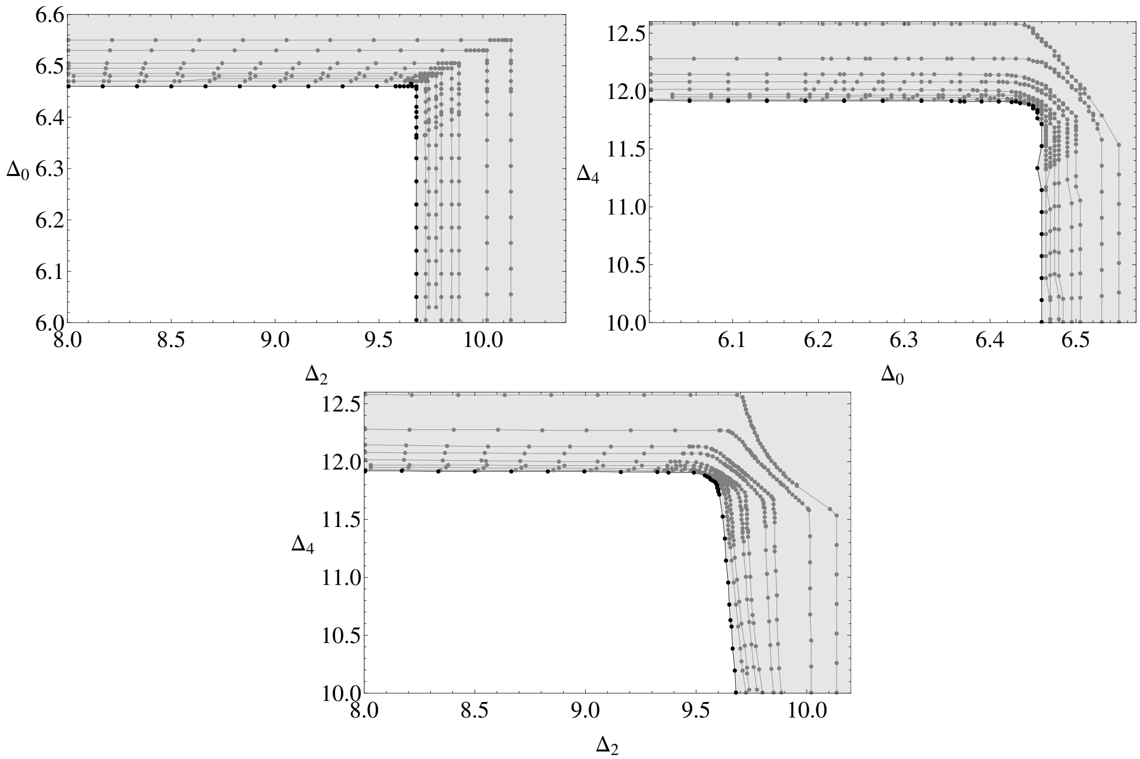


Figure 5.11: Bounds on the spin $0, 2, 4$ superconformal primary dimensions when a gap is imposed in one of the other channels for a cutoff of $\Lambda = 14, \ldots, 22$. These bounds are for the central charge corresponding to the A_1 theory ($c = 25$), and are obtained without the addition of the short multiplet $\mathcal{D}[0, 4]$.

the corner position provides a better estimate for Δ_2.

Bounds on the lowest dimensional scalar operator

We can now go back and see what impact the removal of the $\mathcal{D}[0, 4]$ short multiplet will have on the dimension bounds obtained in the previous subsections. Again, if we are imposing a single gap at a time this will only affect the bounds on the scalar sector.

In Fig. 5.12 we show the bounds on the dimension of the first unprotected spin zero long multiplet, without the $\mathcal{D}[0, 4]$ short. At $c = c_{min}$ the bound coincides with the one shown in Fig. 5.6, but for $c > c_{min}$ the bound without $\mathcal{D}[0, 4]$ is stronger, and as c increases it moves closer to the unitarity bound $\Delta = 6$. In fact extrapolation of these bounds for the central charge corresponding to the A_2 theory suggests that with $\Delta \to \infty$ the bound will be at $\Delta \approx 6$. This means that only for small enough central charges can we get away without the $\mathcal{D}[0, 4]$ multiplet.

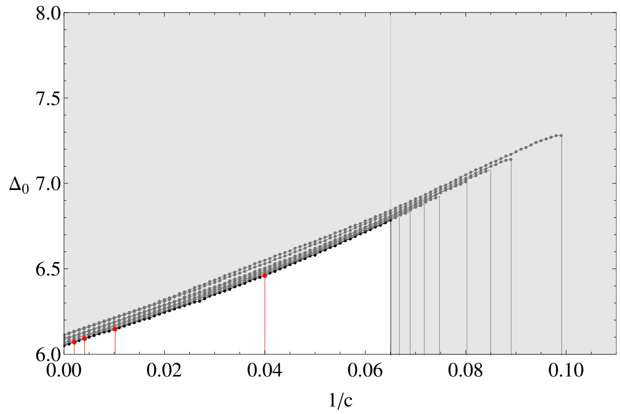


Figure 5.12: Bound on the first spin 0 superconformal primary long dimension as a function of the inverse central charge c, without adding the $\mathcal{D}[0,4]$ short multiplet, for increasing number of derivatives $\Lambda = 14, 15, \dots 22$. The vertical red lines correspond to the lowest central charge of type A_n $(2,0)$ theories.

It is worth noticing however, that the bound in Fig. 5.12 appears to be converging much better than the one in Fig. 5.6. This means that the trick of requiring the absence of the $\mathcal{D}[0,4]$ multiplet does indeed help overcome the slow convergence, and we can obtain bounds on the operator dimensions of the A_1 theory which are close to the optimal values as $\Lambda \to \infty$. Moreover if our claim that $c_{min} \to 25$ as $\Lambda \to \infty$ is correct, the theory at $c = 25$ is indeed unique, and the bounds we obtain are more than just bounds – they correspond to the actual dimensions of operators of the A_1 theory.

An extrapolation of Δ_0 is shown in blue in Fig. 5.13. Of course we expect that as $\Lambda \to \infty$ the absence of $\mathcal{D}[0,4]$ will follow from the numerical results, just like it happens at c_{min} for finite Λ, and therefore the value of the bound at $c = 25$ from 5.6 should also converge to the same limit as the curve in blue. This extrapolation is shown in red in Fig. 5.13. Alternatively we can analyze the bound at c_{min} as the number of derivatives is increased, which should converge to the same value as the previous ones. This is shown in black in Fig. 5.13, and it seems consistent with the limits obtained from the points in blue (without the $\mathcal{D}[0,4]$ short multiplet). In particular both bounds seem to converge to $\Delta_0 \sim 6.4$. The extrapolation from the points with the short multiplet does not agree as well with the other two extrapolations, which perhaps is not surprising since it seems to be the bound which is converging slower. In light of the simultaneous bounds shown in Fig. 5.9 we see that a better estimate would come from the corner position when imposing simultaneously gaps in spin zero and two. We claim this is again just a consequence of working with a finite Λ, and that there is in fact a unique solution to the crossing equations at $c = 25$ which will be recovered numerically if we can go to high enough Λ. Nevertheless at finite Λ the bounds we find are always valid and are obeyed by any $(2,0)$ SCFT with $c = 25$: we can say that if no assumption is made about

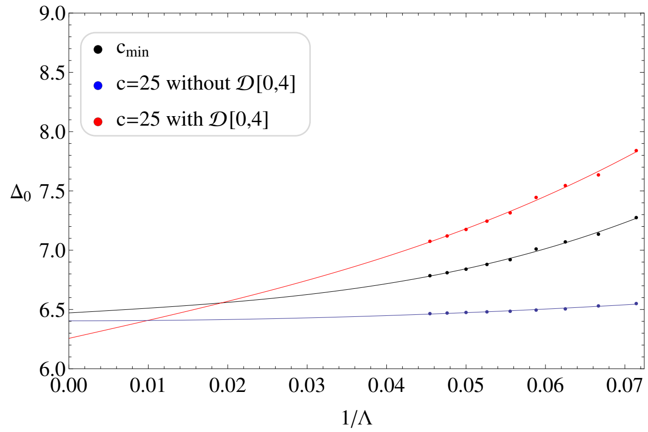


Figure 5.13: Bound on the first spin 0 superconformal primary long dimension as a function of the inverse of the cutoff Λ for the minimum central charge allowed numerically, c_{min}, for that cutoff (black) and for $c = 25$ both with (red) and without (blue) the $\mathcal{D}[0,4]$ short multiplet.

the presence of the $\mathcal{D}[0,4]$ operator, there must exist an operator of dimension *smaller or equal* to the bound one can read off from the red curve in Fig. 5.13.

Bounds on the second lowest dimensional scalar operator

We can then explore what happens if we have a theory with an operator whose dimension does not saturate the bound (which as we have seen appears to overestimate by a lot the dimension of the actual operator present in the A_1 theory) but lies somewhere below it. Denoting the bound on spin zero operators at $c = 25$, allowing for the presence of the short multiplet $\mathcal{D}[0,4]$, by $\Delta^\star$, we assume there exists an operator with dimension Δ_0, varying between the unitarity bound and $\Delta^\star$. Then we ask what is the upper bound on the second smallest dimensional operator with spin zero Δ_0'. This is shown in Fig. 5.14. We observe that the bound starts from $(\Delta_0, \Delta_0') = (6, \Delta^\star)$ and it increases sharply until a "kink", after which the increase is not as prominent, becoming approximately linear in Δ_0'. The position of the "kink" precisely corresponds to Δ_0 being equal to the dimension bound obtained in the absence of $\mathcal{D}[0,4]$ shown in Fig. 5.12. As the number of derivatives is increased we observe that the endpoint of the plot $\Delta_0 = \Delta^\star$ is moving towards the position of the kink, as we should expect from Figs. 5.13, since the bound with and without the short multiplet $\mathcal{D}[0,4]$ appears to converge to similar values. Just as in the aforementioned plots convergence seems to the faster for the bound in the absence of the short: the position of the "kink" seems to be converging faster than the end-point $\Delta^\star$. If the position of the "kink" gives indeed Δ_0 for the A_1 theory, then the bound on Δ_0' at the "kink" gives a bound on the second scalar in the theory, and seems to be converging reasonably well.

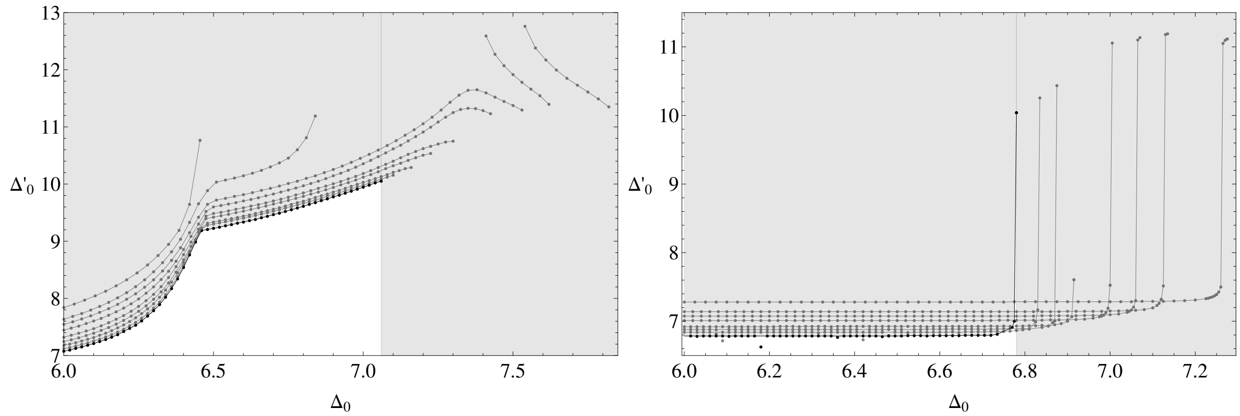


Figure 5.14: Bound on the dimension of the second scalar superconformal primary dimension Δ_0', as a function of the dimension of the dimension of the first scalar superconformal primary Δ_0 for $c = 25$ (left) and $c = c_{min}$ (right). The dimension Δ_0 is allowed to vary from the unitarity bound to the bound obtained in in Fig. 5.6 allowing for the short operator $\mathcal{D}[0, 4]$. The cutoff is increased from $\Lambda = 14$ to 22. The sharp step-like jump in the figure on the right illustrates the uniqueness of solution to the truncated crossing symmetry equations at c_{min}.

It also seems that the increase in Δ_0' to the left of the "kink" is becoming more accentuated, and it seems plausible that with infinitely many derivatives it will give rise to a step-like behavior. This is exactly what would be expected if there were a unique solution to crossing symmetry at $c = 25$ – the one with an operator whose dimension is given by the position of the "kink". Our expectation is that the shape of the bound will converge to the one obtained at finite Λ at $c = c_{min}(\Lambda)$, which is shown on the right side of Fig. 5.14.[12] There since there is a unique solution to our truncated set of crossing equations the first operator should have dimension exactly $\Delta^\star$. This explains what is observed on the right side of 5.14: if we assume there exists an operator of smaller $\Delta_0 < \Delta^\star$ then we find $\Delta_0' = \Delta^\star$, showing that the operator of dimension $\Delta^\star$ must in fact be present.

To provide further evidence that we will in fact find uniqueness at $c = 25$ let us now bound the OPE coefficient of a long with dimension $6 \leq \Delta_0 \leq \Delta^\star$. This is shown on the left side of Fig. 5.15, and the OPE coefficient peaks around the position of the kink. If there were a unique solution to the crossing equations the upper bound on the OPE coefficient squared should be zero, except when Δ_0 corresponds to the dimension of the operator in the unique solution. We do not see such a sharp behavior, but once again it is plausible that with $\Lambda \to \infty$ that will be the case.

[12]The bound shown is actually at $c = c_{min} + 0.01$, since at c_{min} a functional is always found, even if no bound is imposed, whose zeros give us the dimensions of the operators present in the unique solution to the truncated crossing equation. This therefore makes it hard to obtain bounds on operator dimensions.

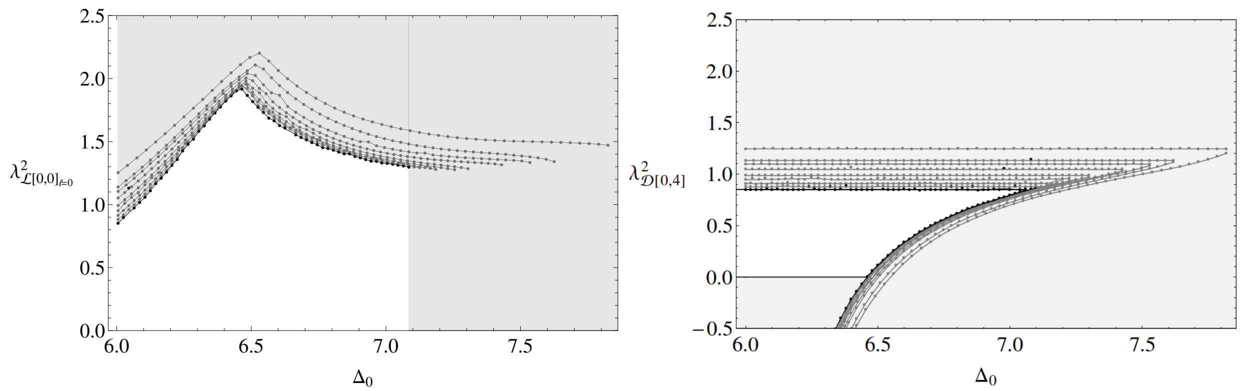


Figure 5.15: Left: Bound on the OPE coefficient (squared) of a scalar operator of dimension Δ_0, varying from the unitarity bound to the dimension bound obtained in Fig. 5.6 (allowing for the short operator $\mathcal{D}[0,4]$) at $c = 25$. Right: Lower and upper bound on the OPE coefficient squared of the $\mathcal{D}[0,4]$ multiplet as a function of the dimension of the dimension of the first scalar superconformal primary Δ_0 and for $c = 25$. The cutoff is increased from $\Lambda = 14$ to 22. The excluded values of the OPE coefficient correspond to the shaded region.

The fact that the position of the "kink" coincides with the bound obtained in the absence of $\mathcal{D}[0,4]$ can be understood by bounding the OPE coefficient of this multiplet after imposing different gaps Δ_0. The short multiplet $\mathcal{D}[0,4]$ is exactly at the spin zero $\mathcal{L}[0,0]$ multiplet unitarity bound, therefore if a gap is imposed in this channel, the short multiplet block becomes isolated. If this is the case we can impose also a lower bound on its OPE coefficient squared by requiring the functional to be -1 on $a^{\text{at}}_{6,0}(z,\bar{z})$. We can then ask the question of what is the lower bound on this OPE coefficient as a function of the gap imposed in the $\ell = 0$ channel, which can be combined with the upper bound for each gap. The resulting bounds for central charge $c = 25$ are shown on the right side of Fig. 5.15, where the excluded values of the OPE coefficient squared are shaded. We see that the OPE coefficient is allowed to be zero precisely for dimensions smaller or equal than that of the "kink" position. (Note that for dimensions smaller than the "kink" the obtained numerical bound is negative, and as such it is no longer relevant, as unitarity requires $\lambda^2_{\mathcal{D}[0,4]} \geq 0$.) Moreover as the number of derivatives is increased it seems that the upper bound is converging to a small number, which could plausibly be zero.

Bounds on the second lowest dimensional spinning operators

Finally let us explore the dimensions of the second operators of spin two and four. This is shown in Fig. 5.16 where we assume there to exist an operator of dimension $\Delta_{2,4}$ in the allowed range and bound the dimension of the second operator $\Delta'_{2,4}$. For spin two the bound for $\Delta_2 = 8$ starts off at $\Delta'_2 = \Delta^\star_2$, and increases for larger Δ_2. There is a "kink"-like feature for a small value of Δ_2 which is most likely an artefact of working at finite Λ, similarly to what was observed in Fig. 5.7. The spin four bounds appear to be closer to the optimal result with $\Lambda \to \infty$, as the bounds appear to have converged. We see that as long as Δ_4 is smaller than its maximum allowed value of $\Delta_4^\star$, the second operator Δ_4' takes a dimension close to $\Delta_4^\star$, then when Δ_4 approaches $\Delta_4^\star$ the value of Δ_4' grows sharply, resembling the plot on the right of Fig. 5.14. This provides further justification of the claim that the optimal bound will resemble the one observed at c_{min}, providing further evidence to our uniqueness claim.

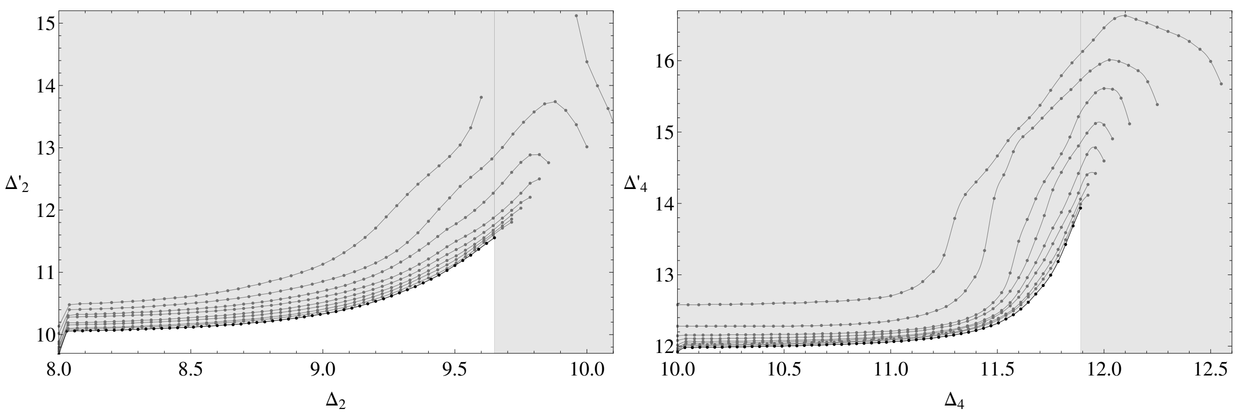


Figure 5.16: Bound on the dimension of the second spin $\ell = 2, 4$ superconformal primary dimension $\Delta_{2,4}'$, as a function of the dimension of the dimension of the first spin $\ell = 2, 4$ superconformal primary $\Delta_{2,4}$ for $c = 25$. The cutoff is increased from $\Lambda = 14$ to 22.

Summary

The results presented in this section are all consistent and seem to support the conjecture that there is a unique solution to crossing symmetry at $c = 25$. Moreover, while limited by working at a finite Λ, by making the right assumptions about the theory we want to study, namely that the is no $\mathcal{D}[0, 4]$ operator in the A_1 theory, we seem to obtain bounds that converge faster. In this way, we are able to obtain what should correspond to the physical value of the dimension of the first spin zero operator in the A_1 theory. Another way to overcome the fact that we are constrained to work at finite Λ is to impose simultaneous bounds. In this spirit we have seen that even if the the $\mathcal{D}[0, 4]$ multiplet is allowed to be present, the corner position of the cube shown on the left side of Fig. 5.10 provides with a better estimate of the spin zero operator dimension. In a similar way the corner of the cube shown in the right side of Fig. 5.10 should give a bound close to the actual dimensions of the A_1 theory. This could help overcome the slow convergence of the spin two dimension bounds. Finally the spin four dimension bounds seem to be converging faster, hinting already at features expected at $\Lambda = \infty$ coming as a consequence of uniqueness. In this way we expect that the bound obtain for the spin four operator is close to the actual value of the A_1 theory.

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Appendix A

Superconformal algebras

This appendix lists useful superconformal algebras that are used in the body of chapter 2. We adopt the convention of working in terms of the complexified version of symmetry algebras. We adopt bases for the complexified algebras such that the restriction to the real form that is relevant for physics in Lorentzian signature is the most natural. In general, the structures described in chapter 2 are insensitive to the spacetime signature of the four-dimensional theory, with the caveat that we will assume that the theories in question, when Wick rotated to Lorentzian signature, are unitary.

A.1 The four-dimensional superconformal algebra

The spacetime symmetry algebra for $\mathcal{N}=2$ superconformal field theories in four dimensions is the superalgebra $\mathfrak{sl}(4\,|\,2)$. The maximal bosonic subalgebra is $\mathfrak{so}(6,\mathbb{C})\times\mathfrak{sl}(2)_R\times\mathbb{C}^*$. The $\mathfrak{so}(6,\mathbb{C})$ conformal algebra, in a spinorial basis with $\alpha,\dot\alpha=1,2$, is given by

$$
\begin{aligned}
[\mathcal{M}_\alpha{}^\beta,\mathcal{M}_\gamma{}^\delta] &= \delta_\gamma{}^\beta\mathcal{M}_\alpha{}^\delta-\delta_\alpha{}^\delta\mathcal{M}_\gamma{}^\beta\ ,\\
[\mathcal{M}^{\dot\alpha}{}_{\dot\beta},\mathcal{M}^{\dot\gamma}{}_{\dot\delta}] &= \delta^{\dot\alpha}{}_{\dot\delta}\mathcal{M}^{\dot\gamma}{}_{\dot\beta}-\delta^{\dot\gamma}{}_{\dot\beta}\mathcal{M}^{\dot\alpha}{}_{\dot\delta}\ ,\\
[\mathcal{M}_\alpha{}^\beta,\mathcal{P}_{\gamma\dot\gamma}] &= \delta_\gamma{}^\beta\mathcal{P}_{\alpha\dot\gamma}-\tfrac{1}{2}\delta_\alpha{}^\beta\mathcal{P}_{\gamma\dot\gamma}\ ,\\
[\mathcal{M}^{\dot\alpha}{}_{\dot\beta},\mathcal{P}_{\gamma\dot\gamma}] &= \delta^{\dot\alpha}{}_{\dot\gamma}\mathcal{P}_{\gamma\dot\beta}-\tfrac{1}{2}\delta^{\dot\alpha}{}_{\dot\beta}\mathcal{P}_{\gamma\dot\gamma}\ ,\\
[\mathcal{M}_\alpha{}^\beta,\mathcal{K}^{\dot\gamma\gamma}] &= -\,\delta_\alpha{}^\gamma\mathcal{K}^{\dot\gamma\beta}+\tfrac{1}{2}\delta_\alpha{}^\beta\mathcal{K}^{\dot\gamma\gamma}\ ,\\
[\mathcal{M}^{\dot\alpha}{}_{\dot\beta},\mathcal{K}^{\dot\gamma\gamma}] &= -\,\delta^{\dot\gamma}{}_{\dot\beta}\mathcal{K}^{\dot\alpha\gamma}+\tfrac{1}{2}\delta^{\dot\alpha}{}_{\dot\beta}\mathcal{K}^{\dot\gamma\gamma}\ ,\\
[\mathcal{H},\mathcal{P}_{\alpha\dot\alpha}] &= \mathcal{P}_{\alpha\dot\alpha}\ ,\\
[\mathcal{H},\mathcal{K}^{\dot\alpha\alpha}] &= -\,\mathcal{K}^{\dot\alpha\alpha}\ ,\\
[\mathcal{K}^{\dot\alpha\alpha},\mathcal{P}_{\beta\dot\beta}] &= \delta_\beta{}^\alpha\delta^{\dot\alpha}{}_{\dot\beta}\mathcal{H}+\delta_\beta{}^\alpha\mathcal{M}^{\dot\alpha}{}_{\dot\beta}+\delta^{\dot\alpha}{}_{\dot\beta}\mathcal{M}_\beta{}^\alpha\ .
\end{aligned}
\tag{A.1}
$$

The $\mathfrak{sl}(2)_R$ algebra has a Chevalley basis of generators $\mathcal{R}^{\pm}$ and $\mathcal{R}$, where

$$[\mathcal{R}^+, \mathcal{R}^-] = 2\mathcal{R} \ , \qquad [\mathcal{R}, \mathcal{R}^{\pm}] = \pm\mathcal{R}^{\pm} \ . \tag{A.2}$$

In Lorentz signature where the appropriate real form of this algebra is $\mathfrak{su}(2)_R$, these generators will obey hermiticity conditions $(\mathcal{R}^+)^\dagger = \mathcal{R}^-$, $\mathcal{R}^\dagger = \mathcal{R}$. The generator of the Abelian factor $\mathbb{C}^*$ is denoted by r and is central in the bosonic part of the algebra. It is also convenient to introduce the basis $\mathcal{R}^{\mathcal{I}}{}_{\mathcal{J}}$, with

$$\mathcal{R}^1{}_2 = \mathcal{R}^+ \ , \qquad \mathcal{R}^2{}_1 = \mathcal{R}^- \ , \qquad \mathcal{R}^1{}_1 = \frac{1}{2}r + \mathcal{R} \ , \qquad \mathcal{R}^2{}_2 = \frac{1}{2}r - \mathcal{R} \ , \tag{A.3}$$

where we follow the conventions of [52] for r, and which obey the commutation relations

$$[\mathcal{R}^{\mathcal{I}}{}_{\mathcal{J}}, \mathcal{R}^{\mathcal{K}}{}_{\mathcal{L}}] = \delta^{\mathcal{K}}{}_{\mathcal{J}}\mathcal{R}^{\mathcal{I}}{}_{\mathcal{L}} - \delta^{\mathcal{I}}{}_{\mathcal{L}}\mathcal{R}^{\mathcal{K}}{}_{\mathcal{J}} \ . \tag{A.4}$$

There are sixteen fermionic generators in this superconformal algebra – eight Poincaré supercharges and eight conformal supercharges – denoted $\{\mathcal{Q}^{\mathcal{I}}_{\alpha},\ \widetilde{\mathcal{Q}}_{\mathcal{I}\dot{\alpha}},\ \mathcal{S}^{\alpha}_{\mathcal{J}},\ \widetilde{\mathcal{S}}^{\mathcal{J}\dot{\alpha}}\}$. The nonvanishing commutators amongst them are as follows,

$$\begin{aligned}
\{\mathcal{Q}^{\mathcal{I}}_{\alpha},\ \widetilde{\mathcal{Q}}_{\mathcal{J}\dot{\alpha}}\} &= \delta^{\mathcal{I}}{}_{\mathcal{J}}\mathcal{P}_{\alpha\dot{\alpha}} \ , \\
\{\widetilde{\mathcal{S}}^{\mathcal{I}\dot{\alpha}},\ \mathcal{S}_{\mathcal{J}}{}^{\alpha}\} &= \delta^{\mathcal{I}}{}_{\mathcal{J}}\mathcal{K}^{\dot{\alpha}\alpha} \ , \\
\{\mathcal{Q}^{\mathcal{I}}_{\alpha},\ \mathcal{S}_{\mathcal{J}}{}^{\beta}\} &= \tfrac{1}{2}\delta^{\mathcal{I}}{}_{\mathcal{J}}\delta_{\alpha}{}^{\beta}\mathcal{H} + \delta^{\mathcal{I}}{}_{\mathcal{J}}\mathcal{M}_{\alpha}{}^{\beta} - \delta_{\alpha}{}^{\beta}\mathcal{R}^{\mathcal{I}}{}_{\mathcal{J}} \ , \\
\{\widetilde{\mathcal{S}}^{\mathcal{I}\dot{\alpha}},\ \widetilde{\mathcal{Q}}_{\mathcal{J}\dot{\beta}}\} &= \tfrac{1}{2}\delta^{\mathcal{I}}{}_{\mathcal{J}}\delta^{\dot{\alpha}}{}_{\dot{\beta}}\mathcal{H} + \delta^{\mathcal{I}}{}_{\mathcal{J}}\mathcal{M}^{\dot{\alpha}}{}_{\dot{\beta}} + \delta^{\dot{\alpha}}{}_{\dot{\beta}}\mathcal{R}^{\mathcal{I}}{}_{\mathcal{J}} \ .
\end{aligned} \tag{A.5}$$

Finally, the commutators of the supercharges with the bosonic symmetry generators are the following:

$$\begin{aligned}
[\mathcal{M}_{\alpha}{}^{\beta}, \mathcal{Q}^{\mathcal{I}}_{\gamma}] &= \delta_{\gamma}{}^{\beta}\mathcal{Q}^{\mathcal{I}}_{\alpha} - \tfrac{1}{2}\delta_{\alpha}{}^{\beta}\mathcal{Q}^{\mathcal{I}}_{\gamma} \ , \\
[\mathcal{M}^{\dot{\alpha}}{}_{\dot{\beta}}, \widetilde{\mathcal{Q}}_{\mathcal{I}\dot{\delta}}] &= \delta^{\dot{\alpha}}{}_{\dot{\delta}}\widetilde{\mathcal{Q}}_{\mathcal{I}\dot{\beta}} - \tfrac{1}{2}\delta^{\dot{\alpha}}{}_{\dot{\beta}}\widetilde{\mathcal{Q}}_{\mathcal{I}\dot{\delta}} \ , \\
[\mathcal{M}_{\alpha}{}^{\beta}, \mathcal{S}_{\mathcal{I}}{}^{\gamma}] &= -\delta_{\alpha}{}^{\gamma}\mathcal{S}_{\mathcal{I}}{}^{\beta} + \tfrac{1}{2}\delta_{\alpha}{}^{\beta}\mathcal{S}_{\mathcal{I}}{}^{\gamma} \ , \\
[\mathcal{M}^{\dot{\alpha}}{}_{\dot{\beta}}, \widetilde{\mathcal{S}}^{\mathcal{I}\dot{\gamma}}] &= -\delta^{\dot{\gamma}}{}_{\dot{\beta}}\widetilde{\mathcal{S}}^{\mathcal{I}\dot{\alpha}} + \tfrac{1}{2}\delta^{\dot{\alpha}}{}_{\dot{\beta}}\widetilde{\mathcal{S}}^{\mathcal{I}\dot{\gamma}} \ , \\
[\mathcal{H}, \mathcal{Q}^{\mathcal{I}}_{\alpha}] &= \tfrac{1}{2}\mathcal{Q}^{\mathcal{I}}_{\alpha} \ , \\
[\mathcal{H}, \widetilde{\mathcal{Q}}_{\mathcal{I}\dot{\alpha}}] &= \tfrac{1}{2}\widetilde{\mathcal{Q}}_{\mathcal{I}\dot{\alpha}} \ , \\
[\mathcal{H}, \mathcal{S}_{\mathcal{I}}{}^{\alpha}] &= -\tfrac{1}{2}\mathcal{S}_{\mathcal{I}}{}^{\alpha} \ , \\
[\mathcal{H}, \widetilde{\mathcal{S}}^{\mathcal{I}\dot{\alpha}}] &= -\tfrac{1}{2}\widetilde{\mathcal{S}}^{\mathcal{I}\dot{\alpha}} \ ,
\end{aligned} \tag{A.6}$$

$$
\begin{aligned}
[\mathcal{R}^{\mathcal{I}}{}_{\mathcal{J}}, \mathcal{Q}^{\mathcal{K}}_{\alpha}] &= \delta_{\mathcal{J}}{}^{\mathcal{K}} \mathcal{Q}^{\mathcal{I}}_{\alpha} - \frac{1}{4}\delta^{\mathcal{I}}_{\mathcal{J}} \mathcal{Q}^{\mathcal{K}}_{\alpha} \ , \\
[\mathcal{R}^{\mathcal{I}}{}_{\mathcal{J}}, \widetilde{\mathcal{Q}}_{\mathcal{K}\dot{\alpha}}] &= -\,\delta_{\mathcal{K}}{}^{\mathcal{I}} \widetilde{\mathcal{Q}}_{\mathcal{J}\dot{\alpha}} + \frac{1}{4}\delta^{\mathcal{I}}_{\mathcal{J}} \widetilde{\mathcal{Q}}_{\mathcal{K}\dot{\alpha}} \ , \\
[\mathcal{K}^{\dot{\alpha}\alpha}, \mathcal{Q}^{\mathcal{I}}_{\beta}] &= \delta_{\beta}{}^{\alpha} \widetilde{\mathcal{S}}^{\mathcal{I}\dot{\alpha}} \ , \\
[\mathcal{K}^{\dot{\alpha}\alpha}, \widetilde{\mathcal{Q}}_{\mathcal{I}\dot{\beta}}] &= \delta_{\dot{\beta}}{}^{\dot{\alpha}} \mathcal{S}_{\mathcal{I}}{}^{\alpha} \ , \\
[\mathcal{P}_{\alpha\dot{\alpha}}, \mathcal{S}_{\mathcal{I}}{}^{\beta}] &= -\,\delta_{\alpha}{}^{\beta} \widetilde{\mathcal{Q}}_{\mathcal{I}\dot{\alpha}} \ , \\
[\mathcal{P}_{\alpha\dot{\alpha}}, \widetilde{\mathcal{S}}^{\mathcal{I}\dot{\beta}}] &= -\,\delta_{\dot{\alpha}}{}^{\dot{\beta}} \mathcal{Q}^{\mathcal{I}}_{\alpha} \ .
\end{aligned}
$$

A.2 The two-dimensional superconformal algebra

The second superalgebra of interest is $\mathfrak{sl}(2|2)$, which corresponds to the right-moving part of the global superconformal algebra in $\mathcal{N} = (0,4)$ SCFTs in two dimensions. The maximal bosonic subgroup is $\mathfrak{sl}(2)\times\mathfrak{sl}(2)_R$, with generators $\{L_0, L_{\pm1}\}$ for $\mathfrak{sl}(2)$ and $\{\mathcal{R}^{\pm}, \mathcal{R}\}$ for $\mathfrak{sl}(2)_R$. The non-vanishing bosonic commutation relations are given by

$$
\begin{aligned}
[\mathcal{R}, \mathcal{R}^{\pm}] &= \pm\mathcal{R}^{\pm} \ , \qquad & [\mathcal{R}^{+}, \mathcal{R}^{-}] &= 2\mathcal{R} \ , \\
[\bar{L}_0, \bar{L}_{\pm1}] &= \mp\bar{L}_{\pm1} \ , \qquad & [\bar{L}_1, \bar{L}_{-1}] &= 2\bar{L}_0 \ .
\end{aligned}
$$

There are additionally right-moving Poincaré supercharges $\mathcal{Q}^{\mathcal{I}}$, $\widetilde{\mathcal{Q}}_{\mathcal{J}}$ and right-moving superconformal charges $\mathcal{S}_{\mathcal{J}}$, $\widetilde{\mathcal{S}}^{\mathcal{I}}$. The commutation relations amongst the fermionic generators are given by

$$
\begin{aligned}
\{\mathcal{Q}^{\mathcal{I}}, \widetilde{\mathcal{Q}}_{\mathcal{J}}\} &= \delta^{\mathcal{I}}_{\mathcal{J}} \bar{L}_{-1} \ , \\
\{\widetilde{\mathcal{S}}^{\mathcal{I}}, \mathcal{S}_{\mathcal{J}}\} &= \delta^{\mathcal{I}}_{\mathcal{J}} \bar{L}_{+1} \ , \\
\{\mathcal{Q}^{\mathcal{I}}, \mathcal{S}_{\mathcal{J}}\} &= \delta^{\mathcal{I}}_{\mathcal{J}} \bar{L}_0 - \mathcal{R}^{\mathcal{I}}_{\mathcal{J}} - \frac{1}{2}\delta^{\mathcal{I}}_{\mathcal{J}} \mathcal{Z} \ , \\
\{\widetilde{\mathcal{Q}}_{\mathcal{J}}, \widetilde{\mathcal{S}}^{\mathcal{I}}\} &= \delta^{\mathcal{I}}_{\mathcal{J}} \bar{L}_0 + \mathcal{R}^{\mathcal{I}}_{\mathcal{J}} + \frac{1}{2}\delta^{\mathcal{I}}_{\mathcal{J}} \mathcal{Z} \ ,
\end{aligned}
$$

where $\mathcal{R}^{\mathcal{I}}{}_{\mathcal{J}}$ are defined as in (A.3), but with r set to zero. Here $\mathcal{Z}$ is a central element, the removal of which gives the algebra $\mathfrak{psl}(2|2)$. The additional commutators of bosonic symmetry generators with the supercharges are given by

$$
\begin{aligned}
[\bar{L}_{-1}\,, \widetilde{\mathcal{S}}^{\mathcal{I}}] &= -\mathcal{Q}^{\mathcal{I}}\ , \\
[\bar{L}_{-1}\,, \mathcal{S}_{\mathcal{I}}] &= -\widetilde{\mathcal{Q}}_{\mathcal{I}}\ , \\
[\bar{L}_{+1}\,, \widetilde{\mathcal{Q}}_{\mathcal{I}}] &= \mathcal{S}_{\mathcal{I}}\ , \\
[\bar{L}_{+1}\,, \mathcal{Q}^{\mathcal{I}}] &= \widetilde{\mathcal{S}}^{\mathcal{I}}\ , \\
[\bar{L}_{0}\quad, \widetilde{\mathcal{S}}^{\mathcal{I}}] &= -\tfrac{1}{2}\widetilde{\mathcal{S}}^{\mathcal{I}}\ , \\
[\bar{L}_{0}\quad, \mathcal{S}_{\mathcal{I}}] &= -\tfrac{1}{2}\mathcal{S}_{\mathcal{I}}\ , \\
[\bar{L}_{0}\quad, \widetilde{\mathcal{Q}}_{\mathcal{I}}] &= \tfrac{1}{2}\widetilde{\mathcal{Q}}_{\mathcal{I}}\ , \\
[\bar{L}_{0}\quad, \mathcal{Q}^{\mathcal{I}}] &= \tfrac{1}{2}\mathcal{Q}^{\mathcal{I}}\ .
\end{aligned}
\tag{A.7}
$$

Appendix B

Unitary representations of the $\mathcal{N} = 2$ superconformal algebra

The representation theory of the four-dimensional $\mathcal{N} = 2$ superconformal algebra plays a central role in this dissertation. The classification of short representations of the four-dimensional $\mathcal{N} = 2$ superconformal algebra [14, 52, 154] plays a major role in the structure of the chiral algebras described in chapter 2.[1] It also plays a central role in our choice of strategy and in the structure of the partial wave analysis of four-point functions of chapter 4. This appendix provides a review of the classification of unitary irreducible representations of $\mathfrak{su}(2,2|2)$ (*cf.* [14, 52, 154]), as well as of the various indices that can be defined on any representation of the algebra that are insensitive to the recombination of collections of short multiplets into generic long multiplets.

Unitary representations of $\mathfrak{su}(2,2|2)$ are highest weight representations and are labelled by quantum numbers (Δ, j_1, j_2, r, R) of the highest weight state also called the *superconformal primary* of the representation. A *generic representation* – also called a *long representation* – is obtained by the action of the eight Poincaré supercharges as well as the momentum generators and $SU(2)_R$ lowering operators on the highest weight state. *Short representations* occur when a superconformal descendant state in what would otherwise be a long representation is rendered null by a conspiracy of quantum numbers. The unitarity bounds for a superconformal primary operator are given by

$$\begin{aligned} &\Delta \geqslant \Delta_i \ , && j_i \neq 0 \ , \\ &\Delta = \Delta_i - 2 \quad \text{or} \quad \Delta \geqslant \Delta_i \ , && j_i = 0 \ , \end{aligned} \tag{B.1}$$

[1] Note that in this appendix we use Δ instead of the E used in chapter 2 to denote the conformal dimension of operators.

where we have defined

$$\Delta_1 := 2 + 2j_1 + 2R + r \ , \qquad \Delta_2 := 2 + 2j_2 + 2R - r \ . \tag{B.2}$$

Short representations occur when one or more of these bounds are saturated. The different ways in which this can happen correspond to different combinations of Poincaré supercharges that will annihilate the superconformal primary state in the representation.

There are two types of shortening conditions, each of which has four incarnations corresponding to an $SU(2)_R$ doublet's worth of conditions for each supercharge chirality:

$$\mathcal{B}^{\mathcal{I}} \quad : \qquad \mathcal{Q}^{\mathcal{I}}_{\alpha}|\psi\rangle = 0 \ , \quad \alpha = 1, 2 \ , \tag{B.3}$$

$$\bar{\mathcal{B}}_{\mathcal{I}} \quad : \qquad \widetilde{\mathcal{Q}}_{\mathcal{I}\dot{\alpha}}|\psi\rangle = 0 \ , \quad \dot{\alpha} = 1, 2 \ , \tag{B.4}$$

$$\mathcal{C}^{\mathcal{I}} \quad : \qquad \begin{cases} \epsilon^{\alpha\beta}\mathcal{Q}^{\mathcal{I}}_{\alpha}|\psi\rangle_{\beta} = 0 \ , & j_1 \neq 0 \ , \\ \epsilon^{\alpha\beta}\mathcal{Q}^{\mathcal{I}}_{\alpha}\mathcal{Q}^{\mathcal{I}}_{\beta}|\psi\rangle = 0 \ , & j_1 = 0 \ , \end{cases} \tag{B.5}$$

$$\bar{\mathcal{C}}_{\mathcal{I}} \quad : \qquad \begin{cases} \epsilon^{\dot{\alpha}\dot{\beta}}\widetilde{\mathcal{Q}}_{\mathcal{I}\dot{\alpha}}|\psi\rangle_{\dot{\beta}} = 0 \ , & j_2 \neq 0 \ , \\ \epsilon^{\dot{\alpha}\dot{\beta}}\widetilde{\mathcal{Q}}_{\mathcal{I}\dot{\alpha}}\widetilde{\mathcal{Q}}_{\mathcal{I}\dot{\beta}}|\psi\rangle = 0 \ , & j_2 = 0 \ . \end{cases} \tag{B.6}$$

The different admissible combinations of shortening conditions that can be simultaneously realized by a single unitary representation are summarized in Table B.1, where we also list the relations that must be satisfied by the quantum numbers of the superconformal primary in such a representation. We also list two common notations used to designate the different representations – one from [52] (DO) and the other from [14] (KMMR).[2]

In the limit where the dimension of a long representation approaches a unitarity bound, it becomes decomposable into a collection of short representations. This fact is often referred to as the existence of *recombination rules* for short representations into a long representation at the unitarity bound. The generic recombination rules are as follows,

$$\begin{aligned} \mathcal{A}^{\Delta \to 2R+r+2+2j_1}_{R,r(j_1,j_2)} &\simeq \mathcal{C}_{R,r(j_1,j_2)} \oplus \mathcal{C}_{R+\frac{1}{2},r+\frac{1}{2}(j_1-\frac{1}{2},j_2)} \ , \\ \mathcal{A}^{\Delta \to 2R-r+2+2j_2}_{R,r(j_1,j_2)} &\simeq \bar{\mathcal{C}}_{R,r(j_1,j_2)} \oplus \bar{\mathcal{C}}_{R+\frac{1}{2},r-\frac{1}{2}(j_1,j_2-\frac{1}{2})} \ , \\ \mathcal{A}^{\Delta \to 2R+j_1+j_2+2}_{R,j_1-j_2(j_1,j_2)} &\simeq \hat{\mathcal{C}}_{R(j_1,j_2)} \oplus \hat{\mathcal{C}}_{R+\frac{1}{2}(j_1-\frac{1}{2},j_2)} \oplus \hat{\mathcal{C}}_{R+\frac{1}{2}(j_1,j_2-\frac{1}{2})} \oplus \hat{\mathcal{C}}_{R+1(j_1-\frac{1}{2},j_2-\frac{1}{2})} \ . \end{aligned} \tag{B.7}$$

In special cases the quantum numbers of the long multiplet at threshold are such that some Lorentz quantum numbers in (B.7) would be negative and unphysical. In these cases the

[2]We are adopting the the R-charge conventions of [52].

Shortening	Quantum Number Relations			DO	KMMR
$\varnothing$	$\Delta \geqslant \max(\Delta_1, \Delta_2)$			$\mathcal{A}^{\Delta}_{R,r(j_1,j_2)}$	$\mathbf{aa}_{\Delta,j_1,j_2,r,R}$
$\mathcal{B}^1$	$\Delta = 2R + r$	$j_1 = 0$		$\mathcal{B}_{R,r(0,j_2)}$	$\mathbf{ba}_{0,j_2,r,R}$
$\bar{\mathcal{B}}_2$	$\Delta = 2R - r$	$j_2 = 0$		$\bar{\mathcal{B}}_{R,r(j_1,0)}$	$\mathbf{ab}_{j_1,0,r,R}$
$\mathcal{B}^1 \cap \mathcal{B}^2$	$\Delta = r$	$R = 0$		$\mathcal{E}_{r(0,j_2)}$	$\mathbf{ba}_{0,j_2,r,0}$
$\bar{\mathcal{B}}_1 \cap \bar{\mathcal{B}}_2$	$\Delta = -r$	$R = 0$		$\bar{\mathcal{E}}_{r(j_1,0)}$	$\mathbf{ab}_{j_1,0,r,0}$
$\mathcal{B}^1 \cap \bar{\mathcal{B}}_2$	$\Delta = 2R$	$j_1 = j_2 = r = 0$		$\hat{\mathcal{B}}_R$	$\mathbf{bb}_{0,0,0,R}$
$\mathcal{C}^1$	$\Delta = 2 + 2j_1 + 2R + r$			$\mathcal{C}_{R,r(j_1,j_2)}$	$\mathbf{ca}_{j_1,j_2,r,R}$
$\bar{\mathcal{C}}_2$	$\Delta = 2 + 2j_2 + 2R - r$			$\bar{\mathcal{C}}_{R,r(j_1,j_2)}$	$\mathbf{ac}_{j_1,j_2,r,R}$
$\mathcal{C}^1 \cap \mathcal{C}^2$	$\Delta = 2 + 2j_1 + r$	$R = 0$		$\mathcal{C}_{0,r(j_1,j_2)}$	$\mathbf{ca}_{j_1,j_2,r,0}$
$\bar{\mathcal{C}}_1 \cap \bar{\mathcal{C}}_2$	$\Delta = 2 + 2j_2 - r$	$R = 0$		$\bar{\mathcal{C}}_{0,r(j_1,j_2)}$	$\mathbf{ac}_{j_1,j_2,r,0}$
$\mathcal{C}^1 \cap \bar{\mathcal{C}}_2$	$\Delta = 2 + 2R + j_1 + j_2$	$r = j_2 - j_1$		$\hat{\mathcal{C}}_{R(j_1,j_2)}$	$\mathbf{cc}_{j_1,j_2,j_2-j_1,R}$
$\mathcal{B}^1 \cap \bar{\mathcal{C}}_2$	$\Delta = 1 + 2R + j_2$	$r = j_2 + 1$		$\mathcal{D}_{R(0,j_2)}$	$\mathbf{bc}_{0,j_2,j_2+1,R}$
$\bar{\mathcal{B}}_2 \cap \mathcal{C}^1$	$\Delta = 1 + 2R + j_1$	$-r = j_1 + 1$		$\bar{\mathcal{D}}_{R(j_1,0)}$	$\mathbf{cb}_{j_1,0,-j_1-1,R}$
$\mathcal{B}^1 \cap \mathcal{B}^2 \cap \bar{\mathcal{C}}_2$	$\Delta = r = 1 + j_2$	$r = j_2 + 1$	$R = 0$	$\mathcal{D}_{0(0,j_2)}$	$\mathbf{bc}_{0,j_2,j_2+1,0}$
$\mathcal{C}^1 \cap \bar{\mathcal{B}}_1 \cap \bar{\mathcal{B}}_2$	$\Delta = -r = 1 + j_1$	$-r = j_1 + 1$	$R = 0$	$\bar{\mathcal{D}}_{0(j_1,0)}$	$\mathbf{cb}_{j_1,0,-j_1-1,0}$

Table B.1: Summary of unitary irreducible representations of the $\mathcal{N} = 2$ superconformal algebra.

following exceptional recombination rules apply,

$$
\begin{aligned}
\mathcal{A}^{2R+r+2}_{R,r(0,j_2)} &\simeq \mathcal{C}_{R,r(0,j_2)} \oplus \mathcal{B}_{R+1,r+\frac{1}{2}(0,j_2)}\ , \\
\mathcal{A}^{2R-r+2}_{R,r(j_1,0)} &\simeq \bar{\mathcal{C}}_{R,r(j_1,0)} \oplus \bar{\mathcal{B}}_{R+1,r-\frac{1}{2}(j_1,0)}\ , \\
\mathcal{A}^{2R+j_2+2}_{R,-j_2(0,j_2)} &\simeq \hat{\mathcal{C}}_{R(0,j_2)} \oplus \mathcal{D}_{R+1(0,j_2)} \oplus \hat{\mathcal{C}}_{R+\frac{1}{2}(0,j_2-\frac{1}{2})} \oplus \mathcal{D}_{R+\frac{3}{2}(0,j_2-\frac{1}{2})}\ , \\
\mathcal{A}^{2R+j_1+2}_{R,j_1(j_1,0)} &\simeq \hat{\mathcal{C}}_{R(j_1,0)} \oplus \hat{\mathcal{C}}_{R+\frac{1}{2}(j_1-\frac{1}{2},0)} \oplus \bar{\mathcal{D}}_{R+1(j_1,0)} \oplus \bar{\mathcal{D}}_{R+\frac{3}{2}(j_1-\frac{1}{2},0)}\ , \\
\mathcal{A}^{2R+2}_{R,0(0,0)} &\simeq \hat{\mathcal{C}}_{R(0,0)} \oplus \mathcal{D}_{R+1(0,0)} \oplus \bar{\mathcal{D}}_{R+1(0,0)} \oplus \hat{\mathcal{B}}_{R+2}\ .
\end{aligned}
\tag{B.8}
$$

The last three recombinations involve multiplets that make an appearance in the associated chiral algebra described in this work. The recombinations that play a role in the analyses of chapter 4 are the last recombinations in (B.7) and (B.8). This is relevant for the partial wave analysis of the moment map four-point function in Section 4.3.1. Note that the $\mathcal{E}$, $\bar{\mathcal{E}}$, $\hat{\mathcal{B}}_{\frac{1}{2}}$, $\hat{\mathcal{B}}_1$, $\hat{\mathcal{B}}_{\frac{3}{2}}$, $\mathcal{D}_0$, $\bar{\mathcal{D}}_0$, $\mathcal{D}_{\frac{1}{2}}$ and $\bar{\mathcal{D}}_{\frac{1}{2}}$ multiplets can never recombine, along with $\mathcal{B}_{\frac{1}{2},r(0,j_2)}$ and

$\bar{\mathcal{B}}_{\frac{1}{2},r(j_1,0)}$.

There exist a variety of trace formulas [13, 14] that can be defined on the Hilbert space of an $\mathcal{N}=2$ SCFT such that the result receives contributions only from states that lie in short representations of the superconformal algebra, with the contributions being such that the indices are insensitive to recombinations. The indices are defined and named as follows:

$$\begin{aligned}
\text{Superconformal Index} &: \quad \mathrm{Tr}_{\mathcal{H}}(-1)^F p^{\frac{1}{2}(\Delta+2j_1-2R-r)} q^{\frac{1}{2}(E-2j_1-2R-r)} t^{R+r}, &\text{(B.9)}\\
\text{Macdonald} &: \quad \mathrm{Tr}_{\mathcal{H}_{\mathrm{M}}}(-1)^F q^{\frac{1}{2}(\Delta-2j_1-2R-r)} t^{R+r}\ , &\text{(B.10)}\\
\text{Schur} &: \quad \mathrm{Tr}_{\mathcal{H}}(-1)^F q^{\Delta-R}\ , &\text{(B.11)}\\
\text{Hall-Littlewood} &: \quad \mathrm{Tr}_{\mathcal{H}_{\mathrm{HL}}}(-1)^F \tau^{2\Delta-2R}\ , &\text{(B.12)}\\
\text{Coulomb} &: \quad \mathrm{Tr}_{\mathcal{H}_{\mathrm{C}}}(-1)^F \sigma^{\frac{1}{2}(\Delta+2j_1-2R-r)} \rho^{\frac{1}{2}(\Delta-2j_1-2R-r)}\ . &\text{(B.13)}
\end{aligned}$$

The specialized Hilbert spaces appearing in the trace formulas above are defined as follows,

$$\begin{aligned}
\mathcal{H}_{\mathrm{M}} &:= \left\{\psi\in\mathcal{H}\ \middle|\ \Delta+2j_1-2R-r=0\right\}\ , &\text{(B.14)}\\
\mathcal{H}_{\mathrm{HL}} &:= \left\{\psi\in\mathcal{H}\ \middle|\ \Delta-2R-r=0\ , j_1=0\right\}\ , &\text{(B.15)}\\
\mathcal{H}_{\mathrm{C}} &:= \left\{\psi\in\mathcal{H}\ \middle|\ \Delta+2j_1+r=0\right\}\ . &\text{(B.16)}
\end{aligned}$$

The different indices are sensitive to different superconformal multiplets. In particular, the Coulomb index counts only $\mathcal{E}$ and $\mathcal{D}_0$ type multiplets. These can be thought of as $\mathcal{N}=1$ chiral ring operators that are $SU(2)_R$ singlets. Similarly, the Hall-Littlewood index counts only $\hat{\mathcal{B}}_R$ and $\mathcal{D}_R$ multiplets, which can be thought of as the consistent truncation of the $\mathcal{N}=1$ chiral ring to operators that are neutral under $U(1)_r$. The Schur and Macdonald indices count only the operators that are involved in the chiral algebras of chapter 2: $\hat{\mathcal{B}}_R$, $\hat{\mathcal{C}}_R$, $\mathcal{D}$, and $\bar{\mathcal{D}}$ multiplets. The full index receives contributions from all of the multiplets appearing in Table B.1.

Appendix C

Kazhdan-Lusztig polynomials and affine characters

Computing the characters of irreducible modules of an affine Lie algebra at a negative integer level is a nontrivial task. For low levels, the multiplicity and norms of states can be found by hand using the mode expansion of the affine currents $J^A(z)$, but this computation quickly becomes rather involved. Fortunately there exists another method to compute these characters, based on the work of Kazhdan and Lusztig [188], which (with the aid of a computer) can produce results to very high order. In this appendix we give a brief introduction to this method. The interested reader is referred to, *e.g.*, [189, 190] for more details.

A generic method to obtain an irreducible representation of any (affine) Lie algebra is to start with the Verma module M built on a certain highest weight state $\psi_{h.w.}$, and then to subtract away all the null states in this module with the correct multiplicities. Let us recall that according to the Poincaré-Birkhoff-Witt theorem, the Verma module is spanned by all the states of the form

$$(E^{-\alpha_1,1})^{n_{1,1}}(E^{-\alpha_1,2})^{n_{1,2}}\ldots(E^{-\alpha_1,m_1})^{n_{1,m_1}}\ldots(E^{-\alpha_2,1})^{n_{2,1}}\ldots(E^{-\alpha_N,m_N})^{n_{N,m_N}}\psi_{h.w.}\ , \tag{C.1}$$

with nonnegative integer coefficients $n_{i,j}$. Here the $E^{-\alpha,k_\alpha}$ are the negative roots with weight $-\alpha$, and the auxiliary index $k_\alpha \in \{1,\ldots,m_\alpha\}$ is only necessary when the multiplicity m_α of the given weight is greater than one. The ordering of the roots in the above equation is arbitrary but fixed. If the highest weight state $\psi_{h.w.}$ has weight μ then the state defined as above has weight

$$\mu - \alpha_1(n_{1,1}+n_{1,2}+\ldots+n_{1,m_1}) - \alpha_2(n_{2,1}+\ldots) - \ldots - \alpha_N(\ldots+n_{N,m_N})\ , \tag{C.2}$$

and with a moment's thought one sees that the character M_μ of the Verma module is given by

$$\mathrm{char}M_\mu = e^\mu \prod_{\alpha>0}(1-e^{-\alpha})^{-\mathrm{mult}(\alpha)}\ . \tag{C.3}$$

This is the *Kostant partition function*. The product is taken over the set of all the positive roots, which is infinite for an affine Lie algebra.

For a given affine Lie algebra there are special values of the highest weights for which the Verma module becomes reducible due to the existence of null states. We need to subtract all these null states to recover the irreducible module. Since any descendant of a null state is also null, the null states are themselves organized into Verma modules and we can subtract away entire modules at a time. This procedure is further complicated by the existence of "nulls of nulls", *i.e.*, null states inside the Verma module that we are subtracting. In general, this leads to a rather intricate pattern of subtractions. It follows that the character of the irreducible module with highest weight λ, which we denote as L_λ, can be obtained from a possibly infinite sum of the form

$$\mathrm{char} L_\lambda = \sum_{\mu \leqslant \lambda} m_{\lambda,\mu} \mathrm{char} M_\mu \ , \tag{C.4}$$

where the integers $m_{\lambda,\mu}$ are not of definite sign and reflect the aforementioned pattern of null states. Of course $m_{\lambda,\lambda} = 1$. The vectors labeled by μ in the above sum are called the *primitive null vectors* of the Verma module M_λ.

This leaves us with the task of determining the weights μ that appear in (C.4) along with their associated multiplicities $m_{\lambda,\mu}$. The first task is accomplished by noting that these weights are necessarily annihilated by all raising operators, and therefore must be highest weight states in themselves. The quadratic Casimir operator of an affine Lie algebra acts simply on highest weight states with weight μ as multiplication by $|\mu + \rho|^2$, where ρ is the Weyl vector with unit Dynkin labels. On the other hand, the eigenvalue should be an invariant of the full representation, which means that the only states μ that can appear in (C.4) have to satisfy

$$|\mu + \rho|^2 = |\lambda + \rho|^2 \ . \tag{C.5}$$

Notice that so far we have made no distinction between unitary representations, where the highest weight λ is dominant integral (*i.e.*, its Dynkin labels are nonnegative integers), and non-unitary representations like the ones in which we are interested. This distinction becomes crucial in the computation of the multiplicities $m_{\lambda,\mu}$.

For the irreducible representations associated to dominant integral weights, the weight multiplicities are invariant under the action of the Weyl group, and correspondingly $\mathrm{char} L_\lambda$ is invariant under the action of the Weyl group on the fugacities. On the other hand, the Kostant partition function is essentially *odd* under this action (*cf.* [189]),

$$w(e^{-\rho-\mu} \mathrm{char} M_\mu) = \mathrm{sign}(w) e^{-\rho-\mu} \mathrm{char} M_\mu \ , \tag{C.6}$$

where the sign of an element w in the Weyl group is simply given by -1 raised to the power of the number of generators used to express w. One can easily convince oneself that the multiplicities $m_{\lambda,\mu}$ therefore necessarily satisfy

$$m_{\lambda,\mu} = \text{sign}(w) m_{\lambda, w\cdot\mu} \ , \tag{C.7}$$

where $w \cdot \mu := w(\mu + \rho) - \rho$ is the shifted action of the Weyl group on the weight μ. All the multiplicities $m_{\lambda,\mu}$ for weights μ on the same shifted Weyl orbit are therefore related by factors of sign(w), and it suffices to know only one multiplicity on each orbit. Happily, if the highest weight λ is dominant integral, then it lies on the shifted Weyl orbit of *any* primitive null vector. This essentially follows from the fact that there is a unique dominant integral weight on every shifted Weyl orbit, and from (C.5) it can be shown that this has to be λ. So, using that $m_{\lambda,\lambda} = 1$, we find that all the weights appearing in (C.4) are given by the shifted Weyl orbit of λ and have multiplicities equal to sign(w). In summary, then,

$$\text{char} L_\lambda = \frac{\sum_{w\in W} \text{sign}(w) e^{w(\rho+\lambda)-\rho}}{\prod_{\alpha>0}(1-e^{-\alpha})^{\text{mult}(\alpha)}} \ , \tag{C.8}$$

which is the famous Weyl-Kac character formula.

Let us return to the case where the λ is not dominant integral. This is the case that interests us: indeed, for $\mathfrak{so}(8)_{-2}$ the vacuum representation has Dynkin labels $[-2\,0\,0\,0\,0]$ and the zeroth Dynkin label is not positive.[1] For non-dominant integral weights the above derivation already fails at the very first step: the weight multiplicities in the irreducible representation are not invariant under the action of the Weyl group. This is most easily seen by taking the infinite irreducible representation of $\mathfrak{su}(2)$ whose highest weight is negative. In this case the single Weyl reflection maps the highest weight, which of course has multiplicity one, to a positive weight, which has multiplicity zero. The derivation of the coefficients $m_{\lambda,\mu}$ now becomes considerably more involved. Since we find qualitative differences depending on the sign of $k + h^\vee$, we will in the remainder of this appendix focus on the relevant case $k + h^\vee > 0$.

For the non-unitary representations considered here it is still true that all the primitive null vectors lie on the shifted Weyl orbit of the highest weight λ, and for $k + h^\vee > 0$ there is still a unique *dominant* weight Λ on the same orbit such that $\Lambda + \rho$ has nonnegative Dynkin labels. For example, for the vacuum module of $\mathfrak{so}(8)_{-2}$ the dominant weight has Dynkin labels $[0\,0\,-1\,0\,0]$ which happens to be related to $[-2\,0\,0\,0\,0]$ by a single elementary reflection. All the weights in (C.4), including λ itself, can thus be written as $\mu = w \cdot \Lambda$ for some Weyl element w. We can therefore alternatively try to label these weights with the corresponding element of the Weyl group w instead of μ. We will see that such a relabeling has great benefits, but first we need to mention two important subtleties.

The first subtlety concerns the fact that we may restrict ourselves to elementary reflections of the Weyl group for which the corresponding Dynkin label in Λ is integral, since it

[1] Recall that the zeroth Dynkin label for a weight vector in an affine Lie algebra $\hat{\mathfrak{g}}$ is given by $k - (\lambda, \theta)$ with λ the part of the weight vector corresponding to the original Lie algebra $\mathfrak{g}$ and θ the highest root of $\mathfrak{g}$.

is only in those cases that null states can possibly appear. These reflections generate a subgroup of the Weyl group that we will denote as W_Λ. In the case of $\mathfrak{so}(8)_{-2}$ the weights are all integral and $W_\Lambda = W$. The second subtlety is the possibility of the existence of a subgroup W_Λ^0 of W_Λ that leaves Λ invariant. This happens precisely when some of the Dynkin labels of $\Lambda + \rho$ are zero - in our case there is a single such zero. It is clear that the weights μ can then at best be uniquely labeled by elements of the coset W_Λ / W_Λ^0.

It is now a deep result that the multiplicities $m_{\lambda,\mu}$ depend on the dominant integral weight Λ *only* through the corresponding elements w and w' of the coset W_Λ / W_Λ^0. We may therefore replace

$$m_{\lambda,\mu} \to m_{w,w'} \ , \tag{C.9}$$

where $\lambda = w \cdot \Lambda$, $\mu = w' \cdot \Lambda$ and w and w' are elements of the coset. The celebrated *Kazhdan-Lusztig conjecture* tells us that the precise form of these multiplicities is given by

$$m_{w,w'} = \tilde{Q}_{w,w'}(1) \ . \tag{C.10}$$

where the Kazhdan-Lusztig polynomial $\tilde{Q}_{w,w'}(q)$ is a single-variable polynomial depending on two elements w and w' of the coset W_Λ / W_Λ^0. These polynomials are determined via rather intricate recursion relations that are explained in detail in [190]. For $k + h^\vee > 0$ and integral weights, which is the case that interests us here, the Kazhdan-Lusztig conjecture was proven in [191, 192].

For the computations mentioned in the main text, we have implemented the recursive definitions of the Kazhdan-Lusztig polynomials on cosets given in [190] in `Mathematica`. Equations (C.3), (C.4), and (C.10) then allow us to compute all the states in the irreducible vacuum character of $\mathfrak{so}(8)_{-2}$ up to level five. The results are shown in Table 2.5.

Appendix D

Affine critical characters and the Schur index

We show how to re-write the superconformal index [14] in the so-called Schur limit [13, 46] in terms of characters of affine Kac-Moody modules at the critical level. The superconformal index of class $\mathcal{S}$ theories was computed in [13, 46, 82, 84], and the characters of affine Kac-Moody algebras at the critical level in [92]. Here we just collect the final expressions and refer the readers to the original work for details.

Our conventions for affine Lie algebras follow those of [93], and here we simply review some notation needed to write the characters. We denote the affine Lie algebra obtained by adding an imaginary root δ to a finite Lie algebra $\mathfrak{g}$ (of rank r) by $\hat{\mathfrak{g}}$. The Cartan subalgebra of $\hat{\mathfrak{g}}$ ($\mathfrak{g}$) is denoted by $\hat{\mathfrak{h}}$ ($\mathfrak{h}$), and the positive roots of $\hat{\mathfrak{g}}$ ($\mathfrak{g}$) by $\hat{\Delta}_+$ (Δ_+). We also denote the real positive roots of the affine Lie algebra, that is positive roots not of the form $n\delta$, by $\hat{\Delta}_+^{\mathrm{re}}$. The character of a critical irreducible highest weight representation $\mathfrak{R}_\lambda$ with highest weight $\hat{\lambda}$, whose restriction to the finite Lie algebra λ is by definition an integral dominant weight is given in [92]. It reads [1]

$$\mathrm{ch}_{\mathfrak{R}_\lambda} = \frac{\sum_{w\in W} \epsilon(w) e^{w(\lambda+\rho)-\rho}}{\prod_{\alpha\in\Delta_+} (1-q^{\langle\lambda+\rho,\alpha^\vee\rangle}) \prod_{\hat{\alpha}\in\hat{\Delta}_+^{\mathrm{re}}} (1-e^{-\hat{\alpha}})}\,, \tag{D.1}$$

where W is the Weyl group of $\mathfrak{g}$, $\epsilon(w)$ is the signature of w, $q = e^{-\delta}$, ρ denotes the Weyl vector, $\langle\cdot,\cdot\rangle$ denotes the Killing inner product and $\alpha^\vee$ is the coroot associated to α.

[1] Here we used that for a critical highest weight $\hat{\lambda} + \hat{\rho} = \lambda + \rho$, and normalized the character to match the standard conventions for a partition function.

The Schur limit of the superconformal index of a T_n theory is given by [13, 46]

$$\mathcal{I}_{T_n}(q;\mathbf{x}_i) = \sum_{\mathfrak{R}_\lambda} \frac{\prod_{i=1}^{3} \mathcal{K}_\Lambda(q;\mathbf{x}_i)\ \chi_{\mathfrak{R}_\lambda}(\mathbf{x}_i)}{\mathcal{K}_{\Lambda^t}(q)\ \dim_q \mathfrak{R}_\lambda}\,, \tag{D.2}$$

with

$$\mathcal{K}_{\Lambda^t}(q) = \text{P.E.}\left[\sum_{j=1}^{n-1} \frac{q^{d_j}}{1-q}\right]\,, \qquad \mathcal{K}_\Lambda(q;\mathbf{x}) = \text{P.E.}\left[\frac{q\,\chi_{\text{adj.}}(\mathbf{x})}{1-q}\right]\,. \tag{D.3}$$

Here $\mathbf{x}_i$ denotes flavor fugacities conjugate to the Cartan generators of the $\mathfrak{su}(n)_i$ flavor group associated with each of the three punctures, Λ and Λ^t are respectively the trivial and principal embeddings of $\mathfrak{su}(2) \hookrightarrow \mathfrak{su}(n)$, and d_j are the degrees of invariants. Furthermore, $\dim_q \mathfrak{R}_\lambda$ is the q-deformed dimension of the representation $\mathfrak{R}_\lambda$, *i.e.*,

$$\dim_q \mathfrak{R}_\lambda = \prod_{\alpha\in\Delta_+} \frac{[\langle \lambda+\rho,\alpha\rangle]_q}{[\langle \rho,\alpha\rangle]_q}\,, \qquad \text{where} \qquad [x]_q = \frac{q^{-\frac{x}{2}} - q^{\frac{x}{2}}}{q^{-\frac{1}{2}} - q^{\frac{1}{2}}}\,. \tag{D.4}$$

As shown in [28], if $\lambda = 0$ is the highest weight of the vacuum module

$$\text{ch}_{\mathfrak{R}_{\lambda=0}} = \frac{\mathcal{K}_\Lambda(q;\mathbf{x}_i)}{\mathcal{K}_{\Lambda^t}(q)}\,. \tag{D.5}$$

The expectation is that the full index for T_n can be re-written as a sum of characters of critical modules[2]. Re-writing (D.1) to make manifest the vacuum module we find

$$\text{ch}_{\mathfrak{R}_\lambda} = \text{ch}_{\mathfrak{R}_{\lambda=0}} \prod_{\alpha\in\Delta_+} \left(\frac{1-q^{\langle\rho,\alpha^\vee\rangle}}{1-q^{\langle\lambda+\rho,\alpha^\vee\rangle}}\right) \frac{\sum_{w\in W}\epsilon(w) e^{w(\lambda+\rho)-\rho}}{\sum_{w\in W}\epsilon(w) e^{w(\rho)-\rho}}\,, \tag{D.6}$$

where we recognize the last term as the character of the representation with highest weight λ of $\mathfrak{g}$,

$$\chi_\lambda(\mathbf{x}) = \frac{\sum_{w\in W}\epsilon(w) e^{w(\lambda+\rho)}}{\sum_{w\in W}\epsilon(w) e^{w(\rho)}}\,. \tag{D.7}$$

After factoring out a $q^{-\langle\lambda,\rho\rangle}$, the middle factor can be written in terms of the q-deformed

[2] Along similar lines, one can rewrite the Schur limit of the superconformal index of the $T_{SO(2n)}$ theory [5, 193] in terms of critical affine $\widehat{\mathfrak{so}(2n)}$ characters.

dimension (D.4) of the same representation:

$$\prod_{\alpha\in\Delta_+}\left(\frac{1-q^{\langle\rho,\alpha^\vee\rangle}}{1-q^{\langle\lambda+\rho,\alpha^\vee\rangle}}\right)=\prod_{\alpha\in\Delta_+}q^{-\langle\lambda,\alpha^\vee\rangle/2}\prod_{\alpha\in\Delta_+}\left(\frac{q^{-\langle\rho,\alpha^\vee\rangle/2}-q^{\langle\rho,\alpha^\vee\rangle/2}}{q^{-\langle\lambda+\rho,\alpha^\vee\rangle/2}-q^{\langle\lambda+\rho,\alpha^\vee\rangle/2}}\right)=q^{-\langle\lambda,\rho\rangle}\frac{1}{\dim_q\mathfrak{R}_\lambda}\,, \tag{D.8}$$

where we used that $\alpha^\vee=\alpha$, for $\mathfrak{su}(n)$, to identify ρ in the last step. In total we thus find

$$\mathrm{ch}_{\mathfrak{R}_\lambda}=\frac{\text{P.E.}\left[\frac{q\,\chi_{\text{adj.}}(\mathbf{x})}{1-q}\right]\chi_\lambda(\mathbf{x})}{q^{\langle\lambda,\rho\rangle}\text{P.E.}\left[\sum_{j=1}^{n-1}\frac{q^{d_j}}{1-q}\right]\dim_q\mathfrak{R}_\lambda}\,. \tag{D.9}$$

Using this result in the expression for the superconformal index (D.2) we obtain (3.1). To obtain (3.2) we also note that the denominator of (D.9) can be rewritten as

$$\begin{aligned} q^{\langle\lambda,\rho\rangle}\ \text{P.E.}\left[\sum_{j=1}^{n-1}\frac{q^{d_j}}{1-q}\right]\ \dim_q\mathfrak{R}_\lambda &= \text{P.E.}\left[\sum_{j=1}^{n-1}\frac{q^{d_j}}{1-q}+\sum_{\alpha\in\Delta_+}q^{\langle\rho,\alpha\rangle}-\sum_{\alpha\in\Delta_+}q^{\langle\lambda+\rho,\alpha\rangle}\right] \\ &= \text{P.E.}\left[\sum_{j=1}^{n-1}\frac{q^{d_j}}{1-q}+\sum_{j=1}^{n-1}(n-j)\,q^j-\sum_{j=2}^{n}\sum_{1\le i<j}q^{\ell_i-\ell_j+j-i}\right]. \end{aligned} \tag{D.10}$$

Appendix E

The T_4 OPEs

In this appendix we give all the OPEs between the generators of the T_4 chiral algebra. Here all OPE coefficients (including the central charges) are already set to the values required by the Jacobi-identities, as described in Section 3.3. Since all generators are both Virasoro and AKM primaries, with the exception of the stress tensor which is neither and the AKM currents which are not AKM primaries, all singular OPEs involving the affine currents and the stress tensor are completely fixed by flavor symmetries and Virasoro symmetry, up to the flavor central charges $(k_{2d})_{i=1,2,3} = -4$ and the Virasoro central charge $c_{2d} = -78$ appearing in the most singular term in their respective self-OPEs. Different AKM currents are taken to have zero singular OPE. As discussed in Section 3.3 we consistently treat the three flavor symmetries on equal footing, in particular we require $k_{2d} \equiv (k_{2d})_1 = (k_{2d})_2 = (k_{2d})_3$. We recall that also the precise values of c_{2d} and k_{2d} central charges are a result of imposing the Jacobi-identities.

The singular OPEs of the $W, \widetilde{W}$ generators among themselves were found to be

$$
\begin{aligned}
W_{a_1a_2a_3}(z)\; W_{b_1b_2b_3}(0) \;&\sim\; \frac{1}{2\,z} V_{[a_1b_1][a_2b_2][a_3b_3]}\,, \\
\widetilde{W}^{a_1a_2a_3}(z)\; \widetilde{W}^{b_1b_2b_3}(0) \;&\sim\; \frac{1}{2\,z}\frac{1}{8}\epsilon^{a_1b_1c_1d_1}\epsilon^{a_2b_2c_2d_2}\epsilon^{a_3b_3c_3d_3} V_{[c_1d_1][c_2d_2][c_3d_3]}\,, \\
W_{a_1a_2a_3}(z)\; \widetilde{W}^{b_1b_2b_3}(0) \;&\sim\; \frac{1}{z^3}\delta_{a_1}^{b_1}\delta_{a_2}^{b_2}\delta_{a_3}^{b_3} - \frac{1}{4\,z^2}\left(\delta_{a_1}^{b_1}\delta_{a_2}^{b_2}(J^3)_{a_3}^{b_3} + \text{perms.}\right) \\
&\quad - \frac{1}{4\,z}\left(\delta_{a_1}^{b_1}\delta_{a_2}^{b_2}\partial(J^3)_{a_3}^{b_3} + \text{perms.}\right) + \frac{1}{16\,z}\left(\delta_{a_1}^{b_1}(J^2)_{a_2}^{b_2}(J^3)_{a_3}^{b_3} + \text{perms.}\right) \\
&\quad + \frac{1}{z}\delta_{a_1}^{b_1}\delta_{a_2}^{b_2}\delta_{a_3}^{b_3}\left(-\frac{1}{16}T - \frac{1}{96}\left((J^1)_{\beta_1}^{\alpha_1}(J^1)_{\alpha_1}^{\beta_1} + 2\text{ more}\right)\right) \\
&\quad + \frac{1}{16\,z}\left(\delta_{a_1}^{b_1}\delta_{a_2}^{b_2}(J^3)_{a_3}^{\alpha_3}(J^3)_{\alpha_3}^{b_3} + \text{perms.}\right)\,,
\end{aligned}
$$

where we have fixed the normalization of W and $\widetilde{W}$ to convenient values. In all these OPEs

"+2 more" means we must add the same term for the remaining two currents, and "+perms." that all independent permutations of the previous term must be added. We also found the OPEs between the $W, \widetilde{W}$ and V generators to be

$$\begin{aligned}
W_{a_1a_2a_3}(z)\ V_{[b_1c_1][b_2c_2][b_3c_3]}(0)\ &\sim \frac{1}{8}\epsilon_{a_1b_1c_1d_1}\epsilon_{a_2b_2c_2d_2}\epsilon_{a_3b_3c_3d_3}\left(-\frac{3}{z^2}\widetilde{W}^{d_1d_2d_3}-\frac{1}{z}\partial\widetilde{W}^{d_1d_2d_3}\right)\\
&\quad-\frac{1}{8}\epsilon_{a_1b_1c_1d_1}\epsilon_{a_2b_2c_2d_2}\epsilon_{a_3b_3c_3d_3}\frac{1}{3\,z}\left((J^1)^{d_1}_{\alpha_1}\widetilde{W}^{\alpha_1d_2d_3}+\text{perms.}\right)\\
&\quad-\frac{1}{8\,z}\left(\epsilon_{\alpha_1b_1c_1d_1}\epsilon_{a_2b_2c_2d_2}\epsilon_{a_3b_3c_3d_3}(J^1)^{d_1}_{a_1}\widetilde{W}^{\alpha_1d_2d_3}+\text{perms.}\right)\,,\\
\widetilde{W}^{a_1a_2a_3}(z)\ V_{[b_1c_1][b_2c_2][b_3c_3]}(0)\ &\sim \delta^{a_1}_{[b_1}\delta^{a_2}_{[b_2}\delta^{a_3}_{[b_3}\left(\frac{3}{z^2}W_{c_1]c_2]c_3]}+\frac{1}{z}\partial W_{c_1]c_2]c_3]}\right)\\
&\quad+\frac{1}{3\,z}\delta^{a_1}_{[b_1}\delta^{a_2}_{[b_2}\delta^{a_3}_{[b_3}\left((J^1)^{\alpha_1}_{c_1]}W_{\alpha_1c_2]c_3]}+\text{perms.}\right)\\
&\quad+\frac{1}{z}\left(\delta^{a_2}_{[b_2}\delta^{a_3}_{[b_3}(J^1)^{a_1}_{[b_1}W_{c_1]c_2]c_3]}+\text{perms.}\right)\,.
\end{aligned}$$

Finally, the singular VV OPE reads

$$\begin{aligned}
&V_{[a_1b_1][a_2b_2][a_3b_3]}(z)V^{[c_1d_1][c_2d_2][c_3d_3]}(0)\\
&\sim\ \delta^{[c_1}_{a_1}\delta^{d_1]}_{b_1}\delta^{[c_2}_{a_2}\delta^{d_2]}_{b_2}\delta^{[c_3}_{a_3}\delta^{d_3]}_{b_3}\left(\frac{6}{z^4}-\frac{1}{2\,z^2}T-\frac{1}{4\,z}\partial T-\frac{1}{24\,z^2}\left((J^1)^{\alpha_1}_{\beta_1}(J^1)^{\beta_1}_{\alpha_1}+2\text{ more}\right)\right.\\
&\qquad-\frac{19}{480\,z}\partial\left((J^1)^{\alpha_1}_{\beta_1}(J^1)^{\beta_1}_{\alpha_1}+2\text{ more}\right)-\frac{37}{20\,z}W_{\alpha_1\beta_1\gamma_1}\widetilde{W}^{\alpha_1\beta_1\gamma_1}\\
&\qquad\left.+\frac{1}{40\,z}\left((J^1)^{\beta_1}_{\alpha_1}(J^1)^{\gamma_1}_{\beta_1}(J^1)^{\alpha_1}_{\gamma_1}+2\text{ more}\right)\right)\\
&\quad+\delta^{[c_2}_{a_2}\delta^{d_2]}_{b_2}\delta^{[c_3}_{a_3}\delta^{d_3]}_{b_3}\left(-\frac{3}{16\,z}\delta^{[d_1}_{[b_1}(J^1)^{|\gamma_1}_{a_1]}(J^1)^{\beta_1|}_{\gamma_1}(J^1)^{c_1]}_{\beta_1}+\frac{33}{80\,z}(J^1)^{\beta_1}_{[a_1}(J^1)^{[d_1}_{|\beta_1|}(J^1)^{c_1]}_{b_1]}\right.\\
&\qquad-\frac{43}{80\,z^2}\partial\left((J^1)^{[d_1}_{[a_1}(J^1)^{c_1]}_{b_1]}\right)-\frac{3}{2\,z^3}(J^1)^{[c_1}_{[a_1}\delta^{d_1]}_{b_1]}+\frac{1}{40\,z^3}T(J^1)^{[c_1}_{[a_1}\delta^{d_1]}_{b_1]}\\
&\qquad-\frac{11}{120\,z}\left((J^1)^{\alpha_1}_{\beta_1}(J^1)^{\beta_1}_{\alpha_1}+2\text{ more}\right)(J^1)^{[c_1}_{[a_1}\delta^{d_1]}_{b_1]}-\frac{5}{4\,z^2}\partial(J^1)^{[c_1}_{[a_1}\delta^{d_1]}_{b_1]}\\
&\qquad+\frac{1}{4\,z^2}(J^1)^{\alpha_1}_{[a_1}(J^1)^{[c_1}_{|\alpha_1|}\delta^{d_1]}_{b_1]}+\frac{17}{40\,z}\partial(J^1)^{\alpha_1}_{[a_1}(J^1)^{[c_1}_{|\alpha_1|}\delta^{d_1]}_{b_1]}\\
&\qquad\left.-\frac{23}{80\,z}\partial^2(J^1)^{[c_1}_{[a_1}\delta^{d_1]}_{b_1]}-\frac{1}{4\,z}(J^1)^{[d_1}_{[a_1}(J^1)^{c_1]}_{b_1]}+\frac{7}{40\,z}(J^1)^{\alpha_1}_{[a_1}\partial(J^1)^{[c_1}_{|\alpha_1|}\delta^{d_1]}_{b_1]}\right)\\
&\qquad+\text{permutations}[1,2,3]
\end{aligned}$$

$$
\begin{aligned}
&+\delta^{[c_3}_{a_3}\delta^{d_3]}_{b_3}\left(\frac{1}{4\,z}\delta^{[d_2}_{[b_2}(J^2)^{c_2]}_{a_2]}(J^1)^{[d_1}_{[a_1}(J^1)^{c_1]}_{b_1]}+\frac{3}{4\,z^2}\delta^{[d_2}_{[b_2}(J^2)^{c_2]}_{a_2]}(J^1)^{[c_1}_{[a_1}\delta^{d_1]}_{b_1]}+\right.\\
&\qquad\left.+\frac{13}{4\,z}\delta^{[d_2}_{[b_2}(J^2)^{c_2]}_{a_2]}\partial(J^1)^{[c_1}_{[a_1}\delta^{d_1]}_{b_1]}-\frac{3}{4\,z}\delta^{[d_2}_{[b_2}(J^2)^{c_2]}_{a_2]}(J^1)^{\alpha_1}_{[a_1}(J^1)^{c_1]}_{|\alpha_1|}\delta^{d_1}_{b_1]}\right)\\
&\quad+\text{permutations}[1,2,3]\\
&+\frac{19}{5\,z}\left(\delta^{[c_2}_{a_2}\delta^{d_2]}_{b_2}\delta^{[c_3}_{a_3}\delta^{d_3]}_{b_3}\delta^{[d_1}_{[b_1}W_{a_1]\beta_2\gamma_3}\widetilde{W}^{c_1]\beta_2\gamma_3}+\delta^{[c_3}_{a_3}\delta^{d_3]}_{b_3}\delta^{[c_1}_{a_1}\delta^{d_1]}_{b_1}\delta^{[d_2}_{[b_2}W_{\beta_1a_2]\gamma_3}\widetilde{W}^{\beta_1c_2]\gamma_3}\right.\\
&\qquad\left.+\delta^{[c_1}_{a_1}\delta^{d_1]}_{b_1}\delta^{[c_2}_{a_2}\delta^{d_2]}_{b_2}\delta^{[d_3}_{[b_3}W_{\beta_1\gamma_2a_3]}\widetilde{W}^{\beta_1\gamma_2c_3]}\right)\\
&-\frac{4}{z}\left(\delta^{[c_3}_{a_3}\delta^{d_3]}_{b_3}\delta^{[d_1}_{[b_1}\delta^{[d_2}_{[b_2}W_{a_1]a_2]\gamma_3}\widetilde{W}^{c_1]c_2]\gamma_3}+\delta^{[c_1}_{a_1}\delta^{d_1]}_{b_1}\delta^{[d_2}_{[b_2}\delta^{[d_3}_{[b_3}W_{\gamma_1a_2]a_3]}\widetilde{W}^{\gamma_1c_2]c_3]}\right.\\
&\qquad\left.+\delta^{[c_2}_{a_2}\delta^{d_2]}_{b_2}\delta^{[d_3}_{[b_3}\delta^{[d_1}_{[b_1}W_{a_1]\gamma_2a_3]}\widetilde{W}^{c_1]\gamma_2c_3]}\right)\\
&-\frac{1}{z}\delta^{[d_1}_{[b_1}\delta^{[d_2}_{[b_2}\delta^{[d_3}_{[b_3}(J^1)^{c_1]}_{a_1]}(J^2)^{c_2]}_{a_2]}(J^3)^{c_3]}_{a_3]}-\frac{16}{z}\delta^{[d_1}_{[b_1}\delta^{[d_2}_{[b_2}\delta^{[d_3}_{[b_3}W_{a_1]a_2]a_3]}\widetilde{W}^{c_1]c_2]c_3]}\,,
\end{aligned}
$$

where the norm of V was also fixed, and for convenience we defined $V^{[c_1d_1][c_2d_2][c_3d_3]}$ through $V_{[a_1b_1][a_2b_2][a_3b_3]}=\frac{1}{8}\epsilon_{a_1b_1c_1d_1}\epsilon_{a_2b_2c_2d_2}\epsilon_{a_3b_3c_3d_3}V^{[c_1d_1][c_2d_2][c_3d_3]}$. Here "permutations[1, 2, 3]" means we must repeat the previous term with all possible permutations of the flavor groups indices.

Appendix F

Superconformal block decompositions for four-dimensional $\mathcal{N} = 2$ theories

This appendix contains a number of technical details pertaining to the superconformal block decompositions of correlators investigated in chapter 2. The conventional conformal blocks of four-dimensional non-supersymmetric CFT make repeated appearances here, and for those we adopt the conventions of [41]. Namely, the conformal block associated to the exchange of an $\mathfrak{so}(4,2)$ conformal family whose primary has dimension Δ and spin ℓ in the four-point function of degenerate scalars is given by $u^{\frac{1}{2}(\Delta-\ell)} G_{\Delta}^{(\ell)}(u,v)$, where

$$G_{\Delta}^{(\ell)}(u,v) := \frac{1}{z-\bar{z}} \left(\left(-\frac{z}{2}\right)^{\ell} z\, {}_2F_1\left(\frac{1}{2}(\Delta+\ell), \frac{1}{2}(\Delta+\ell); \Delta+\ell; z\right) \right. \\ \left. \times\, {}_2F_1\left(\frac{1}{2}(\Delta-\ell-2), \frac{1}{2}(\Delta-\ell-2); \Delta-\ell-2; \bar{z}\right) - z \leftrightarrow \bar{z} \right) . \tag{F.1}$$

Here, as in the main text, we will only ever need to consider operators with $j_1 = j_2 =: j$, for which the spin ℓ is defined as $\ell := 2j$.

F.1 Superconformal blocks for the $\hat{\mathcal{B}}_1$ four-point function

The superconformal blocks relevant to the partial wave decomposition of the $\hat{\mathcal{B}}_1$ four-point function were derived in the beautiful work of [41]. In this subsection we summarize those results. As our starting point we take the selection rule for operators appearing in the OPE of two moment map operators. These selection rules were determined in [171] via an analysis of three-point functions in harmonic superspace.[1] The results can be schematically presented as follows

$$\hat{\mathcal{B}}_1 \times \hat{\mathcal{B}}_1 \quad \sim \quad \mathbf{1} \; + \; \hat{\mathcal{B}}_1 \; + \; \hat{\mathcal{B}}_2 \; + \; \hat{\mathcal{C}}_{0(j,j)} \; + \; \hat{\mathcal{C}}_{1(j,j)} \; + \; \mathcal{A}^{\Delta}_{0,0(j,j)} \; . \tag{F.2}$$

Below we outline the contribution of each of these multiplets in the superconformal partial wave expansion of a moment map four-point function. We do so in two ways. First, we describe the contribution of such a multiplet to the functions $\mathcal{G}_i(z,\bar{z})$ and $f_i(z)$ that appear in the solution of the superconformal Ward identities described in Section 4.3.1. This is the form of the superconformal blocks for the numerical analysis of crossing symmetry described in Section 4.6. In order to make the structure of these contributions more transparent, we also list the contribution of each multiplet to the functions $a_{R,i}(u,v)$ associated with a fixed $SU(2)_R$ channel. Since these expressions are rather lengthy, we have collected them in Table F.1.

We start with the case of long multiplets. For these multiplets only the two-variable functions $\mathcal{G}_i(u,v)$ are non-zero (the $f_i(z)$ is protected and only receives contributions from short and semi-short multiplets). In the long multiplets listed in (F.2), there is a unique conformal primary in the $\mathbf{5}$ of $SU(2)_R$ that can appear in the OPE. This determines the contribution of a long multiplet to $a_{2,i}(u,v)$, which in turn via (4.40) fixes the contribution of long multiplets as follows

$$\mathcal{A}^{\Delta}_{0,0(j,j)} \;\; \text{in} \;\; \mathfrak{R}_i : \begin{cases} \mathcal{G}_i(u,v) & = \quad 6u^{\frac{\Delta-\ell}{2}}\, G^{(\ell)}_{\Delta+2}(u,v) \; , \\ f_i(z) & = \quad 0 \; . \end{cases} \tag{F.3}$$

The full conformal block expansion in the three R-symmetry channels can now be determined by inserting (F.3) back into (4.40) and making use of various identities for hypergeometric functions [41]. The full expansion in terms of conventional conformal blocks is given in Table F.1.

Next we turn to the $\hat{\mathcal{C}}_{0(j,j)}$ and $\hat{\mathcal{B}}_1$ multiplets. These multiplets do not include *any* operators that can contribute in the $R=2$ channel, from which it follows that for these multiplets $\mathcal{G}_i(u,v)=0$. In the $R=1$ channel, each of these multiplets contributes exactly one conformal primary of dimension $\ell+3$ and spin $\ell+1$ (dimension 2 and spin 0 in the $\hat{\mathcal{B}}_1$ case). This allows the values of the single-variable functions for these multiplets to be fixed

[1] These selection rules can also be understood as following a few simple criteria. Namely, a conformal primary can only have a non-zero three point function with two moment map operators if the superconformal primary of the same multiplet does as well. Ordinary Lorentz symmetry and R-symmetry selection rules then constrain the possible superconformal multiplets appearing in the OPE. A further constraint comes from the fact that any R-symmetry quintuplet appearing in the OPE comes from the product of two Higgs branch chiral ring operators, and so must itself be annihilated by the action of $\mathcal{Q}^1_\alpha$ and $\widetilde{\mathcal{Q}}_{2\dot{\alpha}}$.

from (4.40), and we find

$$\hat{\mathcal{C}}_{0(j,j)} \;\text{ in }\; \mathfrak{R}_i : \qquad \begin{cases} \mathcal{G}_i(u,v) &= \; 0 \;, \\ f_i(z) &= \; 2g_{2j+2}(z) \;. \end{cases} \tag{F.4}$$

$$\hat{\mathcal{B}}_1 \;\text{ in }\; \mathfrak{R}_i : \qquad \begin{cases} \mathcal{G}_i(u,v) &= \; 0 \;, \\ f_i(z) &= \; 2g_1(z) \;. \end{cases} \tag{F.5}$$

Again, the contributions of these multiplets to the individual $SU(2)_R$ channels is determined by (4.40), and the subsequent decomposition into conventional conformal blocks follows from identities for hypergeometric functions. The result is displayed in Table F.1. (Another operator that contributes only to $f_i(z)$ is the identity operator, which only arises in the $R=0$ channel and contributes to $f_i(z)$ as a constant.)

The superconformal blocks for the remaining two multiplets can be understood by studying the behavior of a generic long multiplet as it approaches the unitarity bound $\Delta = 2+\ell$. At the unitarity bound, the representation becomes reducible and decomposes according to the relevant rules in (B.7) and (B.8) specialized to the case $R=0$,

$$\begin{aligned} \mathcal{A}^{\Delta=2j+2}_{0,0(j,j)} &\simeq \hat{\mathcal{C}}_{0(j,j)} \oplus \hat{\mathcal{C}}_{\frac{1}{2}(j-\frac{1}{2},j)} \oplus \hat{\mathcal{C}}_{\frac{1}{2}(j,j-\frac{1}{2})} \oplus \hat{\mathcal{C}}_{1(j-\frac{1}{2},j-\frac{1}{2})} \;, \\ \mathcal{A}^{\Delta=2j+2}_{0,0(0,0)} &\simeq \hat{\mathcal{C}}_{0(0,0)} \oplus \mathcal{D}_{1(0,0)} \oplus \bar{\mathcal{D}}_{1(0,0)} \oplus \hat{\mathcal{B}}_2 \;. \end{aligned} \tag{F.6}$$

In each case, only the first and last multiplet are allowed in the four-point function by the selection rules. This simplifies the task of finding superconformal blocks for $\hat{\mathcal{C}}_{1(j,j)}$ and $\hat{\mathcal{B}}_2$ multiplets. Namely, by subtracting six copies of the $\hat{\mathcal{C}}_{0(j,j)}$ block from the long superconformal block with $\Delta = 2+\ell$ one obtains the superconformal block for a $\hat{\mathcal{C}}_{1(j-\frac{1}{2},j-\frac{1}{2})}$ with $j \geqslant \frac{1}{2}$. Similarly, subtracting six copies of the $\hat{\mathcal{C}}_{0(0,0)}$ block from the long superconformal block with $\Delta = 2$ yields the superconformal block for the $\hat{\mathcal{B}}_2$ representation. The result is that these multiplets contribute both to $f_i(z)$ and to $\mathcal{G}_i(u,v)$ as follows,

$$\hat{\mathcal{C}}_{1(j,j)} \;\text{ in }\; \mathfrak{R}_i : \qquad \begin{cases} \mathcal{G}_i(u,v) &= \; 6u\, G^{(\ell+1)}_{\ell+5}(u,v) \;, \\ f_i(z) &= -12g_{2j+3}(z) \;, \end{cases} \tag{F.7}$$

$$\hat{\mathcal{B}}_2 \;\text{ in }\; \mathfrak{R}_i : \qquad \begin{cases} \mathcal{G}_i(u,v) &= \; 6u\, G^{(0)}_{4}(u,v) \;, \\ f_i(z) &= -12g_{2}(z) \;. \end{cases} \tag{F.8}$$

The decomposition in the three $SU(2)_R$ channels of all these superconformal blocks are again displayed in Table F.1.

Finally, there are a few extra selection rules having to do with the representation $\mathfrak{R}_i$ of the flavor symmetry group in which the various multiplets can appear. For example, $\hat{\mathcal{B}}_1$ multiplets are those containing the conserved flavor symmetry currents, so they necessarily

Multiplet in $\mathfrak{R}_i$	Contribution to $a_{R,i}(u,v)$
$\hat{\mathcal{B}}_1$	$a_{0,i}(u,v) = \frac{1}{3}uG_3^{(1)}(u,v)$ $a_{1,i}(u,v) = uG_2^{(0)}(u,v)$ $a_{2,i}(u,v) = 0$
$\hat{\mathcal{B}}_2$	$a_{0,i}(u,v) = \frac{1}{30}u^3G_6^{(0)}(u,v)$ $a_{1,i}(u,v) = \frac{2}{5}u^2G_5^{(1)}(u,v)$ $a_{2,i}(u,v) = u^2G_4^{(0)}(u,v)$
$\hat{\mathcal{C}}_{0(j,j)}$	$a_{0,i}(u,v) = uG_{\ell+2}^{(\ell)}(u,v) + \frac{(\ell+2)^2}{(2\ell+3)(2\ell+5)}uG_{\ell+4}^{(\ell+2)}(u,v)$ $a_{1,i}(u,v) = uG_{\ell+3}^{(\ell+1)}(u,v)$ $a_{2,i}(u,v) = 0$
$\hat{\mathcal{C}}_{1(j,j)}$	$a_{0,i}(u,v) = \frac{1}{2}u^2G_{\ell+5}^{(\ell+1)}(u,v) + \frac{1}{8}u^3G_{\ell+5}^{(\ell-1)}(u,v) + \frac{(\ell+3)^2}{8(2\ell+5)(2\ell+7)}u^3G_{\ell+7}^{(\ell+1)}(u,v)$ $a_{1,i}(u,v) = \frac{3}{2}u^2G_{\ell+4}^{(\ell)}(u,v) + \frac{3}{24}u^3G_{\ell+6}^{(\ell)}(u,v) + \frac{3(\ell+3)^2}{2(2\ell+5)(2\ell+7)}u^2G_{\ell+6}^{(\ell+2)}(u,v)$ $a_{2,i}(u,v) = u^2G_{\ell+5}^{(\ell+1)}(u,v)$
$\mathcal{A}^{\Delta}_{0,0(j,j)}$	$a_{0,i}(u,v) = u^{\frac{\Delta-\ell}{2}}\Big(6G_{\Delta}^{(\ell)}(u,v) + \frac{3(\Delta+\ell+2)^2}{2(\Delta+\ell+1)(\Delta+\ell+3)}G_{\Delta+2}^{(\ell+2)}(u,v)$ $+\frac{3(\Delta-\ell)^2}{32(\Delta-\ell-1)(\Delta-\ell+1)}u^2G_{\Delta+2}^{(\ell-2)}(u,v) + \frac{1}{2}uG_{\Delta+2}^{(\ell)}(u,v)$ $+\frac{3(\Delta+\ell+2)^2(\Delta-\ell)^2}{128(\Delta+\ell+1)(\Delta+\ell+3)(\Delta-\ell-1)(\Delta-\ell+1)}u^2G_{\Delta+4}^{(\ell)}(u,v)\Big)$ $a_{1,i}(u,v) = 3u^{\frac{\Delta-\ell}{2}}\Big(2G_{\Delta+1}^{(\ell+1)}(u,v) + \frac{1}{2}G_{\Delta+1}^{(\ell-1)}(u,v)$ $+\frac{(\Delta+\ell+2)^2}{8(\Delta+\ell+1)(\Delta+\ell+3)}uG_{\Delta+3}^{(\ell+1)}(u,v) + \frac{(\Delta-\ell)^2}{32(\Delta-\ell-1)(\Delta-\ell+1)}u^2G_{\Delta+3}^{(\ell-1)}(u,v)\Big)$ $a_{2,i}(u,v) = u^{\frac{\Delta+2-\ell}{2}}G_{\Delta+2}^{\ell}(u,v)$

Table F.1: Superconformal blocks for the different $\mathfrak{su}(2,2|2)$ representations appearing in the OPE of two moment map operators.

appear only in the adjoint representation $\mathfrak{R} = \text{Adj}$. In a theory with a unique stress tensor, there will be only one $\hat{\mathcal{C}}_{0(0,0)}$ multiplet, so it will necessarily transform in the singlet representation $\mathfrak{R} = \mathbf{1}$. In general, one may take tensor products of multiple SCFTs and violate this kind of selection rule. We will call a theory that is not decomposable as the tensor product of several theories *simple*. The complete set of flavor symmetry selection rules for simple

Multiplet	Possible $\mathfrak{R}_i$ in simple theories
$\hat{\mathcal{B}}_1$	$\mathfrak{R} = \text{Adj.}$
$\hat{\mathcal{B}}_2$	$\mathfrak{R} \in \text{Sym}^2(\text{Adj.})$
$\hat{\mathcal{C}}_{0(j,j)}$	$\mathfrak{R} = \mathbf{1}$ for $\ell = 0$. None for $\ell \geqslant 1$.
$\hat{\mathcal{C}}_{1(j,j)}$	$\mathfrak{R} \in \wedge^2(\text{Adj.})$ for ℓ even. $\mathfrak{R} \in \text{Sym}^2(\text{Adj.})$ for ℓ odd.
$\mathcal{A}^{\Delta}_{0,0(j,j)}$	$\mathfrak{R} \in \text{Sym}^2(\text{Adj.})$ for ℓ even. $\mathfrak{R} \in \wedge^2(\text{Adj.})$ for ℓ odd.

Table F.2: Flavor symmetry selection rules for multiplets appearing in the $\hat{\mathcal{B}}_1 \times \hat{\mathcal{B}}_1$ OPE in simple theories.

theories are displayed in Table F.2.

Protected contributions to the crossing symmetry equation

Here we collect the contributions to the crossing symmetry equation (4.46) coming from short multiplets and that are completely fixed following the discussion in Section 4.3.2.

$\mathfrak{su}(2)$ global symmetry

For the global symmetry $\mathfrak{su}(2)$ the single variable functions $f_i(z)$ are shown in (4.72). From these single variable functions, the spectrum and OPE coefficients of short multiplets contributing to the four-point function can be determined in the manner described in Section 4.3.2. The contributions of these short multiplets to the two-variable functions $\mathcal{G}_i(z,\bar{z})$ are then given by infinite sums of the type displayed on the second line in (4.54). Performing the sums yields the following expressions,

$$\begin{aligned}
\mathcal{G}^{\text{short}}_{\mathbf{1}}(z,\bar{z}) \;\; = \;\; & \frac{\log(1-\bar{z})\,(k(6-z(z(c((z-2)z+2)-6)+12))-8c(z-1)z^2)}{ck(z-\bar{z})(z-1)^2} \\
&+\frac{\log(1-z)\,(k(\bar{z}(\bar{z}(c((\bar{z}-2)\bar{z}+2)-6)+12)-6)+8c(\bar{z}-1)\bar{z}^2)}{ck(z-\bar{z})(\bar{z}-1)^2} \\
&-\frac{6\log(1-z)\log(1-\bar{z})}{cz\bar{z}}\,,
\end{aligned} \tag{F.9}$$

$$\mathcal{G}^{\text{short}}_{\mathbf{3}}(z,\bar{z}) = \frac{(z-2)z(z(kz+4)-4)\log(1-\bar{z})}{k(z-\bar{z})(z-1)^2} - \frac{(\bar{z}-2)\bar{z}(\bar{z}(k\bar{z}+4)-4)\log(1-z)}{k(z-\bar{z})(\bar{z}-1)^2}\,,$$
$$\mathcal{G}^{\text{short}}_{\mathbf{5}}(z,\bar{z}) = \frac{\bar{z}^2(k\bar{z}^2-2(2+k)(\bar{z}-1))\log(1-z)}{k(\bar{z}-1)^2(z-\bar{z})} - \frac{z^2(kz^2-2(2+k)(z-1))\log(1-\bar{z})}{k(z-1)^2(z-\bar{z})}\,.$$

These expressions are part of the input to the "known" part of the amplitude denoted as $a_i(z,\bar{z})$ in (4.90).

$\mathfrak{e}_6$ global symmetry

For $\mathfrak{e}_6$ global symmetry, the single-variable functions $f_i(z)$, obtained by acting with the appropriate projectors on (4.48), are given by

$$\begin{aligned}
f_{\mathbf{1}}(z) &= \frac{k(z(z((z-2)z+80)-156)+78)+48(z-1)z^2}{k(z-1)^2}\,,\\
f_{\mathbf{650}}(z) &= \frac{z^2(k((z-2)z+2)+12(z-1))}{k(z-1)^2}\,,\\
f_{\mathbf{2430}}(z) &= \frac{z^2(k((z-2)z+2)-4z+4)}{k(z-1)^2}\,,\\
f_{\mathbf{78}}(z) &= -\frac{(z-2)z(z(kz+24)-24)}{k(z-1)^2}\,,\\
f_{\mathbf{2925}}(z) &= -\frac{(z-2)z^3}{(z-1)^2}\,.
\end{aligned} \tag{F.10}$$

The functions $\mathcal{G}^{\text{short}}_i(z,\bar{z})$ are again computed by fixing the OPE coefficients for all short multiplets as described in Section 4.3.2 and performing the infinite sums like in (4.54). We find:

$$\begin{aligned}
\mathcal{G}^{\text{short}}_{\mathbf{1}}(z,\bar{z}) &= \frac{\log(1-\bar{z})\left(k(156-z(z(c((z-2)z+2)-156)+312))-48c(z-1)z^2\right)}{ck(z-\bar{z})(z-1)^2}\\
&\quad+\frac{\log(1-z)\left(k(\bar{z}(\bar{z}(c((\bar{z}-2)\bar{z}+2)-156)+312)-156)+48c(\bar{z}-1)\bar{z}^2\right)}{ck(z-\bar{z})(\bar{z}-1)^2}\\
&\quad-\frac{156\log(1-z)\log(1-\bar{z})}{cz\bar{z}}\,,\\
\mathcal{G}^{\text{short}}_{\mathbf{650}}(z,\bar{z}) &= \frac{\bar{z}^2(k\bar{z}^2+2(k-6)(1-\bar{z}))\log(1-z)}{k(\bar{z}-1)^2(z-\bar{z})} - \frac{z^2(kz^2+2(k-6)(1-z))\log(1-\bar{z})}{k(z-1)^2(z-\bar{z})}\,,\\
\mathcal{G}^{\text{short}}_{\mathbf{2430}}(z,\bar{z}) &= \frac{\bar{z}^2(k((\bar{z}-2)\bar{z}+2)-4\bar{z}+4)\log(1-z)}{k(\bar{z}-1)^2(z-\bar{z})}
\end{aligned}$$

$$
- \frac{z^2(k((z-2)z+2)-4z+4)\log(1-\bar{z})}{k(z-1)^2(z-\bar{z})} ,
$$

$$
\mathcal{G}^{\text{short}}_{\mathbf{78}}(z,\bar{z}) = \frac{(z-2)z(z(kz+24)-24)\log(1-\bar{z})}{k(z-\bar{z})(z-1)^2} - \frac{(\bar{z}-2)\bar{z}(\bar{z}(k\bar{z}+24)-24)\log(1-z)}{k(z-\bar{z})(\bar{z}-1)^2} ,
$$

$$
\mathcal{G}^{\text{short}}_{\mathbf{2925}}(z,\bar{z}) = \frac{(z-2)z^3\log(1-\bar{z})}{(z-\bar{z})(z-1)^2} - \frac{(\bar{z}-2)\bar{z}^3\log(1-z)}{(z-\bar{z})(\bar{z}-1)^2} . \tag{F.11}
$$

F.2 Superconformal blocks for the $\mathcal{E}_r$ four-point function

In the case of the four-point function of $\mathcal{N}=2$ chiral operators described in Section 4.4, there are two qualitatively different sets of superconformal blocks corresponding to the *chiral channel* and the *non-chiral channel* for the double OPE (see Fig. 4.1). In the first part of this appendix, we sketch the arguments that lead to the superconformal selection rules for these two OPE channels. It is explained in Section 4.4 that, for the purposes of crossing symmetry, it is useful to change basis and introduce three channels $\hat{1}$, $\hat{2}$, and $\hat{3}$. In the second part of this appendix, we present the superconformal blocks for these different channels.

F.2.1 Selection rules in the non-chiral channel

The set of representations that may appear in an $\mathcal{E}_{r_0} \times \bar{\mathcal{E}}_{-r_0}$ OPE can be determined by means of a simple selection rule. Without loss of generality, we may focus on conformal primary operators. Then let us consider an operator $\mathcal{O}(x)$ that is a conformal primary but a descendant of a superconformal primary $\mathcal{O}'(x)$. The selection rule that we will derive below can then be summarized as follows,

$$
\left\langle \phi(x_1)\bar{\phi}(x_2)\mathcal{O}(x_3)\right\rangle \neq 0 \quad \Longrightarrow \quad \left\langle \phi(x_1)\bar{\phi}(x_2)\mathcal{O}'(x_3)\right\rangle \neq 0 . \tag{F.12}
$$

In other words, for any operator that is a super-descendant to have a nonvanishing three-point function with an $\mathcal{N}=2$ chiral primary and its conjugate, the superconformal primary for that operator must also have such a nonvanishing three-point function.

This selection rule follows from a direct application of superconformal Ward identities. The relevant Ward identities have been derived in [61], and they take the following form,

$$
\psi^\alpha(x_3)\Big\langle \phi(x_1)\bar{\phi}(x_2)\left[\mathcal{Q}^{\mathcal{I}}_\alpha,\mathcal{O}\right\}(x_3)\Big\rangle + \partial_{\alpha\dot{\alpha}}\psi^\alpha(x_3)\Big\langle \phi(x_1)\bar{\phi}(x_2)\left[\widetilde{\mathcal{S}}^{\mathcal{I},\dot{\alpha}},\mathcal{O}\right\}(x_3)\Big\rangle = 0 . \tag{F.13}
$$

As in [61], the commutators appearing in the above expression should be interpreted as meaning that the relevant commutator has been computed at the origin and the resulting operator has been translated to the appropriate insertion point. An analogous identity holds with $\widetilde{\mathcal{Q}}_{\mathcal{I},\dot{\alpha}}$ and $\mathcal{S}^\alpha_{\mathcal{I}}$. Now if $\mathcal{O}(x_3)$ is a superconformal primary operator itself, then the second term in (F.13) vanishes, from which it follows that operators of the form $\left[\mathcal{Q}^{\mathcal{I}}_{\alpha}, \mathcal{O}(x)\right\}$ cannot appear in the $\phi \times \bar{\phi}$ OPE. If instead we take $\mathcal{O}(x) = \left[\widetilde{\mathcal{Q}}_{\mathcal{J},\dot{\beta}}, \mathcal{O}'(x)\right\}$, with $\mathcal{O}'$ being a superconformal primary, then some algebraic manipulations lead to the following form of the Ward identity,[2]

$$
\begin{aligned}
\psi^{\alpha}(x_3)\langle\phi(x_1)\bar{\phi}(x_2) \quad & \left\{\mathcal{Q}^{\mathcal{I}}_{\alpha}, \left[\widetilde{\mathcal{Q}}_{\mathcal{J},\dot{\beta}}, \mathcal{O}'^{\mathcal{K}_1,\ldots\mathcal{K}_n}_{\alpha_1\ldots\alpha_{2j}\dot{\alpha}_1\ldots\dot{\alpha}_{2j}}\right]\right\}(x_3)\rangle = \\
- \quad & \partial_{\alpha\dot{\alpha}}\psi^{\alpha}(x_3)\left(\delta^{\mathcal{I}}_{\mathcal{J}}\left(j\langle\phi(x_1)\bar{\phi}(x_2)\mathcal{O}'^{\mathcal{K}_1,\ldots\mathcal{K}_n}_{\alpha_1\ldots\alpha_{2j}\dot{\beta}(\dot{\alpha}_1\ldots\dot{\alpha}_{2j-1}}(x_3)\rangle\delta^{\dot{\alpha}}_{\dot{\alpha}_{2j})}\right.\right. \\
+ \quad & \left.\left(\tfrac{\Delta-j+r-n}{2}\right)\delta^{\dot{\alpha}}_{\dot{\beta}}\langle\phi(x_1)\bar{\phi}(x_2)\mathcal{O}'^{\mathcal{K}_1,\ldots\mathcal{K}_n}_{\alpha_1\ldots\alpha_{2j}\dot{\alpha}_1\ldots\dot{\alpha}_{2j}}(x_3)\rangle\right) \\
+ \quad & \left.\delta^{\dot{\alpha}}_{\dot{\beta}}\delta^{(\mathcal{K}_1}_{\mathcal{J}}\langle\phi(x_1)\bar{\phi}(x_2)\mathcal{O}'^{\mathcal{K}_2,\ldots\mathcal{K}_n),\mathcal{I}}_{\alpha_1\ldots\alpha_{2j}\dot{\alpha}_1\ldots\dot{\alpha}_{2j}}(x_3)\rangle\right) \ . \qquad \text{(F.14)}
\end{aligned}
$$

where r and Δ are the $U(1)_r$ charge and dimension of $\mathcal{O}'$. It follows from this identity that the three-point function including the superconformal descendant $\left\{\mathcal{Q}^{\mathcal{I}}_{\alpha}, \left[\widetilde{\mathcal{Q}}_{\mathcal{J},\dot{\beta}}, \mathcal{O}'^{\mathcal{K}_1,\ldots\mathcal{K}_n}_{\alpha_1\ldots\alpha_{2j_1}\dot{\alpha}_1\ldots\dot{\alpha}_{2j_2}}\right]\right\}$ is fixed in terms of the three point function of the superconformal primary. Similar results can be derived for all higher descendants of $\mathcal{O}'(x)$ using (F.13) plus the corresponding relation involving the conjugate supercharges. All told, we are left with the selection rule given above in (F.12).

Given these selection rules, the possible superconformal representations that may appear in the $\phi \times \bar{\phi}$ OPE are severely restricted. Namely, only representations for which the superconformal primary has $R = r = 0$ and $j := j_1 = j_2$ may appear. A brief survey of the representations in Appendix B leads to the following list,

$$
\mathcal{E}_{r_0(0,0)} \times \bar{\mathcal{E}}_{-r_0(0,0)} \quad \sim \quad \mathbf{1} \ + \ \hat{\mathcal{C}}_{0(j,j)} \ + \ \mathcal{A}^{\Delta}_{0,0(j,j)} \ . \qquad \text{(F.15)}
$$

We should note that this selection rule has only been derived here for the *superconformal primaries* of the $\mathcal{E}_{r_0(0,0)}$ and $\bar{\mathcal{E}}_{-r_0(0,0)}$ multiplets.

F.2.2 Selection rules in the chiral channel

The selection rules for the chiral OPE can be determined by a generalization of arguments of [34], where the analogous problem for $\mathcal{N} = 1$ SCFTs was considered. Suppose an operator $\mathcal{O}(x)$ appears in the $\phi_{r_0} \times \phi_{r_0}$ OPE. Ordinary non-supersymmetric selection rules imply that $\mathcal{O}$ must be an $SU(2)_R$ singlet with $r_{\mathcal{O}} = 2r_0$ and $j := j_1 = j_2 \in \mathbb{Z}$. There are then additional constraints that come from the supersymmetry properties of the chiral operators that are being multiplied. Namely, we observe that for any x, we have

$$
[\mathcal{Q}^{\mathcal{I}}_{\alpha}, \phi_{r_0}(x)] = 0 \ , \qquad [\widetilde{\mathcal{S}}^{\mathcal{I},\dot{\alpha}}, \phi_{r_0}(x)] = 0 \ . \qquad \text{(F.16)}
$$

[2]In this calculation we have assumed that $\mathcal{O}'(x_3)$ is bosonic. A similar calculation leading to the same conclusion holds in the fermionic case.

The first condition is simply a part of the definition of the $\mathcal{E}_r$ multiplet. The latter is automatic when $x = 0$ because ϕ_{r_0} is the superconformal primary in its representation. For $x \neq 0$, we note the following relation from the $\mathcal{N} = 2$ superconformal algebra,

$$[\mathcal{P}_{\alpha\dot{\alpha}}, \widetilde{\mathcal{S}}^{\mathcal{I},\dot{\beta}}] = \delta_{\dot{\alpha}}^{\dot{\beta}} \mathcal{Q}_{\alpha}^{\mathcal{I}} \ . \tag{F.17}$$

It follows that when ϕ_{r_0} is translated away from the origin, its variation under the action of $\widetilde{\mathcal{S}}^{\mathcal{I},\dot{\alpha}}$ is proportional to its variation under the action of a chiral supercharge, which vanishes.

Thus we see that $\phi_{r_0}(x_1) \times \phi_{r_0}(x_2)$ itself is invariant under the action of $\mathcal{Q}_{\alpha}^{\mathcal{I}}$ and $\widetilde{\mathcal{S}}^{\mathcal{I},\dot{\alpha}}$, and so must be any operator appearing in the corresponding OPE,

$$[\mathcal{Q}_{\alpha}^{\mathcal{I}}, \mathcal{O}(x)] = 0 \ , \qquad [\widetilde{\mathcal{S}}^{\mathcal{I},\dot{\alpha}}, \mathcal{O}(x)] = 0 \ . \tag{F.18}$$

The only superconformal primary operator that can appear in the chiral OPE is therefore that of an $\mathcal{E}_{2r}$ multiplet, and its superconformal descendants are excluded from appearing. Any other operator that appears must be a superconformal descendant obtained by acting on a given superconformal primary with all possible supercharges $\mathcal{Q}_{\alpha}^{\mathcal{I}}$ that do not annihilate it. Thus only one conformal family per superconformal multiplet can contribute, and the superconformal blocks in this channel will be equal to the conventional conformal blocks for that family.

Upon consulting the catalogue of $\mathcal{N} = 2$ superconformal multiplets reviewed in Appendix B, it is straightforward to identify the multiplets that fit the bill. (For simplicity, we temporarily assume that $r_0 > 1$.) To illustrate the procedure, let us consider the case of long multiplets. The above argument implies that a long multiplet may only contribute to this OPE via a descendant of the schematic form $\mathcal{O} = \mathcal{Q}^4 \mathcal{O}'$, where $\mathcal{O}'$ is a superconformal primary. This descendant must be an $SU(2)_R$ singlet with $r_{\mathcal{O}} = r_{\mathcal{O}'} + 2 = 2r_0$ and spin $\ell_{\mathcal{O}} = 2j = \ell_{\mathcal{O}'}$. The relevant long multiplet is therefore of type $\mathcal{A}_{0,2r_0-2(j,j)}$. Unitarity requires that the dimension of the superconformal primary satisfies $\Delta_{\mathcal{O}'} \geqslant 2r_0 + \ell$, so the contributing descendant will have $\Delta_{\mathcal{O}} \geqslant 2r_0 + \ell + 2$.

Similar reasoning leads to the complete list of short multiplets that may contribute to the OPE, with the final selection rule taking the form

$$\mathcal{E}_{r_0(0,0)} \times \mathcal{E}_{r_0(0,0)} \sim \mathcal{E}_{2r_0(0,0)} + \mathcal{C}_{0,2r_0-1(j,j+1)} + \mathcal{B}_{1,2r_0-1(0,0)} + \mathcal{C}_{\frac{1}{2},2r_0-\frac{3}{2}(j,j+\frac{1}{2})} + \mathcal{A}_{0,2r_0-2(j,j)} \ . \tag{F.19}$$

We note that again, this derivation applies only to the OPE for superconformal primaries of the $\mathcal{E}_{r_0(0,0)}$ multiplets. For $r_0 = 1$ we can find additional short multiplets of types

$$\mathcal{D}_{1(0,0)} \qquad \hat{\mathcal{C}}_{\frac{1}{2}(j,j+\frac{1}{2})}, \qquad \hat{\mathcal{C}}_{0(j,j+1)} \ . \tag{F.20}$$

The second of these multiplets contains higher spin conserved currents, as is to be expected since the chiral operator with $r_0 = 1$ is a free scalar field.

F.2.3 Superconformal blocks in the non-chiral channel

The superconformal blocks for the various representations appearing in the non-chiral channel have been determined in [146]. In the language of Section 4.4, these are the superconformal blocks in the $\hat{1}$ channel. They are as follows,

$$
\begin{aligned}
\mathcal{G}_{\hat{1}}^{\mathrm{Id}}(z,\bar{z}) &:= 1\ ,\\
\mathcal{G}_{\hat{1}}^{\hat{\mathcal{C}},\ell}(z,\bar{z}) &:= \frac{z\bar{z}}{z-\bar{z}}\left(\left(-\tfrac{z}{2}\right)^{\ell} z\, {}_2F_1\left(\ell+1,\ell+3;2\ell+4;z\right)\right)-z\leftrightarrow\bar{z}\Big)\ ,\\
\mathcal{G}_{\hat{1}}^{\Delta,\ell}(z,\bar{z}) &:= \frac{(z\bar{z})^{\frac{\Delta-\ell}{2}}}{z-\bar{z}}\Big(\left(-\tfrac{z}{2}\right)^{\ell} z\, {}_2F_1\left(\tfrac{1}{2}(\Delta+\ell),\tfrac{1}{2}(\Delta+\ell+4);\Delta+\ell+2;z\right)\Big)\\
&\qquad\times {}_2F_1\left(\tfrac{1}{2}(\Delta-\ell-2),\tfrac{1}{2}(\Delta-\ell+2);\Delta-\ell;\bar{z}\right)\Big)-z\leftrightarrow\bar{z}\Big)\ ,
\end{aligned}
\tag{F.21}
$$

Note that the superconformal block for the $\hat{\mathcal{C}}_{0(j,j)}$ representation is just the specialization of the superconformal block for a long multiplet to the case $\Delta=\ell+2$. This is to be expected based on the recombination rules of Appendix B. The superconformal block for a long multiplet can be decomposed into ordinary conformal blocks, which makes manifest the collection of conformal families from this multiplet that contribute to the four-point function:

$$
\begin{aligned}
\mathcal{G}_{i=\hat{1}}^{\Delta,\ell}(z,\bar{z}) &= u^{\frac{\Delta-\ell}{2}}G_{\Delta}^{(\ell)}(u,v)+\left(\tfrac{1}{2(\Delta-\ell)}-\tfrac{1}{4}\right)u^{\frac{\Delta-\ell+2}{2}}G_{\Delta+1}^{(\ell-1)}(u,v)-\tfrac{(\Delta+\ell)}{(\Delta+\ell+2)}u^{\frac{\Delta-\ell}{2}}G_{\Delta+1}^{(\ell+1)}(u,v)\\
&+\tfrac{(\Delta+\ell)^2}{4(\Delta+\ell+1)(\Delta+\ell+3)}u^{\frac{\Delta-\ell}{2}}G_{\Delta+2}^{(\ell+2)}(u,v)+\tfrac{(\Delta-\ell-2)(\Delta+\ell)}{4(\Delta-\ell)(\Delta+\ell+2)}u^{\frac{\Delta-\ell+2}{2}}G_{\Delta+2}^{(\ell)}(u,v)\\
&+\tfrac{(\Delta-\ell-2)^2}{64((\Delta-\ell)^2-1)}u^{\frac{\Delta-\ell+4}{2}}G_{\Delta+2}^{(\ell-2)}(u,v)-\tfrac{(\Delta-\ell-2)^2(\Delta+\ell)}{64(\Delta-\ell-1)(\Delta-\ell+1)(\Delta+\ell+2)}u^{\frac{\Delta-\ell+4}{2}}G_{\Delta+3}^{(\ell-1)}(u,v)\\
&-\tfrac{(\Delta-\ell-2)(\Delta+\ell)^2}{16(\Delta-\ell)(\Delta+\ell+1)(\Delta+\ell+3)}u^{\frac{\Delta-\ell+2}{2}}G_{\Delta+3}^{(\ell+1)}(u,v)\\
&+\tfrac{(\Delta-\ell-2)^2(\Delta+\ell)^2}{256(\Delta-\ell-1)(\Delta-\ell+1)(\Delta+\ell+1)(\Delta+\ell+3)}u^{\frac{\Delta-\ell+4}{2}}G_{\Delta+4}^{(\ell)}(u,v)\ .
\end{aligned}
\tag{F.22}
$$

The same multiplets contributing to the non-chiral channel also contribute to the $\hat{3}$ channel via the "braided" version of the above superconformal blocks. The braided version is obtained by replacing each $G_{\Delta}^{(\ell)}$ by $(-1)^{\ell}G_{\Delta}^{(\ell)}$ in (F.22).

F.2.4 Superconformal blocks in the chiral channel

Because the supermultiplets appearing in the chiral channel contribute a single conformal family to the four point function, the superconformal blocks in the chiral channel (or $\hat{2}$ channel in the language of Section 4.4) are just the conventional conformal blocks appropriate to those conformal families. Table F.3 displays the corresponding block for each allowed supermultiplet.

The fourth and fifth lines in Table F.3 correspond to short representations that lie at

Multiplet	Contribution to $\mathcal{G}_{\hat{i}=\hat{2}}(u,v)$	Restrictions
$\mathcal{A}_{0,2r_0-2(j,j)}$	$u^{\frac{\Delta-\ell}{2}}\, G_{\Delta}^{(\ell=2j)}(u,v)$	$\Delta \geqslant 2+2r_0+\ell$
$\mathcal{E}_{2r_0}$	$u^{r_0}\, G_{2r_0}^{(0)}(u,v)$	
$\mathcal{C}_{0,2r_0-1(j,j+1)}$	$u^{r_0}\, G_{2r_0+\ell}^{(\ell=2j+2)}(u,v)$	$\ell \geqslant 2$
$\mathcal{B}_{1,2r_0-1(0,0)}$	$u^{r_0+1}\, G_{2r_0+2}^{(0)}(u,v)$	
$\mathcal{C}_{\frac{1}{2},2r_0-\frac{3}{2}(j,j+\frac{1}{2})}$	$u^{r_0+1}\, G_{2r_0+\ell+2}^{(\ell=2j+1)}(u,v)$	$\ell \geqslant 2$
$\hat{\mathcal{C}}_{\frac{1}{2}(j,j+\frac{1}{2})}$	$u^2\, G_{\Delta=\ell+4}^{(\ell=2j+1)}$	$\ell \geqslant 2;\ r_0=1$
$\hat{\mathcal{C}}_{0(j,j+1)}$	$u\, G_{\Delta=\ell+2}^{(\ell)}$	$\ell \geqslant 2;\ r_0=1$
$\mathcal{D}_{1(0,0)}$	$u^2\, G_{\Delta=42}^{(\ell=0)}$	$r_0=1$

Table F.3: Superconformal blocks for the $\mathcal{E}_{r_0}$ four point function in the $\hat{2}$ channel.

the unitarity bound for long multiplets. Accordingly, their superconformal blocks are simply the specializations of the long multiplet block to appropriate values of Δ and ℓ. On the other hand, the first two classes of short representations are separated from the continuous spectrum of long multiplets by a gap. The last two representations are only present when we relax our assumption that there are no higher spin conserved currents or free fields in the theory.

Appendix G

Semidefinite programming and polynomial inequalities

This appendix is devoted to a review of the methods of [35], whereby the search for a linear functional of the type described in Section 4.5 can be recast as a *semidefinite program*. The principal observation that leads to this reformulation is that, up to a universal prefactor, any derivative of a conformal block for fixed ℓ can be arbitrarily well approximated by a *polynomial* in the conformal dimension Δ, that is

$$\partial_z^m \partial_{\bar{z}}^n G_\Delta^{(\ell)}(z,\bar{z})\big|_{z=\bar{z}=\frac{1}{2}} \;\approx\; \chi(\Delta,\ell)\mathcal{P}_{m,n}^{(\ell)}(\Delta)\ . \tag{G.1}$$

Here $\chi(\Delta,\ell)$ may be complicated, but it is positive for all physical values of Δ and ℓ and is independent of the choice of derivative. On the other hand, $\mathcal{P}_{m,n}^{(\ell)}(\Delta)$ is a finite order polynomial in Δ. For the superconformal blocks appearing in chapter 2, the details of this polynomial approximation are explained below in Appendix H.

With the aid of this approximation, we consider the action of a linear functional on smooth functions of z and $\bar{z}$ of the form

$$\phi[F(z,\bar{z})] = \sum_{m,n=0}^{\Lambda} a_{m,n}\partial_z^m \partial_{\bar{z}}^n F(z,\bar{z})\big|_{z=\bar{z}=\frac{1}{2}}\ . \tag{G.2}$$

Up to the positive prefactor described above, the action of this functional on a conformal block is now given by a finite order polynomial in the conformal dimension,

$$\phi[G_\Delta^{(\ell)}(z,\bar{z})] = \chi(\Delta,\ell)\sum_{m,n=0}^{\Lambda} a_{m,n}\mathcal{P}_{m,n}^{(\ell)}(\Delta) =: \chi(\Delta,\ell)\,\mathcal{P}^{\ell}(\Delta)\ . \tag{G.3}$$

The numerical problem in question (see Section 4.5) is thus transformed into a search in the space of $a_{m,n}\in\mathbb{R}$ such that the polynomial $\mathcal{P}^{\ell}(\Delta)\geqslant 0$ for $\Delta\geqslant\Delta_\ell^\star$ for each ℓ. Note that the range of values of Δ for which the polynomial must be positive is always bounded from below, either by the unitarity bound or by the chosen value $\Delta_\ell^\star$.

A polynomial in Δ that is positive for all $\Delta > \Delta^\star$ can always be decomposed as follows,

$$\mathcal{P}(\Delta) = P(\Delta) + (\Delta - \Delta^*)Q(\Delta) , \tag{G.4}$$

where $P(\Delta)$ and $Q(\Delta)$ are polynomials that are positive for all real Δ. Furthermore, in terms of the monomial vector $\vec{\Delta} := (1, \Delta, \Delta^2, \ldots, \Delta^N)$, such non-negative polynomials can always be written as

$$P(\Delta) = \vec{\Delta}^t P \vec{\Delta} , \qquad Q(\Delta) = \vec{\Delta}^t Q \vec{\Delta} , \tag{G.5}$$

where P and Q are positive semidefinite matrices, which is notated as $P, Q \succeq 0$. We should emphasize that the matrices P and Q are not completely fixed in terms of $P(\Delta)$ and $Q(\Delta)$. There is a redundancy to which we will return shortly.

The action of the functional on conformal blocks will therefore be non-negative above some dimension $\Delta_\ell^\star$ in the spin ℓ channel if and only if there exist two positive semidefinite matrices, $P^{(\ell)}, Q^{(\ell)} \succeq 0$ such that

$$a_{m,n} P_{m,n}^{(\ell)}(\Delta) = \vec{\Delta}^t P^{(\ell)} \vec{\Delta} + (\Delta - \Delta_\ell^*) \vec{\Delta}^t Q^{(\ell)} \vec{\Delta} . \tag{G.6}$$

In words, we are demanding that the left- and right-hand sides of (G.6) be the same polynomial in Δ, which amounts to linear relations between the coefficients of $P^{(\ell)}$ and $Q^{(\ell)}$ and the $a_{m,n}$. Such an equation must hold for each ℓ appearing in the crossing symmetry equation, and if there are multiple flavor symmetry channels then there will be such an equation for each channel. The problem is thus reduced to the search for a set of positive semidefinite matrices whose entries satisfy certain linear constraints. This is a prototypical instance of a semidefinite program, the basic theory of which we review next.

Semidefinite programming

A *semidefinite program* (SDP) is an optimization problem wherein the goal is to minimize a linear objective function over the intersection of the cone of positive semidefinite matrices with an affine space. Such a problem can be described in terms of a vector of real variables x_i as follows,

$$\begin{aligned} &\underset{x_i}{\text{minimize}} \quad (x_i c^i) \\ &\text{such that} \quad X := x_i F^i - F^0 \succeq 0 , \end{aligned} \tag{G.7}$$

where c^i is a fixed *cost vector* that defines the objective function, and F^i and F^0 are some fixed square matrices.

This semidefinite program has a *dual problem* that is defined as the search for a positive semi-definite matrix Y that maximizes an appropriate objective function and satisfies certain linear constraints,

$$\begin{aligned} \underset{Y}{\text{maximize}} \quad & \operatorname{Tr}(F^0 \cdot Y) \\ \text{such that} \quad & Y \succeq 0 \ , \\ & \operatorname{Tr}(F^i \cdot Y) = c^i \ . \end{aligned} \tag{G.8}$$

The original problem written in (G.7) – called the *primal problem* – and the dual problem of (G.8) are not generally guaranteed to be equivalent. Indeed, given a solution x_i to the primal problem and a solution Y to the dual problem, a measure of the inequivalence of the solutions is the *duality gap*:

$$x_i c^i - \operatorname{Tr}(F^0 \cdot Y) = x_i \operatorname{Tr}(F^i \cdot Y) - \operatorname{Tr}(F^0 \cdot Y) = \operatorname{Tr}(X \cdot Y) \geqslant 0 \ , \tag{G.9}$$

where the last line holds because both matrices are positive semidefinite.

The absence of a duality gap, and the existence of an optimal solution to the primal (dual) problem, is guaranteed if the dual (primal) problem is bounded from above (below) and has a *strictly* feasible solution, *i.e.*, there exists a matrix $Y \succ 0$ ($X \succ 0$) satisfying the relevant constraints. This is called *Slater's condition*.

G.1 A toy model for polynomial inequalities

To demonstrate the application of semidefinite programming techniques to the type of crossing symmetry problem being considered in chapter 2, let us consider a simplified model in which the notation is less burdensome. Namely, consider the problem of studying the space of solutions to a "crossing symmetry" equation of the form

$$\sum_k \lambda_k^2 G_{\Delta_k}(z) = c(z) \ , \tag{G.10}$$

where Δ_k are allowed to vary over the entire real line. We will assume that the functions $G_\Delta(z)$ and their derivatives can be well approximated by polynomials in Δ, so we have

$$\partial_z^i G_\Delta(z)\Big|_{z=1/2} \approx \sum_{\alpha=0}^{2N} p_\alpha^i \Delta^\alpha =: \hat{P}^i(\Delta) \ , \tag{G.11}$$

where we have assumed that for a given range of values of i, each such polynomial has degree less than or equal to some fixed even number $2N$.[1]

[1] For the sake of comparison, we note that in the actual crossing symmetry equations encountered in this work we have an additional $\bar{z}$ coordinate, as well as sums over spins and possibly flavor symmetry channels. Also the values of Δ_k are bounded below in a given channel by unitarity bounds. However, these complications do not conceptually change this discussion.

G.1.1 The primal problem: ruling out solutions

To constrain the space of solutions to such a problem, we consider acting with a linear functional ϕ on both sides of the equality and check for contradictions. The problem can be formalized as follows,

$$\begin{aligned} &\underset{\phi}{\text{minimize}} \quad \phi[c(z)] \\ &\text{such that} \quad \phi[G_\Delta(z)] \geqslant 0 \qquad \forall\, \Delta\ . \end{aligned} \tag{G.12}$$

If the minimum turns out to be negative then our toy problem has no solution. Taking $\phi[f(z)] := \sum_{i=0}^{n} a_i \partial_z^i f(z)\big|_{z=1/2}$, we can reformulate the optimization problem as follows

$$\begin{aligned} &\underset{a_i}{\text{minimize}} \quad a_i c_i \\ &\text{such that} \quad a_i \hat{P}^i(\Delta) \geqslant 0 \qquad \forall\, \Delta\ . \end{aligned} \tag{G.13}$$

where we have defined

$$c_i := \partial_z^i c(z)\big|_{z=\frac{1}{2}}\ . \tag{G.14}$$

In terms of the vector $\vec{\Delta} = (1, \Delta, \Delta^2, \ldots \Delta^N)^t$, the second line of (G.13) requires the existence of a symmetric, positive semidefinite matrix $\hat{P}$ such that

$$\hat{P}(\Delta) = \vec{\Delta}^t P \vec{\Delta} \qquad \text{with} \qquad P \succeq 0\ . \tag{G.15}$$

This allows us to reformulate the polynomial inequalities as a semidefinite program.

We begin by introducing two sets of matrices in terms of which the problem is naturally reformulated. For $N > 1$, the matrix P is not completely fixed by (G.15) because there are only $2N+1$ components in $\hat{P}(\Delta)$ whereas P has $(N+1)(N+2)/2$ independent components. This redundancy in P can be parametrized by matrices Q satisfying

$$\vec{\Delta}^t Q \vec{\Delta} = 0 \qquad\qquad \forall\, \Delta\ . \tag{G.16}$$

Examples of such matrices Q are the 3×3 matrices with $(-1, 2, -1)$ on the cross-diagonal, or the 4×4 matrix with $(1, -1, -1, 1)$ on the cross-diagonal. All other matrices Q take a similar form, and the first set of matrices we must introduce is a complete basis for such Q. We denote the elements of this basis as $Q^{\hat{i}}$.

The second set of matrices are in one-to-one correspondence with the polynomials $\hat{P}^i(\Delta)$.

They take the form:

$$P^i := \begin{pmatrix} p_0^i & \frac{1}{2}p_1^i & 0 & 0 & \dots \\ \frac{1}{2}p_1^i & p_2^i & \frac{1}{2}p_3^i & 0 & \dots \\ 0 & \frac{1}{2}p_3^i & p_4^i & \frac{1}{2}p_5^i & \dots \\ 0 & 0 & \frac{1}{2}p_5^i & p_6^i & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} . \tag{G.17}$$

By construction these matrices satisfy the condition

$$\hat{P}^i(\Delta) = \vec{\Delta}^t P^i \vec{\Delta} . \tag{G.18}$$

Armed with these matrices we can write down the most general matrix that, upon contraction from both sides with $\vec{\Delta}$, gives the requisite polynomial:

$$a_i \hat{P}^i(\Delta) = \vec{\Delta}^t \left(a_i P^i + b_{\hat{i}} Q^{\hat{i}} \right) \vec{\Delta} , \tag{G.19}$$

where the $b_{\hat{i}}$ are arbitrary real parameters. The optimization (G.13) can now be rephrased as

$$\begin{aligned} &\underset{a_i, b_{\hat{i}}}{\text{minimize}} \quad a_i c_i \\ &\text{such that} \quad a_i P^i + b_{\hat{i}} Q^{\hat{i}} \succeq 0 , \end{aligned} \tag{G.20}$$

which we recognize to be precisely a semidefinite program of the form given in (G.7), with

$$x_i \sim (a_i, b_{\hat{i}}) , \qquad F^i \sim (P^i, Q^{\hat{i}}) , \qquad F^0 = 0 . \tag{G.21}$$

The constraints in (G.20) are invariant under an overall rescaling of the $(a_i, b_{\hat{i}})$, so the optimal value is either zero or negative infinity. To render the primal formulation bounded we can introduce an additional normalization constraint

$$\text{Tr}(P) = a_i \text{Tr}(P^i) + b_{\hat{i}} \text{Tr}(Q^{\hat{i}}) = 1 . \tag{G.22}$$

This condition is always enforceable because a nonzero, positive semidefinite matrix has strictly positive trace. Although other normalization conditions are possible, we will see that (G.22) is particularly natural from the perspective of the dual problem. In practice, we can simply solve the additional constraint for, say, a_1 to end up with a bounded variation of (G.20).

G.1.2 The dual problem: constructing solutions

Let us now address the dual problem to (G.20) with the additional constraint (G.22). After a little rewriting, the problem is as follows:

$$\begin{aligned} \underset{\lambda, Y}{\text{maximize}} \quad & -\lambda \\ \text{such that} \quad & Y + \lambda I \succeq 0\,, \\ & \mathrm{Tr}(P^i \cdot Y) = c^i \quad \forall\, i, \\ & \mathrm{Tr}(Q^{\hat{i}} \cdot Y) = 0 \quad \forall\, \hat{i}\,. \end{aligned} \tag{G.23}$$

This is a well-known form of a *feasibility* problem, which is the search for a matrix $Y \succeq 0$ subject to linear constraints. If the optimal value of λ comes out non-positive then such a matrix Y exists (*i.e.*, there is a *feasible solution*), otherwise it does not. In standard applications the reason for introducing a variable λ multiplying the identity matrix I is to ensure that a *strictly* feasible solution will always exist, because for $\lambda\gamma 0$ the matrix $Y + \lambda I \succ 0$. Its appearance in (G.23) is a consequence of the trace constraint (G.22) in the primal problem.

Whereas the primal problem amounted to the search for functionals that certify the absence of solutions to crossing symmetry, dual problem is related to constructing solutions to crossing symmetry [102]. Let us observe how this works for these semidefinite programs. We first solve the constraints $\mathrm{Tr}(Q^{\hat{i}} \cdot Y) = 0$. The most general solution is given by

$$Y = y^\alpha Y_\alpha\;, \qquad \alpha = 0, \ldots, 2N, \tag{G.24}$$

with arbitrary coefficients y^α and with matrices Y_α defined as

$$Y_0 = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \quad Y_1 = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots \\ 1 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

$$Y_2 = \begin{pmatrix} 0 & 0 & 1 & 0 & \cdots \\ 0 & 1 & 0 & 0 & \cdots \\ 1 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} , \quad \cdots . \tag{G.25}$$

Now let us choose tuples (λ_k^2, Δ_k) so that $y^\alpha = \sum_k \lambda_k^2 (\Delta_k)^\alpha$. We then have $Y = \sum_k \lambda_k^2 \vec{\Delta}_k \vec{\Delta}_k^t$ and the additional constraints of the form $\mathrm{Tr}(P^i \cdot Y) = c^i$ become

$$\sum_k \lambda_k^2 \hat{P}^i(\Delta_k) = c^i . \tag{G.26}$$

This is precisely the crossing symmetry equation (G.10) after truncating to a finite number of derivatives.

Finally, let us comment on the duality gap and the interpretation of solutions to this problem. The freedom to set λ to a large positive number ensures that the above formulation of the dual problem is strictly feasible. It is, however, not obviously bounded. From the formulation of the problem it is clear that this is related to the existence of solutions to crossing symmetry where $c(z) = 0$. More precisely, the problem is unbounded if there is a positive semidefinite matrix Y that satisfies $\mathrm{Tr}(P^i \cdot Y) = 0$ and $\mathrm{Tr}(Q^{\hat{i}} \cdot Y) = 0$ for all i and $\hat{i}$. In the absence of such solutions the problem is bounded, Slater's condition is satisfied, and there is no duality gap, so for the optimal values we find that $-\lambda = a^i c_i$. This equation makes intuitive sense. Indeed, suppose the dual formulation does not find a solution to crossing symmetry. This happens when $-\lambda = a^i c_i < 0$ and therefore the primal formulation indeed provides a functional that proves that such a solution cannot exist. Similarly, suppose we do find a matrix $Y \succeq 0$ satisfying all the above constraints. In that case $-\lambda = a^i c_i \geqslant 0$, so no functional can be found in the primal problem.

Extremal functionals

In the applications of this framework to study interesting physical theories, there are often additional parameters in the problem such as assumed gaps in the spectrum for certain spins. In such cases we are usually interested in finding the boundary in the space of such parameters between regions where crossing symmetry can and cannot be satisfied. Precisely at the boundary $-\lambda = a^i c_i = 0$. This turns out to imply that the corresponding solution to crossing symmetry is completely determined by the zeroes of the *extremal functional* [34, 123]. This is because the absence of a duality gap implies $\mathrm{Tr}(X \cdot Y) = 0$ which together with the above assumption on the form of Y leads to

$$a_i P^i(\Delta_k) = 0 \; . \tag{G.27}$$

The solution to crossing symmetry encoded in Y therefore involves precisely those values of Δ for which the extremal functional vanishes. This observation leads to the following algorithm for finding the solution to crossing symmetry: one first lists the Δ_k for which the $\vec{\Delta}_k^t X \vec{\Delta}_k = 0$, and then finding the λ_k^2 reduces to solving the linear problem $y^\alpha = \sum_k \lambda_k^2 (\Delta_k)^\alpha$. Note that we require both the X and the Y matrix here.

G.2 Notes on implementation

In this work we have utilized the dual formulation of the semidefinite program associated to crossing symmetry. We first solved all the linear constraints analogous to those appearing in (G.23), leading to a smaller set of independent parameters that we denote $z^{\hat{\alpha}}$ and corresponding matrices $Z_{\hat{\alpha}}$. The nonzero c_i lead to an inhomogeneous term that we may call $Z_{\hat{0}}$. The complete semidefinite program is then as above with

$$x^i \Rightarrow (z^{\hat{\alpha}}, \lambda) \, , \qquad F^i \Rightarrow (Z^{\hat{\alpha}}, I) \, , \qquad F^0 \Rightarrow Z_{\hat{0}} \; , \tag{G.28}$$

and a cost vector such that only λ is extremized. Since we were unable to rigorously show that the dual problem was bounded in all cases, we added an additional constraint $\lambda \geqslant 0$. In the primal problem this additional constraint transforms the trace equality (G.22) into the inequality $\mathrm{Tr}(P) \leqslant 1$. With this condition the optimal value will be zero if a solution exists and no functional is found, or strictly negative if the opposite happens.

We used SDPA and SDPA-GMP solvers [194, 195], which use an interior point method that simultaneously optimizes both the primal and dual problems, and that terminates when the duality gap is below a certain (small) threshold. This requires a strictly feasible solution to both the primal and the dual problem, and our formulation of the problem ensures that such strictly feasible solutions exist. Furthermore, we found that a normalization of the form given in (G.22) improves numerical stability compared to other normalizations such as, *e.g.*, $a^i c_i = 1$. We ascribe this difference to the fact that $a^i c_i$ naturally tends to zero in physically interesting regions, and so setting it to one as a normalization leads to large numbers elsewhere.[2]

In order to achieve maximal numerical stability we 'renormalized' many of the numbers fed into the problem. For example, the polynomials $P^i(\Delta)$ can be redefined by multiplying with an overall (positive) constant, by affine redefinitions of Δ, and by choosing a different basis for the space of derivatives. Altogether these reparametrizations give us the freedom

[2]Our normalization is not suitable for obtaining bounds on OPE coefficients. In that case we need to normalize the functional as described in Section 4.5.

Parameter	Value
maxIteration	1000
epsilonStar	10^{-12}
lambdaStar	10^{8}
omegaStar	10^{6}
lowerBound	-10^{30}
upperBound	10^{30}
betaStar	0.1
betaBar	0.3
gammaStar	0.9
epsilonDash	10^{-12}
precision	200

Table G.1: Parameters used for the SDPA and SDPA-GMP solvers. The 'precision' variable is only relevant for the SDPA-GMP solver.

to transform the problem according to

$$P^i(\Delta) \to M^i_j P^j(a\Delta + b)\,, \qquad\qquad c^i \to M^i_j c^j\ . \tag{G.29}$$

We choose M^i_j, a, and b so as to minimize the potential for numerical inaccuracies. Numerical stability can be further improved by rescaling the normalization condition $\mathrm{Tr}(X) = 1$ to $\mathrm{Tr}(X) = \mu$ for a positive real μ. (In the dual problem μ becomes the cost vector, so this parameter is introduced through the optimization of $\mu\lambda$ instead of λ.) In order to avoid large numerical differences between the primal and the dual formulation, we choose μ large so that X, which is a matrix of size $O(10^3)$, can have $O(1)$ entries on its diagonal.

In previous implementations of the numerical bootstrap as a semidefinite program [35], it was necessary to employ the arbitrary precision solver SDPA-GMP to avoid numerical instabilities. The setup described above, with Slater's condition satisfied and coefficients that are suitably renormalized, has allowed us to use the double precision SDPA program for low and intermediate values of Λ. Since working at machine precision is significantly faster than working at arbitrary precision, we were able to explore a much greater range of the parameter space given our computational resources. We still found it necessary to switch to SDPA-GMP for higher values of Λ, with the exact transition value somewhat dependent on the problem at hand. For example, we had to switch at $\Lambda = 16$ for the bounds on theories with $\mathfrak{e}_6$ flavor symmetry shown in Section 4.6, but were able to obtain reliable results with double precision numerics up to $\Lambda = 22$ for some of the bounds on theories with $\mathfrak{su}(2)$ flavor symmetry. Typical settings for the parameters of both the SDPA and SDPA-GMP solvers can be found in Table G.1.

Appendix H

Polynomial approximations and four-dimensional conformal blocks

The semidefinite programming approach to the numerical bootstrap depends on our ability to approximate conformal blocks of fixed spin ℓ and varying conformal dimension Δ by polynomials in Δ [35, 125]. This appendix includes a brief review of these approximations and some details relevant to the special cases of interest. The goal is to express the conformal blocks and their derivatives in a factorized form, with one factor being a function that can be well approximated by a polynomial in Δ, and the other a non-polynomial term that is strictly positive and independent of the choice of derivative. We denote the polynomial in Δ by $\mathcal{P}^{(\ell)}_{m,n}(\Delta)$ and the non-polynomial term by $\chi(\Delta,\ell)$, so the approximation takes the following form,

$$\partial_z^m \partial_{\bar{z}}^n G_\Delta^{(\ell)}(z,\bar{z})|_{z=\bar{z}=\frac{1}{2}} \;\approx\; \chi(\Delta,\ell)\mathcal{P}^{(\ell)}_{m,n}(\Delta)\ . \tag{H.1}$$

The starting point for this approximation scheme is a recursion relation for derivatives of the hypergeometric functions appearing in conformal blocks,

$$\left[\frac{d^2}{dz^2}+\frac{1-a-b}{z-1}\frac{d}{dz}+\frac{\beta^2-\beta+abz}{z^2(z-1)}\right]\left(z^\beta\, {}_2F_1\left(\beta-a,\beta-b,2\beta,z\right)\right)=0\ . \tag{H.2}$$

This recursion relation follows immediately from the fact that the ${}_2F_1$ hypergeometric function is a solution to Euler's differential equation. Using this relation, any derivative of the above hypergeometric function at fixed z can be expressed as the sum of the zeroth and first order derivatives of the same hypergeometric function, each with some polynomial in β as a prefactor. Thus the only non-polynomial feature of any derivative of the hypergeometric function can be expressed in terms of the value of the hypergeometric function itself and that of its first derivative.

To approximate conventional conformal blocks we follow exactly the same steps as in [35]. From (F.1) any derivative of a conformal block $\partial_z^m\partial_{\bar{z}}^n G_\Delta^{(\ell)}(z,\bar{z})$ can be rewritten, by recursive use of (H.2) with $a=b=0$, in terms of the hypergeometric functions and their first derivatives. These functions encode all of the non-polynomial dependence on Δ. We can then pull out factors out of the blocks such that the leftover expression can be well approximated by polynomials. To start we factor out the following term

$$\frac{1}{\beta\bar{\beta}}\left(\frac{\partial}{\partial z}z^{\beta}{}_2F_1(\beta,\beta,2\beta,z)\right)\Bigg|_{z=\frac{1}{2}}\left(\frac{\partial}{\partial z}z^{\bar{\beta}}{}_2F_1(\bar{\beta},\bar{\beta},2\bar{\beta},z)\right)\Bigg|_{z=\frac{1}{2}} . \tag{H.3}$$

Here we have $\beta = \frac{\Delta+\ell}{2}$, $\bar{\beta} = \frac{\Delta-\ell-2}{2}$. This is positive for all $\beta \geqslant -1$, and so it is positive for any conformal block appearing in a unitary theory. After factoring out this positive non-polynomial term, the remaining non-polynomial dependence is isolated in the following ratio (and a similar one for $\beta \to \bar{\beta}$),

$$K_\beta = \left.\frac{\beta z^{\beta}{}_2F_1(\beta,\beta,2\beta,z)}{\frac{\partial}{\partial z}z^{\beta}{}_2F_1(\beta,\beta,2\beta,z)}\right|_{z=\frac{1}{2}} \simeq \frac{1}{\sqrt{2}}\prod_{j=0}^{M}\frac{(\beta-r_j)}{(\beta-s_j)} \equiv \frac{N_M(\beta)}{D_M(\beta)} . \tag{H.4}$$

The coefficient r_j is the j–th zero of ${}_2F_1(\beta,\beta,2\beta,z)$ and the coefficient s_j the j–th zero of $\frac{\partial}{\partial z}z^{\beta}{}_2F_1(\beta,\beta,2\beta,z)$.[1] The rational function $\frac{N_M(\beta)}{D_M(\beta)}$ is an approximation of K_β obtained by restricting to the first M zeroes of both the numerator and denominator. The approximation becomes arbitrarily good as M is increased, and converges very quickly, as described in [35].

The last step is to multiply by $D(\beta)D(\bar{\beta})$, which is strictly positive for the same range of β and $\bar{\beta}$. In this way we have factored out all of the nonpolynomial dependence of $\partial_z^m\partial_{\bar{z}}^n G_\Delta^{(\ell)}(z,\bar{z})$, which defines $\chi(\Delta,\ell)$ in (H.1), and are left with a polynomial in Δ, $\mathcal{P}_{m,n}^{(\ell)}(\Delta)$, whose degree is controlled by the number of terms M kept in the approximation (H.4). Exactly this approximation is used for the blocks in the $\hat{2}$ channel for the $\mathcal{E}_r$ correlator, and for all the blocks in the $\hat{\mathcal{B}}_1$ correlator (with a shift $\Delta \to \Delta+4$).

For superconformal blocks in the $\hat{1}$ channel given in (4.78) the procedure is analogous. This time we use (H.2), where now $a=1$ and $b=-1$, to write all of the block derivatives in terms of the zeroth and second derivatives of the hypergeometric function. In this case we define $\beta = \frac{\Delta+\ell+2}{2}$ and $\bar{\beta} = \frac{\Delta-\ell}{2}$. The first step is again to factor out

$$\left(\frac{1}{\beta(\beta-1)}\frac{\partial^2}{\partial z^2}z^{\beta}{}_2F_1(\beta,\beta,2\beta,z)\right)\Bigg|_{z=\frac{1}{2}}\left(\frac{1}{\bar{\beta}(\bar{\beta}-1)}\frac{\partial^2}{\partial z^2}z^{\beta}{}_2F_1(\bar{\beta},\bar{\beta},2\bar{\beta},z)\right)\Bigg|_{z=\frac{1}{2}} , \tag{H.6}$$

which is positive for all possible values of β and $\bar{\beta}$ occurring in the relevant OPE ($\beta, \bar{\beta} \geqslant$

[1] In practice we compute the zeros of the latter by making use of the following identity, which relates it to another hypergeometric function

$$\frac{d^n}{dz^n}\left[z^{\beta-a+n-1}{}_2F_1\left(\beta-a,\beta-b,2\beta,z\right)\right] = \left(\beta-a\right)_n z^{\beta-a-1}{}_2F_1\left(\beta-a+n,\beta-b,2\beta,z\right) , \tag{H.5}$$

where in this case we want to use $n=1$, and we have $a=0$.

1). The remaining nonpolynomial dependence is then encoded by ratios of hypergeometric functions and their second derivatives. As it happens, an application of various identities for hypergeometric functions (*cf.* [196]) allows us to express this nonpolynomial quantity in terms of the same function K_β, so we utilize the same approximation of (H.4) and find

$$\frac{{}_2F_1\left(\beta-1,\beta+1,2\beta,\frac{1}{2}\right)}{{}_2F_1\left(\beta+1,\beta+1,2\beta,\frac{1}{2}\right)}=\frac{1+4(\beta-1)K_\beta}{4+8(\beta-1)K_\beta}\simeq\frac{D_M(\beta)+4(\beta-1)N_M(\beta)}{4D_M(\beta)+8(\beta-1)N_M(\beta)}\,. \tag{H.7}$$

Here we used (H.5) to relate the second derivative of the hypergeometric function to a different hypergeometric function. A similar ratio appears for the $\bar{\beta}$ dependent hypergeometric functions, which we approximate in the same way. After approximating K_β by (H.4) we can again factor out another strictly positive denominator $(4D_M(\beta)+8(\beta-1)N_M(\beta))(4D_M(\bar{\beta})+8(\bar{\beta}-1)N_M(\bar{\beta}))$ for the same range of $\beta, \bar{\beta}$.

The approximation for the braided superconformal block goes in the same way. (We will now ignore the $\bar{\beta}$ dependence since it is simply obtained by $\beta\to\bar{\beta}$ in the discussion below.) We start by noting that braiding the block has the following effect on the hypergeometric functions [196]

$${}_2F_1\left(\beta-1,\beta+1,2\beta,\frac{z}{z-1}\right)=(1-z)^{\beta-1}\,{}_2F_1\left(\beta-1,\beta-1,2\beta,z\right)\,. \tag{H.8}$$

The next step is now to write all derivatives in terms of the zeroth and second derivatives of the hypergeometric function by means of (H.2) with $a=1$, $b=1$. We can then again factor out any nonnegative and nonpolynomial terms, beginning with $z^\beta{}_2F_1(\beta-1,\beta-1,2\beta,\frac{1}{2})(\beta-1)\beta$ which is strictly positive for $\beta\geqslant 1$. The residual non-polynomial dependence is then given by

$$\frac{{}_2F_1\left(\beta-1,\beta+1,2\beta,\frac{1}{2}\right)}{{}_2F_1\left(\beta-1,\beta-1,2\beta,\frac{1}{2}\right)}=4\frac{{}_2F_1\left(\beta-1,\beta+1,2\beta,\frac{1}{2}\right)}{{}_2F_1\left(\beta+1,\beta+1,2\beta,\frac{1}{2}\right)}\simeq 4\frac{D_M(\beta)+4(\beta-1)N_M(\beta)}{4D_M(\beta)+8(\beta-1)N_M(\beta)}\,, \tag{H.9}$$

where we have rewritten, through (H.5), the second derivative of the hypergeometric function as $z^{\beta-2}{}_2F_1\left(\beta-1,\beta+1,2\beta,1/2\right)$, and used several hypergeometric identities. For the relevant range of β the denominator in the above equation is strictly positive, and it is the final term to be factored out.

The ratio r_j/s_j tends to one extremely fast, and we observed that truncating the product in (H.4) at $M=4$ was already accurate enough for all $\Lambda\leqslant 22$. For $22<\Lambda\leqslant 30$ we found that $M=5$ was sufficient. In a number of cases we repeated the numerical analysis with $M=6$ and verified that there was no change in the results.

Appendix I

Exact OPE coefficients for the $\mathcal{N} = 2$ chiral ring

The OPE coefficients of Coulomb branch chiral ring operators in four-dimensional $\mathcal{N} = 2$ SCFTs satisfy four-dimensional $tt^\star$ equations [173, 174]. In this appendix we limit our attention to the case of theories with a conformal manifold that has one complex dimension, *i.e.*, theories with just a single $\mathcal{E}_2$ multiplet. In such cases there is a close connection between the chiral ring OPE coefficients and the Zamolodchikov metric on the conformal manifold. After diagonalization of the fields, the OPE of the (unit normalized) chiral operators takes the form

$$\mathcal{E}_2(x)\mathcal{E}_2(0) = \lambda_{\mathcal{E}_4}\mathcal{E}_4(0) + \dots \ , \tag{I.1}$$

and we are interested in the squared OPE coefficient $\lambda^2_{\mathcal{E}_4}$. Precisely this coefficient is part of a solvable subsector of the tt^* equations and it takes the form

$$\lambda^2_{\mathcal{E}_4} = 2 + \frac{\partial_\tau \partial_{\bar\tau} \log(g_{\tau\bar\tau})}{g_{\tau\bar\tau}} = 2 - \frac{1}{2} R[g_{\tau\bar\tau}] \ , \tag{I.2}$$

where $g_{\tau\bar\tau}$ is the only nonvanishing component of the Zamolodchikov metric on the conformal manifold.[1] On the right-hand side we recognize the expression for the scalar curvature of the Zamolodchikov metric. The bounds reported in Section 4.7 for $\lambda^2_{\mathcal{E}_4}$ therefore provide lower and upper bounds on this curvature.

Let us consider a few examples, starting with the theory of n free vector multiplets. The superconformal primary of the flavor singlet $\mathcal{E}_2$ multiplet in this theory is $\varphi_a\varphi_a(x)$, with $\varphi(x)$ the scalar operator in the vector multiplet. We can compute $\lambda^2_{\mathcal{E}_4}$ directly by performing Wick contractions, whereupon we find

$$n \text{ free vector multiplets:} \quad \lambda^2_{\mathcal{E}_4} = 2 + \tfrac{4}{n} = 2 + \tfrac{2}{3c} \ . \tag{I.3}$$

[1] In the notations of [174], this is the metric written as $g_{i\bar\jmath}$. This differs from the true Zamolodchikov metric $G_{i\bar\jmath}$ by a factor of 192.

In the last equality we have used the precise value of the central charge in this theory: $c = n/6$. In any $\mathcal{N} = 2$ superconformal gauge theory with gauge group G, the tree-level value for this OPE coefficient takes the same form,

$$\text{tree level gauge theory:} \quad \lambda^2_{\mathcal{E}_4} = 2 + \tfrac{4}{\dim(G)} \geqslant 2 + \tfrac{2}{3c} \ . \tag{I.4}$$

The inequality is a consequence of the fact that the central charge of a superconformal gauge theory is always greater than that of the vector multiplets alone.

In $\mathcal{N} = 4$ supersymmetric Yang-Mills theory, the central charge is $c = \frac{1}{4}\dim(G)$. In this special case, extended supersymmetry prevents the OPE coefficient in question from being renormalized. Consequently the exact value (for all values of the complex gauge coupling) is given by the tree-level result,

$$\mathcal{N} = 4 \text{ super Yang-Mills:} \ \lambda^2_{\mathcal{E}_4} = 2 + \tfrac{1}{c} \ . \tag{I.5}$$

In many $\mathcal{N} = 2$ SCFTs, this OPE coefficient is made accessible by the relation between the Kähler metric on the conformal manifold and the S^4 partition function [197],

$$g_{i\bar{j}} = \partial_i \partial_{\bar{k}} \log(Z_{S^4}) \ . \tag{I.6}$$

It is frequently the case that the partition function Z_{S^4} can be computed exactly using supersymmetric localization [198]. As an example, consider $\mathcal{N} = 2$ SCQCD with $N_f = 4$ flavors (sometimes referred to in the text as *the* $\mathfrak{so}(8)$ *theory*). The Nekrasov instanton partition function that features in the localization result is related to four-point Virasoro conformal blocks [86]. These in turn are efficiently computed using the recursion relations developed in [199]. Altogether, one ultimately finds the following expression for the S^4 partition function,

$$\log Z_{S^4}(q) = \log\left(\int_{-\infty}^{\infty} da\, a^2 |16q|^{2a^2} \left|\frac{G(1+2ia)^2}{G(1+ia)^8}\right|^2 H(a,q)H(a,\bar{q})\right) + f(\tau) + f(\bar{\tau}) \ , \tag{I.7}$$

where the functions $f(\tau)$ are Kähler transformations that drop out in the computation of the curvature, and $G(z)$ is Barnes' G-function.[2] The function $H(a,q)$ has been defined in [199] by means of a somewhat intricate recursion relation that we will not review here. It is a building block of the Virasoro four-point conformal block with $c = 25$, all four external dimensions equal to one, and internal dimension equal to $1 + a^2$. The first few terms in its series expansion take the form

$$H(a,q) = 1 + \frac{12\,(a^2+2)\,q^2}{(4a^2+9)^2} + \frac{18\,(32a^6+308a^4+955a^2+940)\,q^4}{(4a^2+9)^2\,(4a^2+25)^2} + \cdots \ . \tag{I.8}$$

[2]This function is implemented in `Mathematica` as `BarnesG[z]`.

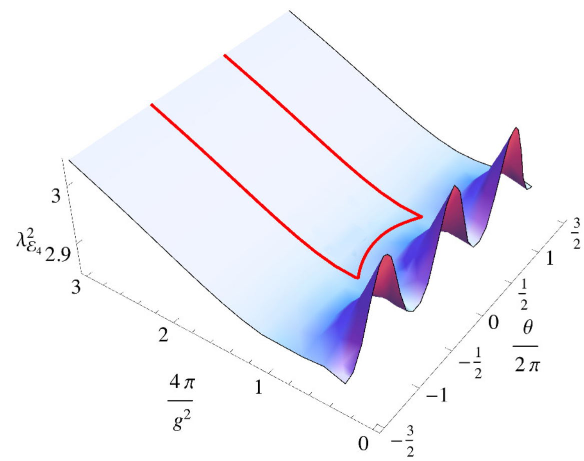

Figure I.1: The value of $\lambda^2_{\mathcal{E}_4}$ for $\mathcal{N} = 2$ SQCD with $N_f = 4$ flavors. The coupling is shown as a function of the exactly marginal complexified gauge coupling $\tau = \frac{\theta}{2\pi} + \frac{4\pi i}{g^2}$, and the fundamental domain for the action of $SL(2, \mathbb{Z})$-duality on the coupling plane is outlined in red.

One should note that the parameter q is $q_{\rm IR}$ which is *not* the same parameter as the parameter $q_{\rm UV}$ used in [198] and in [173, 174].[3] The relation between the two is given in [86], and also in [199],

$$q_{\rm IR} = \exp(i\pi\tau_{\rm IR}) = \exp\left(-\pi\frac{K(1-q_{\rm UV})}{K(q_{\rm UV})}\right) . \tag{I.9}$$

Here $K(m)$ is the complete elliptic integral of the first kind.[4] The explicit form of this transformation is in fact not particularly relevant for our purposes because the scalar curvature is a diffeomorphism invariant. But it is $\tau_{\rm IR}$ that is valued in the fundamental domain for the action of $SL(2, \mathbb{Z})$ on the upper half plane. Namely, under S- and T-transformations we have

$$T: \qquad \tau_{\rm IR} \to -1/\tau_{\rm IR} , \qquad q_{\rm UV} \to 1 - q_{\rm UV} , \tag{I.10}$$

$$S: \qquad \tau_{\rm IR} \to \tau_{\rm IR} + 1 , \qquad q_{\rm UV} \to \frac{q_{\rm UV}}{q_{\rm UV} - 1} . \tag{I.11}$$

The transformations of $q_{\rm UV}$ describe the action of crossing symmetry on the Liouville four-point function.

The value of the OPE coefficient $\lambda^2_{\mathcal{E}_4}(\tau)$ can be computed numerically to arbitrary accuracy at any value of the coupling. The free-field value is given by $\lambda^2_{\mathcal{E}_4}(\tau = \infty) = 10/3$. The

[3] An early discussion of this point can be found in [200].
[4] This function is implemented in `Mathematica` as `EllipticK[m]`.

OPE coefficient decreases monotonically as a function of the gauge coupling and becomes stationary at the self-dual points. To get reasonable accuracy we need to expand $H(q)$ to order q^8, resulting in the following stationary values:

$$\lambda^2_{\mathcal{E}_4}(\tau = i) = 2.8983769\ldots \qquad \lambda^2_{\mathcal{E}_4}(\tau = e^{i\pi/3}) = 2.8940994\ldots \tag{I.12}$$

This OPE coefficient is plotted in Fig. I.1. The stationary point at $\tau = i$ is a saddle point, while the global minimum occurs at $\tau = e^{i\pi/3}$, so the range for this OPE coefficient is given by

$$2.8940994\ldots \leqslant \lambda^2_{\mathcal{E}_4}(\tau) \leqslant \frac{10}{3}\ . \tag{I.13}$$

This is the range of values that appear in Fig. 4.26 of Section 4.7.

Appendix J

Superconformal blocks for six-dimensional $(2,0)$ theories

In this appendix we collect results relevant for the decomposition in superconformal blocks of the stress tensor multiplet four-point function.

The six-dimensional conformal blocks for the decomposition of a four-point function of scalar operators with conformal dimension Δ_i, $i = 1, \ldots, 4$, $\mathcal{G}_{\Delta}^{(\ell)}(\Delta_{12} := \Delta_1 - \Delta_2, \Delta_{34} := \Delta_3 - \Delta_4; z, \bar{z})$, are given by [138, 201]:

$$\begin{aligned}
\mathcal{G}_{\Delta}^{(\ell)}(\Delta_{12}, \Delta_{34}; z, \bar{z}) \;=\;& \mathcal{F}_{00} - \frac{\ell+3}{\ell+1}\mathcal{F}_{-11} + \frac{(\Delta-4)(\ell+3)}{16(\Delta-2)(\ell+1)(\Delta-\ell-\Delta_{34}-4)} \\
& \frac{(\Delta-\ell-\Delta_{12}-4)(\Delta-\ell+\Delta_{12}-4)(\Delta-\ell+\Delta_{34}-4)}{(\Delta-\ell-5)(\Delta-\ell-4)^2(\Delta-\ell-3)}\mathcal{F}_{02} \\
& - \frac{\Delta-4}{\Delta-2}\frac{(\Delta+\ell-\Delta_{12})(\Delta+\ell+\Delta_{12})(\Delta+\ell+\Delta_{34})(\Delta+\ell-\Delta_{34})}{16(\Delta+\ell-1)(\Delta+\ell)^2(\Delta+\ell+1)}\mathcal{F}_{11} \\
& + \frac{2(\Delta-4)(\ell+3)\Delta_{12}\Delta_{34}}{(\Delta+\ell)(\Delta+\ell-2)(\Delta+\ell-4)(\Delta+\ell-6)}\mathcal{F}_{01}\,, \qquad \text{(J.1)}
\end{aligned}$$

where

$$\begin{aligned}
\mathcal{F}_{nm}(z, \bar{z}) \;=\;& \frac{(z\bar{z})^{\frac{\Delta-\ell}{2}}}{(z-\bar{z})^3}\Bigg(\left(-\frac{z}{2}\right)^{\ell} z^{n+3}\bar{z}^{m} \\
& {}_2F_1\left(\frac{\Delta+\ell-\Delta_{12}}{2}+n, \frac{\Delta+\ell+\Delta_{34}}{2}+n, \Delta+\ell+2n, z\right) \\
& {}_2F_1\left(\frac{\Delta-\ell-\Delta_{12}}{2}-3+m, \frac{\Delta-\ell+\Delta_{34}}{2}-3+m, \Delta-\ell-6+2m, \bar{z}\right) \\
& -(z \longleftrightarrow \bar{z})\Bigg)\,.
\end{aligned}$$

The projectors onto the various $\mathfrak{so}(5)_R$ symmetry irreducible representation appearing in the decomposition of the four-point function (5.26) are given by [42, 43]

$$
\begin{aligned}
Y^{[4,0]}(\alpha,\bar{\alpha}) &= \sigma^2+\tau^2+4\sigma\tau-\frac{8(\sigma+\tau)}{9}+\frac{8}{63}\,,\\
Y^{[2,2]}(\alpha,\bar{\alpha}) &= \sigma^2-\tau^2-\frac{4(\sigma-\tau)}{7}\,,\\
Y^{[0,4]}(\alpha,\bar{\alpha}) &= \sigma^2+\tau^2-2\sigma\tau-\frac{2(\sigma+\tau)}{3}+\frac{1}{6}\,,\\
Y^{[0,2]}(\alpha,\bar{\alpha}) &= \sigma-\tau\,,\\
Y^{[2,0]}(\alpha,\bar{\alpha}) &= \sigma+\tau-\frac{2}{5}\,,\\
Y^{[0,0]}(\alpha,\bar{\alpha}) &= 1\,,
\end{aligned}
\tag{J.2}
$$

where $\sigma=\alpha\bar{\alpha}$ and $\tau=(\alpha-1)(\bar{\alpha}-1)$.

The superconformal blocks for the $\mathcal{D}[2,0]$ four-point function decomposition are obtained in [42]. We quote here the results relevant for our purposes. As discussed before, for each $\mathfrak{so}(5)_R$ channel the superconformal blocks can be obtained from two functions $a(z,\bar{z})$ and $h(z)$, through

$$
\begin{aligned}
A_{[4,0]}(z,\bar{z}) &= \frac{1}{6}u^4\Delta_2\left[u^2a(z,\bar{z})\right]\,,\\
A_{[2,2]}(z,\bar{z}) &= \frac{1}{2}u^4\Delta_2\left[u(v-1)a(z,\bar{z})\right]\,,\\
A_{[0,4]}(z,\bar{z}) &= \frac{1}{6}u^4\Delta_2\left[u(3(v+1)-u)a(z,\bar{z})\right]\,,\\
A_{[0,2]}(z,\bar{z}) &= \frac{1}{2}u^4\Delta_2\left[(v-1)\left((v+1)-\frac{3}{7}u\right)a(z,\bar{z})\right]\\
&\quad -u^2\left(\frac{(z-2)zh'(z)+(\bar{z}-2)\bar{z}h'(\bar{z})}{2(z-\bar{z})^2}+(z+\bar{z}-z\bar{z})\frac{h(z)-h(\bar{z})}{(z-\bar{z})^3}\right)\,,\\
A_{[2,0]}(z,\bar{z}) &= \frac{1}{2}u^4\Delta_2\left[\left((v-1)^2-\frac{1}{3}u(v+1)+\frac{2}{27}u^2\right)a(z,\bar{z})\right]\\
&\quad +u^2\left(z\bar{z}\frac{h(z)-h(\bar{z})}{(z-\bar{z})^3}-\frac{z^2h'(z)+\bar{z}^2h'(\bar{z})}{2(z-\bar{z})^2}\right)\,,\\
A_{[0,0]}(z,\bar{z}) &= \frac{1}{4}u^4\Delta_2\left[\left((v+1)^2-\frac{1}{5}(v-1)^2-\frac{3}{5}u(v+1)+\frac{3}{35}u^2\right)a(z,\bar{z})\right]\\
&\quad -u^2\frac{(5(1-z)+z^2)h'(z)+(5(1-\bar{z})+\bar{z}^2)h'(\bar{z})}{5(z-\bar{z})^2}
\end{aligned}
$$

$$+u^2\left(2z\bar{z}+5(1-z)+5(1-\bar{z})\right)\frac{h(z)-h(\bar{z})}{5(z-\bar{z})^3}\,. \tag{J.3}$$

Each $A_{[i,j]}(z,\bar{z})$ admits a decomposition in a finite number of conformal blocks, given in Eq. (J.1) with $\Delta_i = 4$, with positive coefficients. As explained in section 5.3, the relative coefficients between conformal primaries belonging to the same superconformal multiplets are fixed, and there is only one unfixed OPE coefficient per superconformal multiplet. This is apparent from the form of (J.3), where we see we only need to specify how each superconformal multiplet contributes to $a(z,\bar{z})$ and $h(z)$. This information is summarized in table 5.1. To go from the contribution of each superconformal multiplet to $a(z,\bar{z})$ and $h(z)$ to a finite sum over conformal blocks, which includes acting with the differential operator Δ_2, one can make use of the recurrence relations given in appendix D of [42], which were corrected in [133].