Existence of Undominated Subgame Perfect Equilibrium in Extensive Form Games

A Dissertation Presented

by

Ceren G. Yilmaz

to

The Graduate School

in Partial Fulfillment of the

Requirements

For the Degree of

Doctor in Philosophy

in

Economics

Stony Brook University

August 2014

Stony Brook University

The Graduate School

Ceren G. Yilmaz

We, the dissertation committee for the above candidate for the
Doctor in Philosophy degree, hereby recommend
acceptance of this dissertation.

Dr. Sandro Brusco- Dissertation Advisor
Professor

Dr. Yair Tauman - Chairperson of Defense
Professor

Dr. Pradeep Dubey- Dissertation Committee Member
Professor

Dr. Esther Arkin - Outside Committee Member
Professor

This dissertation is accepted by the Graduate School

Charles Taber
Dean of the Graduate School

Abstract of the Dissertation

Existence of Undominated Subgame Perfect Equilibrium in Finite Extensive Form Games

by

Ceren G. Yilmaz

Doctor in Philosophy

in

Economics

Stony Brook University

2014

There are games for which all subgame perfect equilibria are such that some (or all) players use weakly dominated strategies. Surely this is undesirable as it diminishes the credibility of equilibria. It is implausible to expect a player to play a weakly dominated strategy just because it is an 'equilibrium strategy'. We focus on the class of finite extensive form games with complete and perfect information and show that in this class of games there exists an undominated subgame perfect equilibrium; a subgame perfect equilibrium in which no player uses a weakly dominated strategy. The results also provide insight as to why one should restrict the class of mechanisms to finite mechanisms where relevant.

Contents

Acknowledgements

I am indebted to my thesis advisor Dr. Sandro Brusco for his influence, expert guidance and humble patience.

I also thank the other members in my thesis committee Dr.Yair Taumann, Dr. Pradeep Dubey and Dr. Estie Arkin. I thank Dr. Thomas Muench for numerous discussions that brought me clarity.

I owe exceptional gratitude to my undergraduate mentor Dr. Tarik Kara to whom I owe all of the basic skills required to be able to do original research, for inspiring me, teaching me how to think critically and for introducing me to Game Theory.

In addition to the above mentioned, my graduate studies and thesis would not have been possible without the help, support and encouragement of Andrew J Verdile, Joseph Malkoun, Danmo Lin and Grace R. Welch.

Chapter 1

Introduction

1.1 Basic Notation and Definitions

In this section we introduce the basic framework for Finite Perfect Information Extensive Form Games and well known solution concepts. We restrict attention to finite extensive form games with perfect recall and no chance moves. The formal treatment of an extensive form game is standard and the reader is referred to **?** and **?**. The seasoned reader may skip to the next section.

Definition 1 (Finite Perfect Information Extensive Form Game) *A finite perfect informaiton extensive form game (hereon referred to as FPIE) consists of:*

- *A finite set of players, denoted N;*
- *A finite* directed game tree *consisting of* nodes*, denoted H, and* branches *connecting nodes;*
  - *Each node represents a state of the game. Each branch represents an action. A branch from a node h to a node h' represents the situation where an action (represented by*

the branch) taken at state h leads the game to the state h'. Each state represents the sequence of actions taken until that time instance. Hence in a game tree there can't be any loops or any nodes more than one incoming branch.

  - *There is a special node, called the starting node, which represents the beginning of the game. This node does not have any incoming branch.*
  - *Any node which does not have any outgoing branch is called a terminal node. These are the nodes which represents an end state of the game. The set of terminal nodes will be denoted with Z. Any node which is not a terminal node is an* decision node. *At decision nodes an agent must take an action.*

- *An assignment of actions to each branch.*
- *A division of the decision nodes over players. The set of nodes assigned to player i, denoted H_i, are the nodes at which player i must take action.*
- *An assignment of an outcome to each terminal node.*
- *A preference relation over the set of outcomes, hence over the terminal nodes, for each agent.*

Definition 2 (Strategy) *A strategy of a player in a FPIE is a complete plan of action that specifies the action the player will take at each decision node allocated to that player*[1].

A typical strategy of a player i is usually denoted s_i, and the strategy set of a player may be denoted as S_i. A list of possible strategies for every player constructs a *strategy profile* typically denoted as s. The set of all strategy profiles of a game is denoted as S. A strategy profile s is often denoted as (s_i, s_{-i}) where s_i represents player i's strategy and s_{-i} represents all other players' strategies.

[1]Throughout the paper we will consider only pure strategies.

Since we are only considering pure strategies it is enough that we assume ordinal preferences over the terminal nodes. For notational convenience we will use *utility functions*. For any given strategy profile $s \in S$ and any player $i \in N$, a utility function $u_i(s)$ will assign a real value so as to preserve the preference order of player i over all possible outcomes. Hence $u_i : Z \to \Re$ or equivalently $u_i : S \to \Re$ represents of player i's preferences.

Definition 3 (Weakly dominated strategy) *Given any FPIE, $i \in N$, a strategy $s_i \in S_i$, is said to be weakly dominated if there exists $s'_i \in S_i$ such that*

$$\forall s_{-i} \in S_{-i} \quad u_i(s'_i, s_{-i}) \geq u_i(s_i, s_{-i}),$$

and

$$\exists s'_{-i} \in S_{-i} \quad u_i(s'_i, s'_{-i}) > u_i(s_i, s'_{-i}) \ .$$

The interpretation here is that a strategy is weakly dominated if there is another strategy that always performs at least as well, and sometimes strictly better. Intuitively it is not difficult to conclude that when undominated strategies exist, we would expect that players do not play dominated strategies. It is more difficult to determine what we would expect in a game where players have no undominated strategies.

Our final goal in this section is to introduce a well known solution concept called Subgame Perfect Nash Equibrium, or just Subgame Perfect Equilibrium. We will first describe an intuitive algorithm known as *backward induction* and then formally define the equilibrium via the aid of Nash Equilibrium.

Given a FPIE, the backward induction algorithm will produce a strategy for each player. The initial step of the algorithm starts with a final decision node (there may be more than one). The player that is assigned to that decision node is assumed to choose the action that leads to the most preferred outcome (assume for now that there are no ties). Once this step is repeated for all final decision nodes, one can move on to considering decision nodes that are immediate predecessors of final decision nodes. Here we assume that players chose the action that leads to their most preferred outcome given that the other players will play according to the assumption in the first step. We proceed in this manner towards the initial node and the resulting strategy profile is the backward induction solution to the game.

Definition 4 (Nash Equilibrium) *Given a FPIE, a strategy profile $s^* \in S$ is a Nash equilibrium of the game if*

$$\forall i \in N, \forall s_i \in S_i \quad u_i(s_i^*, s_{-i}^*) \geq u_i(s_i, s_{-i}^*) \ ,$$

i.e., no agent can gain by unilaterally deviating from the strategy profile s^*.

Before we define Subgame Perfect we need to introduce the notion of a *subgame*. Although there is a slightly more general interpretation of a subgame than what we will introduce here, our definition will suffice FPIE.

For any FPIE Γ, consider any decision node h. There is a another FPIE $\Gamma(h)$ embedded in the original game that has h as it's initial node. In particular, we can think of each decision node as a subgame of the original game.

Given a FPIE Γ, a strategy profile s and a subgame $\Gamma(h)$, we can think of the *restriction of* s as a strategy profile in $\Gamma(h)$ that agrees with s.

Definition 5 (Subgame Perfect Equilibrium) *Given a FPIE Γ, a strategy profile $s^* \in S$ is a subgame perfect equilibrium of the game if the restriction of s^* is a Nash equilibrium in every subgame of Γ.*

1.2 Related Literature

In the event there is a unique Subgame Perfect Nash Equilibrium, hereon referred to as SPE, is a widely accepted solution concept for extensive form games. Indeed even when there is multiplicity of SPE, one can make a strong argument in favor, as long as these multiple equilibria have the same equilibrium path leading to the same outcome. Unfortunately for a generic extensive form game, this is not always the case. Various refinements to SPE have been introduced so as to provide more definitive answers as to what the ”equilibrium” outcome is in any given game. A comprehensive study of these refinements can be found in van Damme (1991).

A relatively simple and intuitive refinement of SPE is that players play undominated strategies in equilibrium. This has not received much attention. Perhaps the biggest reason is that this refinement, on it’s own, is not enough to guarantee uniqueness. A second reason may be that it is difficult to imagine a SPE may include a weakly dominated strategy for a player. We demonstrate that this can in fact happen with the very simple example in figure 1.1. Although (a, a) is a SPE, clearly the strategy a is a weakly dominated strategy for player 1.

This discussion outlines our motivation. If SPE can be in weakly dominated strategies is there a class of games for which we can guarantee existence of a SPE in undominated strategies? From hereon we will refer to such an equilibrium as USPE (Undominated Subgame Perfect Equilibrium). The next chapter will prove existence of USPE in finite perfect information extensive form games (FPIE).

The problem at hand is especially relevant in the context of the mechanism design literature. Despite it’s drawback in not being able to provide a unique SPE, the requirement that equilibrium strategies are not dominated is a rather crucial one. It is unreasonable to expect any rational player to play a weakly dominated strategy just because it happens to be an ”equilibrium” strategy. Such an equilibrium provides no meaningful prediction on what will be played in any given game. As such the use of such an equilibrium in the context of a constructive proof is problematic.

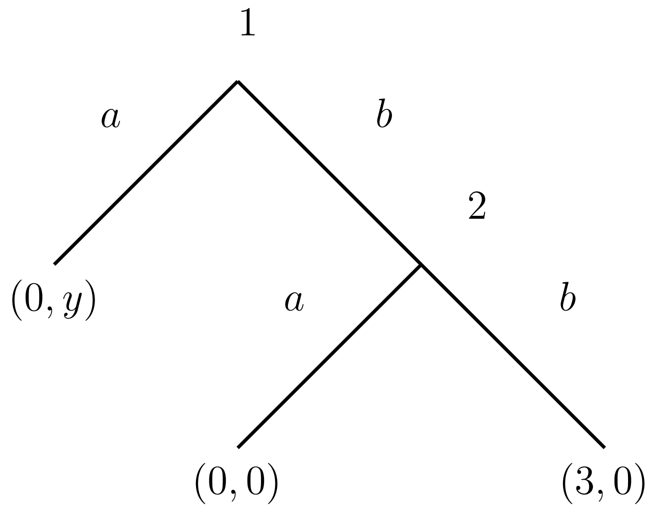

Figure 1.1: There are three SPE, namely (a, a), (b, a) and (b, b). Although (a, a) is a SPE, player 1 is prescribed a weakly strategy.

The situation of weakly dominated strategies being used in SPE becomes more problematic in a game in which all SPE are such that all players use weakly dominated strategies. One such example can be found in Brusco (2002).

To get a basic idea of why this is happening, we will take a look at what is referred in the literature as an 'integer game'. Consider a 2 player normal form game where the strategy set for each player consists of non-negative integers. In the event that at least one of the players announces zero both players receive a payoff of zero. In all other cases the player that announces the higher number is awarded a prize and the other player receives a payoff of zero. Ties are broken in favor of the first player. Observe that the only equilibrium of such a game is when both players announce zero. As such this game has only one equilibrium in which both players use dominated strategies. Observe also that in this game any strategy is weakly dominated. Hence it is not easy to predict how players will play in such a game.

This technique is widely used in the Mechanism Design literature as a tool to break undesirable equilibria. The action of announcing an integer is appended to the players' strategy sets and the payoff structure is manipulated in a way to break the undesirable equilibria. For a critique of this kind of mechanism as used in the mechanism design literature in the context of normal form games, the reader is referred to Jackson (1992).

Our goal will be to identify a class of games for which the undesirable consequence of such integer games - particularly players playing dominated strategies in all reasonable equilibria - does not exist. In the next chapter we will not conclude that FPIE is the largest class of games that guarantee existence of USPE, indeed this remains an open question. Our conjecture is that it is the finiteness and not the perfect information aspect that guarantees USPE.

Chapter 2

Existence of Undominated Subgame Perfect Equilibrium

2.1 Results

Our first results is in the form of a lemma that outlines a fact for games that have a single SPE. For FPIE we know that backward induction and SPE are equivalent. We show that for a FPIE with a single SPE, at each stage of the backward induction procedure, there is a single optimal branch. We will introduce some further notation before we proceed.

We will use the notation (h, a, s) to refer to following terminal history: h is reached followed by a, followed by s being played in the subgame Γ_h everywhere except at h where $a \in A(h)$ is played. Likewise $(h, a_1, ..., a_n, s)$ will refer to the terminal history reached by passing h, playing $a_1, ..., a_n$, and then conforming to s from thereon where $a_1 \in A(h), a_2 \in A((h, a_1))$ and so on. In the subgame starting at h denoted as Γ_h, we will use the notation (a, s) to denote the strategy profile that conforms with s everywhere except at h where a is played.

Lemma 1 *Let Γ be a FPIE. If Γ has a unique subgame perfect equilibrium s^* in pure strategies, then for each decision node h, there is a unique action $a \in A(h)$ that maximizes player $i(h)$'s payoff given s^* will be played thereafter, i.e.,*

$$|\arg\max_{a \in A(h)} u_{i(h)}(h, a, s^*)| = 1$$

Proof Assume not, i.e., there exists $h \in H$ such that there are two (or more) actions that are optimal at h given s^*_h, the restriction of s^* to subgame $\Gamma(h)$, will be played thereafter. Chose h so that either all successors of h are terminal nodes or in any strict subgame of h there is a unique optimal action given the restriction of s^* will be played thereafter. By definition $s^*(h) \in \{\arg\max_{a \in A(h)} u_{i(h)}(h, a, s^*)\}$. Let $a^0 \neq s^*(h)$ be such that $u_{i(h)}(h, s^*(h), s^*) = u_{i(h)}(h, a^0, s^*)$. Now we have that in the subgame starting at h, there are two different SPE; $(s^*(h), s^*)$ and (a^0, s^*). The former is obvious, as s^* is a SPE. The latter follows from the fact that $a^0 \in \{arg\, max_{a \in A(h)} u_{i(h)}(h, a, s^*)\}$ together with the statement of the proposition. For convenience let's refer to those two SPE as s^*_h and s^a_h respectively. Note that since they only differ at h, the appropriate restrictions are also SPE in any subgame for any history that is a successor of h. Now consider h^{-1} and the corresponding player $i(h^{-1})$. Since $A(h^{-1})$ is finite we have that $\{\arg\max_{a \in A(h^{-1})} u_{i(h^{-1})}(h^{-1}, a, s^*)\} \neq \emptyset$. Let $a^* \in \{\arg\max_{a \in A(h^{-1})} u_{i(h)}(h^{-1}, a, s^*)\}$. Likewise let $a^{-1} \in \{\arg\max_{a \in A(h^{-1})} u_{i(h)}(h^{-1}, a, s^*)\}$ Even if $a^{-1} = a^*$ the subgame at h^{-1} has two SPE. In one a^* is played at h^{-1} then s^*_h is played from thereon, in the other a^{-1} is played at h^{-1} and s^a_h is played from thereon. Reasoning in this way towards the beginning of the tree we obtain that there are multiple SPE. This contradicts the original assumption that there was a unique SPE. $\square$

Remark 1 The result does not necessarily hold when there is a unique SPE outcome as it can be supported by multiple SPE.

Remark 2 Given a FPIE game Γ with $SPE(\Gamma) = \{s^*\}$ for any nonterminal history $h \in H$, $SPE(\Gamma(h)) = \{s^*_h\}$.

Remark 3 The result holds only in finite games. Consider the game where Player 1 chooses $s_1 \in [0, 1]$, and after observing x Player 2 chooses $s_2 \in \{yes, no\}$. If (s_1, yes) is observed, then the payoffs are $(s_1, 1-s_1)$ and if (s_1, no) is observed payoffs are $(0, 0)$. In the unique SPE Player 1 plays $s_1 = 1$ and Player 2 responds yes for any s_1. However at the decision node following $s_1 = 1$ Player 2 has two optimal actions, hence Lemma 1 clearly doesn't hold. This is because once we fix the second optimal action $s_2 = no$ for Player 2, there is no optimal action for Player 1, so no such SPE is generated from there.

Next we prove existence of USPE for FPIE with a unique SPE. In other words we show that when the SPE of a FPIE Γ is unique, we have that $SPE(\Gamma) = USPE(\Gamma)$.

Proposition 1 *If Γ is a FPIE with $SPE(\Gamma) = \{s^*\}$ then s^*_i is not weakly dominated for any $i \in N$.*

Proof Suppose that for player $i \in N$ there is a strategy $s^{'}_i \neq s^*_i$ such that $s^{'}_i$ weakly dominates s^*_i. By definition we have the following

$$\exists s^{'}_{-i} \in S_{-i} \quad u_i(s^{'}_i, s^{'}_{-i}) > u_i(s^*_i, s^{'}_{-i}) \tag{2.1}$$

This implies that either $(s^{'}_i, s^{'}_{-i})$ and $(s^*_i, s^{'}_{-i})$ differ at the starting node of Γ or there is a node h reached under both $(s^{'}_i, s^{'}_{-i})$ and $(s^*_i, s^{'}_{-i})$ where their paths differ, i.e. $s^{'}(h) \neq s^*(h)$. Since in either case both strategy profiles $(s^{'}_i, s^{'}_{-i})$ and $(s^*_i, s^{'}_{-i})$ pass through h, we know that $s^{'}_i$ and s^*_i agree until h and $h \in H(s^{'}_i)$. We will refer to these facts later on in the proof. Consider the subgame $\Gamma(h)$. Exactly one the following three is true; $u_i(s^{'}_i|_h, s^*_{-i}|_h) > u_i(s^*_i|_h, s^*_{-i}|_h)$, $u_i(s^{'}_i|_h, s^*_{-i}|_h) = u_i(s^*_i|_h, s^*_{-i}|_h)$ or $u_i(s^*_i|_h, s^*_{-i}|_h) > u_i(s^{'}_i|_h, s^*_{-i}|_h)$.

The first is impossible as it implies s^* is not a Nash equilibrium in the subgame Γ_h.

Let's assume that the second is true. In the subgame $\Gamma(h)$, the paths of $(s_i^{'}|_h, s_{-i}^*|_h)$ and $(s_i^*|_h, s_{-i}^*|_h)$ differ beginning at the initial node h. When player i takes the actions $s_i^{'}(h)$ or $s_i^*(h)$ this leads the game to subgame $\Gamma(h(s_i^{'}))$ or $\Gamma(h(s_i^*))$ respectively. For notational simplicity lets refer to them as $\Gamma(h^{'})$ and $\Gamma(h^*)$ here. Let's first consider $\Gamma(h^{'})$. Notice first that $u_i(s_i^{'}|_h, s_{-i}^*|_h) = u_i(s_i^{'}|_{h'}, s_{-i}^*|_{h'})$ since they both lead to the exact same outcome. Likewise $u_i(s_i^*|_h, s_{-i}^*|_h) = u_i(s_i^*|_{h^*}, s_{-i}^*|_{h^*})$. Notice also that since $s^*|_{h'}$ is a subgame perfect equilibrium in $\Gamma(h')$, either $u_i(s_i^*|_{h'}, s_{-i}^*|_{h'}) > u_i(s_i^{'}|_{h'}, s_{-i}^*|_{h'})$ or $u_i(s_i^*|_{h'}, s_{-i}^*|_{h'}) = u_i(s_i^{'}|_{h'}, s_{-i}^*|_{h'})$ must be true. Similarly either $u_i(s_i^*|_{h^*}, s_{-i}^*|_{h^*}) > u_i(s_i^{'}|_{h^*}, s_{-i}^*|_{h^*})$ or $u_i(s_i^*|_{h^*}, s_{-i}^*|_{h^*}) = u_i(s_i^{'}|_{h^*}, s_{-i}^*|_{h^*})$. So both must hold with equality. But this contradicts Lemma 1.

Therefore the third must be true. However this is a contradiction. Remember that $h \in H(s_i^{'})$. This means that there is a strategy profile for the other players, say $\hat{s_{-i}}$ that brings the game to h when player i plays $s_i^{'}$. Consider a strategy profile for the other agents where that agrees with $\hat{s_{-i}}$ before h and agrees with s_{-i}^* elsewhere and recall that s_i^a and s_i^* agree until h. We have just established that given this strategy profile of the other players s_i^* performs strictly better than $s_i^{'}$. $\square$

Definition 6 *Given FPIE Γ, a history $h \in H$ is called an indifference node for player i with respect to s^* if $i = i(h)$, $s^* \in SPE(\Gamma)$, and there exists $a, b \in A(h)$ with $a \neq b$ such that, $\{a, b\} \subset argmax_{c \in A(h)} u_i(c, s^*)$.*

Figure 2.1 is useful in understanding the above definition.

Remark 4 An intuitive way to think of indifference nodes is to imagine the backward induction algorithm applied to a game. As long as there is a unique optimal action at every step of the algorithm, there are no indifference nodes. Keeping in mind Lemma 1, one may also think of the indifference nodes as SPE generating. In the next proposition we will see that if there are exactly two optimal actions at the only indifference node of the game, there are exactly two subgame perfect equilibria and the indifference node is with respect to both of them. When h is

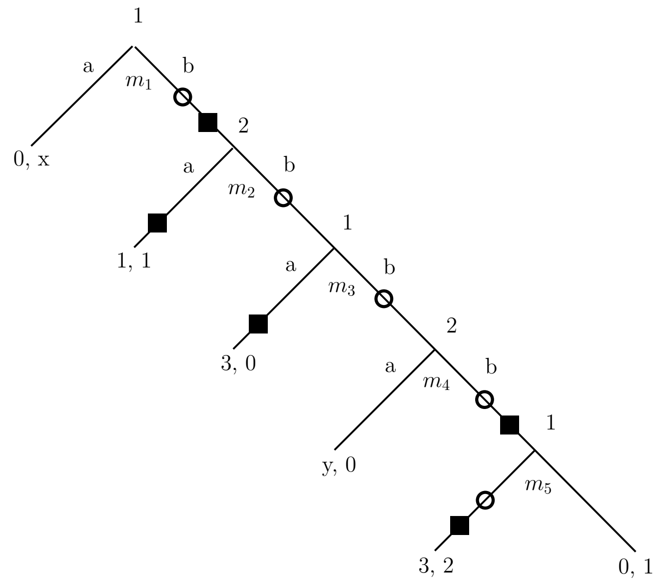

Figure 2.1: In this FPIE there are two SPE; the first prescribes the the action labelled with a circle at each decision node and the second prescribes the action labelled with a square. Although there are two optimal actions available at both m_2 and m_3 for *some* SPE, only at m_3 are there two actions optimal given a *fixed* SPE will be played from thereon.

an indifference node with respect to at least one SPE, we will simply refer to h as an indifference node.

Definition 7 *Let Γ be a FPIE, $i \in N$ and $s_i, s_i^{'} \in S_i$. The strategies s_i, $s_i^{'} \in S_i$ are equivalent for player i if for any $s_{-i} \in S_{-i}$ we have $u_i(s_i, s_{-i}) = u_i(s_i^{'}, s_{-i})$. If two strategies s_i and $s_i^{'}$ are equivalent for player i, we will denote this by $s_i \simeq s_i^{'}$.*

Proposition 2 *Let Γ be a FPIE, $i \in N$. Let $\overline{h}$ be the unique indifference node of the game with $i(\overline{h}) = i$ such that $a, b \in A(\overline{h})$ are the only optimal actions available at $\overline{h}$ given a SPE will be played from thereon. Then;*

- *The game has exactly two SPE, namely s^a and s^b with $s^a(\overline{h}) = a$ and $s^b(\overline{h}) = b$, and they agree at every successor of $\overline{h}$*
- *The history $\overline{h}$ is an indifference node with respect to s^a and s^b*
- *If $j \in N$ with $j \neq i$ then there is no strategy of player j that weakly dominates s_j^a or s_j^b*

- *If $s_i^{'}$ weakly dominates s_i^a, then $s_i^{'} |_{\overline{h}} \simeq s_i^b |_{\overline{h}}$*

Proof We will use the backward induction algorithm to find all the SPE of the game. Denote the set of decision nodes including $\overline{h}$ and it's predecessors by $\overline{H}$, all the successors of $\overline{h}$ by $\overline{H^0}$ and all other nodes by H^0. Assume that $H^0 \cup \overline{H^0} = \emptyset$. In other words there are no non terminal histories that are not in $\overline{H}$. The longest histories in such a game are the ones where $\overline{h}$ is reached and a final action is taken and furthermore there are no decision nodes that are not on the unique path of $\overline{h}$. In particular there is a complete preorder on the set of decision nodes. For such a game it is relatively simple to the backward induction procedure, especially since $\overline{h}$ is the only indifference node. Since at $\overline{h}$ the action $a \in A(\overline{h})$ is optimal, fix this action and continue to solve by backward induction. At the history $\overline{h}^{-1}$, the predecessor of $\overline{h}$, fixing the action a there will be a unique optimal action say a^{-1} as otherwise $\overline{h}^{-1}$ would have been an indifference node. Fixing the path (a^{-1}, a) at the history $\overline{h}^{-2}$, the action $a^{-2} \in A(\overline{h}^{-2})$ will be optimal and so on. Proceeding to the root of the tree in this manner, we can obtain a SPE s^a. Repeating the backward induction algorithm by instead fixing the other optimal action b at the indifference node, we arrive at the second SPE, s^b. Note that it is not necessarily true that $s^a(\overline{h}^{-k}) = s^b(\overline{h}^{-k})$.

If $H^0 \cup \overline{H^0}$ is nonempty, then pick a last decision node $h^0 \in H^0 \cup \overline{H^0}$. Since h^0 is not an indifference node, there will be a unique optimal action. Repeat the same step with all final decision nodes and move to an immediate predecessor that is also in H^0, if one exists. Continuing with the algorithm $\overline{h}$ will be reached. From here the argument in the previous paragraph applies and we have s^a and s^b as the two SPE of the game. This concludes the proof of the first part of the proposition.

Before we prove the second part, remember that we have already established while proving the first part that at in the subgame $\Gamma(\overline{h})$ there are exactly two SPE namely $s_{\overline{h}}^a$ and $s_{\overline{h}}^b$. Furthermore we know that in the subgame $\Gamma(\overline{h})$ $s_{\overline{h}}^a$ and $s_{\overline{h}}^b$ agree everywhere except at $\overline{h}$. Therefore at any subgame starting from a successor of $\overline{h}$ there is a unique equilibrium. Since at $\overline{h}$ both a and

b are optimal given the unique SPE from thereon will be played, $\overline{h}$ is an indifference node with respect to s^a_h and s^b_h.

To prove the third part, let $j \in N$ and $j \neq i$. Assume w.l.o.g. that there exists $s^{'}_j \in S_j$ such that $s^{'}_j$ weakly dominates s^a_j. By definition there exists $h \in H(s^{'}_j)$ for which $s^{'}_j(h) \neq s^a_j(h)$. Using a similar argument to that of Proposition 1, let's consider the subgame $\Gamma(h)$ and compare the values of $u_i(s^{'}_j|_h, s^a_{-j}|_h)$ and $u_i(s^a_j|_h, s^a_{-j}|_h)$. If the left hand side is greater, this contradicts the fact that s^a is a SPE. Assume the left hand side is less. Since $h \in H(s^{'}_j)$, by definition there exists a strategy profile of the other players say $s\hat{_{-j}}$ that brings the game to h when player j plays $s^{'}_j$. Now consider a strategy profile for the other players that agrees with $s\hat{_{-j}}$ on the path of h and agrees with s^a_j elsewhere and recall that s^a_j and $s^{'}_j$ agree until h. We have just established that given this strategy profile of the other players s^a_j performs strictly better than $s^{'}_j$, which leads to a contradiction. Therefore we must have $u_i(s^{'}_j|_h, s^a_{-j}|_h) = u_i(s^a_j|_h, s^a_{-j}|_h)$, yet this also leads to a contradiction. The argument is similar to that of Proposition 1. In the subgame $\Gamma(h)$ let's label the histories following $s^{'}_j(h)$ and $s^a_j(h)$ as $h^{'}$ and h^a respectively. Let's first consider $\Gamma(h^{'})$. Notice first that $u_j(s^{'}_j|_h, s^a_{-j}|_h) = u_j(s^{'}_j|_{h^{'}}, s^a_{-j}|_{h^{'}})$ since they both lead to the exact same outcome. Likewise $u_j(s^a_j|_h, s^a_{-j}|_h) = u_j(s^a_j|_{h^a}, s^a_{-j}|_{h^a})$. Notice also that since $s^a|_{h^{'}}$ is a subgame perfect equilibrium in $\Gamma(h')$, either $u_j(s^a_j|_{h'}, s^a_{-j}|_{h'}) > u_j(s^{'}_j|_{h'}, s^a_{-j}|_{h'})$ or $u_j(s^a_j|_{h'}, s^a_{-j}|_{h'}) = u_j(s^{'}_j|_{h'}, s^a_{-j}|_{h'})$ must be true. Similarly either $u_j(s^a_j|_{h^a}, s^a_{-j}|_{h^a}) > u_i(s^{'}_j|_{h^a}, s^a_{-j}|_{h^a})$ or $u_j(s^a_j|_{h^a}, s^a_{-j}|_{h^a}) = u_j(s^{'}_j|_{h^a}, s^a_{-j}|_{h^a})$. So both must hold with equality. In particular we are interested that $u_j(s^a_j|_{h'}, s^a_{-j}|_{h'}) = u_j(s^a_j|_{h^a}, s^a_{-j}|_{h^a})$ holds which implies that h is an indifference node. This is contradiction since our assumption is that Γ has a single indifference node that belongs to $i \neq j$. This concludes the third part of the proposition.

Now assume that $s^{'}_i$ weakly dominates s^a_i. By definition there exists $s^{'}_{-i} \in S_{-i}$ such that $u_i(s^{'}_i, s^{'}_{-i}) > u_i(s^a_i, s^{'}_{-i})$. Again we know that there exists $h \in H(s^{'}_i)$ where $s^{'}_i(h) \neq s^a_i(h)$. Let h be the first such history, so either $s^{'}_i$ and s^a_i differ at the starting node of Γ or the path of $(s^{'}_i, s^{'}_{-i})$ and $(s^a_i, s^{'}_{-i})$ agree until h. If $u_i(s^{'}_i|_h, s^a_{-i}|_h) > u_i(s^a_i|_h, s^a_{-i}|_h)$ this contradicts the fact that s^a

is a SPE. On the other hand if $u_i(s_i^{'}|_h, s_{-i}^a|_h) < u_i(s_i^a|_h, s_{-i}^a|_h)$ this contradicts that $s_i^{'}$ weakly dominates s_i^a. To see why, remember that $h \in H(s_i^{'})$ which means that there exists a strategy profile for the other players say $s_{-i}^{\hat{}}$ such that $(s_i^{'}, s_{-i}^{\hat{}})$ reaches h. Furthermore $s_i^{'}$ and s_i^a agree until h. So consider the strategy profile of the other agents that agrees with $s_{-i}^{\hat{}}$ until h and agrees with s_{-i}^a elsewhere. Given this strategy profile of the other players s_i^a performs strictly better than $s_i^{'}$ as implied by the inequality. This is impossible if $s_i^{'}$ weakly dominates s_i^a. Again the last remaining possibility is that $u_i(s_i^{'}|_h, s_{-i}^a|_h) = u_i(s_i^a|_h, s_{-i}^a|_h)$. For convenience let's denote the successor of h after action $s_i^{'}(h)$ as h' and the successor after $s_i^a(h)$ as h^a. From here we can use the same argument we made for player j above when proving part three and arrive at $u_i(s_i^a|_{h'}, s_{-i}^a|_{h'}) = u_i(s_i^a|_{h^a}, s_{-i}^a|_{h^a})$. This implies that h is an indifference node for player i and therefore $h = \overline{h}$ and $s_i^{'}(h) = s_i^b(h) = b$.

Notice that to prove part four, it is not necessary to show that $s_i^{'}$ and s_i^b agree at every history. It is sufficient (yet still not necessary) to show that $s_i^{'}$ and s_i^b agree at any history that can be reached via $s_i^{'}$ and some strategy profile of the other agents. This is what we will show here. We will proceed separately or histories in $\overline{H^0}$ and H^0 respectively.

First remember that at $\overline{h}$, $s_i^{'}(\overline{h}) = s_i^b(\overline{h}) \neq s_i^a(\overline{h})$. Denote the history following $\overline{h}$ and $s_i^{'}(\overline{h})$ as $\hat{h}$. Any history in $\overline{H^0} \cap H(s_i^{'})$ must belong to the subgame $\Gamma_{\hat{h}}$. Since in the subgame $\Gamma_{\hat{h}}$ there are no indifference nodes, $s_{\hat{h}}^a = s_{\hat{h}}^b$ is the unique SPE. Consider $u_i(s_i^{'}|_{\overline{h}}, s_{-i}^b|_{\overline{h}})$ and $u_i(s_i^b|_{\overline{h}}, s_{-i}^b|_{\overline{h}})$. Notice that $u_i(s_{\overline{h}}^a) = u_i(s_{\overline{h}}^b)$, $s_i^{'}$ weakly dominates s_i^a and $(s_i^{'}, s_{-i}^{'})$ and $(s_i^a, s_{-i}^{'})$ both reach $\overline{h}$. Then the left hand side must be greater than or equal to the right hand side otherwise we can arrive at a contradiction to the fact that $s_i^{'}$ weakly dominates s_i^a because we can construct a strategy profile of the other players for which s_i^a performs strictly better than $s_i^{'}$. Since $s_{\overline{h}}^b$ is a SPE, this only leaves the possibility of equality. Notice that in the subgame we are considering, there is a unique equilibrium. This means we are in the environment of Proposition 1 once again and this equality leads to the contradiction that there is an indifference node, if we have $s_i^{'}|_{\overline{h}} \neq s_i^b|_{\overline{h}}$.

Hence $s_i^{'}|_{\overline{h}}$ and $s_i^b|_{\overline{h}}$ must agree.

Any subgame originating from a history in H^0 will have a unique SPE since there are no indifference nodes in that subgame. Therefore s_i^a and s_i^b must agree at any history in H^0. At any history in $H^0 \cap H(s_i^{'})$, $s_i^{'}$ and s_i^b must also agree. To see why, assume that $\tilde{h} \in H^0 \cap H(s_i^{'})$, is a first such history with $s_i^{'}(\tilde{h}) \neq s_i^b(\tilde{h})$. Since $s_i^a(\tilde{h}) = s_i^b(\tilde{h})$, we have $s_i^a(\tilde{h}) \neq s_i^{'}(\tilde{h})$. We will once again compare the utilities $u_i(s_i^a|_{\tilde{h}}, s_{-i}^a|_{\tilde{h}})$ and $u_i(s_i^{'}|_{\tilde{h}}, s_{-i}^a|_{\tilde{h}})$. Given that $\tilde{h}$ is not an indifference equality is ruled out again. Notice that the subgame we are considering has a unique SPE, so the same arguments in Proposition 1 hold here. Also the former cannot be less as s^a is a SPE. We are left with $u_i(s_i^a|_{\tilde{h}}, s_{-i}^a|_{\tilde{h}}) > u_i(s_i^{'}|_{\tilde{h}}, s_{-i}^a|_{\tilde{h}})$. However this contradicts that $s_i^{'}$ weakly dominates s_i^a because $s_i^{'}$ and s_i^a agree at $\tilde{h}$ and all it's predecessors. Since $\tilde{h} \in H(s_i^{'})$, by definition there exists $\tilde{s_{-i}}$ such that $(s_i^{'}, \tilde{s_{-i}})$ reaches $\tilde{h}$. Note here that $(s_i^a, \tilde{s_{-i}})$ must also reach $\tilde{h}$ since we have assumed that $s_i^{'}$ and s_i^a agree at all predecessors of $\tilde{h}$. Consider the strategy of the other players s_{-i} in which players play according to $\tilde{s_{-i}}$ at all predecessors of $\tilde{h}$ and according to s_{-i}^a everywhere else. Under this strategy for the other players, s_i^a performs strictly better than $s_i^{'}$ however this is a contradiction. Therefore we conclude that for any $\tilde{h} \in H^0 \cap H(s_i^{'})$, $s_i^{'}(\tilde{h}) = s_i^b(\tilde{h})$.

We've just established that at any history that is reached when $s_i^{'}$ is played, $s_i^{'}$ and s_i^b agree. This concludes the proof of Proposition 2. $\square$

Corollary 1 *Given a FPIE with a single indifference node, the set of USPE is nonempty.*

Corollary 2 *Given a FPIE with exactly two SPE, the set of USPE is nonempty.*

Proposition 3 *Let Γ be a FPIE, $i \in N$. Let $\overline{h}$ be the unique indifference node of the game with $i(\overline{h}) = i$ such that $\{a_1, a_2, ..., a_n\} \in A(\overline{h})$ are the only optimal actions available at $\overline{h}$ given a SPE will be played from thereon. Let $k \in \{1, ..., n\}$.*

- *There are n SPE, namely $s^1, s^2, ..., s^n$ with $s^k(\overline{h}) = a^k$ and they agree at every successor of $\overline{h}$*

- *The history $\overline{h}$ is an indifference node with respect to s^k*

- *Let $j \in N$ with $j \neq i$ then for any k, there is no strategy of player j that weakly dominates s_j^k*

- *For any k, if $s_i^{'}$ weakly dominates s_i^k, then for some $m \in \{1, ..., n\}$ $s_i^{'} \mid_{\overline{h}} \simeq s_i^m \mid_{\overline{h}}$*

Proof We will follow a very similar argument to the first part of Proposition 2 here. Again we will denote the set of decision nodes that include $\overline{h}$ and it's predecessors by $\overline{H}$ and the set of all other decision nodes by H^0. If $H^0 = \emptyset$, it is relatively easy to use the backward induction algorithm. Since $\overline{h}$ is the unique indifference node, we have that $SPE(\Gamma) = \{s^1, ..., s^n\}$ with $s_i^k(\overline{h}) = a^k$ for each $k \leq n$. If H^0 is nonempty then pick a last decision node $h^0 \in H^0$. Since h^0 is not an indifference node, there will be a unique optimal action when following the algorithm. Repeat the process with a new last decision node. Continuing this way eventually $\overline{h}$ will be reached. From here the above argument applies once we reach a game that falls into the category of $H^0 = \emptyset$. This proves the first part.

Note that it follows from the first part that there are exactly n SPE namely $s^1, ..., s^n$. As in part 2 of Proposition 2, in any proper subgame of $\Gamma(\overline{h})$ there is a unique equilibrium which is the appropriate restriction of s^k to the subgame[1].

To prove the third part, let $j \in N$ and $j \neq i$. Analogously to Proposition 2 part 3, assume that there exists $s_j^{'} \in S_j$ such that $s_j^{'}$ weakly dominates s_j^k so there exists $h \in H(s_j^{'})$ for

[1]The restriction of any of the SPE will give the same strategy profile in any proper subgame of $\Gamma(\overline{h})$ since there is a unique equilibrium in any such subgame. So $\overline{h}$ is an indifference node with respect to s^k for any k $k \leq n$.

which $s_j^{'}(h) \neq s_j^k(h)$. Let h be the first such history. Once again we compare the two utilities; $u_i(s_j^{'}|_h, s_{-j}^k|_h)$ and $u_i(s_j^k|_h, s_{-j}^k|_h)$. If the left hand side is greater, this contradicts the fact that s^a is a SPE. Assume the left hand side is less. Just as in part 3 of Proposition 2 we can construct a strategy profile for the other players such that s_j^k performs strictly better than $s_j^{'}$ which is a contradiction. Therefore equality must hold. From previous arguments already made in part 3 of Proposition 2, we know that this implies that h is an indifference node, which is a contradiction since the only indifference node in this game belongs to player $i \neq j$.

To prove part 4 assume that $s_i^{'}$ weakly dominates s_i^k. Therefore there is a strategy profile for the other players $s_{-i} \in S_{-i}$ such that $u_i(s_i^{'}, s_{-i}^{'}) > u_i(s_i^k, s_{-i}^{'})$. Let h be the first history in $H(s_i^{'})$ at which $s_i^{'}(h) \neq s_i^k(h)$ and let's compare $u_i(s_i^{'}|_h, s_{-i}^k|_h)$ and $u_i(s_i^k|_h, s_{-i}^k|_h)$. The left hand side cannot be strictly greater as this contradicts that s^k is a SPE. It cannot be strictly less, since we can construct a strategy profile for the other agents that arrives at h then agrees with s_{-i}^k for which s_i^k performs strictly better than $s_k^{'}$ which contradicts that $s_k^{'}$ weakly dominates s_i^k. Therefore equality must hold. Let's denote the history after $(h, s_i^{'})$ and (h, s_i^k) as $h^{'}$ and h^k respectively. Using an analogous argument as in part 3 made for player j and later used again in part 4 of Proposition 2, we can arrive at the equality $u_i(s_i^k|_{h^{'}}, s_{-i}^k|_{h^{'}}) = u_i(s_i^k|_{h^k}, s_{-i}^k|_{h^k})$, which implies that h is an indifference node therefore $h = \overline{h}$. This is only possible if for some $m \in \{1, ..., n\} \backslash \{k\}$ we have $s_i^{'}(h) = s_i^m(h)$.

Now we will show that $s_i^{'}$ and s_i^m agree at any history that can be reached by $s_i^{'}$ and some strategy profile of the other players. We know that $s_i^{'}(\overline{h}) = s_i^m(\overline{h}) \neq s_i^k(\overline{h})$. Let's denote the history $(\overline{h}, s_i^{'}(\overline{h}))$ as h^m. First we deal with histories in $\overline{H}$. Any history in $\overline{H} \backslash \overline{h} \cap H(s^{'})$ must belong to the subgame Γ_{h^m}. Since in that subgame there are no indifference nodes, $s_{h^m}^m = s_{h^m}^k$.

Now we will compare $u_i(s_i^{'}|_{\overline{h}}, s_{-i}^m|_{\overline{h}})$ and $u_i(s_i^m|_{\overline{h}}, s_{-i}^m|_{\overline{h}})$. Notice that $u_i(s_{\overline{h}}^m) = u_i(s_{\overline{h}}^k)$, $s_i^{'}$ weakly dominates s_i^k, and $(s_i^{'}, s_{-i}^{'})$ and $(s_i^k, s_{-i}^{'})$ both reach $\overline{h}$. Then of the two utilities we are comparing the left hand side cannot be smaller. If it was, then we could construct a strategy profile for the other players for which s_i^k performs strictly better than $s_i^{'}$, which is a contradiction. Given that $s_{\overline{h}}^m$ is a SPE, this leaves equality as the last option so $u_i(s_i^{'}|_{\overline{h}}, s_{-i}^m|_{\overline{h}}) = u_i(s_i^m|_{\overline{h}}, s_{-i}^m|_{\overline{h}})$.

When comparing $u_i(s_i^{'}|_{\overline{h}}, s_{-i}^m|_{\overline{h}})$ and $u_i(s_i^m|_{\overline{h}}, s_{-i}^m|_{\overline{h}})$ as we did above, remember that at $\overline{h}$ $s_i^{'}$ and s_i^m agree. Therefore we also have that $u_i(s_i^{'}|_{h^m}, s_{-i}^m|_{h^m}) = u_i(s_i^m|_{h^m}, s_{-i}^m|_{h^m})$. Remember that Γ_{h^m} has no indifference nodes which means that we are once again in the environment of Proposition 1. If we have $s_i^{'}|_{h^m} \neq s_i^b|_{h^m}$ this would imply that there is an indifference node, ence $s_i^{'}|_{h^m} = s_i^b|_{h^m}$.

Next we deal with histories in $H^0 = H \backslash \overline{H}$. Any subgame in H^0 will have a unique SPE therefore s_i^k and s_i^m agree at any such history. At any history in $H^0 \cap H(s_i^{'})$, $s_i^{'}$ and s_i^m must also agree. To show this fact, assume $\tilde{h} \in H^0 \cap H(s_i^{'})$, $s_i^{'}$ is the first such history for which $s_i^{'}(\tilde{h}) \neq s_i^m(\tilde{h})$. Compare the utilities $u_i(s_i^{'}|_{\overline{h}}, s_{-i}^k|_{\overline{h}})$ and $u_i(s_i^k|_{\overline{h}}, s_{-i}^k|_{\overline{h}})$. Once again we are in the environment of Proposition 1 and we use the by now familiar arguments. The right hand side cannot be less since s^k is a SPE. It also cannot be greater, as we can easily construct a strategy profile for the other players and arrive at a contradiction to the assumption that $s_i^{'}$ weakly dominates s_i^k. Equality is the only remaining possibility but we know that there are no indifference nodes in this part of the game so this is also not possible. We conclude that the assumption that $s_i^{'}(\tilde{h}) \neq s_i^m(\tilde{h})$ must then be false and we therefore have $s_i^{'}(\tilde{h}) = s_i^m(\tilde{h})$ and of course they also agree at any other history in $H^0 \cap H(s_i^{'})$. This concludes the proof of part 4. $\square$

To generalize the result to all FPIE, we will make use of a variation of the well known backward induction algorithm that we will refer to as the *undominated backward induction algorithm*. The mentioned variation of the algorithm can produce multiple results. The algorithm only brings a restriction on the backward induction algorithm at steps where an indifference node is reached. We start the backward induction algorithm by choosing a final decision node. If such a node has a unique optimal action, we assign that to the strategy profile s^* we will construct using the algorithm. If there are multiple optimal actions at a last decision node, we assign one randomly to s^* and continue to the next step of the algorithm moving up the tree as in backward induction. At any step in the algorithm for any history h for which there are multiple optimal actions available for player $i = i(h)$, for each optimal action a, we will be concerned with a set of utilities $U_i(a(h)) = \{u_i \in \mathbb{R} : u_i = u_i(a, s_h), s_h \in S(\Gamma(h))\}$ which are precisely all possible utilities for player i after the action a has been taken at h. If there is a weak domination relationship (defined analogously to the regular weak dominance concept for strategies) then we choose the action associated with the undominated set and assign it to s^*. Such an undominated set exists since there is always only a finite number of sets to compare. The formal definition for set domination is given below. We show below that any strategy profile s^* that is obtained form the algorithm is an USPE.

Definition 8 *Let $A, B \in \mathbb{R}$. Set A weakly dominates set B if for all $(x, y) \in A \times B$ we have $x \geq y$ and there exists $(x, y) \in A \times B$ such that $x > y$.*

Proposition 4 *Let Γ be a FPIE and s^* a strategy profile obtained from the undominated backward induction algorithm. Then $s^* \in USPE(\Gamma)$.*

Proof Assume to the contrary that s_i' weakly dominates s_i^*. We follow a similar argument to that of Proposition 1.

$$\exists s_{-i}^{'} \in S_{-i} \quad u_i(s_i^{'}, s_{-i}^{'}) > u_i(s_i^*, s_{-i}^{'}) \tag{2.2}$$

This implies that either $(s_i^{'}, s_{-i}^{'})$ and $(s_i^*, s_{-i}^{'})$ differ at the starting node of Γ or there is a node h reached under both $(s_i^{'}, s_{-i}^{'})$ and $(s_i^*, s_{-i}^{'})$ where their paths differ, i.e. $s^{'}(h) \neq s^*(h)$. Since in either case both strategy profiles $(s_i^{'}, s_{-i}^{'})$ and $(s_i^*, s_{-i}^{'})$ pass through h, we know that $s_i^{'}$ and s_i^* agree until h and $h \in H(s_i^{'})$. We will refer to these facts later on in the proof. Consider the subgame $\Gamma(h)$. Exactly one the following three is true; $u_i(s_i^{'}|_h, s_{-i}^*|_h) > u_i(s_i^*|_h, s_{-i}^*|_h)$,

$u_i(s_i^{'}|_h, s_{-i}^*|_h) = u_i(s_i^*|_h, s_{-i}^*|_h)$ or $u_i(s_i^*|_h, s_{-i}^*|_h) > u_i(s_i^{'}|_h, s_{-i}^*|_h)$.

The first is impossible as it implies s^* is not a Nash equilibrium in the subgame Γ_h.

Let's assume that the second is true. (And this is where the proof differs from Proposition 1 towards the end of this paragraph). In the subgame $\Gamma(h)$, the paths of $(s_i^{'}|_h, s_{-i}^*|_h)$ and $(s_i^*|_h, s_{-i}^*|_h)$ differ beginning at the initial node h. When player i takes the actions $s_i^{'}(h)$ or $s_i^*(h)$ this leads the game to subgame $\Gamma(h(s_i^{'}))$ or $\Gamma(h(s_i^*))$ respectively. For notational simplicity lets refer to them as $\Gamma(h^{'})$ and $\Gamma(h^*)$ here. Let's first consider $\Gamma(h^{'})$. Notice first that $u_i(s_i^{'}|_h, s_{-i}^*|_h) = u_i(s_i^{'}|_{h'}, s_{-i}^*|_{h'})$ since they both lead to the exact same outcome. Likewise $u_i(s_i^*|_h, s_{-i}^*|_h) = u_i(s_i^*|_{h^*}, s_{-i}^*|_{h^*})$. Notice also that since $s^*|_{h'}$ is a subgame perfect equilibrium in $\Gamma(h')$, either $u_i(s_i^*|_{h'}, s_{-i}^*|_{h'}) > u_i(s_i^{'}|_{h'}, s_{-i}^*|_{h'})$ or $u_i(s_i^*|_{h'}, s_{-i}^*|_{h'}) = u_i(s_i^{'}|_{h'}, s_{-i}^*|_{h'})$ must be true. Similarly either $u_i(s_i^*|_{h^*}, s_{-i}^*|_{h^*}) > u_i(s_i^{'}|_{h^*}, s_{-i}^*|_{h^*})$ or $u_i(s_i^*|_{h^*}, s_{-i}^*|_{h^*}) = u_i(s_i^{'}|_{h^*}, s_{-i}^*|_{h^*})$. So both must hold with equality. This establishes that h is an indifference node with respect to s^* where $s_I'(h)$ and $s_i^*(h)$ are two optimal actions. This along with the assumption that s_i' weakly dominates s_i^*, and the construction of s^* leads us to a contradiction. If the former statements were true, the undominated backward induction algorithm, by definition, would not have assigned $s_i^*(h)$ at the indifference node h.

Therefore the third must be true. However this is a contradiction. Remember that $h \in H(s_i^{'})$. This means that there is a strategy profile for the other players, say $s\hat{_{-i}}$ that brings the game to h when player i plays $s_i^{'}$. Consider a strategy profile for the other agents where that agrees with $s\hat{_{-i}}$ before h and agrees with s_{-i}^* elsewhere and recall that s_i^a and s_i^* agree until h. We have just established that given this strategy profile of the other players s_i^* performs strictly better than $s_i^{'}$. $\square$

Theorem 1 *For any FPIE game USPE is nonempty.*

Theorem 1 is a corollary of Proposition 4.

2.2 Conclusions and Future Research

This thesis brings to the readers attention that in extensive form games we may observe that equilibria are in dominated strategies. Such a phenomenon severely diminishes the stability of such 'equilibria'. This is of particular concern if the game in question is one constructed as a mechanism to implement a particular outcome. While on the surface such a mechanism observes implementability by mere standard of technical definition, with respect to the 'equilibrium', such a conclusion is questionable in practice if players are expected to play dominated strategies.

Further we establish that in the context of finite perfect information extensive form games subgame perfect equilibrium in which no players play dominated strategies exists. We conjecture that the class of games for which we can ensure existence might include finite extensive form games with imperfect information. As such it is our suggession that especially in the context of an otherwise finite environment, the mechanism designer restricts consideration to finite mechanisms. Although such a reduction may very well reduce the set of implementable social choice functions it brings some clarification to what can indeed be implemented by rational players and what can not.

Bibliography

P. Battigalli. On Rationalizability in Extensive Games* 1. *Journal of Economic Theory*, 74(1): 40–61, 1997. ISSN 0022-0531.

S. Brusco. Unique Implementation of Action Profiles: Necessary and Sufficient Conditions*. *International Economic Review*, 43(2):509–532, 2002. ISSN 1468-2354.

J. Duggan. A note on backward induction, iterative elimination of weakly dominated strategies, and voting in binary agendas. *Manuscript, University of Rochester*, 2003.

P. Hummel. Iterative elimination of weakly dominated strategies in binary voting agendas with sequential voting. *Social Choice and Welfare*, 31(2):257–269, 2008. ISSN 0176-1714.

M.O. Jackson. Implementation in undominated strategies: A look at bounded mechanisms. *The Review of Economic Studies*, 59(4):757–775, 1992. ISSN 0034-6527.

H. Kuhn. Extensive games and the problem of information. *Annals of Mathematics Studies*, 28: 193–216, 1953.

G.J. Mailath, L. Samuelson, and J.M. Swinkels. Extensive form reasoning in normal form games. *Econometrica: Journal of the Econometric Society*, 61(2):273–302, 1993. ISSN 0012-9682.

E. Maskin. Nash equilibrium and welfare optimality. *Review of Economic Studies*, 66(1):23–38, 1999. ISSN 1467-937X.

H. Moulin. Dominance solvable voting schemes. *Econometrica: Journal of the Econometric Society*, 47(6):1337–1351, 1979. ISSN 0012-9682.

E. Van Damme. *Stability and perfection of Nash equilibria*. Springer, 1991. ISBN 3540538003.