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SOLUTION SPACES AND CURRENT REGIMES IN TRANSFINITE NETWORKS

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Abstract — The unique current regime in a transfinite network under the finite-power requirement can be found by searching a solution space for the current regime that satisfies a generalized form of Tellegen's equation. However, there are four different spaces that may be searched, and the current regime will depend in general on which space is chosen. This is shown by examples. Furthermore, it is proven herein that node voltages will be unique whenever they exist under two of the solution spaces; this is not so for the other two spaces.

1 Introduction

In a finite resistive network with branches indexed from 1 to n, any vector $\mathbf{i} = (i_1, \ldots, i_n)$ of branch currents satisfying Kirchhoff's current law can be decomposed into a sum of loop currents. To find the unique current regime for the network excited by a given set of sources, we search within the span of all loop currents to find that member of the span whereby Ohm's law and both Kirchhoff laws are fulfilled. This is the essence of a mesh analysis. On the other hand, for an infinite resistive network any current vector has infinitely many branch-current components, and moreover there are several different spaces in which one may search for the solution current vector $\mathbf{i} = \{i_j\}_{j \in J}$, where J is the branch index set. In fact, $\mathbf{i}$ will depend in general upon which solution space is chosen. These several degrees of freedom arise from the fact that infinite networks are mathematical abstractions, which can be constructed in several different ways. Moreover, Kirchhoff's laws fail in general, and the search has to be based upon a more fundamental principle. A generalized form of Tellegen's equation serves this purpose whenever a finite-power regime is required.

So, why deal with infinite networks at all? One reason is that they have a variety of practical applications wherein they are advantageous as compared to finite networks [1, Chapters 1 and 8]. More importantly, their theoretical development creates new knowledge, which in time and in conjunction with other theoretical advances will assuredly feed back practical applications in the

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long run. Basic theoretical research would be crippled if at every step a practical application was demanded. We submit this brief as a contribution to the fundamental theory of electrical circuits.

Our purpose is to point out four different spaces in which one may search for $\mathbf{i}$. Each will yield a different $\mathbf{i}$ for suitably selected networks. Space limitations for a Transactions Brief prevent us from presenting all the definitions and results used herein from the theory of infinite networks. These are all available in either [1], [2], or [3]; however, enough examples are given to provide an understanding of our ideas without recourse to those sources. The "fundamental theorem" we shall be referring to below is stated by [1, Theorem 3.3-5], [2, Theorem 10.2], and [3, Theorem 5.2-8]. The second space $\mathcal{K}$ discussed below has been defined and explained in prior works. The first, third and fourth spaces $\mathcal{L}$, $\mathcal{T}$, and $\mathcal{S}$ are introduced here for the first time. It has been previously established that under $\mathcal{K}$ and (therefore under $\mathcal{L}$) node voltages need not be unique even when they exist [3, Section 5.5]. We show herein that under $\mathcal{T}$ and $\mathcal{S}$ node voltages will always be unique whenever the exist.

Consider a transfinite electrical network $\mathbf{N}$ having countably or uncounbtably many branches with each branch being in the Thevenin form of a positive resistor r_j in series with a voltage source of any real value e_j. Regarding notation, all summations will be over the branch index set J, unless otherwise indicated; thus, $\sum$ will mean $\sum_{j\in J}$. (See [3, Appendix B] for an explanation of series with uncountably many terms.) We assume that $\sum e_j^2 r_j^{-1} < \infty$ in order to ensure a finite-power regime as dictated by the fundamental theorem. $\mathcal{I}$ will denote the set of all branch current vectors $\mathbf{i}$ such that $\sum i_j^2 r_j < \infty$; that is, each $\mathbf{i} \in \mathcal{I}$ dissipates only a finite amount of power but is otherwise unrestricted. For instance, $\mathbf{i} \in \mathcal{I}$ need not satisfy Kirchhoff's current law, nor need the branch voltages corresponding to $\mathbf{i}$ satisfy Kirchhoff's voltage law. $\mathcal{I}$ becomes a Hilbert space when the norm of each current vector is taken to be the square root of the dissipated power.

2 Loop Currents

Let $\mathcal{L}^o$ be the set of all loop currents in $\mathcal{I}$; the loops for those currents may be either finite or transfinite. Let $\mathcal{L}$ be the closure of $\mathcal{L}^o$ in $\mathcal{L}$. $\mathcal{L}$ is a Hilbert space. Exactly the same proof as that for the fundamental theorem shows that there is a unique current vector $\mathbf{i} \in \mathcal{L}$ satisfying a generalized form of Tellegen's equation.

As an example, consider the infinite ladder of Figure 1 having a branch b_0 connected through 1-nodes to its infinite extremities (i.e., to the two 0-tips) corresponding to the two one-way infinite paths along the upper and lower horizontal branches. Sources may occur in any branches. Assume for the moment that every branch resistance is 1 Ω. Then, $\mathcal{L}^o$ consists of loop currents in finite loops only. $\mathcal{L}^o$ is void of any transfinite loop current passing through b_0 because any such current would dissipate infinite power. Therefore, the unique solution $\mathbf{i}$ has 0 A in b_0. On the other hand, if the ladder's resistance values vary geometrically as indicated in Figure 1, then every transfinite loop current will be in $\mathcal{L}^o$, and $\mathbf{i}$ will in general have a nonzero component in b_0.

3 Basic Currents

Depending upon the choice of the network $\mathbf{N}$, $\mathcal{L}$ can in general be expanded by adding *basic currents*; these are precisely defined in the above cited works. They can be intuitively explained as an infinite superposition of loop currents that together permit a spreading and thinning out of a current regime as it flows toward infinity from some node of injection. This might allow such a current regime to dissipate finite power, even when an infinite power dissipation would result were that injected current to flow out to infinity along a single path.

An example of this is shown by the binary tree of Figure 2, wherein a current of 1 A flows from the apex node n^0, spreads out evenly through the tree, and then is gathered through a short at infinity and returned through a source branch to the apex node. When every branch of the binary tree is a 1 Ω resistor, the total power dissipated is finite; were that 1 A current to flow along a single path from n^0 to infinity, the power dissipation would be infinite. The current regime shown in Figure 2 is a basic current and is the one dictated by the fundamental theorem.

For any network, the span of all basic currents in $\mathcal{I}$ is denoted by $\mathcal{K}^0$, and the closure of $\mathcal{K}^0$ in $\mathcal{I}$ is denoted by $\mathcal{K}$. We always have $\mathcal{L} \subset \mathcal{K}$, and in general $\mathcal{K}$ is larger than $\mathcal{L}$, as is the case for Figure 2. In fact, for that figure, $\mathcal{L}$ is void but $\mathcal{K}$ is not void, and the corresponding fundamental theorems yield $\mathbf{i} = \mathbf{0}$ under $\mathcal{L}$ and $\mathbf{i} \neq \mathbf{0}$ under $\mathcal{K}$.

4 Tour Currents

Recall that $\mathcal{L}$ and $\mathcal{K}$ are constructed out of loop currents and that any loop is a tracing through the network that starts and ends at the same node but otherwise never repeats a node. On the other hand, let us define a *track* as a finite sequence $\{P_1, P_2, \ldots, P_m\}$ of oriented two-ended paths such that, for each $k = 1, \ldots, m-1$, the last node of P_k embraces or is embraced by the first node of P_{k+1}. It is not required that the these paths be disjoint. Thus, a track may pass through a node or branch several times — but at most finitely many times. Next, let us define a *tour* to be a track such that the last node of P_m embraces or is embraced by the first node of P_1. Thus, a tour generalizes a loop. Finally, we take a *tour current* to be a constant flow f of current passing along a tour. Thus, if a tour repeats a branch, the corresponding branch current is a multiple of f obtained by adding and/or subtracting f for each passage of f through the oriented branch, addition being used when the flow and branch orientation agree, subtraction otherwise. In fact, the branch current may be 0 by cancellation. Note that a loop current is a special case of a tour current. Figure 3 illustrates a tour current along a transfinite tour. This is the simplest network that must sustain a nonloop tour current if a certain branch is to have a nonzero current, and it is a ubiquitous one because it occurs as a subnetwork in many more complicated networks.

Let $\mathcal{T}^o$ be the span of all tour currents in $\mathcal{I}$, and let $\mathcal{T}$ be the closure of $\mathcal{T}^o$ in $\mathcal{I}$. Then, exactly the same proof as that for the fundamental theorem based on $\mathcal{K}$ establishes a fundamental theorem based on $\mathcal{T}$. Moreover, we now have that $\mathcal{L} \subset \mathcal{T}$. A branch may have a nonzero solution current under $\mathcal{T}$ but a zero solution current under either $\mathcal{L}$ or $\mathcal{K}$. In fact, this is the case for the example of Figure 3, (for whose network $\mathcal{L} = \mathcal{K}$ — a particular but not a general result). Indeed, the current in branch b_0 can be nonzero under $\mathcal{T}$ but must be zero under $\mathcal{L} \doteq \mathcal{K}$ whatever be the choices of the voltage sources. For that network, the voltages at the two 1-nodes are not unique under $\mathcal{L}$ or $\mathcal{K}$ [3, Section 5.5] but are unique under $\mathcal{T}$. The latter points to a general result, which is worth stating explicitly.

In the following, that a track or tour is *permissive* means that the resistances in the track or tour sum to a finite amount. Let n_g and n_0 be any two totally disjoint nodes of the transfinite network $\mathbf{N}$, and let T be a permissive track starting at n_0 and stopping at n_g. We define the *node voltage* v_0 at n_0 with respect to T to be the sum $\sum_{(T)} \pm v_j$, where $\sum_{(T)}$ denotes a sum along the branches of T with an additional term for each occurrence of a branch as T is traced from n_0 to n_g and where the + (resp. −) sign is used with v_j if the orientations of the branch b_j agrees (resp. disagrees) with that tracing of T for the considered occurrence of b_j.

Theorem. Let $\mathbf{N}$ *be a transfinite network with all branches in the Thevenin form and with* $\sum e_j^2 r_j^{-1} < \infty$. *Let the current regime in* $\mathbf{N}$ *be that dictated by the fundamental theorem based on the solution space* $\mathcal{T}$. *Let* n_g *be a chosen ground node and let* n_0 *be any other node. Let* T_1 *and* T_2 *be two permissive tracks in* $\mathbf{N}$ *starting at* n_0 *and stopping at* n_g. *Then, the node voltage assigned to* n_0 *along* T_1 *is the same as that along* T_2. *(That is, under the fundamental theorem based upon* $\mathcal{T}$, *node voltages will be unique whenever they exist.)*

Proof. Let $-T_1$ denote the track T_1 with a reversed orientation — that is, $-T_1$ starts at n_g and stops at n_0. Let $(-T_1) \cup T_2$ denote the tour consisting of $-T_1$ followed by T_2. By the fundamental theorem based upon $\mathcal{T}$, $\sum v_j s_j = 0$ for any $\mathbf{s} \in \mathcal{T}$. Upon choosing $\mathbf{s}$ as a tour current along $(-T_1) \cup T_2$, we find that $\mathbf{s} \in \mathcal{T}$ because of the permissivities of P_1 and P_2, and from this we obtain $\sum_{(T_1)} \pm v_j = \sum_{(T_2)} \pm v_j$. Q.E.D.

5 Splayed Currents

$\mathcal{T}$ can be expanded into a generally larger space $\mathcal{S}$ in much the same way as $\mathcal{L}$ is expanded into $\mathcal{K}$. For certain networks this will permit a thinning out of currents as they flow toward infinity, thereby enabling a finite-power current distribution that may not be available in $\mathcal{T}$. A splayed current is specified as is a basic current [1, page 154], [3, Section 5.2] except that loops are replaced by tours. We define a *splayed current* to be a sum $\mathbf{i} = \sum_{m \in M} \mathbf{i}_m$, where M is a (finite or infinite) index set and each $\mathbf{i}_m$ is a tour current, such that each maximal 0-node and each branch meets no more than finitely many tours of the $\mathbf{i}_m$. Thus, a tour current is a special case of a splayed current. For $\mathbf{i}$ to be in $\mathcal{I}$ it is not required that any of the $\mathbf{i}_m$ be in $\mathcal{I}$; in fact, it is possible for $\mathbf{i}_m \notin \mathcal{I}$ for all m, whereas $\mathbf{i} \in \mathcal{I}$ nonetheless. We now let $\mathcal{S}^o$ be the span of all splayed currents, and let $\mathcal{S}$ be the closure of $\mathcal{S}^o$ in $\mathcal{I}$. This time, we have $\mathcal{K} \subset \mathcal{S}$. Once again, the proof of the fundamental theorem extends to $\mathcal{S}$; just replace $\mathcal{K}^o$ by $\mathcal{S}^o$ and $\mathcal{K}$ by $\mathcal{S}$.

Figure 4 illustrates a particular splayed current $\mathbf{I} = \sum_{m=1}^{\infty} \mathbf{I}_m$ on a quarter-plane square grid along with other branches connected to certain of the grid's extremities (i.e., 0-tips). All branch resistances are 1 Ω. The top three parts of that figure indicate $\mathbf{i}_m$ for $m = 1, 2, 3, 4, 5$. The small circle labeled n_a (resp. n_b) is a 1-node that connects to the 0-tip determined by the lowest (resp. the next lowest) path of horizontal branches. The small circles labeled w_m denote 1-nodes that connect to 0-tips determined by paths that wiggle rectangularly, passing along vertical and horizontal branches. All of these 0-tips and their corresponding 1-nodes w_m are different from each other. For m even (resp. odd), there is a branch at infinity connecting n_a (resp. n_b) to w_m, which allows $\mathbf{i}_m$ to close on itself at infinity. The indicated pattern repeats itself for $m = 6, 7, 8, \ldots$ with the vertical wiggles expanding upward and with the first (i.e., leftmost) wiggle shifting to the right as m increases. The bottom part of Figure 4 shows the total splayed current $\mathbf{i}$ so far as the currents within the grid are concerned. All the branches at infinity carry nonzero currents too, but are not shown in this part. All the tour currents $\mathbf{i}_m$ in $\mathbf{i}$ cancel to zero within the grid except on the squares shown. Note that no $\mathbf{i}_m$ is a member of $\mathcal{I}$ because, by itself, it dissipates infinite power. On the other hand, $\mathbf{i} = \sum \mathbf{i}_m$ is in $\mathcal{I}$. This provides an example of how a splayed current in $\mathcal{I}$ can be a superposition of tour currents not in $\mathcal{I}$. Note also that with respect to $\mathbf{i}$ each current at infinity passes through its branch but goes no further. This is yet another example of how Kirchhoff's current law can be violated in infinite networks.

The following always holds: $\mathcal{L} \subset \mathcal{K} \subset \mathcal{S}$ and $\mathcal{L} \subset \mathcal{T} \subset \mathcal{S}$; strict inclusion will occur for certain networks. As with $\mathcal{T}$ — but in contrast to $\mathcal{L}$ and $\mathcal{K}$, we have the following:

Corollary. *Under the fundamental theorem based on $\mathcal{S}$, node voltages will be unique whenever they exist.*

References

[1] A.H. Zemanian, *Infinite Electrical Networks*, Cambridge University Press, New York, 1991.

[2] A.H. Zemanian, Transfinite graphs and electrical networks, *Trans. Amer. Math. Soc.*, **334** (1992), 1-36.

[3] A.H. Zemanian, *Transfiniteness for Graphs, Electrical Networks, and Random Walks*, Birkhauser, Boston, 1996.

Figure Captions

Figure 1. An infinite ladder with a branch b_0 at infinity connected to the two 0-tips of the upper and lower paths of horizontal branches. The numbers denote branch resistances.

Figure 2. An infinite binary tree fed by a single 1 V source branch that gathers current through a short (i.e., a 1-node) at infinity and feeds it back to the apex node n^0. Every branch has a 1 Ω resistor. The numbers near arrows indicated branch currents.

Figure 3. A transfinite network consisting of a series circuit of infinitely many parallel circuits and a branch b_0 at infinity connected to the two 0-tips induced by the upper and lower branches. The arrows indicate a tour current that passes through each branch once and only once.

Figure 4. The splayed current $\mathbf{i} = \sum_{m \in M} \mathbf{i}_m$ discussed in Section 5. The quarter-plane grid is shown four times to display $\mathbf{i}_m$ for $m = 1, 2, 3, 4, 5$ and also $\mathbf{i}$.

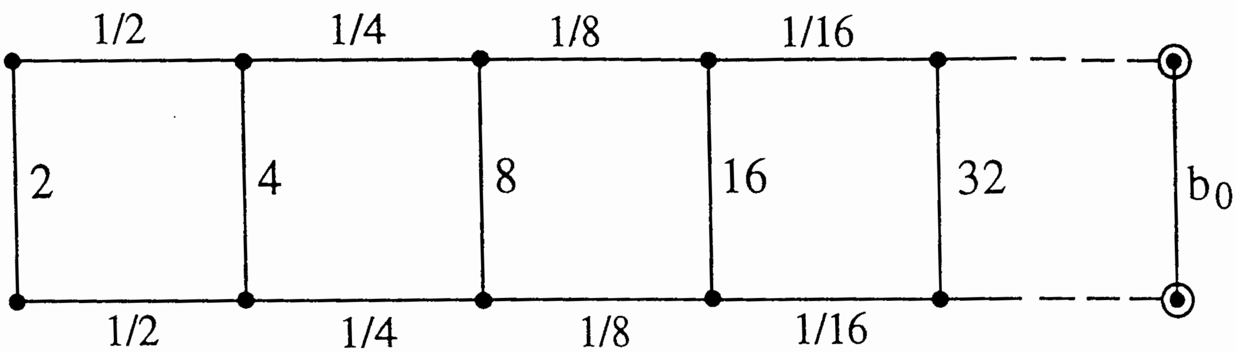

FIG. 1

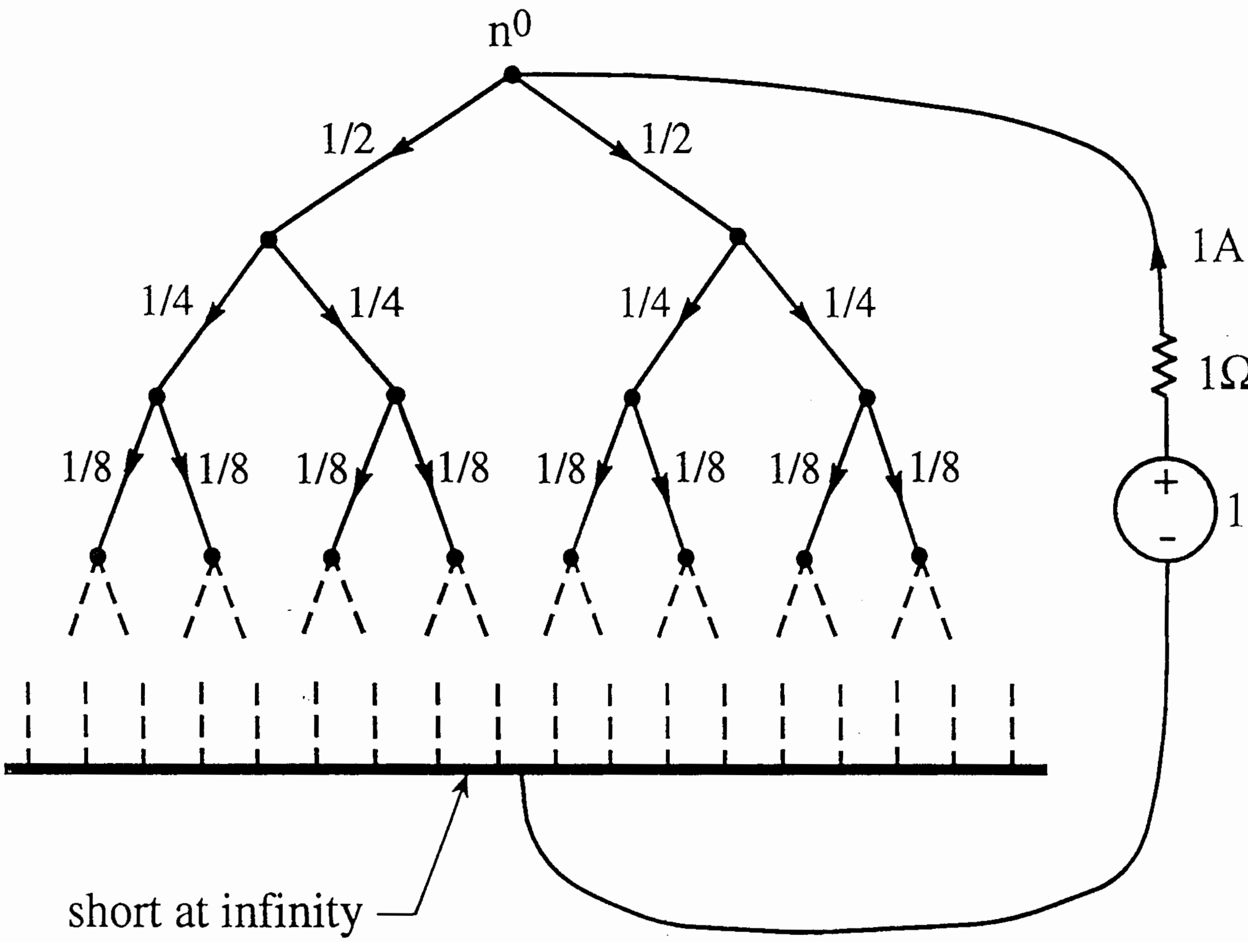
n0
1/2
1/2
1/4
1/4
1/4
1/4
1/8
1/8
1/8
1/8
1/8
1/8
1/8
1/8
1A
1Ω
+
-
1V
short at infinity

FIG. 2

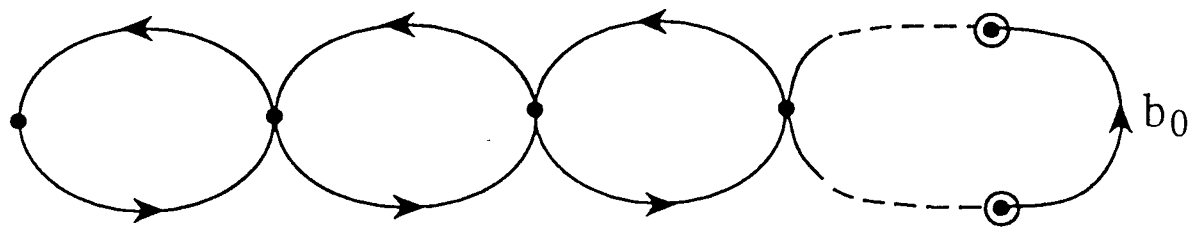

FIG. 3

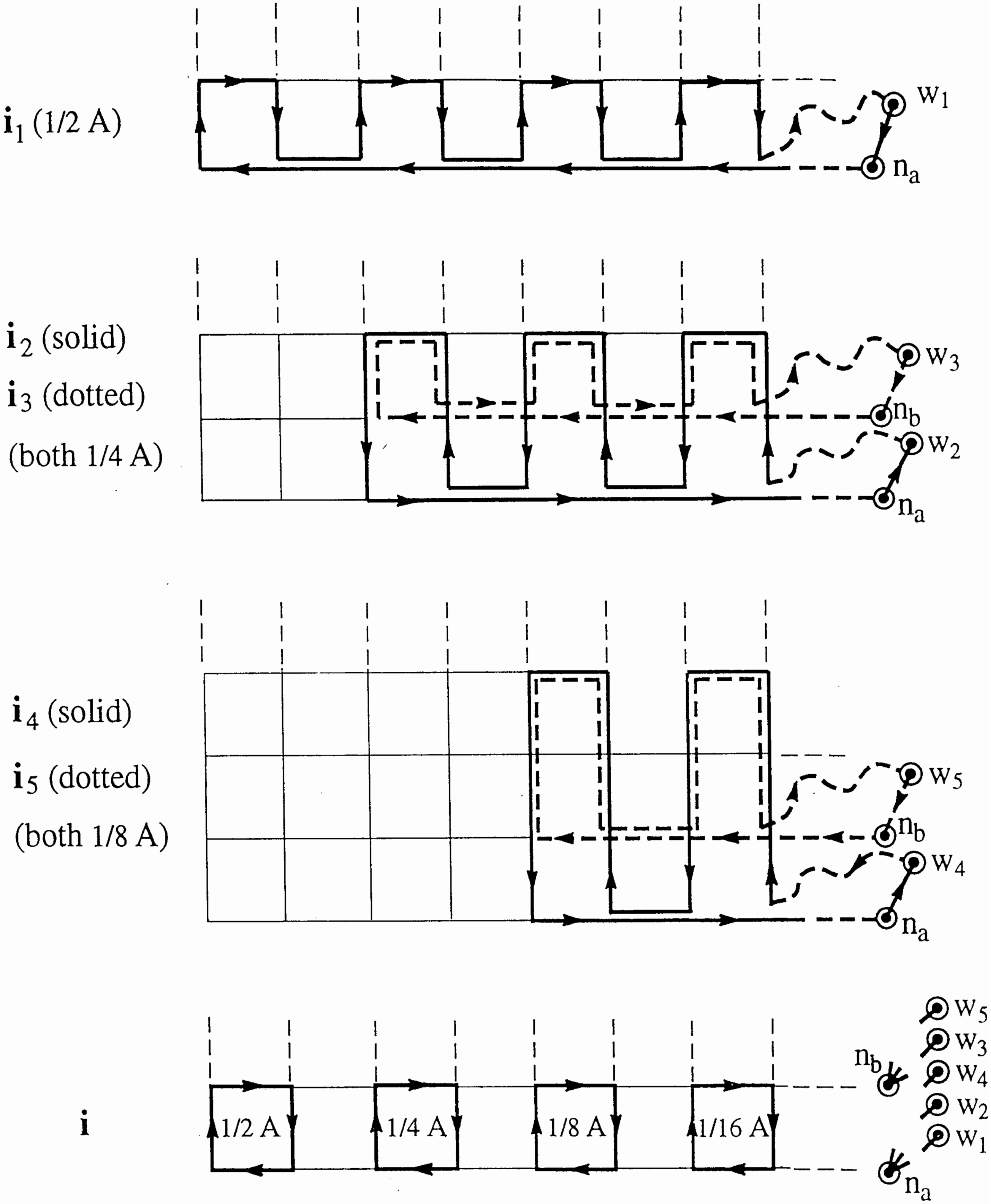
i1 (1/2 A)
w1
na
i2 (solid)
i3 (dotted)
(both 1/4 A)
w3
nb
w2
na
i4 (solid)
i5 (dotted)
(both 1/8 A)
w5
nb
w4
na
i
1/2 A
1/4 A
1/8 A
1/16 A
w5
w3
nb
w4
w2
w1
na

FIG. 4